On an Integral Equation Model for Slender Bodies in Low Reynolds-Number Flows

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Abstract

The interaction of particular slender bodies with low Reynolds-number flows is in the limit "slenderness to 0" described by the linear Fredholm integral equation of the second kind

$$c\,\phi(s) = F(s) + \int_0^1 \frac{\phi(t)-\phi(s)}{|t-s|}dt, \quad s \in [0,1],$$

where c is a real number, F is a continuous function and ϕ is unknown. The integral operator T of this equation has a denumerable set of polynomial eigenfunctions whose corresponding eigenvalues are non-positive and of logarithmic growth. A theorem similiar to a classical result of Plemelj-Privalov for integral operators with Cauchy kernels is proven. In contrast to Cauchy kernel operators, T maps no α-Hölder space into itself. A spectral analysis of an operator $\hat{T}$ (the restriction of T to an appropriate class of analytic functions) is performed. The spectral properties of $\hat{T}$ suggest a collocation-like method to solve the integral equation numerically. For this numerical scheme, convergence is proven and several computations are presented. The numerical computations exhibit the stability of the proposed algorithm.

1 Introduction

The starting point of the subsequent investigations is a model for the shape of a long, slender body (e.g. a fiber) exposed to a normal flow [1]. Under several assumptions (low Reynolds number of the fluid which approaches a constant velocity as the spatial variable tends to infinity, the fiber lies in a plane, has a circular cross-section, the fiber's curvature is $O(1)$ and the center-line of the fiber does not re-approach itself) one deduces from available theoretical results [2, 3, 4, 5, 6, 7, 8, 9, 10] a Fredholm integral equation model for the fiber [1, 9]:

$$c\,\phi(s) = F(s) + \int_0^1 \frac{\phi(t)-\phi(s)}{|t-s|}dt, \quad s \in [0,1]. \tag{1.1}$$

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Here $s \in [0,1]$ is the spatial variable of the one-dimensional fiber. $c \in \mathbb{R}$ is a negative constant which depends on the fiber's geometry. (Loosely speaking, the smaller the radius of the real fiber is, the more negative c becomes, e.g. if the real fiber is an ellipsoid whose s-dependent radius $r(s)$ equals $\varepsilon\sqrt{s(1-s)}$, $\varepsilon \in \mathbb{R}$, then $c = 2\ln\frac{2}{\varepsilon}-1$.) In (1.1) the scaled force per unit length $\phi : [0,1] \to \mathbb{R}$ is unknown. The continuous function $F : [0,1] \to \mathbb{R}$ is the scaled difference of the fiber's velocity and the limiting velocity of the fluid.

Seemingly (1.1) has not been treated in the literature yet [11].

Heuristically, (1.1) is in-between the *Abelian integral equations*

$$c(s)\,\phi(s) + \int_0^1 \frac{\phi(t)}{|s-t|^\alpha}\,dt = F(s), \quad s \in [0,1] \tag{1.2}$$

where $c : [0,1] \to \mathbb{R}$ is continuous and $0 < \alpha < 1$, and integral equations with *Cauchy kernels*

$$c\,\phi(s) + \int_0^1 \frac{\phi(s)-\phi(t)}{s-t}\,dt = F(s), \quad s \in [0,1]. \tag{1.3}$$

The Abelian integral equation is certainly one of the best-known integral equations, see e.g. [12] and the references therein. However, the analysis of Abelian integral equations heavily relies on the integrability of the kernel $|s-t|^{-\alpha}$, $0 < \alpha < 1$. A straight-forward application of the respective theory is therefore out of sight.

The available theory for integral equations with Cauchy kernels $(s-t)^{-1}$ is settled on complex methods which rely on the fact that the kernel of (1.3) allows for an meromorphic extension to the complex plane, see e.g. [13]. However, in our case the kernel is not meromorphic in $\mathbb{C}$. A straight-forward application of the theory for (1.3) is therefore not possible. (Nevertheless it turns out that several aspects of the theory of (1.3) remain valid for (1.1): A weaker version (the Hölder exponent is *not* conserved) of the theorem of Plemelj–Privalov [14] is available and - in analogy to holomorphic methods for (1.3) - several solutions of (1.1) can by represented by power series centered at 1/2.)

A frequently used procedure for integral equations of the second kind is the *method of successive approximations.* In order to solve an integral equation

$$\phi = T[\phi] + F, \tag{1.4}$$

where T is a linear integral operator, one considers for $n \in \mathbb{N}_0$,

$$\phi_{n+1} = T[\phi_n] + F \qquad \text{with } \phi_0 \text{ suitable.} \tag{1.5}$$

If the operator T is bounded with spectral radius less than 1, then the sequence $(\phi_n)_{n\in\mathbb{N}}$ will converge to the unique solution of (1.4) for any initial value ϕ_0. However, this method *cannot* be applied to (1.1) - as it will be shown later on, the integral operator of (1.1) is unbounded on reasonable normed spaces.

As a summary, a theory for (1.1) is not available yet. It is one of the purposes of the present paper to perform that analysis.

The paper is organized as follows. In Section 2 we introduce an integral operator T whose domain is a subset of the space of all continuous functions mapping $[0,1]$ into $\mathbb{R}$. Employing T we re-write (1.1) as operator equation $c\phi = F + T(\phi)$. In Section 2.1 it is shown that T has denumerable many polynomial eigenvectors with non-positive eigenvalues

$$-L_k := -2\sum_{l=1}^{k}\frac{1}{l}, \quad k \in \mathbb{N}_0 .$$

Hence T is not bounded on any reasonable infinite dimensional subspace of its domain. In Section 2.2 the operator T is restricted to spaces of α-Hölder continuous functions. An analogon of a classical result of Plemelj–Privalov is proven. However T maps no space of α-Hölder continuous functions into itself. This result exhibits, on the one hand, an important difference between (1.1) and the integral equations with Cauchy kernels.

On the other hand we conclude that an analysis of (1.1) (and a numerical treatment as well) shall rather not be settled on Hölder spaces but on spaces on which the spectral properties of T are dominated by the polynomial eigenvectors. From a theoretical point of view, a function space consisting of functions possessing expansions in terms of the polynomial eigenfunctions of T would be a rather natural choice. However, from a numerical point of view the polynomial eigenfunctions are with increasing degree less and less tractable. The investigations are therefore settled on another function space which is introduced in Section 3: $\mathcal{L}^\infty$ is the space of all analytic function whose Taylor expansions at 1/2 has uniformly bounded coefficients. The operator $\hat{T}$ - which is the restriction of T to $\mathcal{L}^\infty$ - maps $\mathcal{L}^\infty$ into $\mathcal{L}^\infty_w$ which - loosely speaking - consists of all analytic functions whose Taylor expansions at 1/2 grow at most logarithmically. A rather satisfactoring spectral analysis of $\hat{T}$ is available: The spectrum of $\hat{T}$ is a pure point spectrum $\sigma_p(\hat{T}) = \{-L_k : k \in \mathbb{N}_0\}$. Estimates on the norms of the resolvents of $\hat{T}$ are derived.

Section 4 is concerned with the numerical treatment of (1.1). The question of well-posedness of (1.1) is discussed. It is shown that for all non-eigenvalues $c \in \mathbb{R}$ the projection of (1.1) to spaces of polynomials with degree less than $n \in \mathbb{N}$ gives a (naturally finite-dimensional) matrix equation with upper triangular structure. This matrix equation has for each $n \in \mathbb{N}$ a unique solution $\phi^{[n]} \in \mathcal{L}^\infty$. Assuming a (mild) growth condition on the Taylor coefficients of F at 1/2, the functions $\phi^{[n]}$ converge uniformly on $[0,1]$ to the unique solution of (1.1) as $n \to \infty$. Numerical computations are presented. The investigated model problems exhibit the stability of the proposed numerical approximation even in cases where the theoretical results of Section 3 do not apply.

The proofs of the results are deferred to the Appendix.

2 The Integral Operator T

We shall make use of the following notations. C is the (real) vector space of all continuous functions $f : [0,1] \to \mathbb{R}$. C is equipped with its canonical norm,

$$\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}, \quad f \in \mathsf{C}.$$

For $\alpha \in (0,1]$ we denote by $\mathsf{C}^{0,\alpha}$ the (real) vector space of all α-Hölder continuous functions $g : [0,1] \to \mathbb{R}$. We equip $\mathsf{C}^{0,\alpha}$ with its canonical norm,

$$\|g\|_{0,\alpha} = \|g\|_\infty + \sup\left\{\frac{|g(s)-g(t)|}{|s-t|^\alpha} : 0 \le s,t \le 1, s \ne t\right\}, \quad g \in \mathsf{C}^{0,\alpha}.$$

For $N \in \mathbb{N}_0$ let

$$\mathcal{P}_N = \{p \mid [0,1] : p \in \mathbb{R}^\mathbb{R} \text{ is a polynomial of degree } \le N\}.$$

We shall also make use of the real sequence spaces

$$\ell^1 := \left\{(x_k)_{k\in\mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0} : \sum_{k=1}^\infty |x_k| < \infty\right\},$$

$$\ell^\infty := \left\{(z_k)_{k\in\mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0} : \sup\{|z_k| : k \in \mathbb{N}_0\} < \infty\right\},$$

equipped with the respective canonical norms

$$|(x_k)_{k\in\mathbb{N}_0}|_1 := \sum_{k=1}^\infty |x_k|, \quad (x_k)_{k\in\mathbb{N}_0} \in \ell^1,$$

$$|(z_k)_{k\in\mathbb{N}_0}|_\infty := \sup\{|z_k| : k \in \mathbb{N}_0\}, \quad (z_k)_{k\in\mathbb{N}_0} \in \ell^\infty.$$

Our first goal is to define properly the integral operator arising in (1.1). The physical model from which (1.1) originates requires continuous solutions. The investigations are therefore settled on subspaces of C.

The integrand of (1.1) makes it adviseful to introduce for $\psi \in \mathsf{C}$ and $s \in [0,1]$ the function

$$Q_s[\psi] : [0,1] \to \mathbb{R}, \quad Q_s[\psi](t) = \begin{cases} \dfrac{\psi(t)-\psi(s)}{|t-s|} & , t \ne s \\ 0 & , \text{ else} \end{cases}.$$

The integral of (1.1) has a well-defined meaning iff $Q_s[\psi]$ is Lebesgue-integrable over $[0,1]$, i.e.

$$T[\psi](s) := \int_0^1 Q_s[\psi](t)\, dt,$$

well-defined iff $\psi \in \mathsf{dom}$ where

$$\mathsf{dom} := \{\psi \in \mathsf{C} \mid \forall s \in [0,1] : Q_s[\psi] \text{ is Lebesgue-integrable over } [0,1]\}.$$

This observation makes it adviseful to introduce T via

$$T : \mathsf{dom} \to \mathbb{R}^{[0,1]}, \quad \left\{ \begin{array}{rcl} T[\psi] : [0,1] & \to & \mathbb{R}, \\ s & \mapsto & T[\psi](s) = \displaystyle\int_0^1 \frac{\psi(s)-\psi(t)}{|s-t|}\, dt \end{array} \right. .$$

dom is certainly the largest subspace of C on which T is well-defined. We easily verify

- Each α-Hölder continuous function belongs to dom, i.e $\bigcup_{\alpha\in(0,1]} \mathsf{C}^{0,\alpha} \subseteq \mathsf{dom}$.
- dom is a proper subset of C, i.e. $\mathsf{dom} \subseteq \mathsf{C}$, but $\mathsf{dom} \neq \mathsf{C}$, because, e.g., the continuous function

$$g : [0,1] \to \mathbb{R}, \quad g(t) = \begin{cases} 0 & ,t = 0 \\ -[\log(t)]^{-1} & ,0 < t \leq 1/e \\ 1 & ,1/e < t \leq 1 \end{cases}$$

  does not belong to dom.
- We have $\mathsf{C}^{0,1} \subseteq \mathsf{dom} \subseteq \mathsf{C}$ and $\mathsf{C}^{0,1}$ is dense in C. Hence dom is dense in C, too.

Having gained some insight into dom we turn our attention to the question whether T is bounded on a reasonable subspace of dom. The discussion becomes rather transparent when T is restricted to spaces of polynomials.

2.1 T Restricted to Spaces of Polynomials

We recall

$$L_k := \sum_{l=1}^{k} \frac{2}{l}, \quad k \in \mathbb{N}_0$$

where we observe that $L_0 = 0$. For later reference we note - according to well-known properties of the harmonic serie -

$$\forall k \in \mathbb{N}_0 : \quad 0 < L_k - 2\ \log(1+k) < 2. \tag{2.6}$$

We define for $k \in \mathbb{N}_0$ by recursion the coefficients

$$a_k^k := 1 \tag{2.7}$$

$$a_j^k := \frac{-1}{L_k - L_j} \sum_{l=j+1}^{k} \frac{a_l^k}{l-j} \quad , \qquad j = k-1, \dots, 1 \tag{2.8}$$

$$a_0^k := -\frac{1}{L_k} \sum_{l=1}^{k} \frac{a_l^k}{l} \quad , \qquad k \neq 0, \tag{2.9}$$

and introduce for $k \in \mathbb{N}_0$ the polynomials

$$q_k : [0,1] \to \mathbb{R}, \quad q_k(s) = s^k + a_{k-1}^k s^{k-1} + \ldots + a_1^k s + a_0^k.$$

Furthermore, we denote by T_N, $N \in \mathbb{N}_0$, the restriction of T to $\mathcal{P}_N$.

Then we have

Lemma 1. *For all $N \in \mathbb{N}_0$,*

a) $T_N[\mathcal{P}_N] \subseteq \mathcal{P}_N$.

b) $\mathsf{kn}(T_N) = \mathcal{P}_0$.

c) *The eigenvalues of* T_N *are* $-L_0, -L_1, \dots, -L_N$ *with respective one-dimensional eigenspaces* $\mathsf{span}[q_0], \mathsf{span}[q_1], \dots, \mathsf{span}[q_N]$.

Due to $\bigcup_{N\in\mathbb{N}_0} \mathcal{P}_N \subseteq \mathsf{dom}$ and with the aid of (2.6) (to prove c)) it is easy to deduce

Corollary 1. *a)* $-L_0, -L_1, -L_2, \dots$ *are eigenvalues of* T *whose respective eigenspaces contain* $\mathsf{span}[q_0], \mathsf{span}[q_1], \mathsf{span}[q_2], \dots$.

b) *For all linear subspaces* U *of* C *with* $T[U] \subseteq \mathsf{C}$ *and* $\bigcup_{N\in\mathbb{N}_0} \mathcal{P}_N \subseteq U$*:*

$$T \downharpoonright U : U \to \mathsf{C} \textit{ is } \text{not} \textit{ continuous (with respect to the norm } \|.\|_\infty).$$

Corollary 1 shows that T is not bounded on any reasonable infinite dimensional subspace of C.

2.2 T Restricted to $\mathsf{C}^{0,\alpha}$

Due to Lemma 1 the operator T maps $\mathcal{P}_N$ into itself. The question arises whether T also leaves any of the spaces $\mathsf{C}^{0,\alpha}$, $\alpha \in (0,1]$, invariant. (This is the case, e.g. for the Cauchy kernel operators.) This is not the case.

Proposition 2. *For all* $\alpha \in (0,1]$*:* $T[\mathsf{C}^{0,\alpha}]$ *is* not *contained in* $\mathsf{C}^{0,\alpha}$.

On the other hand we have the following extension of a classical result of Plemelj–Privalov (see e.g. [14], weitere Referenzen) which originally applies to the operator $\psi \mapsto \int_0^1 \frac{\psi(t)-\psi(s)}{t-s}\, dt$:

Lemma 3. *For all* $\alpha \in (0,1]$ *and all* $\alpha' \in [0,\alpha)$*:*

$$T[\mathsf{C}^{0,\alpha}] \subseteq \mathsf{C}^{0,\alpha'}.$$

We draw the following conclusions from Lemma 3: The operator T maps $\bigcup_{\alpha\in(0,1]} \mathsf{C}^{0,\alpha}$ into itself and therefore

$$\bigcup_{\alpha\in(0,1]} \mathsf{C}^{0,\alpha} \subseteq \mathsf{dom}_\mathsf{C} := \{\psi \in \mathsf{dom} : T[\psi] \in \mathsf{C}\}.$$

Hence dom_C is a dense subset of C. By definition we have $T[\mathsf{dom}_\mathsf{C}] \subseteq \mathsf{C}$ such that one may try to apply the BLT Theorem on $T \downharpoonright \mathsf{dom}_\mathsf{C}$ to extend T to C. However, this procedure is not performable: Due to Corollary 1, the operator $T \downharpoonright \mathsf{dom}_\mathsf{C}$ is *not* continuous with respect to the norm $\|.\|_\infty$.

Since T is not bounded with respect to the canonical norm of C one may ask whether $T \downharpoonright \mathsf{C}^{0,\alpha} : \mathsf{C}^{0,\alpha} \to \mathsf{C}$, $\alpha \in (0,1]$, where $\mathsf{C}^{0,\alpha}$ is equipped with the α-Hölder norm, is bounded. This is indeed the case. A simple calculation gives

$$\|T \downharpoonright \mathsf{C}^{0,\alpha}\|_{\mathsf{C}^{0,\alpha}\to\mathsf{C}} \le \frac{2^{1-\alpha}}{\alpha}.$$

3 The Operator $\hat{T}$

We conclude from the previous investigations that an analysis of (1.1) (and a numerical treatment as well) shall rather not be settled on Hölder spaces but on spaces on which the spectral properties of T are dominated by the polynomial eigenvectors.

From a theoretical point of view, a function space consisting of functions possessing expansions in terms of the polynomial eigenfunctions of T would be a rather natural choice. However, due to (2.7) the nth eigenfunction has n non-vanishing and alternating coefficients. Hence the employment of the polynomial eigenfunctions is not attractive from a numerical point of view.

These observations make it adviseful to settle the analysis on another function space, call it "$\mathsf{dom}(\hat{T})$" for the moment. The choice of $\mathsf{dom}(\hat{T})$ is guided by several aspects.

Firstly, $\mathsf{dom}(\hat{T})$ shall be rather "large" such that a discussion of (1.1) for a large class of functions F is possible. Hence $\mathsf{dom}(\hat{T})$ shall be infinite dimensional.

On the other hand, $\mathsf{dom}(\hat{T})$ shall be chosen such that the corresponding theory for (1.1) is of immediate use for a numerical treatment, i.e. the operator T shall have a transparent representation on $\mathsf{dom}(\hat{T})$. This suggests the employment of a more tractable Schauder basis of C, e.g. monomials σ^k, $k \in \mathbb{N}_0$, centered at $1/2$:

$$\sigma^k : [0,1] \to \mathbb{R}, \quad \sigma^k(s) = \left(s - \frac{1}{2}\right)^k .$$

But one has to be careful with expansions in terms of a Schauder basis (or in terms of T's eigenfunctions). T is not continuous with respect to the $\|.\|_\infty$ norm. Hence one may loose one of the most promising features of such expansions, namely the possibility to interchange limits.

One can expect that the interchange of "applying T" with the Schauder expansion will go through if one imposes growth conditions on the expansions' coefficients.

Hence $\mathsf{dom}(\hat{T})$ shall consist of functions whose Schauder coefficients (with respect to $\{\sigma^k : k \in \mathbb{N}_0\}$) satisfy some growth conditions.

Indeed there are several possibilities to choose such a $\mathsf{dom}(\hat{T})$. The choice of $\mathsf{dom}(\hat{T})$ we consider here is taken for the sake of transparency. Other choices may also be possible. We set

$$\mathsf{dom}(\hat{T}) := \mathcal{L}^\infty := \left\{ \sum_{l=0}^{\infty} \psi_l \, \sigma^l : (\psi_l)_{l \in \mathbb{N}_0} \in \ell^\infty \right\},$$

i.e. $\mathcal{L}^\infty$ is the space of all analytic functions whose Taylor expansions at $1/2$ have uniformly bounded coefficients.

Remark 1. *a) One immediately realizes that $(\psi_l)_{l \in \mathbb{N}_0} \in \ell^\infty$ implies convergence with respect to the $\|.\|_\infty$-norm of the serie $\left(\sum_{l=0}^{n} \psi_l \, \sigma^l\right)_{n \in \mathbb{N}_0}$. Hence the*

elements of $\mathcal{L}^\infty$ are well-defined analytic functions whose domain is $[0,1]$.
b) dom *contains non-analytic functions (recall that* $\mathsf{C}^{0,1} \subseteq \mathsf{dom}$*). Therefore* $\mathcal{L}^\infty \subseteq \mathsf{dom}$ *but* $\mathcal{L}^\infty \neq \mathsf{dom}$.

We set

$$\hat{T} := T \downharpoonleft \mathcal{L}^\infty$$

and equip $\mathcal{L}^\infty$ with its canonical norm

$$\left\| \sum_{l=0}^{\infty} \psi_l \, \sigma^l \right\|_{1/2} := |(\psi_l)_{l \in \mathbb{N}_0}|_\infty .$$

It can be left to the reader to verify that $\|.\|_{1/2}$ is well-defined. Furthermore, it is easy to see that $(\mathcal{L}^\infty, \|.\|_{1/2})$ is a Banach space which is continuously embedded in $(\mathsf{C}, \|.\|_\infty)$.

It will turn out that $\hat{T}$ does not map $\mathcal{L}^\infty$ into itself but into

$$\mathcal{L}_w^\infty := \left\{ \sum_{l=0}^{\infty} g_l \, \sigma^l : (g_l)_{l \in \mathbb{N}_0} \in \ell_w^\infty \right\},$$

where

$$\ell_w^\infty := \left\{ (g_l)_{l \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}_0} : \sup \left\{ \frac{|g_l|}{3 + 2\, \log(1+l)} : l \in \mathbb{N}_0 \right\} < \infty \right\}.$$

We equip ℓ_w^∞ with its canonical norm

$$|(g_l)_{l \in \mathbb{N}_0}|_{w,\infty} := \sup \left\{ \frac{|g_l|}{3 + 2\, \log(1+l)} : l \in \mathbb{N}_0 \right\}, \quad (g_l)_{l \in \mathbb{N}_0} \in \ell_w^\infty,$$

and introduce the corresponding norm $\|.\|_{w,1/2}$ on $\mathcal{L}_w^\infty$,

$$\left\| \sum_{l=0}^{\infty} g_l \, \sigma^l \right\|_{w,1/2} := |(g_l)_{l \in \mathbb{N}_0}|_{w,\infty}, \quad \sum_{l=0}^{\infty} g_l \, \sigma^l \in \mathcal{L}_w^\infty.$$

It can be left to the reader to verify that $(\ell_w^\infty, |.|_{w,\infty})$ and $(\mathcal{L}_w^\infty, \|.\|_{w,1/2})$ are (isometric) Banach spaces.

Remark 2. *a) One immediately realizes that the spaces $\mathcal{L}^\infty$ and $\mathcal{L}_w^\infty$ have several properties in common: $(g_l)_{l \in \mathbb{N}_0} \in \ell_w^\infty$ implies convergence with respect to the $\|.\|_\infty$-norm of the serie $\left(\sum_{l=0}^{n} g_l \, \sigma^l\right)_{n \in \mathbb{N}_0}$. Hence the elements of $\mathcal{L}_w^\infty$ are well-defined analytic functions whose domain is $[0,1]$ and whose Taylor expansions around 1/2 have coefficients of at most logarithmic growth. Furthermore, one can argue as in b) of Remark 1 to deduce $\mathcal{L}_w^\infty \subseteq \mathsf{dom}$ but $\mathcal{L}_w^\infty \neq \mathsf{dom}$.*
b) Certainly, $\mathcal{L}^\infty \subseteq \mathcal{L}_w^\infty$ but $\mathcal{L}^\infty \neq \mathcal{L}_w^\infty$.

Occasionally we shall make use of the projection operators ($n \in \mathbb{N}_0$),

$$\mathsf{P}_0^n : \mathcal{L}_w^\infty \to \mathcal{L}_w^\infty, \quad \mathsf{P}_0^n \left[\sum_{l=0}^{\infty} \psi_l \, \sigma^l \right] = \sum_{l=0}^{n} \psi_l \, \sigma^l,$$

$$\mathsf{P}_{n+1}^{\infty}:\mathcal{L}_w^{\infty}\to\mathcal{L}_w^{\infty},\quad \mathsf{P}_{n+1}^{\infty}\left[\sum_{l=0}^{\infty}\psi_l\,\sigma^l\right]=\sum_{l=n+1}^{\infty}\psi_l\,\sigma^l.$$

We certainly have: $\mathsf{P}_0^n+\mathsf{P}_{n+1}^{\infty}$ is the identity on $\mathcal{L}_w^{\infty}$.

Although $\hat{T}$ has no diagonal structure with respect to the Schauder basis $\{\sigma^k : k\in\mathbb{N}_0\}$, a rather tranparent representation of $\hat{T}$ with respect to $\{\sigma^k : k\in\mathbb{N}_0\}$ is available. We observe

Proposition 4. *For all $l\in\mathbb{N}_0$:*

$$\hat{T}\left[\sigma^l\right]=\sum_{\nu=0}^{l}B_\nu^l\,\sigma^\nu,\ \textit{where}\ \begin{cases} B_l^l &:= \; -L_l \\ B_\nu^l &:= \; \dfrac{1+(-1)^{l-\nu}}{2^{l-\nu}\,(l-\nu)}, \quad \nu=0,\ldots,l-1. \end{cases}.$$

Furthermore, for all $\nu\in\mathbb{N}_0$,

$$\sum_{l=\nu+1}^{\infty}B_\nu^l=\sum_{k=1}^{\infty}\frac{1+(-1)^k}{2^k\,k}=\log(4/3). \tag{3.10}$$

With the aid of proposition 4 we obtain

Theorem 5. *Let $\psi=\sum_{l=0}^{\infty}\psi_l\,\sigma^l\in\mathcal{L}^{\infty}$. Then*

$$\hat{T}[\psi]=\sum_{l=0}^{\infty}\left[\sum_{\nu=l}^{\infty}B_l^\nu\,\psi_\nu\right]\sigma^l\in\mathcal{L}_w^{\infty},$$

and

$$\lim_{n\to\infty}\left\|\hat{T}[\psi]-\mathsf{P}_0^n[\hat{T}[\psi]]\right\|_\infty=\lim_{n\to\infty}\left\|\hat{T}[\psi]-\sum_{l=0}^{n}\left[\sum_{\nu=l}^{\infty}B_l^\nu\,\psi_\nu\right]\sigma^l\right\|_\infty=0,$$

as well as

$$\lim_{n\to\infty}\left\|\hat{T}[\psi]-(\mathsf{P}_0^n\circ\hat{T}\circ\mathsf{P}_0^n)[\psi]\right\|_\infty=\lim_{n\to\infty}\left\|\hat{T}[\psi]-\sum_{l=0}^{n}\left[\sum_{\nu=l}^{n}B_l^\nu\,\psi_\nu\right]\sigma^l\right\|_\infty=0.$$

Remark 3. *With the notations of Theorem 5 we have for $n\in\mathbb{N}_0$,*

$$\psi_n:=(\mathsf{P}_0^n\circ\hat{T}\circ\mathsf{P}_0^n)[\psi]=(\hat{T}\circ\mathsf{P}_0^n)[\psi]=\sum_{l=0}^{n}\left[\sum_{\nu=l}^{n}B_l^\nu\,\psi_\nu\right]\sigma^l,$$

and

$$\lim_{n\to\infty}\|\psi-\psi_n\|_\infty=0,\quad \lim_{n\to\infty}\|T[\psi]-T[\psi_n]\|_\infty=0.$$

Nevertheless, T is not $\|.\|_\infty$-$\|.\|_\infty$-continuous.

According to Theorem 5 we have for each $\psi = \sum_{l=0}^{\infty} \psi_l \, \sigma^l \in \mathcal{L}^\infty$ the identity

$$\hat{T}[\psi] = \sum_{l=0}^{\infty} \Gamma_l[\psi] \, \sigma^l, \quad \Gamma_l[\psi] := \sum_{\nu=l}^{\infty} B_l^\nu \, \psi_\nu,$$

where

$$\forall l \in \mathbb{N}_0 : \quad \Gamma_l[\psi] \leq (3 + 2\log(1+l)) \, \|\psi\|_{1/2}.$$

We readily deduce

Corollary 2. *$\hat{T}$ maps $\mathcal{L}^\infty$ into $\mathcal{L}_w^\infty$ with operator norm*

$$\left\| \hat{T} : \mathcal{L}^\infty \to \mathcal{L}_w^\infty \right\| = 1.$$

Remark 4. *The estimate $\left\| \hat{T} : \mathcal{L}^\infty \to \mathcal{L}_w^\infty \right\| \leq 1$ is obvious. To obtain equality let us consider the coefficients $\Gamma_l[\psi]$ of the series expansion of $\hat{T}[\psi]$, $\psi = (1, 1, 1, \dots) \in \mathcal{L}^\infty$. Then*

$$\lim_{l \to \infty} \frac{|\Gamma_l[(1, 1, 1, \dots)]|}{2\,\log(1+l)} = 1,$$

such that $\left\| \hat{T} : \mathcal{L}^\infty \to \mathcal{L}_w^\infty \right\| \geq 1$ as well.

We are now in the position to pass to a matrix representation for $\hat{T}$ which not only simplifies the presentation but will be of great importance for the numerical approximations of (1.1). We give

Definition 1. *For $l, m \in \mathbb{N}_0$ let $A_{l,m} \in \mathbb{R}$. Let $(F_l)_{l \in \mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0}$ and let $(x_l)_{l \in \mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0}$. Then $(x_l)_{l \in \mathbb{N}_0}$ is said to be a "solution of the matrix equation"*

$$\begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} & \cdots \\ A_{1,0} & A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & \cdots \\ A_{2,0} & A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & \cdots \\ A_{3,0} & A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ \vdots \end{pmatrix},$$

iff for all $l \in \mathbb{N}_0$:

$$\left(\sum_{\nu=0}^{n} A_{l,\nu} \, x_\nu \right)_{n \in \mathbb{N}_0} \in \ell^1, \quad \textit{and} \quad \sum_{\nu=0}^{\infty} A_{l,\nu} \, x_\nu = F_l.$$

Combining definition 1 with theorem 5 it is easy to deduce

Corollary 3. *For all $c \in \mathbb{R}$, all $\phi = \sum_{l=0}^{\infty} \phi_l \, \sigma^l \in \mathcal{L}^\infty$ and all $F = \sum_{l=0}^{\infty} F_l \, \sigma^l \in \mathcal{L}_w^\infty$ the propositions i) and ii) are equivalent.*

i) $c \, \phi = F + \hat{T}[\phi]$.

ii) $(\phi_l)_{l \in \mathbb{N}_0}$ *is a solution of the matrix equation*

$$\begin{pmatrix} c+L_0 & -B_0^1 & -B_0^2 & -B_0^3 & -B_0^4 & -B_0^5 & \cdots \\ 0 & c+L_1 & -B_1^2 & -B_1^3 & -B_1^4 & -B_1^5 & \cdots \\ 0 & 0 & c+L_2 & -B_2^3 & -B_2^4 & -B_2^5 & \cdots \\ 0 & 0 & 0 & c+L_3 & -B_3^4 & -B_3^5 & \cdots \\ 0 & 0 & 0 & 0 & c+L_4 & -B_4^5 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \end{pmatrix} \tag{3.11}$$

3.1 The Spectrum of $\hat{T}$

We shall investigate the spectrum of $\hat{T}$ now. It will turn out that the resolvent of $\hat{T}$ is exactly $\mathbb{R} \setminus \{-L_l : l \in \mathbb{N}_0\}$. For $\rho \in \mathbb{R} \setminus \{-L_l : l \in \mathbb{N}_0\}$, an estimate of the norm of $[\rho - \hat{T}]^{-1} : \mathcal{L}_w^\infty \to \mathcal{L}^\infty$ will be given in terms of an apropriately defined norm of the inverse of a matrix $A_{n;\rho}$ representing the canonical finite-dimensional approximation $\mathsf{P}_0^n \circ (\rho - \hat{T}) \circ \mathsf{P}_0^n = (\rho - \hat{T}) \circ \mathsf{P}_0^n$ of $\rho - \hat{T}$. We put for $n \in \mathbb{N}$, and $\rho \in \mathbb{R} \setminus \{-L_l : l \in \mathbb{N}_0\}$,

$$A_{n;\rho} := \begin{pmatrix} \rho+L_0 & -B_0^1 & -B_0^2 & -B_0^3 & \cdots & -B_0^n \\ 0 & \rho+L_1 & -B_1^2 & -B_1^3 & \cdots & -B_1^n \\ 0 & 0 & \rho+L_2 & -B_2^3 & \cdots & -B_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \rho+L_{n-1} & -B_{n-1}^n \\ 0 & 0 & 0 & \ldots & 0 & \rho+L_n \end{pmatrix}. \tag{3.12}$$

By a slight abuse of notation, we identify the matrix $A_{n;\rho}$ with the finite-dimensional operator induced by $A_{n;\rho}$ which maps $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+1}$. Certainly, for $\rho \in \mathbb{R} \setminus \{-L_l : l \in \mathbb{N}_0\}$, this operator has a bounded inverse $A_{n;\rho}^{-1}$ whose norm is

$$\begin{aligned} p_1(n;\rho) &:= \sup\Big\{ \, \big| \, A_{n;\rho}^{-1} [\eta] \, \big|_{n+1} : |\eta|_{w,n+1} \le 1 \Big\} \\ &= \left\| \Big[(\rho - \hat{T}) \circ \mathsf{P}_0^n \Big]^{-1} : \mathsf{P}_0^n[\mathcal{L}_w^\infty] \to \mathsf{P}_0^n[\mathcal{L}^\infty] \right\|, \end{aligned} \tag{3.13}$$

where for $n \in \mathbb{N}_0$ and $\eta = (\eta_0, \ldots, \eta_n) \in \mathbb{R}^{n+1}$,

$$|\eta|_{n+1} := \max\{|\eta_0|, \ldots, |\eta_n|\},$$

and

$$|\eta|_{w,n+1} := \max\left\{\frac{|\eta_0|}{3+2\,\log(1+0)},\dots,\frac{|\eta_n|}{3+2\,\log(1+n)}\right\}.$$

Corollary 3 contains most of the important informations to prove

Theorem 6. *$\hat{T} : \mathcal{L}_w^\infty \supseteq \mathcal{L}^\infty \to \mathcal{L}_w^\infty$ has a pure point spectrum,*

$$\sigma(\hat{T}) = \sigma_p(\hat{T}) = \{-L_l : l \in \mathbb{N}_0\}.$$

Furthermore,

a) the eigenspace of $-L_l$, $l \in \mathbb{N}$, is one-dimensional and spanned by q_l;

b) for all $\rho \in \mathbb{R} \setminus \sigma_p(\hat{T})$,

$$\|(\rho - \hat{T})^{-1} : \mathcal{L}_w^\infty \to \mathcal{L}^\infty\| \le p_1(n;\rho)(1 + p_2(n;\rho)), \quad n \in \mathbb{N}, n \ge N_1(\rho), \tag{3.14}$$

where

$$N_1(\rho) := \min\{k \in \mathbb{N} : L_{k+1} > \log(4/3) - \rho\},$$

$p_1(n;\rho)$ is as in (3.13) and

$$p_2(n;\rho) := \frac{\log(4/3)}{1 - \frac{\log(4/3)}{\rho + L_{n+1}}} \sup\left\{\frac{3+2\,\log(1+l)}{\rho + L_{l+1}} : l \in \mathbb{N}, n \le l\right\}.$$

4 The Integral Equation and its Numerical Approximation

In this section we are concerned with the integral equation

$$c\,\phi = F + T[\phi], \quad \phi \in \mathcal{L}^\infty, \quad F \in \mathcal{L}_w^\infty, \tag{4.15}$$

with $c < 0$.

We deduce from Theorem 6: (4.15) has a unique solution iff $c \notin \{-L(l) : l \in \mathbb{N}_0\}$ and if $c \notin \{-L_l : l \in \mathbb{N}_0\}$, then

$$\phi = (c - \hat{T})^{-1}[F]$$

with $\mathcal{L}_w^\infty$-$\mathcal{L}^\infty$ continuous $(c - \hat{T})^{-1}$.

However, if $c = -L_l$ for some $l \in \mathbb{N}$, then (4.15) has either no solution or the set of all solutions of (4.15) is a one-dimensional affine manifold.

The question arises whether (4.15) is a well-posed problem or not. From a theoretical point of view this is certainly not the case: the number of values for c in - let's say - intervals $[-k-1, -k]$ - for which (4.15) is not uniquely solvable - increases exponentially with k. Hence a slight perturbance of $c \approx -k$, k "large", may entirely change the solvability of (4.15).

From a numerical point of view however the situation is much less dramatic. In practical reasons only finitely many digits of the number c can be handeled. But such c (unless in $\{0,-1,-3/2\}$), never equals $-L_l$ for any $l \in \mathbb{N}$, i.e. for "realistic" values of c, $c \notin \{0,-1,-3/2\}$, (4.15) is always uniquely solvable. In other words: (4.15) is a well-posed problem from a numerical point of view.

Let us turn our attention to a numerical approximation of (4.15) in case of $c \notin \{-L_l : l \in \mathbb{N}_0\}$ now. Due to the spectral properties of $\hat{T}$ it is convenient to project (4.15) to the space of all polynomials of degree less or equal $n \in \mathbb{N}$ and let n tend to ∞ then.

The corresponding $n+1$-dimensional matrix approximation of (4.15) reads

$$A_{n;-c}[(\phi_0^{[n]}, \phi_1^{[n]}, \dots, \phi_n^{[n]})^{\mathrm{t}}] = (F_0, F_1, \dots, F_n)^{\mathrm{t}}, \tag{4.16}$$

where $A_{n;-c}$ is the matrix as in (3.12), $(\phi_0^{[n]}, \phi_1^{[n]}, \dots, \phi_n^{[n]}) \in \mathbb{R}^{n+1}$ is unknown and

$$F = \sum_{l=0}^{\infty} F_l\, \sigma^l.$$

Then we easily deduce the following approximation result.

Theorem 7. *Let $F = \sum_{l=0}^{\infty} F_l\, \sigma^l \in \mathcal{L}_w^\infty$. Let $c \in \mathbb{R} \setminus \{-L_l : l \in \mathbb{N}\}$. Then we have for all $n \in \mathbb{N}$,*

1. *The operator equation*

$$\left((c-\hat{T}) \circ \mathsf{P}_0^n\right)[\phi^{[n]}] = \mathsf{P}_0^n[F]$$

has a unique solution $\phi^{[n]} = \sum_{l=0}^{n} \phi_l^{[n]}\, \sigma^l \in \mathsf{P}_0^n[\mathcal{L}^\infty]$.

2. *$(\phi_0^{[n]}, \phi_1^{[n]}, \dots, \phi_n^{[n]}) \in \mathbb{R}^{n+1}$ is the unique solution of (4.16).*
3. *If $\phi = \sum_{l=0}^{\infty} \phi_l\, \sigma^l \in \mathcal{L}^\infty$ is the unique solution of (4.15), then*

$$\phi - \phi^{[n]} = (c-\hat{T})^{-1}\left[\sum_{l=n+1}^{\infty} F_l\, \sigma^l\right],$$

i.e.

$$\text{If } \lim_{n\to\infty} |(F_{n+1}, F_{n+2}, F_{n+3}, \dots)|_{w,\infty} = 0, \text{ then } \lim_{n\to\infty} \|\phi - \phi^{[n]}\|_{1/2} = 0,$$
$$\text{in particular } \lim_{n\to\infty} \phi_l^{[n]} = \phi_l, \quad \text{uniformly in } l \in \mathbb{N}_0.$$

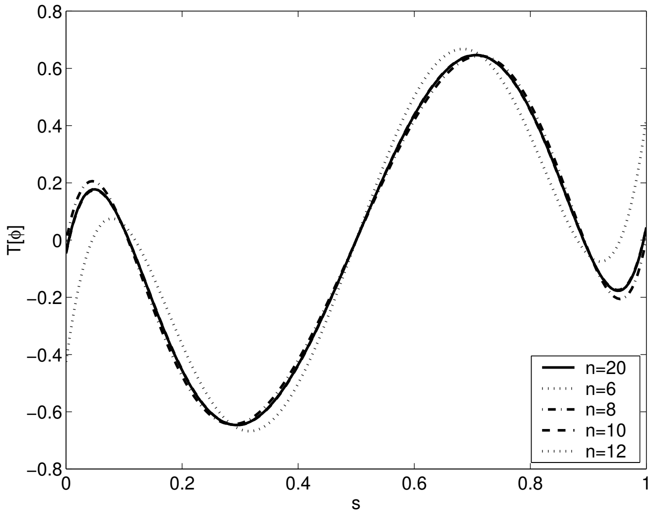

Figure 1: Solutions to (4.15) for $c = -5$ and $F(s) = \sin(2\pi\ s)$

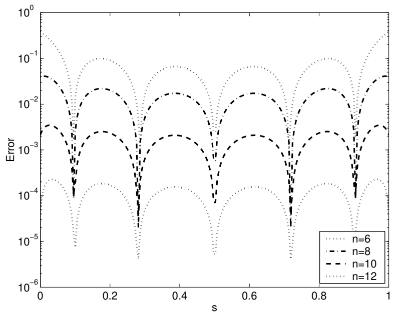

Figure 2: Differences to the solution for $n = 20$

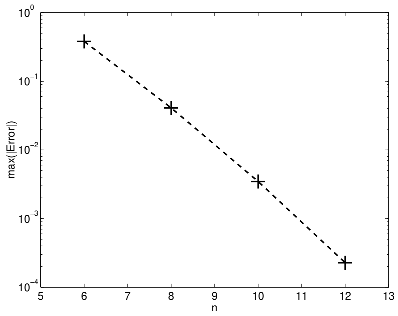

Figure 3: ∞–norm of the difference in figure 2

5 Numerics

In this section we give some numerical examples for the numerical scheme proposed above. The implementations are based on `MATLAB`.

As a first example we consider the problem (4.15) with given data $c = -5$ and $F(s) = \sin(2\pi\ s)$. Figure 1 shows the results obtained using different degrees n of the polynomials. Since the given data and the equation are symmetric around $s = 1/2$, we expect the solution to be symmetric, too. The numerical scheme conserves this symmetry. The coefficients of the even powers of $\sigma = s - \frac{1}{2}$ are computed to zero up to machine precision.
This figure shows also a very good agreement between the low order solutions and the most accurate one, which uses polynomials up to degree $n = 20$. The differences between the most accurate one ($n = 20$) and the lower degree solutions are shown in figure 2. The ∞–norm of these differences is given (in logarithmic scaling) in figure 3, exhibiting clearly an exponential dependence

between the error and the maximal degree of the used polynomials.

In the next example we chose again $c = -5$ and $F(s) = e^{3s} + \ln(\varepsilon + s)(1 + \varepsilon - s)$, where $\varepsilon = \frac{1}{10}$. The results analogously to the previous example are given in the figures 4, 5 and 6. Here the function $F(s)$ is not symmetric around $s = 1/2$ and we can not expect any symmetry for the solution. This is reflected by the numerical scheme: The approximations to the solutions are also not symmetric.

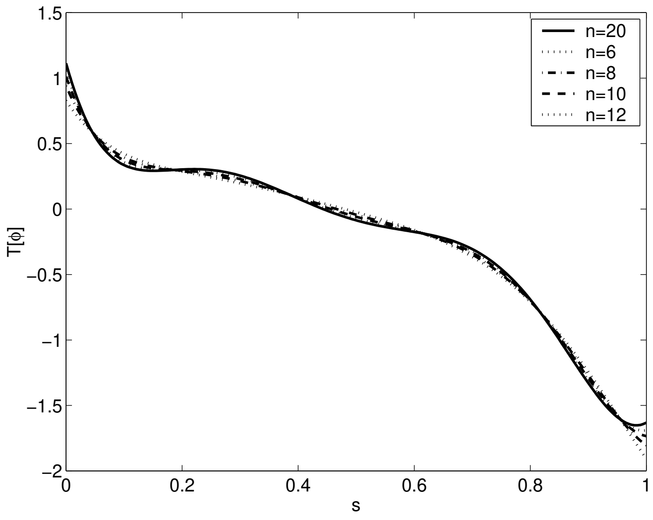

Figure 4: Solutions to (4.15) for $c = -5$ and $F(s) = e^{3s} + \ln(\varepsilon + s)(1 + \varepsilon - s)$, $\varepsilon = 0.1$

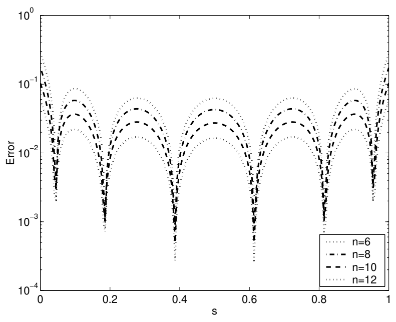

Figure 5: Differences to the solution for $n = 20$

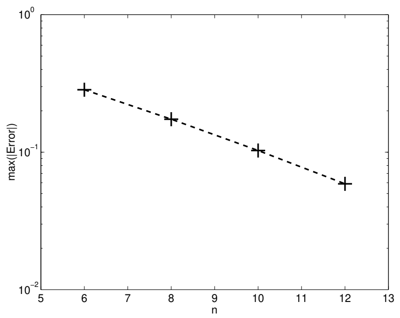

Figure 6: ∞–norm of the difference in figure 5

As a last example we consider $c = -5$ and $F(s) = \ln s(1 + s)$. This case is not covered by the theory of the previous sections, in particular there is no existence result. Nevertheless the algorithm is rather stable (see figures 8 and 9). The solutions computed for polynomial degrees up to $n = 50$ are given in figure 7.

Rather surprisingly the differences given in the figures 2, 5 and 8 oscillate with a frequence independent of the polynomial degree n. The *non*–occurance

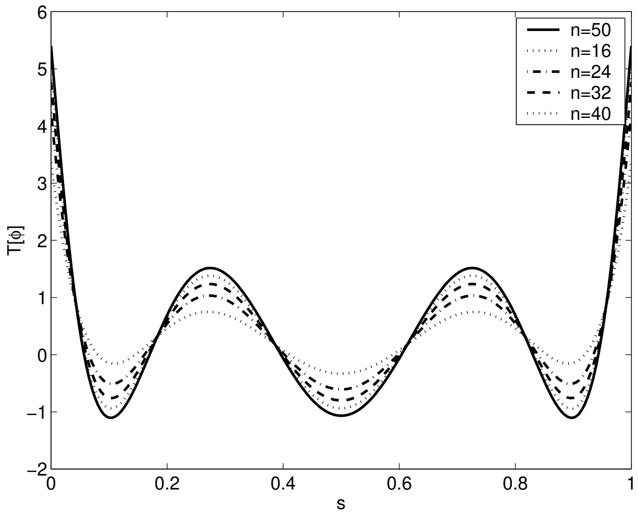

Figure 7: Solutions to (4.15) for $c = -5$ and $F(s) = \ln s(1-s)$

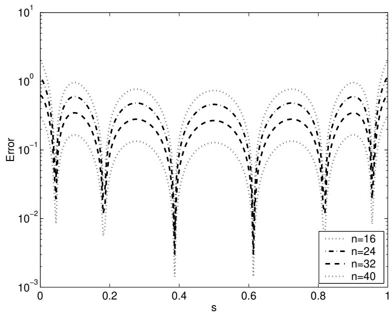

Figure 8: Differences to the solution for $n = 50$

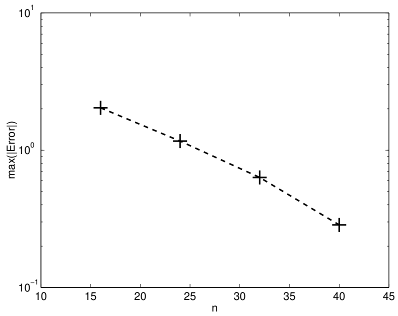

Figure 9: ∞–norm of the difference in figure 8

of high–frequency oscillations indicates the robustness of the numerical scheme beyond the theoretical results derived so far.

6 Appendix: Proofs

Lemma 1

Proof. a) We introduce for $k \in \mathbb{N}_0$,

$$\phi_k : [0,1] \to \mathbb{R}, \quad \phi_k(s) = s^k, \quad \text{and} \quad p_k : [0,1] \to \mathbb{R}, \quad p_k(s) := \sum_{l=0}^{k} \frac{s^{k-l}}{l+1}.$$

Furthermore we set

$$p_{-1} : [0,1] \to \mathbb{R}, \quad p_{-1}(s) = 0.$$

We calculate for all $k \in \mathbb{N}$, and all $s \in [0,1]$,

$$\begin{aligned} T_N[\phi_k](s) &= -\int_0^s \frac{t^k - s^k}{t-s}\,dt + \int_s^1 \frac{t^k - s^k}{t-s}\,dt \\ &= -\sum_{i=0}^{k-1} s^{k-1-i} \int_0^s t^i\,dt + \sum_{i=0}^{n-1} s^{n-1-i} \int_s^1 t^i\,dt \\ &= -s^k \sum_{i=1}^{k} \frac{1}{i} - s^k \sum_{i=1}^{k} \frac{1}{i} + \sum_{i=0}^{k-1} \frac{s^{k-1-i}}{i+1} = -L_k s^k + p_{k-1}(s), \end{aligned}$$

while for all $s \in [0,1]$, $T_N[\phi_0](s) = p_{-1}(s)$. Hence for all $k \in \mathbb{N}_0$,

$$T_N[\phi_k] = -L_k \phi_k + p_{k-1}, \tag{6.17}$$

which immediately gives a).

c) Let $j \in \{0, \dots, N\}$. It sufficies to prove $T_N[q_j] = -L_j\, q_j$. We have $q_j = \phi_j + a^j_{j-1}\phi_{j-1} + \ldots + a^j_1\phi_1 + a^j_0\phi_0$. Hence due to (6.17)

$$\begin{aligned} T_N[q_j] &= \sum_{l=0}^{j} a^j_l T[\phi_l] = \sum_{l=0}^{j} a^j_l\,(-L_l\,\phi_l + p_{l-1}) \\ &= -\sum_{l=0}^{j} a^j_l L_l\,\phi_l + \sum_{l=0}^{j} a^j_l p_{l-1} = -\sum_{l=1}^{j} a^j_l L_l\,\phi_l + \sum_{l=1}^{j}\sum_{\nu=0}^{l-1} \frac{a^j_l\,\phi_\nu}{l-\nu} \\ &= -\sum_{l=1}^{j} a^j_l L_l\,\phi_l + \sum_{l=1}^{j}\sum_{\nu=1}^{l} \frac{a^j_l\,\phi_{\nu-1}}{l+1-\nu} = -\sum_{l=1}^{j} a^j_l L_l\,\phi_l + \sum_{\nu=1}^{j}\sum_{l=\nu}^{j} \frac{a^j_l\,\phi_{\nu-1}}{l+1-\nu} \\ &= -L_j\,\phi_j + \sum_{l=1}^{j-1}\left[-L_l\,a^j_l + \sum_{\nu=l+1}^{j} \frac{a^j_\nu}{\nu-l}\right]\phi_l + \sum_{\nu=1}^{j} \frac{a^j_\nu}{\nu}\phi_0 \\ &= -L_j\,\phi_j + \sum_{l=1}^{j-1}\left[-L_l\,a^j_l + (L_l - L_j)\,a^j_l\right]\phi_l - L_j\,a^j_0\,\phi_0 \\ &= -L_j\,\left(\phi_j + a^j_{j-1}\phi_{j-1} + \ldots + a^j_0\,\phi_0\right) = -L_j\,q_j. \end{aligned}$$

b) follows from c). $\square$

Proposition 2

Proof. If $\alpha \in (0,1)$ consider $\psi(s) = s^\alpha$, $s \in [0,1]$. Then $\psi \in \mathsf{C}^{0,\alpha}$ and we obtain for all $s \in (0,1]$

$$\begin{aligned} \frac{T[\psi](s) - T[\psi](0)}{s^\alpha} &= -\int_0^1 \frac{1-\sigma^\alpha}{1-\sigma}\,d\sigma + \int_1^{1/s} \frac{\sigma^\alpha - 1}{\sigma - 1}\,d\sigma - \frac{1}{\alpha\, s^\alpha} \\ &= \frac{1}{\alpha} - \int_0^1 \frac{1-\sigma^\alpha}{1-\sigma}\,d\sigma + \int_1^{1/s} \frac{\sigma^{\alpha-1} - 1}{\sigma - 1}\,d\sigma, \end{aligned}$$

such that

$$\lim_{s\to 0}\left|\frac{T[\psi](s)-T[\psi](0)}{s^\alpha}\right| = \infty,$$

i.e. $T[\psi]$ is *not* α-Hölder continuous.

In case of $\alpha = 1$ consider $\psi(s) := |s-(1/2)| \in \mathsf{C}^{0,1}$. We immediately obtain

$$T[\psi](s) = \begin{cases} 2\,s + (2\,s-1)\,\log\left(\frac{2-2\,s}{1-2\,s}\right) & , \quad 0 \le s < 1/2 \\ 1 & , \quad s = 1/2 \\ 2 - 2\,s + (1-2\,s)\,\log\left(\frac{2\,s}{2\,s-1}\right) & , \quad 1/2 < s \le 1 \end{cases},$$

i.e. $T[\psi]$ is certainly *not* a Lipschitz-continuous function. $\square$

Lemma 3

Proof. We can assume $\alpha \in (0,1)$. It sufficies to consider $\alpha' \in (0,\alpha)$. Let $s, s' \in [0,1]$ with $s \neq s'$ and $\delta = 2|s-s'|$. We observe $0 < \delta \le 2$. Let $U_1 = [0,1] \cap [s-\delta, s+\delta]$ and $U_2 = [0,1] \setminus [s-\delta, s+\delta]$. For $t \neq s, s'$ let $q(t) := \frac{\psi(t)-\psi(s)}{|t-s|} - \frac{\psi(t)-\psi(s')}{|t-s'|}$. We set without loss of generality $q(s) = q(s') = 0$. By $K_1, K_2, \ldots$ we denote positive constants which may depend on ϕ, but are independent of s, s'.
We have

$$\left|T[\psi](s) - T[\psi](s')\right| = \left|\int_0^1 q(t)\;dt\right| \le \int_{U_1} |q(t)|\;dt + \int_{U_2} |q(t)|\;dt.$$

By Hölder-continuity of ψ we have

$$\int_{U_1} |q(t)|\;dt \le K_1 \int_{U_1} |t-s|^{\alpha-1}\;dt + K_1 \int_{U_1} |t-s'|^{\alpha-1}\;dt,$$

where we note that

$$\int_{U_1} |t-s|^{\alpha-1}\;dt \le \int_{s-\delta}^{s+\delta} |t-s|^{\alpha-1}\;dt = \frac{2}{\alpha}\delta^\alpha.$$

On the other hand we have $U_1 \subseteq [s-\delta, s+\delta] \subseteq [s'-2\,\delta, s'+2\,\delta]$, such that

$$\int_{U_1} |t-s'|^{\alpha-1}\;dt \le \int_{s'-2\,\delta}^{s'+2\,\delta} |t-s'|^{\alpha-1}\;dt = \frac{2^{1+\alpha}}{\alpha}\delta^\alpha,$$

and we obtain due to $|s-s'|^{\alpha-\alpha'} \le 1$,

$$\int_{U_1} |q(t)|\;dt \le K_2\,|s-s'|^\alpha \le K_2\,|s-s'|^{\alpha'}.$$

Now we consider the integration over U_2. We introduce for $t \neq s, s'$, $q_1(t) := \frac{\psi(s')-\psi(s)}{|t-s|}$ and $q_2(t) := (\psi(t)-\psi(s'))\left(\frac{1}{|t-s|} - \frac{1}{|t-s'|}\right)$, and put without loss of generality $q_{1,2}(s) = q_{1,2}(s') = 0$, such that $q(t) = q_1(t) + q_2(t)$. We have due to Hölder-continuity and due to $\delta \leq 2$,

$$\int_{U_2} |q_1(t)|\, dt = |\psi(s') - \psi(s)| \int_{U_2} \frac{dt}{|t-s|}$$

$$\leq K_3\, |s-s'|^\alpha\, 2 \int_\delta^{\max(s,1-s)} \frac{d\,t}{t} \;\leq K_3\, |s-s'|^\alpha\, 2 \int_\delta^2 \frac{d\,t}{t}$$

$$= 2\, K_3\, |s-s'|^\alpha (\log(2) - \log(\delta)) = \; 2\, K_3\, |s-s'|^\alpha (\log(2) - \log(2\, |s-s'|))$$

$$\leq K_4\, |s-s'|^{\alpha'}.$$

Concerning $\int_{U_2} |q_2(t)|\, dt$ we have for all $t \in U_2$ the estimate

$$|q_2|(t) \leq \big|\psi(t) - \psi(s')\big| \frac{|s-s'|}{|t-s||t-s'|} \leq K_5 |s-s'||t-s'|^{\alpha-1}|t-s|^{-1}.$$

Since $t \in U_2$ we have $|t-s| \geq \delta = 2\,|s-s'|$. Hence $|t-s'| \geq ||t-s| - |s-s'|| = |t-s| - |s-s'| \geq |t-s|/2$. We obtain for all $t \in U_2$, $|t-s'|^{\alpha-1} \leq 2^{1-\alpha}|t-s|^{\alpha-1}$ and therefore due to $\delta \leq 2$,

$$\int_{U_2} |q_2(t)|\, dt \leq K_6\, |s-s'| \int_{U_2} |t-s|^{\alpha-2}\, dt$$

$$\leq 2\, K_6\, |s-s'| \int_\delta^3 |t|^{\alpha-2}\, dt = K_7\, |s-s'|\; \left(-3^{\alpha-1} + \delta^{\alpha-1}\right)$$

$$\leq K_8\, |s-s'| + K_9\, |s-s'|^{1+\alpha-1} \leq K_{10}\, |s-s'|^{\alpha'}.$$

By addition we obtain

$$\big|T[\psi](s) - T[\psi](s')\big| \leq (K_2 + K_4 + K_{10})|s-s'|^{\alpha'}.$$

Since $K_1, \ldots, K_{10}$ are independent of s, s' we have $T[\psi] \in \mathsf{C}^{\alpha'}$ with

$$\sup_{s,s' \in [0,1],\, s \neq s'} \frac{|T[\psi](s) - T[\psi](s')|}{|s-s'|^{\alpha'}} \leq K_2 + K_4 + K_{10}.$$

□

Proposition 4

Proof. For all $l \in \mathbb{N}$ (the case $l = 0$ can be left to the reader) and all $s \in [0,1]$

we have

$$
\begin{aligned}
&T[\sigma^l](s)\\
&= -\int_0^s \frac{(s-(1/2))^l-(t-(1/2))^l}{s-t}\,dt + \int_s^1 \frac{(t-(1/2))^l-(s-(1/2))^l}{t-s}\,dt\\
&= -\int_0^s \frac{(s-(1/2))^l-(t-(1/2))^l}{(s-(1/2))-(t-(1/2))}\,dt + \int_s^1 \frac{(t-(1/2))^l-(s-(1/2))^l}{(t-(1/2))-(s-(1/2))}\,dt\\
&= -\sum_{\nu=0}^{l-1}(s-(1/2))^{l-1-\nu}\int_0^s (t-(1/2))^\nu\,dt + \sum_{\nu=0}^{l-1}(s-(1/2))^{l-1-\nu}\int_s^1 (t-(1/2))^\nu\,dt\\
&= -\sum_{\nu=0}^{l-1}\frac{(s-(1/2))^{l-1-\nu}}{1+\nu}\left[(s-(1/2))^{1+\nu}-(-1/2)^{1+\nu}\right]\\
&\quad + \sum_{\nu=0}^{l-1}\frac{(s-(1/2))^{l-1-\nu}}{1+\nu}\left[(1/2)^{1+\nu}-(s-(1/2))^{1+\nu}\right]\\
&= -2\,(s-(1/2))^l\sum_{\nu=0}^{l-1}\frac{1}{1+\nu} + \sum_{\nu=0}^{l-1}(s-(1/2))^{l-1-\nu}\,\frac{(1/2)^{1+\nu}+(-1/2)^{1+\nu}}{1+\nu}\\
&= -L_l\,\sigma^l(s) + \sum_{\nu=0}^{l-1}\frac{1+(-1)^{l-\nu}}{2^{l-\nu}\,(l-\nu)}\,\sigma^\nu(s).
\end{aligned}
$$

The verification of (3.10) can be left to the reader. □

Theorem 5

Proof. We have due to (3.10), (2.6) for all $l \in \mathbb{N}_0$ and all $n \in \mathbb{N}_0$ the estimate

$$
\begin{aligned}
\sum_{\nu=l}^{l+n} |B_l^\nu\,\psi_\nu| \le \|\psi\|_{1/2}\sum_{\nu=l}^{l+n} B_l^\nu &\le \|\psi\|_{1/2}\ (L_l+\log(4/3))\\
&\le \|\psi\|_{1/2}\ (L_l+1) \le \|\psi\|_{1/2}\ (3+2\,\log(1+l)).
\end{aligned}
$$

We note that this estimate is independent of n. Hence for all $l \in \mathbb{N}_0$,

$$\sum_{\nu=l}^{\infty} |B_l^\nu\,\psi_\nu| \le (3+2\,\ln(1+l))\,\|\psi\|_{1/2}.$$

For $n \in \mathbb{N}_0$ let $\psi_n := \mathsf{P}_0^n[\psi]$. We immediately verify

$$\hat{T}\,[\psi_n] = \sum_{l=0}^{n}\left[\sum_{\nu=l}^{n} B_l^\nu\,\psi_\nu\right]\sigma^l = (\mathsf{P}_0^n\circ\hat{T})[\psi_n].$$

This settles $\mathsf{P}_0^n\circ\hat{T}\circ\mathsf{P}_0^n = \hat{T}\circ\mathsf{P}_0^n$.
Now we introduce for $s \in [0,1]$ and $n \in \mathbb{N}$,

$$g_s^{(n)} : [0,1] \to \mathbb{R},\quad g_s^{(n)}(t) = \sum_{l=0}^{n-1}\left(\sum_{\nu=l+1}^{n}\psi_\nu\,(\sigma(s))^{\nu-l-1}\right)\,(\sigma(t))^l,$$

i.e.

$$g_s^{(n)}(t) = \begin{cases} \dfrac{\psi_n(s) - \psi_n(t)}{s-t} & , \quad t \neq s \\ \dfrac{d\,\psi_n}{d\,s}(t) & , \quad t = s \end{cases},$$

such that for all $s \in [0,1]$ and all $n \in \mathbb{N}$,

$$\hat{T}[\psi_n](s) = -\int_0^s g_s^{(n)}(t)\;dt + \int_s^1 g_s^{(n)}(t)\;dt.$$

We need an auxiliary result:

Proposition 8. *For $n \in \mathbb{N} \cup \{\infty\}$ let $(\gamma_l^{(n)})_{l \in \mathbb{N}_0}$ be a real sequence. Assume*

a) There is $K \in \mathbb{R}^+$ such that for all $l \in \mathbb{N}_0$ and all $n \in \mathbb{N}$:

$$|\gamma_l^{(n)} - \gamma_l^{(\infty)}| \leq K,$$

b) $\limsup\limits_{l\to\infty} \sqrt[l]{|\gamma_l^{(\infty)}|} < 2$.

c) For all $l \in \mathbb{N}_0$: $\lim_{n\to\infty} \gamma_l^{(n)} = \gamma_l^{(\infty)}$.

Then:

1. The serie $\left(\sum_{l=0}^{m} \gamma_l^{(\infty)}\,\sigma^l\right)_{m\in\mathbb{N}_0}$ converges in C wrt $\|.\|_\infty$.

2. $\lim\limits_{n\to\infty} \left\| \sum_{l=0}^{n} \gamma_l^{(n)}\,\sigma^l - \sum_{l=0}^{\infty} \gamma_l^{(\infty)}\,\sigma^l \right\|_\infty = 0.$

Proof. 1. immediately follows from b).
2. We observe that a) and b) imply for all $n \in \mathbb{N}_0$:

$$\limsup_{l\to\infty} \sqrt[l]{|\gamma_l^{(n)}|} \leq \max\left\{1, \limsup_{l\to\infty} \sqrt[l]{|\gamma_l^{(\infty)}|}\right\} < 2,$$

such that $\left(\sum_{l=0}^{n} \gamma_l^{(n)}\,\sigma^l\right)_{n\in\mathbb{N}_0}$ converges uniformly on $[0,1]$. Let $\varepsilon > 0$. Then there is $N_1(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N_1(\varepsilon)$:

$$\left\| \sum_{l=n+1}^{\infty} \gamma_l^{(\infty)}\,\sigma^l \right\|_\infty \leq \frac{\varepsilon}{3}.$$

Furthermore, we can take $N_2(\varepsilon) \in \mathbb{N}$, $N_1(\varepsilon) \leq N_2(\varepsilon)$, such that for all $n \geq N_2(\varepsilon)$:

$$\sup_{s\in[0,1]} \sum_{l=n+1}^{\infty} |\sigma(s)|^l \leq \frac{\varepsilon}{3\,K}.$$

Finally we can take $N_3(\varepsilon) \in \mathbb{N}$, $N_2(\varepsilon) \leq N_3(\varepsilon)$, using assumption c), such that for all $n \geq N_3(\varepsilon)$:

$$\left\| \sum_{l=0}^{N_2(\varepsilon)} (\gamma_l^{(n)} - \gamma_l^{(\infty)})\, \sigma^l \right\|_\infty \leq \frac{\varepsilon}{3}.$$

We therefore obtain for all $s \in [0,1]$ and all $n \in \mathbb{N}$, $n \geq N_3(\varepsilon)$,

$$\begin{aligned}
&\left| \sum_{l=0}^{n} \gamma_l^{(n)}\, (s-(1/2))^l - \sum_{l=0}^{\infty} \gamma_l^{(\infty)}\, (s-(1/2))^l \right| \\
&\leq \left| \sum_{l=0}^{N_2(\varepsilon)} (\gamma_l^{(n)} - \gamma_l^{(\infty)})\, (s-(1/2))^l \right| + \left| \sum_{l=N_2(\varepsilon)+1}^{n} (\gamma_l^{(n)} - \gamma_l^{(\infty)})\, (s-(1/2))^l \right| \\
&\qquad + \left| \sum_{l=n+1}^{\infty} \gamma_l^{(\infty)}\, (s-(1/2))^l \right| \\
&\qquad \leq \frac{\varepsilon}{3} + \sum_{l=N_2(\varepsilon)+1}^{n} |\gamma_l^{(n)} - \gamma_l^{(\infty)}|\, |s-(1/2)|^l + \frac{\varepsilon}{3} \leq \varepsilon.
\end{aligned}$$

□

We apply Proposition 8 (for fixed $s \in [0,1]$) with

$$\gamma_l^{(n)} := \sum_{\nu=l+1}^{n} \psi_\nu\, (\sigma(s))^{\nu-l-1} \text{ (for } 0 \leq l \leq n-1) \text{ and } \gamma_l^{(n)} := 0 \text{ (for } l \geq n)$$

and

$$\gamma_l^{(\infty)} := \sum_{\nu=l+1}^{\infty} \psi_\nu\, (\sigma(s))^{\nu-l-1}$$

to obtain

$$\lim_{n\to\infty} \sup_{t\in[0,1]} \left| g_s^{(n)}(t) - g_s^{(\infty)}(t) \right| = 0,$$

where

$$g_s^{(\infty)} : [0,1] \to \mathbb{R}, \quad g_s^{(\infty)}(t) = \sum_{l=0}^{\infty} \left(\sum_{\nu=l+1}^{\infty} \psi_\nu\, (\sigma(s))^{\nu-l-1} \right) (\sigma(t))^l,$$

i.e.

$$g_s^{(\infty)}(t) = \begin{cases} \dfrac{\psi(s)-\psi(t)}{s-t} & , \quad t \neq s \\[2ex] \dfrac{d\,\psi}{d\,s}(t) & , \quad t = s \end{cases},$$

Hence for all $s \in [0,1]$,

$$\hat{T}[\psi](s) = -\int_0^s \frac{\psi(s)-\psi(t)}{s-t}\,dt + \int_s^1 \frac{\psi(s)-\psi(t)}{s-t}\,dt$$
$$= -\lim_{n\to\infty}\int_0^s g_s^{(n)}(t)\,dt + \lim_{n\to\infty}\int_s^1 g_s^{(n)}(t)\,dt$$
$$= \lim_{n\to\infty}\hat{T}[\psi_n](s).$$

We furthermore have for each $n \in \mathbb{N}$,

$$-\int_0^s g_s^{(n)}(t)\,dt + \int_s^1 g_s^{(n)}(t)\,dt = \sum_{l=0}^{n}\left[\sum_{\nu=l}^{n} B_l^{\nu}\,\psi_\nu\right](\sigma(s))^l,$$

such that we can Proposition 8 again to obtain

$$\lim_{n\to\infty}\left\|\sum_{l=0}^{n}\left[\sum_{\nu=l}^{n} B_l^{\nu}\,\psi_\nu\right]\sigma^l - \sum_{l=0}^{\infty}\left[\sum_{\nu=l}^{\infty} B_l^{\nu}\,\psi_\nu\right]\sigma^l\right\|_\infty = 0.$$

We set for $l \in \mathbb{N}_0$, $\gamma_l := \sum_{\nu=l}^{\infty} B_l^{\nu}\,\psi_\nu$. Then we have due to a)

$$\limsup_{l\to\infty}\sqrt[l]{|\gamma_l|} \le 1,$$

such that the (formal) power series $(\sum_{l=0}^{n}\gamma_l\sigma^l)_{n\in\mathbb{N}_0}$ converges for each $\theta\in]0,1[$ uniformly on $[-1/2+\theta, 3/2-\theta]$. This implies convergence of

$$\left(\sum_{l=0}^{n}\left[\sum_{\nu=l}^{\infty} B_l^{\nu}\,\psi_\nu\right]\sigma^l\right)_{n\in\mathbb{N}_0}$$

in C wrt $\|.\|_\infty$ with limiting function $\sum_{l=0}^{\infty}[\sum_{\nu=l}^{\infty} B_l^{\nu}\,\psi_\nu]\,\sigma^l = \hat{T}[\psi]$. □

Theorem 6

Proof. We shall discuss the operator equation $(\lambda-\hat{T})[\psi] = F$ with unknown $\psi = \sum_{l=0}^{\infty}\psi_l\,\sigma^l \in \mathcal{L}^\infty$ and given $F = \sum_{l=0}^{\infty}F_l\,\sigma^l \in \mathcal{L}_w^\infty$, $\lambda\in\mathbb{R}$.

We observe: $\psi\in\mathcal{L}^\infty$ satisfies

$$\hat{T}_\lambda[\psi] := (\lambda-\hat{T})[\psi] = F \tag{6.18}$$

iff $\psi\in\mathcal{L}^\infty$ satisfies

$$(\mathsf{P}_0^n\circ\hat{T}_\lambda\circ\mathsf{P}_0^n)[\psi] + (\mathsf{P}_0^n\circ\hat{T}_\lambda\circ\mathsf{P}_{n+1}^\infty)[\psi] = \mathsf{P}_0^n[F],$$

$$(\mathsf{P}_{n+1}^\infty\circ\hat{T}_\lambda\circ\mathsf{P}_{n+1}^\infty)[\psi] = \mathsf{P}_{n+1}^\infty[F],$$

where we make use of the fact that $\mathsf{P}_{n+1}^\infty\circ\hat{T}_\lambda\circ\mathsf{P}_0^n$ is the zero operator on $\mathcal{L}^\infty$.

Hence (6.18) has a solution $\psi \in \mathcal{L}^\infty$ iff $\psi = \eta + \xi$, $\eta \in \mathsf{P}_0^n[\mathcal{L}^\infty]$, $\xi \in \mathsf{P}_{n+1}^\infty[\mathcal{L}^\infty]$ and

$$\left.\begin{aligned} (\mathsf{P}_0^n \circ \hat{T}_\lambda)[\eta] + (\mathsf{P}_0^n \circ \hat{T}_\lambda)[\xi] = \mathsf{P}_0^n[F], \\ (\mathsf{P}_{n+1}^\infty \circ \hat{T}_\lambda)[\xi] = \mathsf{P}_{n+1}^\infty[F]. \end{aligned}\right\} \tag{6.19}$$

This observation shows that it is adviseful to introduce for $n \in \mathbb{N}$ the operators

$$\mathsf{A}_n := \mathsf{P}_0^n \circ \hat{T}_\lambda \downharpoonleft \mathsf{P}_0^n[\mathcal{L}^\infty],$$
$$\mathsf{B}_n := \mathsf{P}_0^n \circ \hat{T}_\lambda \downharpoonleft \mathsf{P}_{n+1}^\infty[\mathcal{L}^\infty],$$
$$\mathsf{D}_n := \mathsf{P}_{n+1}^\infty \circ \hat{T}_\lambda \downharpoonleft \mathsf{P}_{n+1}^\infty[\mathcal{L}^\infty],$$

such that $(\eta, \xi) \in \mathsf{P}_0^n[\mathcal{L}^\infty] \times \mathsf{P}_{n+1}^\infty[\mathcal{L}^\infty]$ is a solution of (6.19) iff

$$\mathsf{A}_n[\eta] + \mathsf{B}_n[\xi] = \mathsf{P}_0^n[F], \tag{6.20}$$
$$\mathsf{D}_n[\xi] = \mathsf{P}_{n+1}^\infty[F]. \tag{6.21}$$

It is easy to verify: For $n \in \mathbb{N}_0$,

$$\sum_{l=0}^{n} \eta_l\, \sigma^l \in \mathsf{P}_0^n[\mathcal{L}^\infty], \qquad \sum_{l=n+1}^{\infty} \xi_l\, \sigma^l \in \mathsf{P}_{n+1}^\infty[\mathcal{L}^\infty],$$

the matrix representation of $\mathsf{A}_n[\eta]$ is

$$\begin{pmatrix} \lambda + L_0 & -B_0^1 & -B_0^2 & -B_0^3 & \cdots & -B_0^n & 0 & \cdots \\ 0 & \lambda + L_1 & -B_1^2 & -B_1^3 & \cdots & -B_1^n & 0 & \cdots \\ 0 & 0 & \lambda + L_2 & -B_2^3 & \cdots & -B_2^n & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \cdots \\ 0 & 0 & 0 & \ldots & \lambda + L_{n-1} & -B_{n-1}^n & 0 & \cdots \\ 0 & 0 & 0 & \ldots & 0 & \lambda + L_n & 0 & \cdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \tag{6.22}$$

the matrix representation of $\mathsf{B}_n[\xi]$ is

$$\begin{pmatrix} 0 & \cdots & 0 & -B_0^{n+1} & -B_0^{n+2} & -B_0^{n+3} & \cdots \\ 0 & \cdots & 0 & -B_1^{n+1} & -B_1^{n+2} & -B_1^{n+3} & \cdots \\ 0 & \cdots & 0 & -B_2^{n+1} & -B_2^{n+2} & -B_2^{n+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & \cdots & 0 & -B_{n-1}^{n+1} & -B_{n-1}^{n+2} & -B_{n-1}^{n+3} & \cdots \\ 0 & \cdots & 0 & -B_n^{n+1} & -B_n^{n+2} & -B_n^{n+3} & \cdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \xi_{n+1} \\ \xi_{n+2} \\ \xi_{n+3} \\ \xi_{n+4} \\ \vdots \end{pmatrix}, \tag{6.23}$$

and the matrix representation of $\mathsf{D}_n[\xi]$ is

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \lambda + L_{n+1} & -B_{n+1}^{n+2} & -B_{n+1}^{n+3} & -B_{n+1}^{n+4} & \cdots \\ 0 & \cdots & 0 & 0 & \lambda + L_{n+2} & -B_{n+2}^{n+3} & -B_{n+2}^{n+4} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \lambda + L_{n+3} & -B_{n+3}^{n+4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_{n+1} \\ \xi_{n+2} \\ \xi_{n+3} \\ \vdots \end{pmatrix}. \tag{6.24}$$

Due to (6.24) we have $\mathsf{D}_n = \Lambda_n + \mathsf{E}_n$, where the operators

$$\Lambda_n : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}], \quad \mathsf{E}_n : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}]$$

are defined by their respective matrix representations, namely for

$$\xi = \sum_{l=n+1}^{\infty} \xi_l \, \sigma^l \in \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}],$$

the matrix representation of $\Lambda_n[\xi]$ is

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \lambda + L_{n+1} & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \lambda + L_{n+2} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \lambda + L_{n+3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_{n+1} \\ \xi_{n+2} \\ \xi_{n+3} \\ \vdots \end{pmatrix}, \tag{6.25}$$

and the matrix representation of $\mathsf{E}_n[\xi]$ is

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & -B_{n+1}^{n+2} & -B_{n+1}^{n+3} & -B_{n+1}^{n+4} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & -B_{n+2}^{n+3} & -B_{n+2}^{n+4} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & -B_{n+3}^{n+4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_{n+1} \\ \xi_{n+2} \\ \xi_{n+3} \\ \vdots \end{pmatrix}. \tag{6.26}$$

Now it can be left to the reader to verify

Proposition 9. *Let $\lambda \in \mathbb{R}$. Let*

$$N_1(\lambda) := \min\{k \in \mathbb{N} : L_{k+1} > \log(4/3) - \lambda\}.$$

Then

- $0 < \lambda + L_{N(\lambda)+1} < \lambda + L_{N(\lambda)+2} < \ldots,$
- *and for all $n \in \mathbb{N}_0$ with $N_1(\lambda) \leq n$, the linear operator Λ_n has a bounded inverse with*

$$\begin{aligned}\left\|\Lambda_n^{-1} : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}]\right\| \\ \leq \sup\left\{\frac{3 + 2\,\log(1+l)}{\lambda + L_l} : l \in \mathbb{N}, n+1 \leq l\right\} < \infty,\end{aligned}$$

- *and for all $n \in \mathbb{N}_0$ with $N_1(\lambda) \leq n$,*

$$\left\|\Lambda_n^{-1} \circ \mathsf{E}_n : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}]\right\| \leq \frac{\log(4/3)}{\lambda + L_{n+1}} < 1.$$

Furthermore,

$$\limsup_{n\to\infty} \left\|\Lambda_n^{-1} : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}]\right\| \leq 2.$$

Due to Proposition 9 the operator Λ_n has a bounded inverse Λ_n^{-1} and

$$\left\|\Lambda_n^{-1} \circ \mathsf{E}_n : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}]\right\| < 1. \tag{6.27}$$

Hence, we can re-write equation (6.21) for $n \in \mathbb{N}$ with $N_1(\lambda) \leq n$ as

$$[1 + \Lambda_n^{-1} \circ E_n][\xi] = \Lambda_n^{-1}[\mathsf{P}_{n+1}^{\infty}[F]]. \tag{6.28}$$

Now it readily follows from (6.27) that (6.28) has for all $n \in \mathbb{N}$ with $N_1(\lambda) \leq n$ a unique solution $\xi \in \mathcal{L}^{\infty}$, i.e. equation (6.21) has for all $n \in \mathbb{N}$ with $N_1(\lambda) \leq n$

a unique solution $\xi \in \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}]$. Furthermore, we deduce from (6.28) for all $n \in \mathbb{N}_0$ with $N_1(\lambda) \leq n$ the estimate

$$\|\xi\|_{1/2} \leq \frac{\left\|\Lambda_n^{-1} : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}]\right\|}{1 - \frac{\log(4/3)}{\lambda + L_{n+1}}} \|F\|_{w,1/2}. \tag{6.29}$$

Inserting ξ in equation (6.20) we obtain

$$\mathsf{A}_n[\eta] = \mathsf{P}_0^n[F] - \mathsf{B}_n[\xi], \quad n \in \mathbb{N}, n \geq N_1(\lambda), \tag{6.30}$$

where we note that the right-hand side of this equation belongs to the finite-dimensional space $\mathsf{P}_0^n[\mathcal{L}_w^{\infty}]$. Due to the diagonal structure of A_n, two possibilities arise:

1) $\lambda \notin \{-L(l) : l \in \mathbb{N}_0\}$. Then A_n is invertible for any $n \in \mathbb{N}_0$ and we obtain for all $n \in \mathbb{N}$ with $N_1(\lambda) \leq n$ the estimate

$$\|\eta\|_{1/2} \leq \left\|\mathsf{A}_n^{-1} : \mathsf{P}_0^n[\mathcal{L}_w^{\infty}] \to \mathsf{P}_0^n[\mathcal{L}^{\infty}]\right\| \left(\|F\|_{w,1/2} + \log(4/3)\,\|\xi\|_{1/2}\right),$$

where we made use of the fact that

$$\left\|\mathsf{B}_n : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}^{\infty}] \to \mathsf{P}_0^n[\mathcal{L}_w^{\infty}]\right\| \leq \log(4/3).$$

(3.13) follows from the estimates given so far.

2) $\lambda = -L(l)$ for some $l \in \mathbb{N}_0$. In this case the defect of the operator A_n is - due to the diagonal structure of its matrix representation - one. It can be left to the reader to verify that λ is an eigenvalue of $\hat{T}$ whose eigenspace is spanned by q_l. □

Theorem 7

Proof. We employ the notations of the proof of Theorem 6. We observe: $x^n \in \mathsf{P}_0^n[\mathcal{L}^{\infty}]$ for any $n \in \mathbb{N}$ and

$$\mathsf{A}_n[x^n] = \mathsf{P}_0^n[f].$$

Hence, due to (6.20) and (6.21),

$$\mathsf{A}_n[\eta - x^n] - \mathsf{B}_n[\xi] = 0, \quad \mathsf{D}_n[\xi] = \mathsf{P}_{n+1}^{\infty}[f],$$

where $\eta = \mathsf{P}_0^n[\phi]$ and $\xi = \mathsf{P}_{n+1}^{\infty}[\phi]$. Hence

$$\phi - x^n = (\eta - x^n) + \xi$$

solves

$$(c - \hat{T})[\phi - x^n] = \mathsf{P}_{n+1}^{\infty}[f].$$

The verification of the several statements of the Theorem can be left to the reader now. □

References

[1] T. Götz. *Interaction of Fibers with Airflow.* PhD thesis, Math. Dep., University of Kaiserslautern, to be finished.

[2] J. Happel and H. Brenner. *Low Reynolds number hydrodynamics.* Noordhoff International Publishing, 1973.

[3] A.T. Chwang and T.Y. Wu. Hydromechanics of low–Reynolds–number flow. Part2. Singularity methods for Stokes flow. *Journal of Fluid mechanics*, 67(4):787—815, 1975.

[4] S.H. Lee and L.G. Leal. Low–Reynolds–number flow past cylindrical bodies of arbitrary cross–sectional shape. *Journal of Fluid mechanics*, 164:401–427, 1986.

[5] H. Ockendon and E.L. Terril. A mathematical model for the wet–spinning process. *Euro. J. Appl. Math.*, 4:341–360, 1993.

[6] C.S. Peskin and D.M. McQueen. A three–dimensional computational method for blood flow in the heart. I. Immersed elastic fibers in a viscous incompressible fluid. *J. Comp. Physics*, 81:372–405, 1989.

[7] J. Lighthill. *Mathematical Biofluiddynamics.* SIAM, 1975.

[8] C. Pozrikdis. *Boundary integrals & singularity methods for linearized viscous flow*, volume 8 of *Cambridge texts in applied mathematics.* Cambridge University Press, 1992.

[9] J.B. Keller and S.I. Rubinow. Slender–body theory for slow viscous flow. *Journal of Fluid mechanics*, 75(4):705–714, 1976.

[10] R.G. Cox. The motion of long slender bodies in a viscous fluid. Part1. General theory. *Journal of Fluid mechanics*, 44(4):791–810, 1970.

[11] A.D. Polyanin and A.V. Manzhirov. *Handbook of integral equations.* Spektrum, 1999.

[12] R. Gorenflo and S. Vessella. *Abel Integral equatons*, volume 1461 of *Lecture Notes in Mathematics.* Springer, 1991.

[13] N.I. Muskhelishvili. *Singular Integral Equations.* Wolters–Noordhoff, 1972.

[14] W. Hackbusch. *Integralgleichungen : Theorie und Numerik.* Teubner, 1989.