

# Scale Continuous, Scale Discretized and Scale Discrete Harmonic Wavelets for the Outer and the Inner Space of a Sphere and Their Application to an Inverse Problem in Geomathematics

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## Abstract

In this paper we construct a multiscale solution method for the gravimetry problem, which is concerned with the determination of the earth's density distribution from gravitational measurements. For this purpose isotropic scale continuous wavelets for harmonic functions on a ball and on a bounded outer space of a ball, respectively, are constructed. The scales are discretized and the results of numerical calculations based on regularization wavelets are presented. The obtained solutions yield topographical structures of the earth's surface at different levels of localization ranging from continental boundaries to local structures such as Ayer's Rock and the Amazonas area.

**Keywords:** Wavelet, Multiscale Methods, Inverse Problem, Gravimetry, Regularization, Isotropy

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## 1 Introduction

The increasing interest in advanced methods in signal and image processing motivated the development of Euclidean wavelets (see for example Mallat [14] (1989)). For more details on Euclidean wavelets and their application in signal and image processing see Mallat's presentation at the International Congress of Mathematicians 1998 ([15]) and the references herein. Later on the advantages of wavelets became of interest for geomathematical problems such as the representation of the earth's gravitational field. Since the common operators in geodesy and geophysics are isotropic (cf. for example Svensson [20] (1983)), the introduced scale continuous spherical wavelets and their discretizations (see Freeden, Windheuser [11] (1996)) were chosen to be product kernels

$$\Psi_\rho(\xi, \eta) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (\Psi_\rho)^\wedge(n) Y_{n,j}(\xi) Y_{n,j}(\eta), \quad (1)$$

where  $\xi, \eta$  are unit vectors and  $\{Y_{n,j}\}_{n \in \mathbb{N}; j \in \{1, \dots, 2n+1\}}$  is an orthonormal system of homogeneous harmonic polynomials. This isotropy allows the application of the addition theorem for spherical harmonics (see for example [8], p. 37) which enables the representation of these kernels as one-dimensional functions

$$\Psi_\rho(\xi, \eta) = \sum_{n=0}^{\infty} (\Psi_\rho)^\wedge(n) \frac{2n+1}{4\pi} P_n(\xi \cdot \eta), \quad (2)$$

where  $P_n$  is the Legendre polynomial of degree  $n$ . Moreover, pyramid schemes are available as fast numerical algorithms (see Schreiner [19] (1996) for the band-limited case, Freeden [7] (1999) for the non-band-limited case and Beth [4] (2000) for the generalization to vector fields including error control). For more details on spherical wavelets and their application to geomathematical problems we refer to Freeden et al. ([8] (1998)), Freeden ([7] (1999)) and the references in the monographs.

A group theoretical approach for the construction of scale continuous spherical wavelets is used by Vandergheynst ([21] (1998), see also Antoine, Vandergheynst [1] (1999)). These wavelets are in general not isotropic such that they are not suitable for most of the geoscientific problems but might be appropriate in image processing. However, scale discrete versions of these wavelets have not been constructed and they have not been applied to real problems yet, such that their numerical properties and their applicability still have to be investigated.

Since a regular surface is a better approximation to the earth's surface than

a sphere, Freedon, Schneider ([10] (1998)) developed scale discrete wavelets on these generalized domains.

However, mathematical models in the geosciences do not always use functions, whose domain is a surface. An example of such an interesting geomathematical problem is the gravimetry problem, which is concerned with the inversion of Newton's gravitational potential

$$\rho \mapsto \int_{earth} \frac{\rho(x)}{|x - \cdot|} dx. \quad (3)$$

The reconstruction of the density distribution of the earth has numerous applications, for example in geoexploration and seismology. A characteristic drawback of this inverse problem is the non-uniqueness of the solution. More precisely, only the harmonic part of an  $\mathcal{L}^2$ -density function can be reconstructed from the gravitational potential. The orthogonal space of so-called anharmonic functions is the null space of the corresponding Fredholm integral operator of first kind. Therefore, solution strategies for the gravimetry problem should be aware of the fact that only the restriction to harmonic functions yields a uniquely solvable problem. The anharmonic part of the earth's density distribution must be determined from non-gravitational a priori information (see e.g. Weck [24] (1972), Ballani, Stromeyer [3] (1990), Ballani et al. [2] (1993) and Michel [16] (1999)). Therefore, we restrict our attention to the construction of a multiresolution for harmonic functions in this work.

In Michel ([16]) scale discrete wavelets and scaling functions for harmonic functions on the inner space of a sphere and a bounded outer space (for example the area from the earth's surface to a satellite orbit), respectively, are constructed for the development of a multiscale solution method for the gravimetry problem.

The standard technique in physical geodesy is an expansion of the investigated functions in a basis of homogeneous harmonic polynomials. This has obvious disadvantages. The most essential drawback is the global support of the basis, such that a highly localized resolution is only obtainable by increasing the maximum degree of the truncated singular value decomposition to extreme sizes, which is from the numerical point of view very expensive and severely unstable. Uncertainty principles in this context (see for example Freedon, Michel [9] (1999)) show that the product of the variances in space and momentum must be larger than a given positive constant. Therefore, a purely momentum localizing technique like the calculation of Fourier coefficients with respect to homogeneous polynomials has no space localization, such that, as an improvement, kernels are needed which have a positive vari-

ance in space as well as in momentum. Wavelets satisfy this desired property. Moreover, the choice between band-limited and non-band-limited wavelets allows the variation of the intensities of the localizations.

Whereas the usual approach to scale discrete wavelets, i.e. the kernels  $\Psi_J$  are only defined for values  $J \in \mathbb{N}$  or  $J \in \mathbb{Z}$ , begins with the introduction of scale continuous wavelets, i.e. kernels  $\Psi_\rho$  with  $\rho \in \mathbb{R}^+$  (see for example Freedon, Windheuser [11] and Freedon et al. [8], p. 227), which are then discretized, the approach in Michel ([16]), such as the approaches in Freedon, Schneider ([10]) and Freedon ([7]), directly constructs scale discrete wavelets. We will see in this work that it is also possible to develop a continuous wavelet theory for the gravimetry problem, from which scale discrete wavelets can be derived, which are very similar to those introduced in Michel ([16]). Since the Fredholm integral operator in (3) is isotropic, the constructed wavelets are chosen to be isotropic, too. Here, we restrict our attention to the bilinear case, since this approach allows to represent a function via a wavelet coefficient scheme. Finally, we discuss the application of the introduced wavelets and scaling functions to the gravimetry problem and show some results of numerical calculations for the density on the earth's surface from the NASA, GSFC, and NIMA Earth Geopotential Model (EGM96).

## 2 Harmonic Functions

In this section we will present the definitions and important well-known properties of harmonic functions. By  $\mathbb{N}$  we denote the set of all non-negative integers.  $\mathbb{Z}$  is the symbol for the set of integers and  $\mathbb{R}$  is the symbol for the set of real numbers, such that  $\mathbb{R}_0^+$  represents all non-negative real numbers and  $\mathbb{R}^+$  represents all positive real numbers. The Euclidean scalar product on  $\mathbb{R}^n$  is denoted by  $x \cdot y := \sum_{i=1}^n x_i y_i$ ,  $x, y \in \mathbb{R}^n$ .

Every integral used in this work is a Lebesgue integral. As usual  $\mathcal{L}^2(D)$  denotes the set of all classes of almost everywhere identical squareintegrable functions on the measurable set  $D \subset \mathbb{R}^n$ .

**Definition 2.1** ([5], p. 496) *Let  $D \subset \mathbb{R}^n$  be connected. A function  $F \in C^{(2)}(D)$  is called harmonic iff  $\Delta F(x) = 0$  for all  $x \in \text{int } D$ . The set of all harmonic functions in  $C^{(2)}(D)$  is denoted by  $\text{Harm}(D)$ .*

As domains of the considered functions we use the unit sphere

$$\Omega := \{ \xi \in \mathbb{R}^3 \mid |\xi| = 1 \} \quad (4)$$

and the closures of the bounded outer space

$$B_{\text{ext}}^\gamma := \{x \in \mathbb{R}^3 \mid \beta < |x| < \gamma\} \quad (5)$$

( $\gamma > \beta$  sufficiently large) and the inner space

$$B_{\text{int}} := \{x \in \mathbb{R}^3 \mid |x| < \beta\} \quad (6)$$

of the sphere  $B$  with radius  $\beta > 0$ .  $B$  is supposed to be an approximation to the earth's surface. One can use for example the maximum radius of a satellite orbit as  $\gamma$ .

The introduction of the restriction  $|x| \leq \gamma$  in  $\overline{B_{\text{ext}}^\gamma}$  allows us to use outer harmonics (see Definition 2.2) of degree  $-1$  in  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ . In physical geodesy it is usual to avoid coefficients of this degree by choosing an appropriate earth's fixed coordinate system (cf. [12]).

As already mentioned  $\{Y_{n,j}\}_{n \in \mathbb{N}, j \in \{1, \dots, 2n+1\}}$  denotes a complete orthonormal system in  $\text{Harm}(\Omega)$ , such that  $Y_{n,j}$  is the restriction of a homogeneous harmonic polynomial of degree  $n$  to  $\Omega$ .

**Definition 2.2** *Let a system of spherical harmonics  $\{Y_{n,j}\}_{n \in \mathbb{N}; j=1, \dots, 2n+1}$  be given as introduced above. Then we define the outer harmonics by*

$$H_{-n-1,j}^{\text{ext}}(\beta; x) := \sqrt{\frac{2n-1}{\beta^3 - \beta^{2n+2}\gamma^{-2n+1}}} \left(\frac{\beta}{|x|}\right)^{n+1} Y_{n,j}\left(\frac{x}{|x|}\right), \quad x \in \overline{B_{\text{ext}}^\gamma}, \quad (7)$$

where  $n \in \mathbb{N}$  and  $j \in \{1, \dots, 2n+1\}$  and

$$c_{\text{ext}}(n, \beta) := \sqrt{\frac{2n-1}{\beta^3 - \beta^{2n+2}\gamma^{-2n+1}}}. \quad (8)$$

**Definition 2.3** ([10]; [16], p. 15) *Let  $n \in \mathbb{N}$ . Then we define*

$$\begin{aligned} \text{Harm}_n(\overline{B_{\text{ext}}^\gamma}) &:= \text{span} \{H_{-n-1,j}^{\text{ext}}(\beta; \cdot) \mid j \in \{1, \dots, 2n+1\}\}, \\ \text{Harm}_{0 \dots n}(\overline{B_{\text{ext}}^\gamma}) &:= \text{span} \{H_{-m-1,k}^{\text{ext}}(\beta; \cdot) \mid m \in \{0, \dots, n\}, \\ &\quad k \in \{1, \dots, 2m+1\}\}, \\ \text{Harm}_{0 \dots \infty}(\overline{B_{\text{ext}}^\gamma}) &:= \text{span} \{H_{-m-1,k}^{\text{ext}}(\beta; \cdot) \mid m \in \mathbb{N}, k \in \{1, \dots, 2m+1\}\}. \end{aligned}$$

Moreover, we define  $\text{Harm}_{0 \dots m}(\overline{B_{\text{ext}}^\gamma}) := \{0\}$ , if  $m < 0$ .

Using the well known theorem of orthogonal decompositions in Hilbert spaces (cf. [25], p. 82) we define the following operators.

**Definition 2.4** Let  $n \in \mathbb{N}$ . Then  $\mathcal{P}_n : \mathcal{L}^2(\overline{B_{\text{ext}}^\gamma}) \rightarrow \text{Harm}_{0\dots n}(\overline{B_{\text{ext}}^\gamma})$  is the projection operator on  $\text{Harm}_{0\dots n}(\overline{B_{\text{ext}}^\gamma})$ .

**Theorem 2.5**

$$\overline{\text{Harm}_{0\dots\infty}(\overline{B_{\text{ext}}^\gamma})}^{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} = \text{Harm}(\overline{B_{\text{ext}}^\gamma}). \quad (9)$$

Concerning the proof we refer to [6]. This property has an immediate consequence.

**Corollary 2.6** The outer harmonics are a complete orthonormal system in  $\text{Harm}(\overline{B_{\text{ext}}^\gamma})$  with respect to  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ . Hence, for all  $F \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  there exists a sequence  $\{F^\wedge(-n-1, j)\}_{n \in \mathbb{N}; j=1, \dots, 2n+1}$  such that

$$F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \quad (10)$$

in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ , i.e.

$$\lim_{N \rightarrow \infty} \left\| F - \sum_{n=0}^N \sum_{j=1}^{2n+1} F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} = 0, \quad (11)$$

where

$$F^\wedge(-n-1, j) = (F, H_{-n-1, j}^{\text{ext}}(\beta; \cdot))_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} \quad (12)$$

for all  $n \in \mathbb{N}$  and all  $j \in \{1, \dots, 2n+1\}$ .

### 3 Product Kernels

For discussing wavelets and filters we have to introduce product kernels and convolutions (see also [16], p. 60ff).

**Definition 3.1** A function  $K \in \mathcal{L}^2(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma})$  with

$$K(x, y) = \sum_{n=0}^{\infty} K^\wedge(-n-1) \sum_{j=1}^{2n+1} H_{-n-1, j}^{\text{ext}}(\beta; x) H_{-n-1, j}^{\text{ext}}(\beta; y) \quad (13)$$

is called (harmonic outer) product kernel. The sequence  $\{K^\wedge(-n-1)\}_{n \in \mathbb{N}}$  is called symbol of the product kernel. (cf. [10]; [16], p. 60)

The convergence of the series is understood in the  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ -topology and in pointwise sense for all  $x, y \in \overline{B_{\text{ext}}^\gamma}$ .

Note that the restrictions of these product kernels to spheres are isotropic. Therefore, they can be represented by a one-dimensional function using the addition theorem for spherical harmonics (cf. for example [8], p. 37 and [16], p. 61).

Now we can define convolutions.

**Definition 3.2** Let  $K, L \in \mathcal{L}^2\left(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma}\right)$  be two product kernels. Furthermore, let  $F \in \text{Harm}\left(\overline{B_{\text{ext}}^\gamma}\right)$  be a given function. We define the convolutions by

$$(K * F)(x) := \int_{\overline{B_{\text{ext}}^\gamma}} K(x, y)F(y) dy \forall x \in \overline{B_{\text{ext}}^\gamma}, \quad (14)$$

$$(K * L)(x, y) := \int_{\overline{B_{\text{ext}}^\gamma}} K(x, z)L(z, y) dz \forall x, y \in \overline{B_{\text{ext}}^\gamma}. \quad (15)$$

**Theorem 3.3** Let  $K, L \in \mathcal{L}^2\left(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma}\right)$  be two product kernels with the symbols  $\{K^\wedge(-n-1)\}_{n \in \mathbb{N}}$  and  $\{L^\wedge(-n-1)\}_{n \in \mathbb{N}}$ , respectively. Furthermore, let  $F \in \text{Harm}\left(\overline{B_{\text{ext}}^\gamma}\right)$  be a given function. Then  $K * L$  is a product kernel in  $\mathcal{L}^2\left(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma}\right)$  with

$$(K * L)^\wedge(-n-1) = K^\wedge(-n-1)L^\wedge(-n-1) \forall n \in \mathbb{N}, \quad (16)$$

and  $K * F$  is a function in  $\text{Harm}\left(\overline{B_{\text{ext}}^\gamma}\right)$  with

$$(K * F)^\wedge(-n-1, j) = K^\wedge(-n-1)F^\wedge(-n-1, j) \quad \forall n \in \mathbb{N} \forall j \in \{1, \dots, 2n+1\}. \quad (17)$$

**Definition 3.4** Let  $K \in \mathcal{L}^2\left(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma}\right)$  be a product kernel. The product kernel  $K^{(m)}$ ,  $m \in \mathbb{N} \setminus \{0\}$ , where

$$K^{(m+1)} := K * K^{(m)}, K^{(1)} := K, \quad (18)$$

is called  $m$ th iterated kernel of  $K$ .

Consequently, we get a corollary of Theorem 3.3.

**Corollary 3.5** Let  $K \in \mathcal{L}^2\left(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma}\right)$  be a product kernel with the corresponding symbol  $\{K^\wedge(-n-1)\}_{n \in \mathbb{N}}$ . Then

$$(K^{(m)})^\wedge(-n-1) = (K^\wedge(-n-1))^m \quad (19)$$

for all  $m \in \mathbb{N} \setminus \{0\}$  and all  $n \in \mathbb{N}$ .

## 4 Continuous Wavelets of Order $m$

First of all we want to define the wavelets as a family of product kernels.

**Definition 4.1** *Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a positive integrable function and  $m \in \mathbb{N} \cup \{-1\}$  be a given number. We call  $w$  a weight function. Furthermore, let  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be a family of harmonic product kernels. They are called (outer) (scale continuous) wavelets of order  $m$ , if their symbols satisfy the following conditions:*

(i) For every  $n \in \mathbb{N}$  with  $n > m$

$$\int_0^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho = 1. \quad (20)$$

(ii) For every  $\rho \in \mathbb{R}^+$  and every  $n \in \{\nu \in \mathbb{N} \mid 0 \leq \nu \leq m\}$

$$(\Psi_\rho)^\wedge(-n-1) = 0. \quad (21)$$

(iii) For all  $R \in \mathbb{R}^+$

$$\sum_{n=m+1}^\infty (2n+1)^2 \int_R^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho < +\infty. \quad (22)$$

In this case the kernel  $\Psi := \Psi_1$  is called mother wavelet.

Note that  $\int_0^\infty$  symbolizes the Lebesgue integral on  $\mathbb{R}^+$ .

The introduction of the order  $m$  allows the combination of the multiresolution analysis with an expansion in outer harmonics up to degree  $m$ . Note that in the case  $m = -1$  some expressions and conditions in this work can be omitted. The symbol  $\sum_{n=0}^{-1}$  represents, for example, the empty sum and is defined to be equal to zero.

**Lemma 4.2** *Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a weight function and let  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be a family of outer scale continuous wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . Then*

$$\sum_{n=m+1}^\infty (2n+1)^2 \int_R^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho = \quad (23)$$

$$= \int_R^\infty \sum_{n=m+1}^\infty (2n+1)^2 ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \quad (24)$$

for all  $R \in \mathbb{R}^+$ .



**Proof:** The Beppo Levi Theorem (see [23], p. 331f) allows us to interchange summation and integration here, **q.e.d.**

For the wavelets we can now define a dilation and a translation in an abstract sense.

**Definition 4.3** Let  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  consist of outer scale continuous wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . The dilation operator  $D_\rho$  and the shifting operator  $R_y$  are defined by

$$D_\rho \Psi := \Psi_\rho, \quad (25)$$

$$(R_y \Psi_\rho)(x) := \Psi_\rho(y, x), \quad (26)$$

where  $\rho \in \mathbb{R}^+$  and  $y \in \overline{B_{\text{ext}}^\gamma}$ . Furthermore, we use the denotation

$$\Psi_{\rho; y} := R_y \Psi_\rho. \quad (27)$$

Now we want to define a generalization of  $\mathcal{L}^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})$ .

**Definition 4.4** Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a weight function. We define the space

$$L_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma}) := \left\{ F : \mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma} \rightarrow \mathbb{R} \left| \int_{\overline{B_{\text{ext}}^\gamma}} \int_0^\infty F(\rho; y)^2 w(\rho) d\rho dy < \infty \right. \right\} \quad (28)$$

and the subspace

$$N_w^2 := \left\{ F \in L_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma}) \left| \int_{\overline{B_{\text{ext}}^\gamma}} \int_0^\infty F(\rho; y)^2 w(\rho) d\rho dy = 0 \right. \right\} \quad (29)$$

and introduce

$$\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma}) := L_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma}) / N_w^2. \quad (30)$$

For the space  $\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})$  we define

$$(F, G)_{\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})} := \int_{\overline{B_{\text{ext}}^\gamma}} \int_0^\infty F(\rho; y) G(\rho; y) w(\rho) d\rho dy, \quad (31)$$

where  $F, G \in \mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})$ .

The proof of the following lemma is easy.

**Lemma 4.5** *Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a weight function.*

*Then the mapping  $(\cdot, \cdot)_{\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})}$  is a scalar product on  $\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})$ .  
 $(\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma}), (\cdot, \cdot)_{\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})})$  is a Hilbert space.*

**Lemma 4.6** *Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a weight function and let the product kernels in  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . Furthermore, let  $F \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  be a given arbitrary function. Then the identity*

$$\|\Psi \cdot * F\|_{\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})} \leq \|F\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} \quad (32)$$

*holds and consequently*

$$\Psi \cdot * F \in \mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma}). \quad (33)$$

**Proof:** From Theorem 3.3 we know that

$$\Psi_\rho * F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (\Psi_\rho)^\wedge(-n-1) F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \quad (34)$$

in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ . If we compute the square of the  $\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})$ -norm of this convolution we get

$$\begin{aligned} \int_0^\infty \int_{\overline{B_{\text{ext}}^\gamma}} ((\Psi_\rho * F)(x))^2 dx w(\rho) d\rho &= \int_0^\infty \|\Psi_\rho * F\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 w(\rho) d\rho \\ &= \int_0^\infty \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} ((\Psi_\rho)^\wedge(-n-1))^2 (F^\wedge(-n-1, j))^2 w(\rho) d\rho. \end{aligned} \quad (35)$$

The Beppo Levi Theorem allows us to interchange the integration and the summation over  $n$ .

$$\begin{aligned} &\|\Psi \cdot * F\|_{\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})}^2 \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_0^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho (F^\wedge(-n-1, j))^2 \\ &= \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} F^\wedge(-n-1, j)^2 \\ &\leq \|F\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2, \quad \mathbf{q.e.d.} \end{aligned} \quad (36)$$

More precisely, we obtained that

$$\|\Psi \cdot * F\|_{\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})} = \left\| \mathcal{P}_{\text{Harm}_{m+1 \dots \infty}(\overline{B_{\text{ext}}^\gamma})} \mathcal{L}^2(\overline{B_{\text{ext}}^\gamma}) F \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}, \quad (37)$$

where the norm on the right hand side is taken over a function which is the projection of  $F$  onto the closure of the space

$$\begin{aligned} & \text{Harm}_{m+1 \dots \infty}(\overline{B_{\text{ext}}^\gamma}) \\ & := \text{span} \{ H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \mid n \in \mathbb{N}; n \geq m+1; j \in \{1, \dots, 2n+1\} \}. \end{aligned} \quad (38)$$

The following lemma gives us an interesting connection between the scalar products in  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$  and in  $\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})$ .

**Lemma 4.7** *Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a weight function and let the product kernels in  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . Furthermore, let  $F_1, F_2 \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  be given functions. Then*

$$(F_1, F_2)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} = (\mathcal{P}_m F_1, \mathcal{P}_m F_2)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} + (\Psi \cdot * F_1, \Psi \cdot * F_2)_{\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})}, \quad (39)$$

where  $\mathcal{P}_{-1}G := 0$  for all  $G \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$ .

**Proof:** We get

$$\begin{aligned} & (F_1, F_2)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} - (\mathcal{P}_m F_1, \mathcal{P}_m F_2)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} \\ & = \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} (F_1)^\wedge(-n-1, j) (F_2)^\wedge(-n-1, j) \\ & = \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} (F_1)^\wedge(-n-1, j) (F_2)^\wedge(-n-1, j) \cdot \\ & \quad \cdot \int_0^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \\ & = \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} (F_1)^\wedge(-n-1, j) (F_2)^\wedge(-n-1, j) \cdot \\ & \quad \cdot \lim_{R \rightarrow 0^+} \int_R^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho. \end{aligned} \quad (40)$$

As

$$\begin{aligned}
& \left| (F_1)^\wedge(-n-1, j)(F_2)^\wedge(-n-1, j) \int_R^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right| \\
& \leq |(F_1)^\wedge(-n-1, j)(F_2)^\wedge(-n-1, j)| \int_0^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \\
& = |(F_1)^\wedge(-n-1, j)(F_2)^\wedge(-n-1, j)| \tag{41}
\end{aligned}$$

for all  $n \in \mathbb{N}$  with  $n > m$  and all  $R \in \mathbb{R}^+$  and the series

$$\sum_{n=m+1}^\infty \sum_{j=1}^{2n+1} |(F_1)^\wedge(-n-1, j)(F_2)^\wedge(-n-1, j)| \tag{42}$$

converges absolutely, the Weierstraß criterion for uniform convergence (see [22], p. 142) is satisfied and we may interchange  $\lim_{R \rightarrow 0^+}$  and  $\sum_{n=m+1}^\infty$  (see [22], p. 141) in (40). The result is

$$\begin{aligned}
& (F_1, F_2)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} - (\mathcal{P}_m F_1, \mathcal{P}_m F_2)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} \\
& = \lim_{R \rightarrow 0^+} \left( \sum_{n=m+1}^\infty \sum_{j=1}^{2n+1} (F_1)^\wedge(-n-1, j)(F_2)^\wedge(-n-1, j) \cdot \right. \\
& \quad \left. \cdot \int_R^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right). \tag{43}
\end{aligned}$$

The Beppo Levi Theorem now allows us to interchange summation and integration in (43). We get

$$\begin{aligned}
& (F_1, F_2)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} - (\mathcal{P}_m F_1, \mathcal{P}_m F_2)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} \\
& = \lim_{R \rightarrow 0^+} \int_R^\infty \sum_{n=m+1}^\infty \sum_{j=1}^{2n+1} (F_1)^\wedge(-n-1, j)(F_2)^\wedge(-n-1, j) \cdot \\
& \quad \cdot ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \\
& = \int_0^\infty (\Psi_\rho * F_1, \Psi_\rho * F_2)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} w(\rho) d\rho \\
& = (\Psi * F_1, \Psi * F_2)_{\mathcal{L}_w^2(\mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma})}, \quad \mathbf{q.e.d.} \tag{44}
\end{aligned}$$

Now we can introduce the continuous wavelet transform.

**Definition 4.8** Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a weight function and let  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be a family of wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . Then the bilinear wavelet transform

$$(\mathcal{WT}) : \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right) \rightarrow \mathcal{L}_w^2 \left( \mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma} \right) \quad (45)$$

is defined by

$$(\mathcal{WT})(G)(\rho; z) := \int_{\overline{B_{\text{ext}}^\gamma}} \Psi_{\rho; z}(x) G(x) dx. \quad (46)$$

Note that we use the notation  $\mathcal{WT}$  for the continuous wavelet transform and  $WT$  for the discrete wavelet transform in [16]. Lemma 4.6 shows that the images of  $\mathcal{WT}$  are really in  $\mathcal{L}_w^2 \left( \mathbb{R}^+ \times \overline{B_{\text{ext}}^\gamma} \right)$ .

For this wavelet transform we can give a reconstruction formula.

**Theorem 4.9 (Reconstruction Formula)**

Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a weight function and let  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be a family of wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . Furthermore, let  $F \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right)$  with

$$F^\wedge(-n-1, j) = 0 \quad (47)$$

for all  $n \in \{\nu \in \mathbb{N} \mid 0 \leq \nu \leq m\}$  and all  $j \in \{1, \dots, 2n+1\}$  be given. Then

$$F = \int_{\overline{B_{\text{ext}}^\gamma}} \int_0^\infty (\mathcal{WT})(F)(\rho; y) \Psi_{\rho; y} w(\rho) d\rho dy \quad (48)$$

in the sense of  $\mathcal{L}^2 \left( \overline{B_{\text{ext}}^\gamma} \right)$ .

**Proof:** Let  $R \in \mathbb{R}^+$  and  $x \in \overline{B_{\text{ext}}^\gamma}$  be arbitrary. We have

$$\begin{aligned} & \int_{\overline{B_{\text{ext}}^\gamma}} \int_R^\infty (\mathcal{WT})(F)(\rho; y) \Psi_{\rho; y}(x) w(\rho) d\rho dy \\ &= \int_{\overline{B_{\text{ext}}^\gamma}} \int_R^\infty \int_{\overline{B_{\text{ext}}^\gamma}} F(z) \Psi_\rho(z, y) dz \Psi_\rho(y, x) w(\rho) d\rho dy \\ &= \int_{\overline{B_{\text{ext}}^\gamma}} F(z) \left( \int_R^\infty \int_{\overline{B_{\text{ext}}^\gamma}} \Psi_\rho(z, y) \Psi_\rho(y, x) dy w(\rho) d\rho \right) dz. \quad (49) \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
& \int_R \int_{\overline{B_{\text{ext}}^\gamma}} \Psi_\rho(z, y) \Psi_\rho(y, x) dy w(\rho) d\rho \\
&= \int_R \Psi_\rho^{(2)}(z, x) w(\rho) d\rho \\
&= \int_R \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (\Psi_\rho^\wedge(-n-1))^2 H_{-n-1,j}^{\text{ext}}(\beta; z) H_{-n-1,j}^{\text{ext}}(\beta; x) w(\rho) d\rho \\
&= \int_R \sum_{n=0}^{\infty} (\Psi_\rho^\wedge(-n-1))^2 \left( \frac{\beta^2}{|z||x|} \right)^{n+1} (c_{\text{ext}}(n, \beta))^2 \cdot \\
&\quad \frac{2n+1}{4\pi} P_n \left( \frac{z}{|z|} \cdot \frac{x}{|x|} \right) w(\rho) d\rho \\
&= \sum_{n=0}^{\infty} \int_R (\Psi_\rho^\wedge(-n-1))^2 \left( \frac{\beta^2}{|z||x|} \right)^{n+1} (c_{\text{ext}}(n, \beta))^2 \cdot \\
&\quad \frac{2n+1}{4\pi} P_n \left( \frac{z}{|z|} \cdot \frac{x}{|x|} \right) w(\rho) d\rho
\end{aligned} \tag{50}$$

for all  $z \in \overline{B_{\text{ext}}^\gamma}$ , since

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left| \int_R (\Psi_\rho^\wedge(-n-1))^2 \left( \frac{\beta^2}{|z||x|} \right)^{n+1} (c_{\text{ext}}(n, \beta))^2 \frac{2n+1}{4\pi} P_n \left( \frac{z}{|z|} \cdot \frac{x}{|x|} \right) w(\rho) d\rho \right| \\
&\leq \sum_{n=0}^{\infty} \int_R (\Psi_\rho^\wedge(-n-1))^2 w(\rho) d\rho (c_{\text{ext}}(n, \beta))^2 \frac{2n+1}{4\pi} < \infty, \tag{51}
\end{aligned}$$

as  $|P_n(t)| \leq 1$  for all  $t \in [-1, 1]$  and  $(c_{\text{ext}}(n, \beta))^2 = O(n)$ .

Now we have

$$\int_{\overline{B_{\text{ext}}^\gamma}} \int_R (\mathcal{WT})(F)(\rho; y) \Psi_{\rho;y}(x) w(\rho) d\rho dy \tag{52}$$

$$\begin{aligned}
&= \int_{\frac{B_{\text{ext}}^\gamma}{B_{\text{ext}}^\gamma}} F(z) \sum_{n=0}^{\infty} \int_R^{\infty} (\Psi_\rho^\wedge(-n-1))^2 w(\rho) d\rho \left( \frac{\beta^2}{|z||x|} \right)^{n+1} (c_{\text{ext}}(n, \beta))^2 \cdot \\
&\quad \cdot \frac{2n+1}{4\pi} P_n \left( \frac{z}{|z|} \cdot \frac{x}{|x|} \right) dz.
\end{aligned}$$

Let  $H_R \in \mathcal{L}^2(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma})$  be given by

$$H_R^\wedge(-n-1) := \left( \int_R^\infty (\Psi_\rho^\wedge(-n-1))^2 w(\rho) d\rho \right)^{\frac{1}{2}}, \quad n \in \mathbb{N}. \quad (53)$$

Note that

$$\sum_{n=0}^{\infty} (2n+1) (H_R^\wedge(-n-1))^2 = \sum_{n=m+1}^{\infty} (2n+1) \int_R^\infty (\Psi_\rho^\wedge(-n-1))^2 w(\rho) d\rho < +\infty. \quad (54)$$

Using the product kernel  $H_R$  we find

$$\begin{aligned}
&\int_{\frac{B_{\text{ext}}^\gamma}{B_{\text{ext}}^\gamma}} \int_R^\infty (\mathcal{WT})(F)(\rho; y) \Psi_{\rho; y} w(\rho) d\rho dy \\
&= F * H_R^{(2)} \\
&= \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} F^\wedge(-n-1, j) \int_R^\infty (\Psi_\rho^\wedge(-n-1))^2 w(\rho) d\rho H_{-n-1, j}^{\text{ext}}(\beta; \cdot)
\end{aligned} \quad (55)$$

in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ . Thus,

$$\begin{aligned}
&\lim_{R \rightarrow 0^+} \left\| F - F * H_R^{(2)} \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 \\
&= \lim_{R \rightarrow 0^+} \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} (F^\wedge(-n-1, j))^2 \left( 1 - \int_R^\infty (\Psi_\rho^\wedge(-n-1))^2 w(\rho) d\rho \right)^2.
\end{aligned} \quad (56)$$

Since

$$(F^\wedge(-n-1, j))^2 \left( 1 - \int_R^\infty (\Psi_\rho^\wedge(-n-1))^2 w(\rho) d\rho \right)^2 \leq (F^\wedge(-n-1, j))^2 \quad (57)$$

for every arbitrary  $R \in \mathbb{R}^+$  and all  $n \in \mathbb{N} \setminus \{\nu \in \mathbb{N} | 0 \leq \nu \leq m\}$ ,  $j \in \{1, \dots, 2n+1\}$  and

$$\sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} (F^\wedge(-n-1, j))^2 \leq \|F\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 < +\infty, \quad (58)$$

we get

$$\begin{aligned} & \lim_{R \rightarrow 0^+} \left\| F - F * H_R^{(2)} \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 \\ &= \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} (F^\wedge(-n-1, j))^2 \lim_{R \rightarrow 0^+} \left( 1 - \int_R^\infty (\Psi_\rho^\wedge(-n-1))^2 w(\rho) d\rho \right)^2 \\ &= 0. \end{aligned} \quad (59)$$

Hence,

$$F = \int_{\frac{B_{\text{ext}}^\gamma}{0}}^\infty \int_0^\infty (\mathcal{WT})(F)(\rho; y) \Psi_{\rho; y} w(\rho) d\rho dy \quad (60)$$

in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ , **q.e.d.**

## 5 Scaling Functions

In the scale discrete theory of [16] we first defined scaling functions and then used them in order to define wavelets. Here we do it the other way round.

**Definition 5.1** *Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a weight function and let the kernels in  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . The corresponding (scale continuous) scaling functions  $\{\Phi_R\}_{R \in \mathbb{R}^+}$  are defined by*

$$(\Phi_R)^\wedge(-n-1) := \begin{cases} 1 & \text{if } n \leq m \\ \left( \int_R^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right)^{\frac{1}{2}} & \text{if } n > m \end{cases}, \quad n \in \mathbb{N}. \quad (61)$$

**Lemma 5.2** *The scaling functions defined in Definition 5.1 are product kernels in  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma})$ .*



**Proof:** We obtain

$$\begin{aligned}
& \|\Phi_R\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma})}^2 & (62) \\
&= \sum_{n=0}^{\infty} (2n+1) \left( (\Phi_R)^\wedge(-n-1) \right)^2 \\
&= \sum_{n=0}^m (2n+1) + \sum_{n=m+1}^{\infty} (2n+1) \int_R^\infty \left( (\Psi_\rho)^\wedge(-n-1) \right)^2 w(\rho) d\rho.
\end{aligned}$$

According to (22) this is finite, **q.e.d.**

The scale continuous scaling functions also yield a kind of approximate identity.

**Theorem 5.3** *Let the kernels in  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be wavelets of order  $m \in \mathbb{N} \cup \{-1\}$  and let  $\{\Phi_R\}_{R \in \mathbb{R}^+}$  be the corresponding family of scaling functions. Then for all  $F \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  the identity*

$$F = \lim_{R \rightarrow 0^+} (\Phi_R)^{(2)} * F \quad (63)$$

holds in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ .

Moreover, if the kernels  $\left\{ \Phi_R^{(2)} \right\}_{R \in \mathbb{R}^+}$  are uniformly bounded, i.e. there exists  $T \in \mathbb{R}^+$ , such that

$$\left\| \Phi_R^{(2)} \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma})}^2 \leq T \quad (64)$$

for all  $R \in \mathbb{R}^+$ , and  $K \subset \overline{B_{\text{ext}}^\gamma}$  is a compact subset, then identity (63) also holds in the sense of  $C^{(0)}(K)$ .

**Proof:** We have

$$\begin{aligned}
(\Phi_R)^{(2)} * F &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left( (\Phi_R)^\wedge(-n-1) \right)^2 F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \\
&= \sum_{n=0}^m \sum_{j=1}^{2n+1} F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) + \\
&\quad + \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} \int_R^\infty \left( (\Psi_\rho)^\wedge(-n-1) \right)^2 w(\rho) d\rho \cdot \\
&\quad \cdot F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \quad (65)
\end{aligned}$$

and

$$F - (\Phi_R)^{(2)} * F = \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} \left( 1 - \int_R^{\infty} ((\Psi_{\rho})^{\wedge}(-n-1))^2 w(\rho) d\rho \right) \cdot F^{\wedge}(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot). \quad (66)$$

This implies

$$\begin{aligned} & \left\| F - (\Phi_R)^{(2)} * F \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^{\gamma}})}^2 \\ &= \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} \left( 1 - \int_R^{\infty} ((\Psi_{\rho})^{\wedge}(-n-1))^2 w(\rho) d\rho \right)^2 (F^{\wedge}(-n-1, j))^2. \end{aligned} \quad (67)$$

We see that for all  $R \in \mathbb{R}^+$  the inequality

$$\begin{aligned} 0 &\leq \left( 1 - \int_R^{\infty} ((\Psi_{\rho})^{\wedge}(-n-1))^2 w(\rho) d\rho \right)^2 (F^{\wedge}(-n-1, j))^2 \\ &\leq (F^{\wedge}(-n-1, j))^2 \end{aligned} \quad (68)$$

holds because of (20). As the series

$$\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (F^{\wedge}(-n-1, j))^2 = \|F\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^{\gamma}})}^2 \quad (69)$$

converges, the Weierstraß criterion (see [22], p. 142) says that the series in (67) converges uniformly. Hence, we are allowed to interchange  $\lim_{R \rightarrow 0^+}$  with the summation over  $n$  in (67) (see [22], p. 141). We get

$$\begin{aligned} & \lim_{R \rightarrow 0^+} \left\| F - (\Phi_R)^{(2)} * F \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^{\gamma}})}^2 \\ &= \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} \lim_{R \rightarrow 0^+} \left( 1 - \int_R^{\infty} ((\Psi_{\rho})^{\wedge}(-n-1))^2 w(\rho) d\rho \right)^2 \cdot (F^{\wedge}(-n-1, j))^2 \\ &= 0. \end{aligned} \quad (70)$$

Concerning the uniform convergence on  $K$  we consider a sphere  $\Gamma_r$  with radius  $r$ , where  $\beta < r < \min\{|x| : x \in K\} =: \alpha$ . We observe that for all  $x \in \Gamma_r$  and all  $R \in \mathbb{R}^+$  we obtain

$$\left| \frac{\int \Phi_R^{(2)}(x, y) F(y) dy}{B_{\text{ext}}^\gamma} \right| \leq \left( \frac{\int \left( \Phi_R^{(2)}(x, y) \right)^2 dy}{B_{\text{ext}}^\gamma} \right)^{\frac{1}{2}} \left( \frac{\int F(y)^2 dy}{B_{\text{ext}}^\gamma} \right)^{\frac{1}{2}}. \quad (71)$$

Moreover, we find

$$\begin{aligned} & \frac{\int \left( \Phi_R^{(2)}(x, y) \right)^2 dy}{B_{\text{ext}}^\gamma} \\ &= \left\| \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left( \Phi_R^{(2)} \right)^\wedge (-n-1) H_{-n-1, j}^{\text{ext}}(\beta; x) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \right\|_{\mathcal{L}^2(B_{\text{ext}}^\gamma)} \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left( \left( \Phi_R^{(2)} \right)^\wedge (-n-1) \right)^2 \left( H_{-n-1, j}^{\text{ext}}(\beta; x) \right)^2 \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left( \left( \Phi_R^{(2)} \right)^\wedge (-n-1) \right)^2 \left( c_{\text{ext}}(n, \beta) \right)^2 \left( \frac{\beta}{r} \right)^{2n+2} \left( Y_{n, j} \left( \frac{x}{|x|} \right) \right)^2 \\ &\leq \max_{n \in \mathbb{N}} \left( \left( c_{\text{ext}}(n, \beta) \right)^2 \left( \frac{\beta}{r} \right)^{2n+2} \right) \sum_{n=0}^{\infty} \left( \left( \Phi_R^{(2)} \right)^\wedge (-n-1) \right)^2 \frac{2n+1}{4\pi} \\ &\leq \max_{n \in \mathbb{N}} \left( \left( c_{\text{ext}}(n, \beta) \right)^2 \left( \frac{\beta}{r} \right)^{2n+2} \right) T \cdot \frac{1}{4\pi}, \end{aligned} \quad (72)$$

since  $P_n(1) = 1$ .

Let  $\lambda_n$  be the  $n$ -dimensional Lebesgue measure. We now want to show that there exists  $r \in ]\beta, \alpha[$ , such that

$$\lim_{R \rightarrow 0^+} \left\| F - \Phi_R^{(2)} * F \right\|_{\mathcal{L}^2(\Gamma_r)}^2 = 0. \quad (73)$$

Assume that such a radius does not exist. Then for every  $r \in ]\beta, \alpha[$  there exists a subset  $E_r \subset \Gamma_r$  with  $\lambda_2(E_r) > 0$ , such that  $\left( \Phi_R^{(2)} * F \right)(x)$  does not converge to  $F(x)$  for all  $x \in E_r$ . Otherwise, the pointwise convergence almost everywhere in combination with (71) and (72) would yield the  $\mathcal{L}^2(\Gamma_r)$ -convergence according to the dominated convergence theorem. Consequently,  $\left( \Phi_R^{(2)} * F \right)(x)$  does not converge to  $F(x)$  for all  $x \in E := \bigcup_{\beta < r < \alpha} E_r$ , where

$\lambda_3(E) > 0$ . However, the  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ -convergence implies the pointwise convergence almost everywhere in  $\overline{B_{\text{ext}}^\gamma}$ , such that we obtain a contradiction. Let  $\Gamma_r$  be a sphere such that Eq. (73) holds. Since  $F$  and  $\Phi_R^{(2)} * F$  are harmonic in  $B_{\text{ext}}^\gamma$  (cf. Theorem 5.6) and in particular continuous on  $\Gamma_r$ , the unique solution of the corresponding outer Dirichlet problem on  $(\Gamma_r)_{\text{ext}} := \{x \in \mathbb{R}^3 \mid |x| > r\}$  is given by

$$F - \Phi_R^{(2)} * F = \int_{\Gamma_r} G(\cdot, y) \left( F - \Phi_R^{(2)} * F \right) (y) dy, \quad (74)$$

where  $G$  is the Green's function for this problem (see e.g. [13]). Hence,

$$\begin{aligned} & \lim_{R \rightarrow 0^+} \left\| F - \Phi_R^{(2)} * F \right\|_{C^{(0)}(K)} \\ & \leq \left\| \int_{\Gamma_r} (G(\cdot, y))^2 dy \right\|_{C^{(0)}(K)}^{\frac{1}{2}} \lim_{R \rightarrow 0^+} \left\| F - \Phi_R^{(2)} * F \right\|_{\mathcal{L}^2(\Gamma_r)} \\ & = 0, \quad \mathbf{q.e.d.} \end{aligned} \quad (75)$$

We can also, like in the scale discrete case, start with a scaling function and then come to the wavelets.

**Theorem 5.4** (cf. [8], p. 239) *Let  $\{\Phi_R\}_{R \in \mathbb{R}^+}$  be a family of uniformly bounded scaling functions of order  $m \in \mathbb{N} \cup \{-1\}$  with weight function  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , such that*

$$\sum_{n=0}^{\infty} (2n+1)^2 (\Phi_R^\wedge(-n-1))^2 < +\infty \quad (76)$$

for all  $R \in \mathbb{R}^+$ . Furthermore, let the symbols

$$\left\{ (\Phi_R)^\wedge(-n-1) \right\}_{n \in \mathbb{N}, R \in \mathbb{R}^+}, \quad (77)$$

as functions of  $R$  be differentiable and monotonically decreasing for every  $n \in \mathbb{N}$ , such that for all  $R \in \mathbb{R}^+$  they satisfy

$$(\Phi_R)^\wedge(-n-1) = 1, \quad (78)$$

if  $n \in \{\nu \in \mathbb{N} \mid 0 \leq \nu \leq m\}$ , and

$$\lim_{R \rightarrow \infty} (\Phi_R)^\wedge(-n-1) = 0, \quad (79)$$

if  $n \in \mathbb{N}$  with  $n > m$ .

If these kernels generate approximate identities, i.e. if for arbitrary  $F \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$

$$F = \lim_{R \rightarrow 0^+} (\Phi_R)^{(2)} * F \quad (80)$$

in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$  and if there exist product kernels  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+} \subset \mathcal{L}^2(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma})$  with

$$(\Psi_\rho)^\wedge(-n-1) = \left( -\frac{1}{w(\rho)} \frac{d}{d\rho} ((\Phi_\rho)^\wedge(-n-1))^2 \right)^{\frac{1}{2}} \quad (81)$$

for all  $n \in \mathbb{N}$  and all  $\rho \in \mathbb{R}^+$ , then  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  are the associated wavelets of order  $m$ .

**Proof:** As  $(\Phi_\rho)^\wedge(-n-1)$  and consequently  $((\Phi_\rho)^\wedge(-n-1))^2$  is monotonically decreasing in  $\rho$ , the expression  $-\frac{1}{w(\rho)} \frac{d}{d\rho} ((\Phi_\rho)^\wedge(-n-1))^2$  is non-negative. First of all, we get

$$\begin{aligned} & \int_R^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \\ &= - \int_R^\infty \frac{1}{w(\rho)} \left( \frac{d}{d\rho} ((\Phi_\rho)^\wedge(-n-1))^2 \right) w(\rho) d\rho \\ &= - \int_R^\infty \frac{d}{d\rho} ((\Phi_\rho)^\wedge(-n-1))^2 d\rho \\ &= ((\Phi_R)^\wedge(-n-1))^2 \end{aligned} \quad (82)$$

for all  $R \in \mathbb{R}^+$  and all  $n \in \mathbb{N}$  with  $n > m$ . Hence, (61) is valid.

As  $(\Phi_R)^\wedge(-n-1)$  is constant in  $R$  for  $n \in \{\nu \in \mathbb{N} | 0 \leq \nu \leq m\}$  we have  $(\Psi_\rho)^\wedge(-n-1) = 0$  for all  $n \in \{\nu \in \mathbb{N} | 0 \leq \nu \leq m\}$  and all  $\rho \in \mathbb{R}^+$ . This is property (ii).

Now we come to the remaining properties of the wavelets.

(i)

$$\int_0^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho = \lim_{R \rightarrow 0^+} ((\Phi_R)^\wedge(-n-1))^2 = 1 \quad (83)$$

for  $n > m$  because of (80): The uniform boundedness of the kernels, i.e.

$$\|\Phi_R\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma})}^2 \leq T \quad (84)$$

for all  $R \in \mathbb{R}^+$ , where  $T$  is independent of  $R$ , and the monotonicity of the symbols imply the convergence here and identity (80) implies that the limit is 1.

(iii) We get

$$\begin{aligned} & \sum_{n=m+1}^{\infty} (2n+1)^2 \int_R^{\infty} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \\ & \leq \sum_{n=m+1}^{\infty} (2n+1)^2 ((\Phi_R)^\wedge(-n-1))^2 \\ & < +\infty, \quad \mathbf{q.e.d.} \end{aligned} \quad (85)$$

We can introduce scale spaces and detail spaces like in the scale discrete case.

**Definition 5.5** *Let the kernels in  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be wavelets of order  $m \in \mathbb{N} \cup \{-1\}$  with the corresponding family of scaling functions  $\{\Phi_R\}_{R \in \mathbb{R}^+}$ . For  $\rho, R \in \mathbb{R}^+$  we define the scale spaces by*

$$V_R := \left\{ (\Phi_R)^{(2)} * G \mid G \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right) \right\} \quad (86)$$

and the detail spaces by

$$W_\rho := \left\{ (\Psi_\rho)^{(2)} * G \mid G \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right) \right\}. \quad (87)$$

Also here the scale spaces are the images of low-pass filters as condition (iii) implies that for all  $R \in \mathbb{R}^+$  the sequence

$$\left\{ \int_R^{\infty} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right\}_{n \in \mathbb{N}} \quad (88)$$

converges to 0 for  $n \rightarrow \infty$ .

Conditions (i) and (ii) show that in the convolution with  $\Psi_\rho^{(2)}$  only the frequencies with  $n > m$  are considered. Hence, the detail spaces are images of band-pass filters as the Fourier coefficients of the wavelets must converge to 0 for  $n \rightarrow \infty$ .

Another property that we obtain is the multiresolution.

**Theorem 5.6 (Multiresolution)** *Let  $\{\Phi_R\}_{R \in \mathbb{R}^+}$  be scaling functions of order  $m \in \mathbb{N} \cup \{-1\}$ . Then the following properties are satisfied by the scale spaces:*

a)

$$\text{Harm}_{0\dots m}(\overline{B_{\text{ext}}^\gamma}) \subset V_{R_1} \subset V_{R_2} \subset \text{Harm}(\overline{B_{\text{ext}}^\gamma}) \quad (89)$$

for  $0 < R_2 \leq R_1$ ,

b)

$$\overline{\bigcup_{R>0} V_R}^{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} = \text{Harm}(\overline{B_{\text{ext}}^\gamma}). \quad (90)$$

**Proof:** a) The fact that

$$(\Phi_R)^\wedge(-n-1) = 1 \quad (91)$$

for all  $R \in \mathbb{R}^+$ , if  $n \leq m$ , implies

$$(\Phi_R)^{(2)} * F = F \quad (92)$$

for all  $R \in \mathbb{R}^+$  and  $F \in \text{Harm}_{0\dots m}(\overline{B_{\text{ext}}^\gamma})$ . Hence, the inclusion

$$\text{Harm}_{0\dots m}(\overline{B_{\text{ext}}^\gamma}) \subset V_R \quad (93)$$

holds for all  $R \in \mathbb{R}^+$ .

Let  $R_1, R_2 \in \mathbb{R}^+$  with  $R_2 \leq R_1$  be given and let  $F \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  be an arbitrary function. Let  $\mathcal{N} := \{n \in \mathbb{N} \mid (\Phi_{R_2})^\wedge(-n-1) = 0\}$ . We define  $G \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  by

$$G^\wedge(-n-1, j) := \begin{cases} 0 & \text{if } n \in \mathcal{N} \\ \left( \frac{(\Phi_{R_1})^\wedge(-n-1)}{(\Phi_{R_2})^\wedge(-n-1)} \right)^2 F^\wedge(-n-1, j) & \text{else} \end{cases}. \quad (94)$$

$G$  is in  $\text{Harm}(\overline{B_{\text{ext}}^\gamma})$ , because

$$\begin{aligned} \|G\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (G^\wedge(-n-1, j))^2 \\ &= \sum_{n \in \mathbb{N} \setminus \mathcal{N}} \sum_{j=1}^{2n+1} \left( \frac{(\Phi_{R_1})^\wedge(-n-1)}{(\Phi_{R_2})^\wedge(-n-1)} \right)^4 (F^\wedge(-n-1, j))^2. \end{aligned} \quad (95)$$

By construction  $(\Phi_R)^\wedge(-n-1)$  is monotonically decreasing in  $R$  for all  $n \in \mathbb{N}$ . Hence,

$$0 \leq (\Phi_{R_1})^\wedge(-n-1) \leq (\Phi_{R_2})^\wedge(-n-1) \quad (96)$$

for all  $n \in \mathbb{N}$  and consequently

$$\begin{aligned} \|G\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 &\leq \sum_{n \in \mathbb{N} \setminus \mathcal{N}} \sum_{j=1}^{2n+1} (F^\wedge(-n-1, j))^2 \\ &\leq \|F\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2. \end{aligned} \quad (97)$$

Because of the above mentioned monotonicity and the non-negativity of the symbols of all  $\Phi_R$  we have

$$(\Phi_{R_1})^\wedge(-n-1) = 0 \quad (98)$$

for all  $n \in \mathcal{N}$ . This implies

$$\begin{aligned} (\Phi_{R_1})^{(2)} * F &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} ((\Phi_{R_1})^\wedge(-n-1))^2 F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \\ &= \sum_{n \in \mathbb{N} \setminus \mathcal{N}} \sum_{j=1}^{2n+1} ((\Phi_{R_2})^\wedge(-n-1))^2 \left( \frac{(\Phi_{R_1})^\wedge(-n-1)}{(\Phi_{R_2})^\wedge(-n-1)} \right)^2 \\ &\quad \cdot F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \\ &= (\Phi_{R_2})^{(2)} * G. \end{aligned} \quad (99)$$

Hence,  $V_{R_1} \subset V_{R_2}$ . Finally, every function in  $V_R$  is harmonic for all  $R \in \mathbb{R}^+$ , because

$$(\Phi_R)^{(2)} * F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} ((\Phi_R)^\wedge(-n-1))^2 F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \quad (100)$$

in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$  with

$$\begin{aligned} \left\| (\Phi_R)^{(2)} * F \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} ((\Phi_R)^\wedge(-n-1))^4 (F^\wedge(-n-1, j))^2 \\ &\leq \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} 1 \cdot (F^\wedge(-n-1, j))^2 \\ &= \|F\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 \end{aligned} \quad (101)$$



for all  $F \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right)$ .

b) For  $F \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right)$  we consider the sequence  $\left\{ \left( \Phi_{\frac{1}{n+1}} \right)^{(2)} * F \right\}_{n \in \mathbb{N}}$ . We have

$$\left( \Phi_{\frac{1}{n+1}} \right)^{(2)} * F \in V_{\frac{1}{n+1}} \quad (102)$$

for all  $n \in \mathbb{N}$ . Thus,

$$\left\{ \left( \Phi_{\frac{1}{n+1}} \right)^{(2)} * F \right\}_{n \in \mathbb{N}} \subset \bigcup_{R>0} V_R \quad (103)$$

and consequently Theorem 5.3 says that

$$\lim_{n \rightarrow \infty} \left( \Phi_{\frac{1}{n+1}} \right)^{(2)} * F = F \in \overline{\bigcup_{R>0} V_R}^{\|\cdot\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}}, \quad \mathbf{q.e.d.} \quad (104)$$

As usual we have a possibility to come from one scale space to another by adding detail information.

**Theorem 5.7** *Let  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a weight function and let the product kernels in the family  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . Furthermore, let  $\{\Phi_R\}_{R \in \mathbb{R}^+}$  be the corresponding family of scaling functions. Then for all  $F \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right)$*

$$\left( \Phi_{R_2} \right)^{(2)} * F = \left( \Phi_{R_1} \right)^{(2)} * F + \int_{R_2}^{R_1} \left( \Psi_\rho \right)^{(2)} * F w(\rho) d\rho \quad (105)$$

(with respect to  $\mathcal{L}^2 \left( \overline{B_{\text{ext}}^\gamma} \right)$ ) for all  $R_1, R_2 \in \mathbb{R}^+$  with  $R_2 \leq R_1$ .

**Proof:** We have

$$\begin{aligned} \left( \Phi_R \right)^{(2)} * F &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left( \left( \Phi_R \right)^\wedge (-n-1) \right)^2 F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \\ &= \sum_{n=0}^m \sum_{j=1}^{2n+1} F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \\ &\quad + \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} \left( \int_R^{\infty} \left( \left( \Psi_\rho \right)^\wedge (-n-1) \right)^2 w(\rho) d\rho \right) \\ &\quad \cdot F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \end{aligned} \quad (106)$$

in  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ . Consequently,

$$\begin{aligned} & (\Phi_{R_2})^{(2)} * F - (\Phi_{R_1})^{(2)} * F \\ &= \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} \left( \int_{R_2}^{R_1} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right) F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot). \end{aligned} \quad (107)$$

The Beppo Levi Theorem implies

$$\begin{aligned} & (\Phi_{R_2})^{(2)} * F - (\Phi_{R_1})^{(2)} * F \\ &= \int_{R_2}^{R_1} \left( \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} ((\Psi_\rho)^\wedge(-n-1))^2 F^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \right) w(\rho) d\rho \\ &= \int_{R_2}^{R_1} \left( (\Psi_\rho)^{(2)} * F \right) w(\rho) d\rho \end{aligned} \quad (108)$$

in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ , **q.e.d.**

## 6 Examples

In [8], p. 242f examples for spherical scale continuous wavelets are given. As the conditions for the wavelets refer to the symbols we can transfer the spherical wavelets to our wavelets.

### Example 6.1 (Abel-Poisson Wavelet of Order m)

We define

$$(\Psi_\rho)^\wedge(-n-1) := \begin{cases} 0 & \text{if } n \leq m \\ \sqrt{\frac{2n}{w(\rho)}} e^{-n\rho} & \text{if } n > m \end{cases}, \quad (109)$$

where  $m \geq 0$ .

It is easy to check the conditions (i) to (iii).

### Example 6.2 (Gauß-Weierstraß Wavelet of Order m)

We define

$$(\Psi_\rho)^\wedge(-n-1) := \begin{cases} 0 & \text{if } n \leq m \\ \sqrt{\frac{2}{w(\rho)}} \sqrt{n(n+1)} e^{-n(n+1)\rho} & \text{if } n > m \end{cases}, \quad (110)$$

where  $m \geq 0$ .

Again conditions (i) to (iii) are satisfied.

## 7 Scale Discretized Wavelets

In analogy to [8] we now introduce scale discretized wavelets. Let  $\{\rho_j\}_{j \in \mathbb{Z}}$  be a strictly monotonically decreasing sequence with the properties

$$\lim_{j \rightarrow \infty} \rho_j = 0 \text{ and } \lim_{j \rightarrow -\infty} \rho_j = \infty. \quad (111)$$

An example of such a sequence is  $\rho_j = a^{-j}$ , where  $a > 1$ . The usual choice is  $a = 2$ .

This allows us now to define wavelet packets in the usual way.

**Definition 7.1** *Let  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  consist of wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . The product kernels  $\Psi_j^P$ ,  $j \in \mathbb{Z}$ , defined by*

$$\left(\Psi_j^P\right)^\wedge(-n-1) := \left( \int_{\rho_{j+1}}^{\rho_j} \left(\Psi_\rho^\wedge(-n-1)\right)^2 w(\rho) d\rho \right)^{\frac{1}{2}}, \quad n \in \mathbb{N}, \quad (112)$$

are called P-wavelet packets.

## 8 Scale Discrete Wavelets

In [16] scale discrete harmonic wavelets have been introduced for the bounded outer space and the inner space of the sphere  $B$ . We will see now that the scale continuous and scale discretized wavelets introduced in this paper can be used to derive scale discrete wavelets similar to the type treated in [16].

**Definition 8.1** *Let  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  be a family of scale continuous wavelets of order  $m \in \mathbb{N} \cup \{-1\}$ . The corresponding family of wavelet packets with respect to a series  $\{\rho_j\}_{j \in \mathbb{Z}}$  be denoted by  $\{\Psi_j^P\}_{j \in \mathbb{Z}}$ .*

*We define the corresponding scale discrete scaling functions  $\Phi_J^D$ ,  $J \in \mathbb{Z}$ , by*

$$\left(\Phi_J^D\right)^\wedge(-n-1) := \left(\Phi_{\rho_J}\right)^\wedge(-n-1), \quad n \in \mathbb{N}, \quad (113)$$

where  $\{\Phi_R\}_{R \in \mathbb{R}^+}$  is the system of scaling functions referring to the scale continuous wavelets.

Furthermore, the elements of two systems of product kernels  $\{\Psi_J^D\}_{J \in \mathbb{Z}}$  and  $\{\tilde{\Psi}_J^D\}_{J \in \mathbb{Z}}$  are called scale discrete primal and dual wavelets, respectively, if they satisfy the following equation

$$\left(\Psi_J^D\right)^\wedge(-n-1) \left(\tilde{\Psi}_J^D\right)^\wedge(-n-1) = \left(\left(\Psi_j^P\right)^\wedge(-n-1)\right)^2 \quad (114)$$

for all  $J \in \mathbb{Z}$  and all  $n \in \mathbb{N}$ .

We define the scale discrete scale spaces  $V_J^D$ ,  $J \in \mathbb{Z}$ , and detail spaces  $W_J^D$ ,  $J \in \mathbb{Z}$ , by

$$V_J^D := \left\{ (\Phi_J^D)^{(2)} * F \mid F \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right) \right\}, \quad (115)$$

$$W_J^D := \left\{ \Psi_J^D * \tilde{\Psi}_J^D * F \mid F \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right) \right\}. \quad (116)$$

We see that the P-wavelet packets represent a special case of scale discrete wavelets. Note that

$$\begin{aligned} (\Phi_J^D)^\wedge(-n-1) &= (\Phi_{\rho_J})^\wedge(-n-1) \\ &= \left( \int_{\rho_J}^{\infty} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right)^{\frac{1}{2}} \\ &= \left( \sum_{j=-\infty}^{J-1} \int_{\rho_{j+1}}^{\rho_j} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right)^{\frac{1}{2}} \\ &= \left( \sum_{j=-\infty}^{J-1} \left( (\Psi_j^P)^\wedge(-n-1) \right)^2 \right)^{\frac{1}{2}} \\ &= \left( \sum_{j=-\infty}^{J-1} (\Psi_J^D)^\wedge(-n-1) (\tilde{\Psi}_J^D)^\wedge(-n-1) \right)^{\frac{1}{2}}, \end{aligned} \quad (117)$$

if  $n > m$ .

The following theorem shows the similarities to the scale discrete kernels in [16], which also satisfy the conditions a) to h) for all  $J \in \mathbb{N}$  in case of the order  $m = 0$ .

**Theorem 8.2** *Let  $\{\Phi_J^D\}_{J \in \mathbb{Z}}$  be a family of scale discrete scaling functions of order  $m \in \mathbb{N}$  and  $\{\Psi_J^D\}_{J \in \mathbb{Z}}$  and  $\{\tilde{\Psi}_J^D\}_{J \in \mathbb{Z}}$  be corresponding families of scale discrete primal and dual wavelets, respectively. Then the following properties are satisfied.*

- a)  $(\Phi_J^D)^\wedge(-n-1) = 1$  for all  $J \in \mathbb{Z}$ , if  $n \in \{\nu \in \mathbb{N} \mid 0 \leq \nu \leq m\}$ .
- b)  $(\Phi_{J_1}^D)^\wedge(-n-1) \leq (\Phi_{J_2}^D)^\wedge(-n-1)$  for all  $J_1, J_2 \in \mathbb{Z}$  with  $J_1 \leq J_2$ .
- c) *admissibility condition*

$$\sum_{n=0}^{\infty} (2n+1)^2 \left( (\Phi_J^D)^\wedge(-n-1) \right)^2 < +\infty \quad (118)$$

for all  $J \in \mathbb{Z}$ .

d)  $\lim_{J \rightarrow \infty} (\Phi_J^D)^\wedge (-n-1) = 1$  for all  $n \in \mathbb{N}$ .

e) *approximate identity*:  $\lim_{J \rightarrow \infty} \left\| F - (\Phi_J^D)^{(2)} * F \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 = 0$  for all  $F \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$ .

f) *multiresolution*:

(i)  $\text{Harm}_{0\dots m}(\overline{B_{\text{ext}}^\gamma}) \subset V_{J_1}^D \subset V_{J_2}^D \subset \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  for all  $J_1, J_2 \in \mathbb{Z}$  with  $J_1 \leq J_2$ .

(ii)  $\overline{\bigcup_{J \in \mathbb{Z}} V_J^D}^{\|\cdot\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}} = \text{Harm}(\overline{B_{\text{ext}}^\gamma})$ .

g) *refinement equation*

$$\begin{aligned} & (\Psi_J^D)^\wedge (-n-1) \left( \tilde{\Psi}_J^D \right)^\wedge (-n-1) \\ &= \left( (\Phi_{J+1}^D)^\wedge (-n-1) \right)^2 - \left( (\Phi_J^D)^\wedge (-n-1) \right)^2 \end{aligned} \quad (119)$$

for all  $J \in \mathbb{Z}$ .

h) *scale step property*:

$$(\Phi_{J_2}^D)^{(2)} * F = (\Phi_{J_1}^D)^{(2)} * F + \sum_{J=J_1}^{J_2-1} \Psi_J^D * \tilde{\Psi}_J^D * F \quad (120)$$

(in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ ) for all  $F \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  and all  $J_1, J_2 \in \mathbb{Z}$  with  $J_1 < J_2$ .

**Proof:** Let the scale discrete kernels be derived from the families of wavelets  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$ , wavelet packets  $\{\Psi_j^P\}_{j \in \mathbb{Z}}$  and scaling functions  $\{\Phi_R\}_{R \in \mathbb{R}^+}$ . We remember that we defined

$$(\Phi_R)^\wedge (-n-1) := \begin{cases} 1 & \text{if } n \leq m \\ \left( \int_R^\infty ((\Psi_\rho)^\wedge (-n-1))^2 w(\rho) d\rho \right)^{\frac{1}{2}} & \text{if } n > m \end{cases} \quad (121)$$

Hence, property a) is obviously satisfied.

Now let  $J_1, J_2 \in \mathbb{Z}$  with  $J_1 \leq J_2$  be given. Note that  $\{\rho_J\}_{J \in \mathbb{Z}}$  is strictly

monotonically decreasing. We obtain

$$\begin{aligned}
(\Phi_{J_1}^D)^\wedge(-n-1) &= (\Phi_{\rho_{J_1}})^\wedge(-n-1) \\
&= \left( \int_{\rho_{J_1}}^{\infty} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right)^{\frac{1}{2}} \\
&\leq \left( \int_{\rho_{J_2}}^{\infty} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right)^{\frac{1}{2}} \\
&= (\Phi_{J_2}^D)^\wedge(-n-1), \tag{122}
\end{aligned}$$

if  $n > m$ . Concerning property c) it is easy to see that

$$\sum_{n=m+1}^{\infty} (2n+1)^2 \left( (\Phi_J^D)^\wedge(-n-1) \right)^2 \tag{123}$$

$$= \sum_{n=m+1}^{\infty} (2n+1)^2 \left( \int_{\rho_J}^{\infty} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right) < +\infty \tag{124}$$

for all  $J \in \mathbb{Z}$  because of (22).

Property d) is trivially satisfied, if  $n \leq m$ . Now let  $n > m$ . Then

$$\begin{aligned}
\lim_{J \rightarrow \infty} (\Phi_J^D)^\wedge(-n-1) &= \lim_{J \rightarrow \infty} \left( \int_{\rho_J}^{\infty} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right)^{\frac{1}{2}} \\
&= \left( \int_0^{\infty} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho \right)^{\frac{1}{2}} \\
&= 1. \tag{125}
\end{aligned}$$

e) Since  $\lim_{R \rightarrow 0+} \left\| F - \Phi_R^{(2)} * F \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 = 0$  for all  $F \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  and  $\lim_{J \rightarrow \infty} \rho_J = 0$ , where  $\rho_J > 0$  for all  $J \in \mathbb{Z}$ , we have

$$\begin{aligned}
\lim_{J \rightarrow \infty} \left\| F - (\Phi_J^D)^{(2)} * F \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 &= \lim_{J \rightarrow \infty} \left\| F - \Phi_{\rho_J}^{(2)} * F \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 \tag{126} \\
&= \lim_{R \rightarrow 0+} \left\| F - \Phi_R^{(2)} * F \right\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2 = 0
\end{aligned}$$

for all  $F \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right)$ .

The multiresolution f) is an immediate result of the scale continuous multiresolution: Let  $J_1, J_2 \in \mathbb{Z}$  with  $J_1 \leq J_2$ . Then  $\rho_{J_1} \geq \rho_{J_2}$  and

$$\text{Harm}_{0\dots m} \left( \overline{B_{\text{ext}}^\gamma} \right) \subset V_{J_1}^{\text{D}} = V_{\rho_{J_1}} \subset V_{\rho_{J_2}} = V_{J_2}^{\text{D}} \subset \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right). \quad (127)$$

Property (ii) is a consequence of e).

The refinement equation can be proved by the following considerations. If  $n \leq m$ , both sides of the equation vanish. If  $n > m$ , we get

$$\begin{aligned} & \left( (\Phi_{J+1}^{\text{D}})^\wedge (-n-1) \right)^2 - \left( (\Phi_J^{\text{D}})^\wedge (-n-1) \right)^2 \\ &= \int_{\rho_{J+1}}^{\infty} \left( (\Psi_\rho)^\wedge (-n-1) \right)^2 w(\rho) d\rho - \int_{\rho_J}^{\infty} \left( (\Psi_\rho)^\wedge (-n-1) \right)^2 w(\rho) d\rho \\ &= \int_{\rho_{J+1}}^{\rho_J} \left( (\Psi_\rho)^\wedge (-n-1) \right)^2 w(\rho) d\rho \\ &= \left( (\Psi_J^{\text{P}})^\wedge (-n-1) \right)^2 \\ &= (\Psi_J^{\text{D}})^\wedge (-n-1) \left( \tilde{\Psi}_J^{\text{D}} \right)^\wedge (-n-1). \end{aligned} \quad (128)$$

Finally, the scale step property can be proved by using the scale continuous version of it. Let  $J_1, J_2 \in \mathbb{Z}$  with  $J_1 < J_2$ . Then  $\rho_{J_1} > \rho_{J_2}$ . In analogy to the proof of the reconstruction formula we find

$$\begin{aligned} & (\Phi_{J_2}^{\text{D}})^{(2)} * F = (\Phi_{\rho_{J_2}})^{(2)} * F \\ &= (\Phi_{\rho_{J_1}})^{(2)} * F + \int_{\rho_{J_2}}^{\rho_{J_1}} \Psi_\rho^{(2)} * F w(\rho) d\rho \\ &= (\Phi_{\rho_{J_1}})^{(2)} * F + \sum_{J=J_1}^{J_2-1} \int_{\rho_{J+1}}^{\rho_J} \Psi_\rho^{(2)} * F w(\rho) d\rho \\ &= (\Phi_{\rho_{J_1}})^{(2)} * F + \sum_{J=J_1}^{J_2-1} \int_{\rho_{J+1}}^{\rho_J} \int_{\overline{B_{\text{ext}}^\gamma}} \Psi_\rho^{(2)}(\cdot, x) F(x) dx w(\rho) d\rho \\ &= (\Phi_{\rho_{J_1}})^{(2)} * F + \sum_{J=J_1}^{J_2-1} \int_{\overline{B_{\text{ext}}^\gamma}} F(x) \int_{\rho_{J+1}}^{\rho_J} \Psi_\rho^{(2)}(\cdot, x) w(\rho) d\rho dx \end{aligned} \quad (129)$$

$$\begin{aligned}
&= (\Phi_{\rho_{J_1}})^{(2)} * F + \sum_{J=J_1}^{J_2-1} \int_{\overline{B_{\text{ext}}^\gamma}} F(x) \cdot \\
&\quad \cdot \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_{\rho_{J+1}}^{\rho_J} (\Psi_\rho^\wedge(-n-1))^2 w(\rho) d\rho H_{-n-1,j}^{\text{ext}}(\beta;x) H_{-n-1,j}^{\text{ext}}(\beta;\cdot) dx \\
&= (\Phi_{\rho_{J_1}})^{(2)} * F + \sum_{J=J_1}^{J_2-1} \int_{\overline{B_{\text{ext}}^\gamma}} F(x) \left( \Psi_J^{\text{D}} * \tilde{\Psi}_J^{\text{D}} \right) (x, \cdot) dx \\
&= (\Phi_{J_1}^{\text{D}})^{(2)} * F + \sum_{J=J_1}^{J_2-1} \Psi_J^{\text{D}} * \tilde{\Psi}_J^{\text{D}} * F \tag{130}
\end{aligned}$$

for all  $F \in \text{Harm} \left( \overline{B_{\text{ext}}^\gamma} \right)$  in the sense of  $\mathcal{L}^2 \left( \overline{B_{\text{ext}}^\gamma} \right)$ .

Hence, all properties are satisfied, **q.e.d.**

Note that the scale continuous mother wavelets  $\Psi_1$  correspond to the scale discrete mother wavelets  $\Psi_0^{\text{D}}$  and  $\tilde{\Psi}_0^{\text{D}}$  in case of a sequence  $\rho_J = a^{-J}$  with  $a > 1$ . In [16] the dilation is constructed for  $a = 2$ .

**Example 8.3** *The scale discrete scaling functions in the Abel-Poisson case are given by*

$$(\Phi_J^{\text{D}})^\wedge(-n-1) = e^{-n\rho_J}, \tag{131}$$

and in the Gauß-Weierstraß case they satisfy

$$(\Phi_J^{\text{D}})^\wedge(-n-1) = e^{-n(n+1)\rho_J}. \tag{132}$$

Note that a direct construction of scale discrete wavelets and scaling functions for our domain without the need of a scale continuous theory is also possible (see [16]). In this context band-limited product kernels like the cp (cubic polynomial) scaling functions, defined by

$$(\Phi_J^{\text{D}})^\wedge(-n-1) = \begin{cases} (1-n)^2(1+2n), & \text{if } 0 \leq n \leq (\rho_J)^{-1} \\ 0, & \text{if } n > (\rho_J)^{-1} \end{cases} \tag{133}$$

(cf. for instance [19]), are used besides non-band-limited product kernels like in Example 8.3. If the cp scaling functions were constructed out of a system of scale continuous wavelets  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$ , the equation

$$\int_{\mathbb{R}} ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho = \begin{cases} (1-n)^4(1+2n)^2, & \text{if } 0 \leq n \leq R^{-1} \\ 0, & \text{if } n > R^{-1} \end{cases} \tag{134}$$



would have to be satisfied for every  $R \in \mathbb{R}^+$ . It appears to be impossible to find such wavelets. At least formula (81) cannot be applied, since  $(\Phi_R)^\wedge(-n-1)$  is not differentiable with respect to  $R$  in  $n^{-1}$ , if  $n > 0$ .

**Remark 8.4** *We can find a solution to this problem, if we use distributions. Defining*

$$(\Psi_\rho)^\wedge(-n-1) := (1-n)^2 \frac{1+2n}{\sqrt{w(\rho)}} \delta(\rho - n^{-1}), \quad (135)$$

where  $\delta$  is the delta distribution, we obtain in the sense of distributions, using the common notation in physics and assuming that  $\delta^2 := \delta$ ,

$$\begin{aligned} \int_R^\infty ((\Psi_\rho)^\wedge(-n-1))^2 w(\rho) d\rho &= \int_R^\infty (1-n)^4 (1+2n)^2 \delta(\rho - n^{-1}) d\rho \\ &= (1-n)^4 (1+2n)^2 \cdot \begin{cases} 1, & \text{if } n^{-1} \in [R, \infty[ \\ 0, & \text{else} \end{cases} \\ &= ((\Phi_R)^\wedge(-n-1))^2. \end{aligned} \quad (136)$$

Even Eq. (81) holds if we read it in the context of distributions. The Heaviside function  $H : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$H(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}. \quad (137)$$

Therefore, we can write

$$((\Phi_R)^\wedge(-n-1))^2 = (1-n)^4 (1+2n)^2 (1 - H(R - n^{-1})) \quad (138)$$

and obtain

$$\begin{aligned} ((\Psi_\rho)^\wedge(-n-1))^2 &= -\frac{1}{w(\rho)} \frac{d}{d\rho} ((1-n)^4 (1+2n)^2 (1 - H(\rho - n^{-1}))) \\ &= \frac{1}{w(\rho)} (1-n)^4 (1+2n)^2 \delta(\rho - n^{-1}) \end{aligned} \quad (139)$$

in the sense of a distributional derivative. A further investigation is a challenge for future work.

**Remark 8.5 (Inner Wavelets)** *The scale continuous, the scale discretized and the scale discrete concepts for harmonic functions on  $\overline{B_{\text{ext}}^\gamma}$  can analogously be constructed for harmonic functions on the closed inner space  $\overline{B_{\text{int}}}$  using the inner harmonics*

$$H_{n,j}^{\text{int}}(\beta;x) := \sqrt{\frac{2n+3}{\beta^3}} \left(\frac{|x|}{\beta}\right)^n Y_{n,j}\left(\frac{x}{|x|}\right) \quad (140)$$

as  $\mathcal{L}^2(\overline{B_{\text{int}}})$ -orthonormal basis (see also [16] for the scale discrete multiresolution and wavelets).

## 9 Application to the Gravimetry Problem

The gravimetry problem is concerned with the inversion of Newton's gravitational potential

$$TF = \int_{\overline{B_{\text{int}}}} \frac{F(x)}{|x - \cdot|} dx. \quad (141)$$

The investigation of the earth's interior is one of the oldest problems in science history. In the late 1930s the increasing need for oil motivated a prospecting technique based on gravitational measurements (see for example [17] and [18]). A series of further investigations of this problem followed, where the determination of the earth's density distribution from measurements of the gravitational potential or related quantities was discussed from theoretical as well as practical point of view. For an exemplary overview of publications concerned with the gravimetry problem see [16] and the references therein.

However, the investigation of the earth's interior is not only needed for prospecting for oil and other rare resources. It has a series of other applications, among which seismology certainly plays an important role. A better understanding of the geophysical wave propagation is only possible if current models of the earth's composition are improved.

Since the second usual method of investigating the earth's interior is the seismic inversion, which reconstructs reflections of seismic waves on density discontinuities from delays in the arrival of the waves at seismographs, density models obtained via this technique should, from the scientific point of view, not be used for developing a seismological and tectonical model for the earth. Therefore, methods for the determination of the earth's inner structures that are independent of seismological measurements, such as the discussion of the gravimetry problem, are needed.

We can generalize Eq. (141) by defining the operator  $T : \mathcal{L}^2(\overline{B_{\text{int}}}) \rightarrow \mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$  by

$$TF = \int_{\overline{B_{\text{int}}}} k(x, \cdot) F(x) dx, \quad (142)$$

where  $k \in \mathcal{L}^2(\overline{B_{\text{int}}} \times \overline{B_{\text{ext}}^\gamma})$  is the integral kernel with

$$k(x, y) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} k^\wedge(n) H_{n,j}^{\text{int}}(\beta; x) H_{-n-1,j}^{\text{ext}}(\beta; y) \quad (143)$$

for all  $x \in B_{\text{int}}$  and all  $y \in B_{\text{ext}}^\gamma$ . We assume that  $k^\wedge(n) \neq 0$  for all  $n \in \mathbb{N}$ . In case of the Newton potential we have

$$k^\wedge(n) = \frac{4\pi}{(2n+1)\beta c_{\text{int}}(n, \beta) c_{\text{ext}}(n, \beta)} \quad (144)$$

for every  $n \in \mathbb{N}$  (see [8], p. 44 and [16], p. 32).

This inverse problem  $TF = P$  is ill-posed for the following reasons:

- Existence: A solution exists iff  $P \in \text{Harm}(\overline{B_{\text{ext}}^\gamma})$  with

$$\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left( \frac{P^\wedge(-n-1, j)}{k^\wedge(n)} \right)^2 < +\infty. \quad (145)$$

- Uniqueness: The solution is not unique, more precisely

$$\ker T = (\text{Harm}(\overline{B_{\text{int}}}))^{\perp_{\mathcal{L}^2(\overline{B_{\text{int}}})}} =: \text{Anharm}(\overline{B_{\text{int}}}), \quad (146)$$

where  $\dim \text{Anharm}(\overline{B_{\text{int}}}) = \infty$ . Consequently, only the harmonic part, i.e. the projection on  $\text{Harm}(\overline{B_{\text{int}}})$ , of the solution  $F$  can be uniquely reconstructed from  $P$ .

- Stability: The inverse operator

$$\left( T \Big|_{\text{Harm}(\overline{B_{\text{int}}})} \right)^{-1} : T(\text{Harm}(\overline{B_{\text{int}}})) \rightarrow \text{Harm}(\overline{B_{\text{int}}}) \quad (147)$$

is not continuous. Hence, the inversion is unstable, which means that small errors in the measurements of  $P$  can yield a completely different solution  $F_{\text{harm}}$ . Therefore, a regularization is required, such that a stable determination of an approximation of  $F_{\text{harm}}$  becomes possible.

For the proofs and further details we refer to [16].

For our considerations we now choose an arbitrary family of scale continuous wavelets  $\{\Psi_\rho\}_{\rho \in \mathbb{R}^+}$  and construct the corresponding families of scale continuous scaling functions  $\{\Phi_R\}_{R \in \mathbb{R}^+}$ , P-wavelet packets  $\{\Psi_j^{\text{P}}\}_{j \in \mathbb{Z}}$  (with respect to a sequence  $\{\rho_j\}_{j \in \mathbb{Z}}$ ), scale discrete scaling functions  $\{\Phi_J^{\text{D}}\}_{J \in \mathbb{Z}}$  and scale

discrete primal and dual wavelets  $\{\Psi_J^D\}_{J \in \mathbb{Z}}$ ,  $\{\tilde{\Psi}_J^D\}_{J \in \mathbb{Z}}$ . These kernels are constructed on  $\overline{B_{\text{int}}} \times \overline{B_{\text{int}}}$  as well as  $\overline{B_{\text{ext}}^\gamma} \times \overline{B_{\text{ext}}^\gamma}$ .

We denote the Fourier coefficients of a function  $F \in \text{Harm}(\overline{B_{\text{int}}})$  by

$$F^\wedge(n, j) := (F, H_{n,j}^{\text{int}}(\beta; \cdot))_{\mathcal{L}^2(\overline{B_{\text{int}}})}. \quad (148)$$

Furthermore, we will use the indices “int” and “ext”, if it needs to be stressed whether the considered functions are defined on  $\overline{B_{\text{int}}}$  or  $\overline{B_{\text{ext}}^\gamma}$ .

The following theorem gives us an important information about the construction of the harmonic solution of the gravimetry problem and approximations to it.

**Theorem 9.1** *Let  $P \in T(\mathcal{L}^2(\overline{B_{\text{int}}}))$  be a given function. Then the unique harmonic solution  $F \in \text{Harm}(\overline{B_{\text{int}}})$  of  $TF = P$  is given by*

$$F^\wedge(n, j) = \frac{P^\wedge(-n-1, j)}{k^\wedge(n)} \quad \forall n \in \mathbb{N} \forall j \in \{1, \dots, 2n+1\}. \quad (149)$$

Furthermore,  $H_J = (\Phi_{\text{int}; J}^D)^{(2)} * F$  is the unique scale space solution of  $TH_J = (\Phi_{\text{ext}; J}^D)^{(2)} * P$ ,  $H_J \in V_{\text{int}; J}^D$ , for every  $J \in \mathbb{Z}$  and  $\hat{H}_J = \Psi_{\text{int}; J}^D * \tilde{\Psi}_{\text{int}; J}^D * F$  is the unique detail space solution of  $T\hat{H}_J = \Psi_{\text{ext}; J}^D * \tilde{\Psi}_{\text{ext}; J}^D * P$ ,  $\hat{H}_J \in W_{\text{int}; J}^D$ , for every  $J \in \mathbb{Z}$ . Moreover,

$$H_{J_2} = H_{J_1} + \sum_{J=J_1}^{J_2-1} \hat{H}_J \quad (150)$$

for all  $J_1, J_2 \in \mathbb{Z}$  with  $J_1 < J_2$  and

$$\lim_{J \rightarrow \infty} H_J = F \quad (151)$$

in  $\mathcal{L}^2(\overline{B_{\text{int}}})$ .

For reasons of brevity we only give a sketch of a proof.

**Sketch of a Proof:** It is clear that

$$\begin{aligned} TF &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} F^\wedge(n, j) k^\wedge(n) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} P^\wedge(-n-1, j) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \end{aligned} \quad (152)$$

(in the sense of  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ ) iff (149) is satisfied.

The uniqueness of the solutions of the filtered equations is given, since only harmonic functions are considered. We have

$$\begin{aligned} TH_J &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \varphi_J(n)^2 \frac{P^\wedge(-n-1, j)}{k^\wedge(n)} k^\wedge(n) H_{-n-1, j}^{\text{ext}}(\beta; \cdot) \\ &= (\Phi_{\text{ext}; J}^{\text{D}})^{(2)} * P \end{aligned} \quad (153)$$

and analogously

$$T\hat{H}_J = \Psi_{\text{ext}; J}^{\text{D}} * \tilde{\Psi}_{\text{ext}; J}^{\text{D}} * P \quad (154)$$

with respect to  $\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})$ . The scale step property and the approximate identity (cf. Theorem 8.2) imply (150) and (151), **q.e.d.**

This result means that we are able to construct an arbitrarily good approximation to the exact harmonic solution by determining a scale space solution, i.e. by solving the filtered equation

$$TH_J = (\Phi_{\text{ext}; J}^{\text{D}})^{(2)} * P. \quad (155)$$

Moreover, we can improve the approximation by increasing the scale. This can, for example, be realized by adding consecutive detail space solutions.

Two difficulties are still left. First, we need a regularization in order to have a stable, i.e. continuous, determination of the approximations to the exact harmonic solution. Second, we may not assume that  $P$  is given in terms of Fourier coefficients, such that we need a method that can treat a discrete data set of the potential.

Both problems are solved by the following theorem.

**Theorem 9.2** *Let  $\{\Phi_J^{\text{D}}\}_{J \in \mathbb{Z}} \subset \mathcal{L}^2(\overline{B_{\text{int}}} \times \overline{B_{\text{int}}})$  be scale discrete inner scaling functions satisfying*

$$\sum_{n=0}^{\infty} (2n+1)^3 \left| \frac{\left( (\Phi_J^{\text{D}})^\wedge(n) \right)^2}{k^\wedge(n)} \right| < +\infty \quad (156)$$

for every  $J \in \mathbb{Z}$ . Moreover, let the operators  $T_J : T(\mathcal{L}^2(\overline{B_{\text{int}}})) \rightarrow V_{\text{int}; J}^{\text{D}}$ ,  $J \in \mathbb{Z}$ , be given by

$$T_J P := \int_{\overline{B_{\text{ext}}^\gamma}} K_J(\cdot, y) P(y) dy, \quad P \in T(\mathcal{L}^2(\overline{B_{\text{int}}})) , \quad (157)$$

where the kernel  $K_J : \overline{B_{\text{int}}} \times \overline{B_{\text{ext}}^\gamma} \rightarrow \mathbb{R}$  is defined by

$$K_J(x, y) := \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{\left( (\Phi_J^{\text{D}})^\wedge(n) \right)^2}{k^\wedge(n)} H_{n,j}^{\text{int}}(\beta; x) H_{-n-1,j}^{\text{ext}}(\beta; y), \quad (158)$$

$x \in \overline{B_{\text{int}}}$ ,  $y \in \overline{B_{\text{ext}}^\gamma}$ . Then every  $K_J$ ,  $J \in \mathbb{Z}$ , is a function in  $\mathcal{L}^2\left(\overline{B_{\text{int}}} \times \overline{B_{\text{ext}}^\gamma}\right)$  and  $C^{(0)}\left(\overline{B_{\text{int}}} \times \overline{B_{\text{ext}}^\gamma}\right)$ . Moreover, if  $P \in T\left(\mathcal{L}^2\left(\overline{B_{\text{int}}}\right)\right)$ , then  $H_J = T_J P$  is the unique scale space solution at scale  $J$  of Theorem 9.1 and  $T_J$  is continuous for all  $J \in \mathbb{Z}$ .

Again we only give a sketch of a proof for reasons of brevity.

**Sketch of a Proof:** We have

$$\begin{aligned} & \left| \sum_{n=N}^{\infty} \sum_{j=1}^{2n+1} \frac{\left( (\Phi_J^{\text{D}})^\wedge(n) \right)^2}{k^\wedge(n)} H_{n,j}^{\text{int}}(\beta; x) H_{-n-1,j}^{\text{ext}}(\beta; y) \right| \\ & \leq \sum_{n=N}^{\infty} \sum_{j=1}^{2n+1} \frac{\left( (\Phi_J^{\text{D}})^\wedge(n) \right)^2}{k^\wedge(n)} O(n) \left| Y_{n,j}\left(\frac{x}{|x|}\right) Y_{n,j}\left(\frac{y}{|y|}\right) \right| \\ & \leq \sum_{n=N}^{\infty} (2n+1) \frac{\left( (\Phi_J^{\text{D}})^\wedge(n) \right)^2}{k^\wedge(n)} O(n) \frac{2n+1}{4\pi} \rightarrow 0 \end{aligned} \quad (159)$$

for  $N \rightarrow \infty$ . Hence,  $K_J \in C^{(0)}\left(\overline{B_{\text{int}}} \times \overline{B_{\text{ext}}^\gamma}\right) \subset \mathcal{L}^2\left(\overline{B_{\text{int}}} \times \overline{B_{\text{ext}}^\gamma}\right)$ . Moreover, the Parseval identity implies

$$\begin{aligned} (T_J P)(x) &= (K(x, \cdot), P)_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})} \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{\left( (\Phi_J^{\text{D}})^\wedge(n) \right)^2}{k^\wedge(n)} H_{n,j}^{\text{int}}(\beta; x) P^\wedge(-n-1, j). \end{aligned} \quad (160)$$

Thus,  $T_J P = H_J \in V_{\text{int}; J}^{\text{D}}$ .

$T_J$  is continuous, since

$$\|T_J P\|_{\mathcal{L}^2(\overline{B_{\text{int}}})}^2 \leq \sup_{n \in \mathbb{N}} \frac{\left( (\Phi_J^{\text{D}})^\wedge(n) \right)^4}{(k^\wedge(n))^2} \|P\|_{\mathcal{L}^2(\overline{B_{\text{ext}}^\gamma})}^2, \quad \mathbf{q.e.d.}$$

The conditions on the symbols are for example satisfied by the Abel-Poisson scaling functions and the Gauß-Weierstraß scaling functions in case of the

gravimetry problem. Figures 1 and 2 show the results of the determination of Abel-Poisson scale space and detail space solutions at different scales, plotted on the surface of the earth. The corresponding sequence is  $\rho_J = 2^{-J}$ . EGM96 was used as potential and the reconstruction of the solutions was based on formula (149).

We see that the space localization increases if the scale increases. This observation corresponds to the uncertainty principles in [8] and [9].

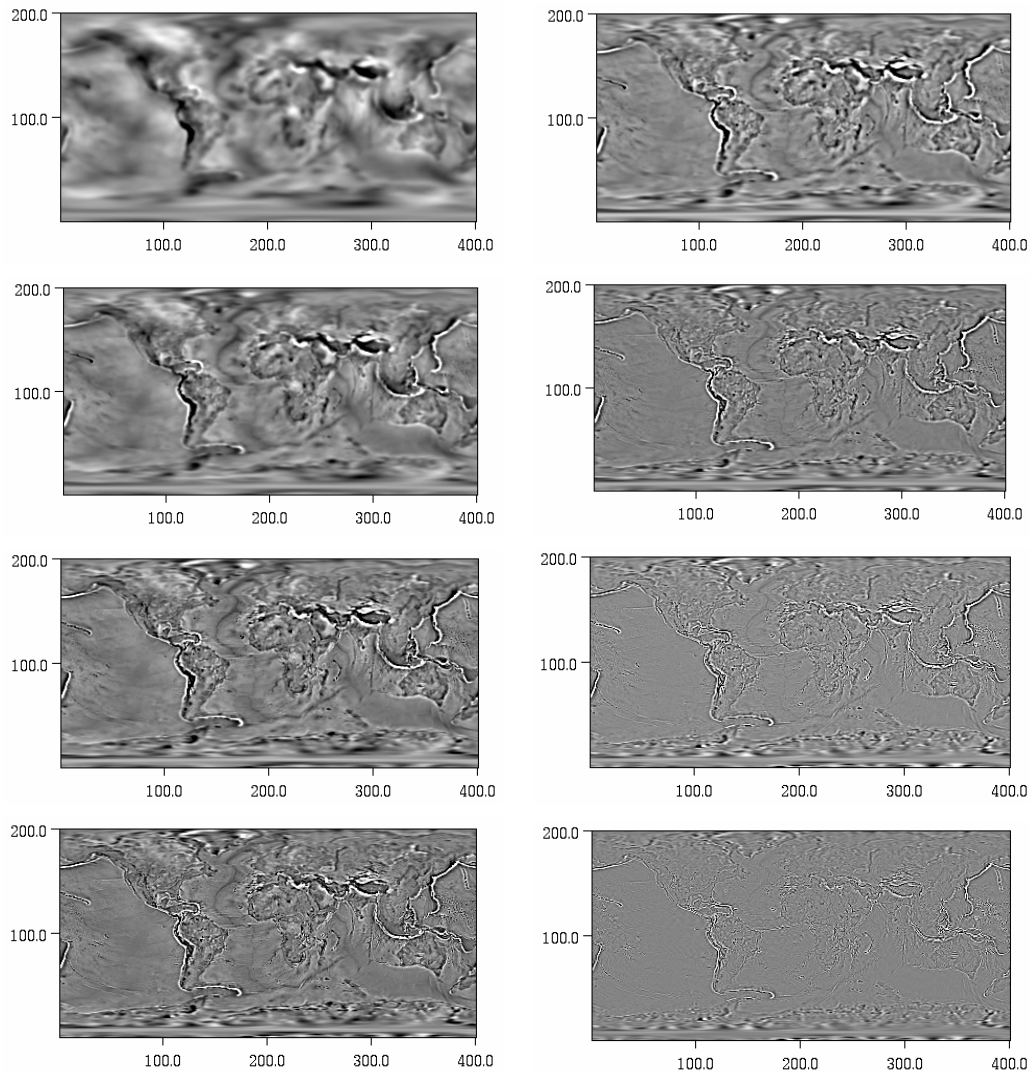


Figure 1: harmonic scale space (left column) and detail space (right column) solutions of the gravimetry problem at scales 4 (top) to 7 (bottom)

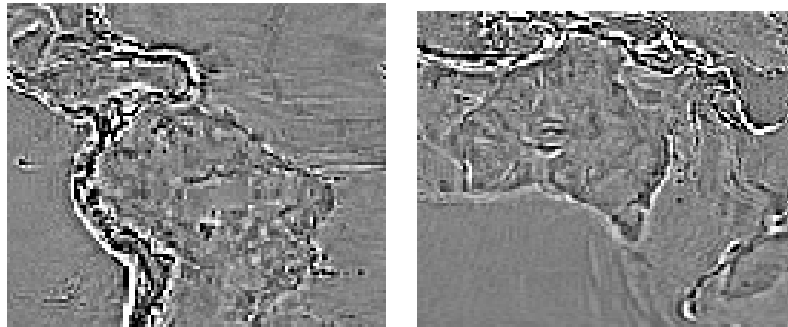


Figure 2: harmonic scale space solution at scale 10 plotted around the Amazonas area (left) and Australia (right), respectively – mind the high resolution of local details like Ayer's Rock

The images of the harmonic solutions on the surface yield many interesting topographical details, such as the Amazonas area and Ayer's Rock in Figure 2.

We know that only the harmonic part of the earth's density distribution can be reconstructed from the gravitational potential. The method developed in this work allows a multiscale regularization of this solution, i.e. we are able to determine an arbitrarily good approximation to the exact harmonic solution by solving a well-posed problem. We can improve the resolution by increasing the scale of the regularization. Whereas former approaches only dealt with truncated polynomial expansions having no space localization, the new multiresolution technique allows the separation of structures with different scales of (space as well as momentum) localization.

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