On Singular Limits of Mean-Field Equations

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Abstract

Mean field equations arise as steady state versions of convection-diffusion systems where the convective field is determined as solution of a Poisson equation whose right hand side is affine in the solutions of the convection-diffusion equations. In this paper we consider the repulsive coupling case for a system of 2 convection-diffusion equations. For general diffusivities we prove the existence of a unique solution of the mean field equation by a variational technique. Also we analyse the small-Debye-length limit and prove convergence to either the so-called charge-neutral case or to a double obstacle problem for the limiting potential depending on the data.

1 Introduction

Consider two species of oppositely charged particles (e.g., negatively charged conduction electrons and positively charged holes in a semiconductor crystal or negatively and positively charged ions in a plasma) whose motion is described macroscopically be the drift-diffusion-Poisson system

$$
\begin{aligned}
n_t &= \operatorname{div}(\nabla f(n) - n\nabla\Phi) && (1)\\
p_t &= \operatorname{div}(\nabla f(p) + p\nabla\Phi) && (2)\\
\lambda\Delta\Phi &= n - p - C(x). && (3)
\end{aligned}
$$

Here $n \geq 0$ and $p \geq 0$ are the position densities of the negatively and, repectively, positively charged particles, Φ is the (self-consistent) Coulomb potential and $C(x) \in L^\infty(\Omega)$ is the difference of background ion position densities. $\lambda > 0$ is a (usually small) parameter representing the scaled Debye-length of the particle system. $f = f(s)$ is the (nonlinear) pressure function satisfying $f'(s) > 0$ for $s > 0$. For the sake of simplicity we assume in this paper equal pressure-density equations of state for both particles.

The equations (1),(2),(3) are posed in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, where the particles are assumed to be confined. We assume zero-outflux conditions:

$$
(\nabla f(n) - n\nabla\Phi)\cdot\boldsymbol{\nu} = (\nabla f(p) + p\nabla\Phi)\cdot\boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega, \tag{4}
$$

and a zero-outward electric field

$$
\nabla\Phi\cdot\boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega, \tag{5}
$$

were $\boldsymbol{\nu}$ denotes the (formal) exterior outward unit vector normal of $\partial\Omega$.

Remark 1. *The subsequent analysis can easily be extended when* (3) *is supplied with homogeneous Dirichlet and homogeneous Dirichlet-Neumann boundary conditions.*

Also we supplement the equations by initial conditions for the densities

$$
\begin{aligned}
n(x, t = 0) = n_\circ(x) \geq 0 \quad , \quad \int_\Omega n_\circ(x)\, dx =: N && (6)\\
p(x, t = 0) = p_\circ(x) \geq 0 \quad , \quad \int_\Omega p_\circ(x)\, dx =: P && (7)
\end{aligned}
$$

N and P are the total negative and, respectively, positive charges, which are conserved by the evolution of (1),(2). Therefore we do have to require total charge neutrality of the system

$$
N - P = \int_\Omega C(x)\, dx. \tag{8}
$$

As basic reference for drift-diffusion-Poisson systems with linear diffusion we cite [MRS90], where many further references can be found. Recently, the limit $t \to \infty$ has been investigated for the bounded domain [Gaj85, GaGr89, CJMTU00, Ott00] and for the whole space problem [BiDo99, BDM99, AMT98, AMTU98, CJMTU00, Ott00] and the limit $\lambda \to 0$ of the parabolic- elliptic system ws analysed in [Gas, GLMS99] (where results for time intervals of $O(1)$-length were obtained).

We reiterate that in this paper we consider mean field equations for two (or one) types of repulsive particles (electrostatic Coulomb interaction). The theory of mean-field equations in the attractive (gravitational) case is decisively different as, in general, solutions do not exist for arbitrarily large mass-parameters (cf., e.g. [CLMP]). This corresponds to finite-time blow-up of solutions of the parabolic-elliptic system of convection-diffusion equations coupled attractively to the Poisson equation, as known from , e.g., turbulence modelling and chemotaxis.

In this paper we are interested in the limit $\lambda \to 0+$ of the stationary system corresponding to (1)-(8). To compute the steady state of a nonlinear (possibly degenerate) convection-diffusion equation with given potential $V = V(x) \in L^\infty(\Omega)$,

$$u_t = \mathrm{div}(\nabla f(u) + u\nabla V(x)), \qquad \int_\Omega u\,dx = M \geq 0, \quad u \geq 0,$$

we write the flux as

$$\nabla f(u) + u\nabla V(x) = u\nabla(h(u) + V(x)),$$

where the enthalpy $h = h(s)$ is defined by

$$h(s) = \int_1^s \frac{f'(r)}{r}\,dr. \tag{9}$$

Then for the equilibrium state $u = u_\infty(x)$ we find that $h(u_\infty(x)) + V(x)$ is constant on connected components of $\{u_\infty > 0\}$. It turns out that the unique equilibrium state, which minimizes the corresponding entropy [CJMTU00], is given by

$$u_\infty(x) = g(c - V(x)), \tag{10}$$

where g is the generalized inverse of the enthalpy,

$$g(\sigma) = \left\{ \begin{array}{ccc} 0 & , & \sigma \leq \underline{h} := h(0+) \\ h^{-1}(t) & , & \underline{h} < \sigma < \overline{h} := h(\infty) \end{array} \right\}, \quad \forall \sigma \in (-\infty, \overline{h}). \tag{11}$$

The constant c in (10) is determined such that

$$M = \int_\Omega g(c - V(x))\,dx. \tag{12}$$

Note that a unique constant c satisfying (12) exists (at least) for

$$M \in \left(0, \overline{M}\right), \quad \overline{M} = \begin{cases} \infty & \text{if} \quad \overline{h} = \infty \\ \displaystyle\int_\Omega g(\overline{h} + \operatorname*{ess\,inf}_\Omega V - V(x))\,dx & \text{if} \quad \overline{h} < \infty \end{cases}$$

For the typical choice

$$f(s) = s^m, \quad \begin{cases} m > 1 & : \text{ porous medium case} \\ m = 1 & : \text{ linear case} \\ m < 1 & : \text{ fast diffusion case} \end{cases} \tag{13}$$

we calculate

$$h(s) = \begin{cases} \frac{m}{m-1}\left(s^{m-1} - 1\right) & , \quad m > 1 \quad \left(-\infty < \underline{h} = -\frac{m}{m-1}, \overline{h} = \infty\right) \\ \log(s) & , \quad m = 1 \quad \left(-\infty = \underline{h}, \overline{h} = \infty\right) \\ \frac{m}{1-m}\left(1 - s^{-(1-m)}\right) & , \quad m < 1 \quad \left(-\infty = \underline{h}, \overline{h} = \frac{m}{1-m} < \infty\right) \end{cases} \tag{14}$$

and

$$g(\sigma) = \begin{cases} \left(\left(1 + \sigma\frac{m-1}{m}\right)^+\right)^{\frac{1}{m-1}} & , \quad m > 1 \\ \exp(\sigma) & , \quad m = 1 \\ \left(1 - \sigma\frac{1-m}{m}\right)^{-\frac{1}{1-m}} & , \quad m < 1 \end{cases} \tag{15}$$

Another physically important case is given by the Fermi-Dirac distribution [Groe86, GaGr89]

$$f(s) = sF^{-1}(s) - \int_0^s F^{-1}(\xi)\,d\xi \tag{16}$$

with

$$h(s) = F^{-1}(s) - F^{-1}(1), \quad -\infty = \underline{h}, \overline{h} = \infty \tag{17}$$

and

$$g(\sigma) = F\left(\sigma + F^{-1}(1)\right).,\tag{18}$$

where

$$F(\sigma) = \int_{\mathbb{R}^d} \frac{d\,\boldsymbol{v}}{\varepsilon + \exp\left(\frac{|\boldsymbol{v}|^2}{2} - \sigma\right)}.$$

By formally applying the solution formula (10) to (1)-(7), we obtain the steady state drift-diffusion-Poisson system (called the "mean field equation" in the sequel) for the equilibrium potential $\Phi = \Phi_\infty$

$$\begin{aligned} \lambda\Delta\Phi &= n[\Phi] - p[\Phi] - C(x), \quad x \in \Omega & (19)\\ n[\Phi] &= g(\alpha[\Phi] + \Phi) & (20)\\ p[\Phi] &= g(\beta[\Phi] - \Phi) & (21)\end{aligned}$$

subject to homogeneous Neumann conditions

$$\nabla\Phi \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega. \tag{22}$$

The constants $\alpha[\Phi], \beta[\Phi]$ (so-called Fermi-levels), which determine the particle densities $n_\infty = n[\Phi]$, $p_\infty = p[\Phi]$, are given by the normalisations

$$\int_\Omega g(\alpha[\Phi] + \Phi)\, dx = N, \tag{23}$$

$$\int_\Omega g(\beta[\Phi] - \Phi)\, dx = P, \tag{24}$$

where (8) is assumed to hold for consistency reasons.

We refer to [GaGr89, CDMS99] for an analysis of (19)-(22) in the "linear" case $g(\sigma) = \exp(\sigma)$ (Boltzmann distribution) and to [BDM99, Unt97] for the model in "nonlinear" cases.

It turns out that (19)-(21) is a critical point of the functional

$$\begin{aligned} J_\lambda[\Phi] = \frac{\lambda}{2}\int_\Omega |\nabla\Phi|^2\, dx - \int_\Omega C(x)\Phi\, dx \\ + \int_\Omega G(\alpha[\Phi] + \Phi)\, dx + \int_\Omega G(\beta[\Phi] - \Phi)\, dx \\ - N\alpha[\Phi] - P\beta[\Phi], \end{aligned} \tag{25}$$

where G is a primitive of g. We shall show in the case $\overline{h} = \infty$ that J_λ is a strictly convex, weakly lower semicontinuous, bounded from below functional on an apropriate convex set. The solution of (19)-(21) is therefore uniquely determined as the minimum of J_λ.

Actually, the constraints (23), (24) which determine α and β makes the problem nonlocal and it turns out that a transformed functional is easier to handle (in case of $\overline{h} = \infty$ and for carrying out the limit $\lambda \to 0$, see Appendix A). This new functional reads

$$E_\lambda(n, p) = \int_\Omega H(n)\, dx + \int_\Omega H(p)\, dx + \frac{1}{2\lambda}\int_\Omega |\nabla V|^2\, dx$$

where $(H')^{-1} = g_{|(0,+\infty)}$ (i.e. $H' = h$) and where $V = V[n-p-C]$ is given by

$$-\Delta V = n - p - C$$

(with homogeneous Neuman boundary conditions) and has now to be minimized under the constraints

$$N = \int_\Omega n(x)\,dx \quad \text{and} \quad P = \int_\Omega p(x)\,dx, \quad n \geq 0, p \geq 0.$$

The constants α and β now simply appear as Lagrange multipliers associated to the normalization constraints for the L^1-norms of n and p.

This allows us to state our first result.

Theorem. A. *With the above notations,* (19)-(21) *has a unique solution.*

The functional E_λ is the one which is used to study in L^1 the asymptotic behaviour of drift-diffusion problems corresponding to a nonlinear diffusion (see [CJMTU00, BDM99] for the case with a Poisson coupling; see [AMTU98, AMT98, BiDo99] for the case with Poisson coupling and a linear diffusion). Such a framework is also especially convenient for the study of the so-called *insulator limit* (see [CDMS99] for a justification in terms of physical quantities) corresponding to $\lambda \to 0$, and as in [CDMS99] (this paper is basically a generalization of [CDMS99] emphasizing the abstract structure of the problem), we have to distinguish two regimes.

Theorem. B. *If either $N > \int_\Omega C^+\,dx$ or (equivalently) $P > \int_\Omega C^-\,dx$, then the solution of* (19)-(21) *converges as $\lambda \to 0$ to the solution of* (19)-(21) *with $\lambda = 0$, which is unique. In the other case, there exist two measurable subsets of Ω, Ω_n and Ω_p, respectively, with $\Omega_n \subseteq \{C \geq 0\}$, $\Omega_p \subseteq \{C \leq 0\}$, and a unique solution (n_0, p_0, V_0), which is the limit of (n, p, V) as $\lambda \to 0$, such that*

$$-\Delta V_0 = n_0 - p_0 - C, \quad \int_\Omega V_0\,dx = 0, \quad \int_\Omega n_0\,dx = N, \quad \int_\Omega p_0\,dx = P,$$

$$\begin{array}{lll} V_0 = \operatorname{ess\,inf}_\Omega V_0 & on & \Omega_n \\ \Delta V_0 = C & on & \Omega \setminus (\Omega_n \cup \Omega_p) \\ V_0 = \operatorname{ess\,sup}_\Omega V_0 & on & \Omega_p \end{array},$$

$$n_0 = \left\{ \begin{array}{lll} C^+ & on & \Omega_n \\ 0 & on & \Omega \setminus \Omega_n \end{array} \right. , \quad p_0 = \left\{ \begin{array}{lll} C^- & on & \Omega_p \\ 0 & on & \Omega \setminus \Omega_p \end{array} \right.$$

The conditions $N > \int_\Omega C^+ \, dx$ and $P > \int_\Omega C^- \, dx$ are equivalent because of the global charge neutrality (25)). In the second case, $\Delta V_0 = C$ on $\Omega \setminus (\Omega_n \cup \Omega_p)$ and v_0 is constant on Ω_n and Ω_p where it reaches its maximum and its minimum respectively. Note that this double obstacle for V is uniquely determined by the side condition $\int_{\Omega_n} C^+ \, dx = N$.

The paper is organized as follows. The remainder of the first section is devoted to some preliminary results. A more detailed statement of Theorem A and its proof are given in Section 2, and Section 3 deals with the *insulator limit*. Remarks on the Legendre transform of the functional can be found in Appendix A and an extension to unbounded domains (corresponding to one species of particles, or two species when one looks for intermediate asymptotics) is given in Appendix B.

Before going further, let us fix some notations and state the detailed assumptions we shall use throughout this paper.

A.1 $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is a bounded, nonvoid domain.

A.2 $h : J \to \mathbb{R}$, $J = \mathbb{R}^+ := (0, \infty)$ or $J = \mathbb{R}_0^+ := [0, \infty)$, is continuous and strictly increasing.
$\underline{h} := \inf_J h$, $\overline{h} := \sup_J h$. $J = \mathbb{R}^+$ iff $\underline{h} = -\infty$.

A.3 $H : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous, differentiable on $\mathbb{R}^+$ and

$$\forall s \in \mathbb{R}^+ : \quad H'(s) = h(s).$$

A.4 $g : (-\infty, \overline{h}) \to J$ is the generalized inverse of h as defined in (11).

A.5 $C \in L^\infty(\Omega)$. $\underline{C} = \operatorname{ess\,inf}_\Omega C$, $\overline{C} = \operatorname{ess\,sup}_\Omega C$.

Remark 2. *a) H is strictly convex.*
b) g is continuous and increasing, strictly increasing on $(\underline{h}, \overline{h})$ and $\lim_{t \to \overline{h}} g(t) = \infty$.

In the sequel we shall assume that assumptions **A.1** and **A.5** are always satisfied.

Let $f \in L^1(\Omega)$ with $\int_\Omega f \, dx = 0$. If there is $V \in H^1(\Omega)$ with $\int_\Omega V \, dx = 0$ and

$$\forall \phi \in H^1(\Omega) : \quad \int_\Omega \nabla V \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx,$$

- which is, e.g., the case for all $f \in L^1(\Omega) \cap H^{-1}(\Omega)$ with $\int_\Omega f \, dx = 0$ - then V will be uniquely determined (by f) and we will set $V = V[f]$ and certainly

$$\int_\Omega |\nabla V[f]|^2 \, dx < \infty.$$

We observe: $V[f]$ is a weak solution of

$$-\Delta V = f,$$

subject to homogeneous Neumann boundary conditions. However, if, for a given f, no such $V \in H^1(\Omega)$ exists, then we set

$$\int_\Omega |\nabla V[f]|^2 \, dx = \infty.$$

We introduce for $N, P \in \mathbb{R}^+$ with $N - P = \int_\Omega C \, dx$ the set

$$\mathcal{C} := \left\{ (n,p) \in L^1_+(\Omega) \times L^1_+(\Omega) : \int_\Omega n \, dx = N, \int_\Omega p \, dx = P \right\},$$

and we define for $\lambda \in \mathbb{R}^+$,

$$\begin{aligned} &E_\lambda : \mathcal{C} \to \mathbb{R} \cup \{\infty\}, \\ &\qquad E_\lambda(n,p) = \int_\Omega H(n) \, dx + \int_\Omega H(p) \, dx + \frac{1}{2\lambda} \int_\Omega |\nabla V[n - p - C]|^2 \, dx, \end{aligned}$$

where we make use of Jensen's inequality ensuring that E_λ is bounded below.

1.1 A Semilinear Elliptic PDE of Order Two

In this section we shall derive several auxiliary results.

Proposition 1. *Assume* **A.1** *and* **A.5** *and let $\lambda \in \mathbb{R}^+$. Let $I \subseteq \mathbb{R}$ be a nonvoid, open interval and let $m : I \to \mathbb{R}$ be continuous and increasing with*

$$-\infty \le \inf_I m < \underline{C} \le 0, \quad 0 \le \overline{C} < \sup_I m \le \infty.$$

Then there is exactly one $\Phi \in H^1(\Omega)$ with $\Phi(x) \in I$ for almost all $x \in \Omega$ and

$$m(\Phi)\, \phi \in L^1(\Omega), \quad -\lambda \int_\Omega \nabla \Phi \cdot \nabla \phi = \int_\Omega m(\Phi) \phi \, dx - \int_\Omega C \, \phi \, dx,$$

for all $\phi \in H^1(\Omega)$. Furthermore, Φ has the following properties:

1. $\Phi \in L^\infty(\Omega), \quad \underline{C} \le m\left(\operatorname{ess\,inf}_\Omega \Phi\right) \le m\left(\operatorname{ess\,sup}_\Omega \Phi\right) \le \overline{C},$
   in particular: there is a constant $\hat{\Phi} \in \mathbb{R}^+$ such that

$$\forall \lambda \in \mathbb{R}^+ : \quad \|\Phi\|_{L^\infty(\Omega)} \le \hat{\Phi}.$$

2. Φ *is the unique weak solution (in $H^1(\Omega)$) of the semilinear elliptic PDE*

$$\lambda \Delta \Phi = m(\Phi) - C,$$

subject to homogeneous Neumann boundary conditions.

3. $\displaystyle\int_\Omega m(\Phi)\, dx = \int_\Omega C\, dx.$

Proof. Let

$$M : I \to \mathbb{R}, \quad M(t) := \int_{t_0}^{t} m(s)\, ds,$$

where $t_0 \in I$ with $m(t_0) = 0$. Since $m(t(< 0$ for $t < t_0$ and since $m(t) > 0$ for $t > t_0$, the function M is a non-negative primitive of m. Furthermore, since m is increasing, M is convex. We introduce the functional

$$F_\lambda : H^1(\Omega) \to \mathbb{R} \cup \{\infty\},$$
$$F_\lambda(\eta) = \begin{cases} \dfrac{\lambda}{2} \displaystyle\int_\Omega |\nabla \eta|^2\, dx + \int_\Omega M(\eta)\, dx - \int_\Omega C\, \eta\, dx & \text{if} \quad \eta \in \mathcal{C}_0 \\ \qquad\qquad\qquad \infty & \text{if} \quad \eta \notin \mathcal{C}_0 \end{cases},$$

where

$$\mathcal{C}_0 = \{\eta \in H^1(\Omega) :\ \eta(x) \in I \text{ for almost all } x \in \Omega\}.$$

We certainly have $\inf_{\mathcal{C}_0} F_\lambda \in \mathbb{R}$. Let $(\eta_k)_{k\in\mathbb{N}}$ be a minimizing sequence of F_λ. By assumption we have for almost all $x \in \Omega$,

$$\lim_{t\to\inf I} (m(t) - C(x)) < 0, \quad \lim_{t\to\sup I} (m(t) - C(x)) > 0,$$

hence there is a compact interval $[a, b] \subseteq I$ (which is, by the way, independent of λ) such that $t \mapsto M(t) - C(x)t$ is decreasing on $(\inf I, a)$ and increasing on $(b, \sup I)$, i.e. for almost all $x \in \Omega$:

$$\forall t \in (\inf I, a) : \quad M(t) - C(x)\, t \geq M(a) - C(x)\, a,$$

$$\forall t \in (b, \sup I) : \quad M(t) - C(x)\, t \geq M(b) - C(x)\, b.$$

Now we set for $k \in \mathbb{N}$,

$$[\eta_k] := \min\{b, \max\{a, \eta_k\}\}.$$

Then $([\eta_k])_{k\in\mathbb{N}}$ is a sequence in $H^1(\Omega)$ because cutting maps $H^1(\Omega)$ (continuously) into itself. Furthermore, cutting does not increase the $L^2(\Omega)$-norms of the gradients [Zie89]. Hence

$$\forall k \in \mathbb{N}: \quad F_\lambda([\eta_k]) \leq F_\lambda(\eta_k),$$

and therefore $([\eta_k])_{k\in\mathbb{N}}$ is a minimizing sequence of F_λ as well. By passing to a subsequence (but without changing notation) we have

$$[\eta_k] \rightharpoonup \Phi \in L^\infty(\Omega) \text{ weak* in } L^\infty(\Omega) \text{ as } k \to \infty,$$

and therefore - since $L^\infty(\Omega) \subseteq L^2(\Omega)$ and since the $L^2(\Omega : \mathbb{R}^d)$-norms of $(\nabla[\eta_k])_{k\in\mathbb{N}}$ are uniformly bounded - we have

$$[\eta_k] \rightharpoonup \Phi \in H^1(\Omega) \text{ weakly in } H^1(\Omega) \text{ as } k \to \infty.$$

We readily deduce from a lower semicontinuity argument that Φ is a minimizer of F_λ in $H^1(\Omega)$. Since F_λ is strictly convex due to the quadratic leading term, Φ is the unique minimizer of F_λ in $H^1(\Omega)$. The uniform bounds $\eta_\lambda(x) \in [a,b] \subseteq I$ for almost all $x \in \Omega$ give $m(\Phi) \in L^\infty(\Omega)$ and the associated Euler-Lagrange equations

$$\forall \phi \in H^1(\Omega): -\lambda \int_\Omega \nabla\Phi \cdot \nabla\phi = \int_\Omega m(\Phi)\phi \, dx - \int_\Omega C\, \phi \, dx$$

easily follow. The proof of the remaining statements is left to the reader. □

In particular cases the result of Proposition 1 can be extended as follows.

Proposition 2. *Assume* **A.1** *and* **A.5** *and let $\lambda \in \mathbb{R}^+$. Let $I \subset \mathbb{R}$ be an open, nonvoid interval with* $\inf I \in \mathbb{R}$ or $\sup I \in \mathbb{R}$. *Let $m : I \to \mathbb{R}$ be continuous and increasing. Assume furthermore* one *of the following conditions*

1. *If* $\inf I \in \mathbb{R}$ *and if*

$$\inf_I m = \underline{C} \leq 0, \quad 0 \leq \overline{C} < \sup_I m \leq \infty,$$

then we set

$$m_0 : [\inf I, \sup I) \to [\underline{C}, \infty), \quad m_0(\rho) = \begin{cases} \underline{C} & \text{if} \quad \inf_I = \rho \\ m(\rho) & \text{if} \quad \inf_I < \rho \end{cases}$$

2. *If* $\sup I \in \mathbb{R}$ *and if*

$$-\infty \leq \inf_I m < \underline{C} \leq 0, \quad 0 \leq \overline{C} = \sup_I m,$$

then we e set

$$m_0 : (\inf I, \sup I] \to (-\infty, \overline{C}], \quad m_0(\rho) = \begin{cases} m(\rho) & \text{if} \quad \rho < \sup I \\ \overline{C} & \text{if} \quad \rho = \sup I \end{cases}$$

3. *If* $\inf I, \sup I \in \mathbb{R}$ *and if*

$$\inf_I m = \underline{C} \leq 0, \quad 0 \leq \overline{C} = \sup_I m,$$

then we set

$$m_0 : [\inf I, \sup I] \to [\underline{C}, \infty), \quad m_0(\rho) = \begin{cases} \underline{C} & \text{if} \quad \rho = \inf_I \\ m(\rho) & \text{if} \quad \inf_I < \rho \\ \overline{C} & \text{if} \quad \rho = \sup I \end{cases}$$

Then there is exactly one $\Phi \in H^1(\Omega)$ *with* $\Phi(x) \in \mathsf{dom}(m_0)$ *for almost all* $x \in \Omega$ *and*

$$m_0(\Phi)\ \phi \in L^1(\Omega), \quad -\lambda \int_\Omega \nabla\Phi \cdot \nabla\phi = \int_\Omega m_0(\Phi)\phi\ dx - \int_\Omega C\,\phi\ dx,$$

for all $\phi \in H^1(\Omega)$. *Furthermore,* Φ *has the following properties:*

1. $\Phi \in L^\infty(\Omega), \quad \underline{C} \leq m_0\left(\operatorname{ess\,inf}_\Omega \Phi\right) \leq m_0\left(\operatorname{ess\,sup}_\Omega \Phi\right) \leq \overline{C},$

   in particular: there is a constant $\hat{\Phi} \in \mathbb{R}^+$ *such that*

$$\forall\lambda \in \mathbb{R}^+ : \quad \|\Phi\|_{L^\infty(\Omega)} \leq \hat{\Phi}.$$

2. Φ *is the unique weak solution (in* $H^1(\Omega)$*) of the semilinear elliptic PDE*

$$\lambda\Delta\Phi = m_0(\Phi) - C,$$

   subject to homogeneous Neumann boundary conditions.

3. $\displaystyle\int_\Omega m_0(\Phi)\ dx = \int_\Omega C\ dx.$

Proof. We consider case 1. Cases 2. and 3. can be treated in analogy. We define the function

$$m_1 : (-\infty, \sup I) \to \mathbb{R},$$

$$m_1(\rho) = \begin{cases} \underline{C} + (\rho - \inf I) & \text{if} \quad \rho \in (-\infty, \inf I] \\ m(\rho) & \text{if} \quad \rho \in (\inf I, \sup I) \end{cases}$$

Since $\inf m_1 = -\infty < \underline{C}$, we can apply proposition 1 to obtain:
There is exactly one $\Phi_1 \in H^1(\Omega)$ such that $\Phi_1(x) \in (-\infty, \sup I)$ for almost all $x \in \Omega$ and

$$m_1(\Phi_1)\ \phi \in L^1(\Omega), \quad -\lambda \int_\Omega \nabla\Phi_1 \cdot \nabla\phi = \int_\Omega m_1(\Phi_1)\phi\ dx - \int_\Omega C\,\phi\ dx,$$

for all $\phi \in H^1(\Omega)$. Furthermore, Φ_1 has the following properties:

1. $\Phi_1 \in L^\infty(\Omega), \quad \underline{C} \le m_1\left(\operatorname{ess\,inf}_{\Omega} \Phi_1\right) \le m_1\left(\operatorname{ess\,sup}_{\Omega} \Phi_1\right) \le \overline{C}$,

2. Φ_1 is the unique weak solution (in $H^1(\Omega)$) of the semilinear elliptic PDE

$$\lambda \Delta \Phi_1 = m_1(\Phi_1) - C,$$

   subject to homogeneous Neumann boundary conditions.

3. $\displaystyle\int_\Omega m_1(\Phi_1)\, dx = \int_\Omega C\, dx.$

$$m_1\left(\Phi_1\right) \phi \in L^1(\Omega), \quad -\lambda \int_\Omega \nabla \Phi_1 \cdot \nabla \phi = \int_\Omega m_1(\Phi_1)\phi\, dx - \int_\Omega C\, \phi\, dx,$$

We deduce from estimate 1.: $m_1(\Phi_1) = m_0(\Phi_1)$. Hence for all $\phi \in H^1(\Omega)$,

$$m_0(\Phi_1)\, \phi \in L^1(\Omega), \quad -\lambda \int_\Omega \nabla \Phi_1 \cdot \nabla \phi = \int_\Omega m_0(\Phi_1)\phi\, dx - \int_\Omega C\, \phi\, dx,$$

$\Phi_1(x) \in \mathsf{dom}(m_0) = [\inf I, \sup I)$ for almost all $x \in \Omega$,

$$\Phi_1 \in L^\infty(\Omega), \quad \underline{C} \le m_0\left(\operatorname{ess\,inf}_{\Omega} \Phi_1\right) \le m_0\left(\operatorname{ess\,sup}_{\Omega} \Phi_1\right) \le \overline{C},$$

Φ_1 is a weak solution (in $H^1(\Omega)$) of the semilinear elliptic PDE

$$\lambda \Delta \Phi_1 = m_0(\Phi_1) - C,$$

subject to homogeneous Neumann boundary conditions, and
$\displaystyle\int_\Omega m_0(\Phi_1)\, dx = \int_\Omega C\, dx.$

Now let $\Phi^*(x) \in \mathsf{dom}(m_0)$ for almost all $x \in \Omega$ and

$$m_0\left(\Phi^*\right) \phi \in L^1(\Omega), \quad -\lambda \int_\Omega \nabla \Phi^* \cdot \nabla \phi = \int_\Omega m_0(\Phi^*)\phi\, dx - \int_\Omega C\, \phi\, dx.$$

Then $m_0(\Phi^*) = m_1(\Phi^*)$ as well and we deduce $\Phi^* = \Phi_1$ from the uniqueness result cited above.

The verification that the semilinear elliptic PDE

$$\lambda \Delta \Phi = m_0(\Phi) - C,$$

subject to homogeneous Neumann boundary conditions and
$\displaystyle\int_\Omega m_0(\Phi)\, dx = \int_\Omega C\, dx$, has a unique solution in $H^1(\Omega)$ is left to the reader. $\square$

1.2 An Abstract Variational Problem

In this section we consider the limit $\lambda \to 0$ of an abstract minimization problems with a class of functionals containing E_λ.

Theorem 3. *Let* $(\mathfrak{B}, \|.\|)$ *be a Banach space and let*

$$\mathcal{C} \subseteq \mathfrak{B}$$

be nonvoid, convex and weakly closed in $\mathfrak{B}$. *Let*

$$E, F : \mathcal{C} \to \mathbb{R} \cup \{\infty\}, \text{ be bounded below with } \inf_{\mathcal{C}} E < \infty, \inf_{\mathcal{C}} F < \infty.$$

We set

$$\mathcal{C}^* := \{x \in \mathcal{C} : E(x) < \infty\}.$$

For $\lambda \in \mathbb{R}^+$ *let* $x_\lambda \in \mathcal{C}$.
Assume

1. x_λ *is for each* $\lambda \in \mathbb{R}^+$ *a minimizer of* $E_\lambda := E + \lambda^{-1} F$ *in* $\mathcal{C}$.

2. $x_\lambda \rightharpoonup x_0$ *weakly in* $\mathfrak{B}$ *as* $\lambda \to 0$.

Then

a) $\limsup_{\lambda \to 0} F(x_\lambda) \le \inf_{\mathcal{C}^*} F$.

b) If F *is weakly lower sequentially continuous at* x_0, *then*

$$F(x_0) \le \inf_{\mathcal{C}^*} F.$$

c) If F *is weakly lower sequentially continuous at* x_0 *and if* $E(x_0) < \infty$, *then* x_0 *is a minimizer of* F *in* $\mathcal{C}^*$, *i.e.*

$$F(x_0) = \inf_{\mathcal{C}^*} F.$$

d) If x^* *is a minimizer of* F *in* $\mathcal{C}^*$, *then*

$$\limsup_{\lambda \to 0} E(x_\lambda) \le E(x^*).$$

e) If x^* *is a minimizer of* F *in* $\mathcal{C}^*$ *and if* E *is weakly lower sequentially continuous at* x_0, *then*

$$E(x_0) \le E(x^*).$$

f) *If E and F are weakly lower sequentially continuous at x_0 and if $E(x_0) < \infty$, then x_0 is a minimizer of F in $\mathcal{C}^*$ whose "energy" $E(x_0)$ is less or equal the energy $E(x^*)$ of any minimizer x^* of F in $\mathcal{C}^*$.*

Proof. a) Let $-\rho \in \mathbb{R}$ be a lower bound of E in $\mathcal{C}$. Let $x \in \mathcal{C}$ with $E(x) < \infty$. Since $\lambda > 0$ we have

$$\lambda(E(x_\lambda) + \rho) + F(x_\lambda) \leq \lambda(E(x) + \rho) + F(x).$$

Hence by non-negativity of $E(x_\lambda) + \rho$,

$$\limsup_{\lambda \to 0} F(x_\lambda) \leq F(x).$$

b) follows from a).
c) If $E(x_0) < \infty$, then x_0 is by b) a minimizer of F in $\mathcal{C}^*$.
d) Let $-\rho \in \mathbb{R}$ be a lower bound of E, F. Let $x_* \in \mathcal{C}^*$ be a minimizer of F in $\mathcal{C}^*$. Then $E(x^*) < \infty$ and we obtain for all $\lambda \in \mathbb{R}^+$ the estimate

$$(E(x_\lambda) + \rho) + \frac{1}{\lambda}(F(x_\lambda) + \rho) \leq (E(x_*) + \rho) + \frac{1}{\lambda}(F(x_*) + \rho),$$

from which we deduce $E(x_\lambda) < \infty$, hence $x_\lambda \in \mathcal{C}^*$, and due to $F(x_*) \leq F(x_\lambda)$,

$$E(x_\lambda) \leq E(x_*).$$

e) follows from d) and f) follows from a)-e). □

2 The Main Results

2.1 $\lambda \in \mathbb{R}^+$

The main result of this section is

Theorem 4. *Assume* **A.1**-**A.5**. *Let $\lambda \in \mathbb{R}^+$ and let $N, P \in \mathbb{R}^+$ with*

$$N - P = \int_\Omega C \, dx.$$

Then:

1. *The functional E_λ has a unique minimizer (n_λ, p_λ) in $\mathcal{C}$.*

2. *$n_\lambda, p_\lambda \in L^\infty(\Omega)$ with*

$$n_\lambda \leq \overline{C} + \frac{P}{\text{meas}(\Omega)}, \quad p_\lambda \leq -\underline{C} + \frac{N}{\text{meas}(\Omega)}.$$

3. There are $\alpha_\lambda, \beta_\lambda \in \mathbb{R}$ such that

$$n_\lambda = g\left(\alpha_\lambda - \lambda^{-1} V_\lambda\right),$$

$$p_\lambda = g\left(\beta_\lambda + \lambda^{-1} V_\lambda\right),$$

where $V_\lambda = V[n_\lambda - p_\lambda - C]$, hence if one sets $\Phi_\lambda := -\lambda^{-1} V_\lambda$, then

$$\lambda \Delta \Phi_\lambda = g\left(\alpha_\lambda + \Phi_\lambda\right) - g\left(\beta_\lambda - \Phi_\lambda\right) - C, \quad \int_\Omega \Phi_\lambda \, dx = 0,$$

subject to homogeneous Neumann boundary conditions and

$$\int_\Omega g(\alpha_\lambda + \Phi_\lambda) \, dx = N, \quad \int_\Omega g(\beta_\lambda - \Phi_\lambda) \, dx = P.$$

Proof. Due to the lack of coercitivity (in case of $\overline{h} < \infty$) of E_λ it is not obvious that E_λ has a minimizer in $\mathcal{C}$. The argumentation will be based settled on the semilinear equation

$$\lambda \Delta \Psi = g(\alpha + \Psi) - g(\beta - \Psi) - C \tag{26}$$

subject to homogeneous Neumann boundary conditions. In (26) the constants α, β are a priori unknown. Our program is to prove: There are $\alpha, \beta \in \mathbb{R}$ such that $\int_\Omega g(\alpha + \Psi) \, dx = N$, $\int_\Omega g(\beta - \Psi) \, dx = P$.

We consider for $c \in (-\infty, 2\overline{h})$ the semilinear elliptic PDE

$$\lambda \Delta \Phi = g(c + \Phi) - g(-\Phi) - C \tag{27}$$

subject to homogeneous Neumann boundary conditions. This problem fits with

$$m(.) = g(c + .) - g(-.), \quad I = (-\overline{h}, \overline{h} - c)$$

to Proposition 1: there is a unique $\Phi_c \in H^1(\Omega)$ which solves (27) subject to homogeneous Neumann boundary conditions. Furthermore, Φ_c belongs to $L^\infty(\Omega)$ and if we set

$$n_c = g(c + \Phi_c), \quad p_c = g(-\Phi_c),$$

$$\overline{\Phi_c} := \operatorname*{ess\,sup}_\Omega \Phi_c, \quad \underline{\Phi_c} := \operatorname*{ess\,inf}_\Omega \Phi_c$$

then we deduce from (27) the estimates

$$g\left(c + \overline{\Phi_c}\right) - g\left(-\overline{\Phi_c}\right) \leq \overline{C}, \quad g\left(c + \underline{\Phi_c}\right) - g\left(-\underline{\Phi_c}\right) \geq \underline{C}. \tag{28}$$

Hence the functions n_c, p_c belong to $L^\infty(\Omega)$. Let us introduce

$$N_c = \int_\Omega n_c \, dx = \int_\Omega g(c + \Phi_c) \, dx, \quad P_c = \int_\Omega p_c \, dx = \int_\Omega g(-\Phi_c) \, dx.$$

Then $N_c - P_c = \int_\Omega C \, dx$ and we have

$$\|n_c\|_{L^\infty(\Omega)} \leq \overline{C} + \frac{P_c}{\mathrm{meas}(\Omega)}, \quad \|p_c\|_{L^\infty(\Omega)} \leq -\underline{C} + \frac{N_c}{\mathrm{meas}(\Omega)}.$$

We have to prove: there is $c \in \mathbb{R}$ such that $N_c = N$, $P_c = P$. This is shown in several steps.

$\underline{c \mapsto (N_c, P_c) \text{ is continuous}}$. Indeed, let $c \in (-\infty, 2\overline{h})$ and let $(c_k)_{k\in\mathbb{N}}$ be a sequence in $(-\infty, 2\overline{h})$ with $\lim_{k\to\infty} c_k = c$ and let $(c_{k_1(k)})_{k\in\mathbb{N}}$ be a subsequence of $(c_k)_{k\in\mathbb{N}}$. We have to prove: there is a subsequence $(c_{k_2(k)})_{k\in\mathbb{N}}$ of $(c_{k_1(k)})_{k\in\mathbb{N}}$ with

$$\lim_{k\to\infty} N_{c_{k_2(k)}} = N_c, \quad \lim_{k\to\infty} P_{c_{k_2(k)}} = P_c.$$

We deduce from (28): There are $K_0, K_1 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\|n_{c_k}\|_{L^\infty(\Omega)}, \|p_{c_k}\|_{L^\infty(\Omega)} \leq K_1, \quad k \geq K_0.$$

Hence there is a subsequence $(k_3(k))_{k\in\mathbb{N}}$ of $(k_1(k))_{k\in\mathbb{N}}$ and there are $n_0, p_0 \in L^\infty(\Omega)$ with

$$n_{k_3(k)} \rightharpoonup n_0, p_{k_3(k)} \rightharpoonup n_0 \quad \text{weak* in } L^\infty(\Omega) \text{ as } k \to \infty.$$

We deduce:

$$\Phi_{k_3(k)} \to \Phi_0 \quad \text{strongly in } H^1(\Omega) \text{ as } k \to \infty$$

where

$$\lambda \Delta \Phi_0 = n_0 - p_0 - C,$$

subject to homogeneous Neumann boundary conditions. Hence there is a subsequence $(k_2(k))_{k\in\mathbb{N}}$ of $(k_3(k))_{k\in\mathbb{N}}$ with

$$\Phi_{k_2(k)}(x) \to \Phi_0(x) \quad \text{for almost all } x \in \Omega \text{ as } k \to \infty.$$

We deduce

$$\begin{aligned} &n_{k_2(k)}(x) \\ &\quad = g(c_{k_2(k)} + \Phi_{k_2(k)})(x) \to g(c + \Phi_0)(x) \text{ for almost all } x \in \Omega \text{ with } k \to \infty, \end{aligned}$$

$$p_{k_2(k)}(x) = g(-\Phi_{k_2(k)})(x) \to g(-\Phi_0)(x) \quad \text{for almost all } x \in \Omega \text{ with } k \to \infty.$$

Hence $n_0 = n_c$, $\Phi_0 = \Phi_c$, $p_0 = p_c$ and therefore due to weak* convergence in $L^\infty(\Omega)$, $\lim_{k\to\infty} N_{c_{k_2(k)}} = N_c$, $\lim_{k\to\infty} P_{c_{k_2(k)}} = P_c$.

$\underline{\lim_{c\to 2\overline{h}} N_c = \infty, \lim_{c\to 2\overline{h}} P_c = \infty.}$
Here we only consider $\overline{h} < \infty$. The case $\overline{h} = \infty$ can be treated similiarily.
Due to $N_c = P_c + \int_\Omega C\ dx$ it sufficies to prove: $\lim_{c\to\infty} P_c = \infty$. According to $c + \Phi_c(x) < \overline{h}$ and $-\Phi_c(x) < \overline{h}$ for almost all $x \in \Omega$, we deduce

$$c - \overline{h} < -\Phi_c(x) < \overline{h} \quad \text{for almost all } x \in \Omega.$$

Hence

$$\lim_{c\to 2\overline{h}} P_c = \lim_{c\to 2\overline{h}} \int_\Omega g\left(-\Phi_c\right)\ dx \geq \lim_{c\to 2\overline{h}} g(c - \overline{h})\text{meas}(\Omega) = \infty.$$

$\underline{\text{If } c < c_1 < 2\overline{h}, \text{ then } c + \Phi_c \leq c_1 + \Phi_{c_1} \text{ and } \Phi_c \geq \Phi_{c_1}}$. Indeed we have

$$\lambda\Delta\Phi_c = g(c + \Phi_c) - g(-\Phi_c) - C, \quad \lambda\Delta\Phi_{c_1} = g(c_1 + \Phi_{c_1}) - g(-\Phi_{c_1}) - C.$$

Substraction and testing with $[(c + \Phi_c) - (c_1 + \Phi_{c_1})]^+ \in H^1(\Omega)$ gives

$$\begin{aligned} &-\lambda \int_\Omega \left|\nabla[(c + \Phi_c) - (c_1 + \Phi_{c_1})]^+\right|^2\ dx \\ =& \int_\Omega (g(c+\Phi_c) - g(-\Phi_c) - (g(c_1+\Phi_{c_1}) - g(-\Phi_{c_1}))[c+\Phi_c) - (c_1+\Phi_{c_1})]^+\ dx \geq 0, \end{aligned}$$

because g is increasing and $c < c_1$ implies $-\Phi_c(x) \leq -\Phi_{c_1}(x)$ for all $x \in \Omega$ with $c + \Phi_c(x) \geq c_1 + \Phi_{c_1}(x)$. Hence $c + \Phi_c(x) \leq c_1 + \Phi_{c_1}(x)$ for almost all $x \in \Omega$. The estimate $\Phi_c(x) \geq \Phi_{c_1}(x)$ for almost all $x \in \Omega$ follows from a similiar argument by using the test function $[\Phi_{c_1} - \Phi_c]^+ \in H^1(\Omega)$.

In particular we deduce

$$\underline{\text{If } c < c_1 < 2\overline{h}, \text{ then } N_c \leq N_{c_1},\ P_c \leq P_{c_1}}$$

$\underline{\text{Behaviour of } n_c,\ p_c \text{ as } c \to -\infty}$ Let $c_\circ \in (-\infty, 2\overline{h})$ be fixed. Then due to previous estimates we have for all $c \in (-\infty, c_\circ]$,

$$\|n_c\|_{L^\infty(\Omega)} \leq \overline{C} + \frac{P_c}{\text{meas}(\Omega)} \leq \|C\|_{L^\infty(\Omega)} + \frac{N_{c_\circ} + P_{c_\circ}}{\text{meas}(\Omega)} = K_\circ,$$

where $K_\circ \in \mathbb{R}^+$ is independent of $c \in (-\infty, c_\circ]$. In analogy we obtain

$$\|p_c\|_{L^\infty(\Omega)} \leq K_\circ.$$

We deduce: there is a sequence $(c_k)_{k\in\mathbb{N}} \in (-\infty, c_\circ]$ with $\lim_{k\to\infty} c_k = -\infty$ and there are $n_-, p_- \in L^\infty(\Omega)$ with

$$n_{c_k} \rightharpoonup n_-, p_{c_k} \rightharpoonup p_- \quad \text{weak* in } L^\infty(\Omega) \text{ as } k \to \infty,$$

and certainly

$$\lim_{k\to\infty} N_{c_k} = N_-, \quad \lim_{k\to\infty} P_{c_k} = P_-.$$

We also have for all $k \in \mathbb{N}$,

$$\lambda \Delta \Phi_{c_k} = n_{c_k} - p_{c_k} - C,$$

such that

$$\|\nabla \Phi_{c_k}\|_{L^2(\Omega:\mathbb{R}^d)} \leq K_1$$

for some $K_1 \in \mathbb{R}^+$ independent of $k \in \mathbb{N}$. Hence - possibly after passing to a subsequence but without changing notations -

$$\Phi_{c_k} - \frac{1}{\text{meas}(\Omega)} \int_\Omega \Phi_{c_k}\, dx \to V_- \in H^1(\Omega) \quad \text{strongly in } H^1(\Omega) \text{ as } k \to \infty,$$

where $\int_\Omega V_-\, dx = 0$. We introduce

$$\omega := \lim_{k\to\infty} \int_\Omega \Phi_{c_k}\, dx \in \mathbb{R} \cup \{\infty\},$$

where we made use of the fact that $c \mapsto \int_\Omega \Phi_c\, dx$ is increasing. Now it is easy to see: Either $\omega = \infty$ with

$$\Phi_{c_k} \to \infty \quad \text{almost everywhere on } \Omega \text{ as } k \to \infty,$$

or

$$\Phi_{c_k} \to \Phi_- \quad \text{strongly in } H^1(\Omega) \text{ and almost everywhere on } \Omega \text{ as } k \to \infty,$$

where

$$\lambda \Delta \Phi_- = n_- - p_- - C,$$

subject to homogeneous Neumann boundary conditions.We also introduce

$$\omega^* := \lim_{k\to\infty} \int_\Omega (c_k + \Phi_{c_k})\, dx \in \mathbb{R} \cup \{-\infty\},$$

where we made use of the fact that $c \mapsto \int_\Omega (c + \Phi_c)\, dx$ is decreasing. We deduce as above: Either $\omega^* = -\infty$ with

$$c_k + \Phi_{c_k} \to -\infty \quad \text{almost everywhere on } \Omega \text{ as } k \to \infty,$$

or

$$c_k + \Phi_{c_k} \to \Phi_* \quad \text{strongly in } H^1(\Omega) \text{ and almost everywhere on } \Omega \text{ as } k \to \infty,$$

where

$$\lambda \Delta \Phi_* = n_- - p_- - C,$$

subject to homogeneous Neumann boundary conditions.

Existence of $c \in (-\infty, 2\overline{h})$ with $N_c = N, P_c = P$. Due to the previous results it remains to prove: $\lim_{k\to\infty} N_{c_k} = 0$ or $\lim_{k\to\infty} P_{c_k} = 0$. We distinguish with respect to $\int_\Omega C\,dx$.

$\int_\Omega C\,dx > 0$: In that case $\omega \in \mathbb{R}$ is *not* possible, because

$$\begin{aligned}\liminf_{k\to\infty} \int_\Omega n_{c_k}\,dx &= \liminf_{k\to\infty} \int_\Omega g(c_k + \Phi_{c_k})\,dx \\ &= 0 = -\liminf_{k\to\infty} \int_\Omega p_{c_k}\,dx - \int_\Omega C\,dx < 0\end{aligned}$$

would follow. Hence $\omega = \infty$ and therefore by Lebesgue's dominated convergence Theorem,

$$\lim_{k\to-\infty} P_{c_k} = \int_\Omega g(-\Phi_{c_k})\,dx = 0.$$

$\int_\Omega C\,dx = 0$: It is easy to see: If $\omega = \infty$, then $\lim_{k\to\infty} P_{c_k} = 0$, or if $\omega \in \mathbb{R}$, then $\lim_{k\to\infty} N_{c_k} = 0$.

$\int_\Omega C\,dx < 0$: We consider the value of ω^*, proceed as in case of $\int_\Omega C\,dx > 0$ and conclude $\lim_{k\to\infty} N_{c_k} = 0$.

End of proof. We have shown: there is $c \in \mathbb{R}$ such that

$$\lambda \Delta \Phi_c = g(c + \Phi_c) - g(-\Phi_c) - C$$

subject to homogeneous Neumann boundary conditions has a solution $\Phi_c \in H^1(\Omega)$ with $\int_\Omega g(c+\Phi_c)\,dx = N$, $\int_\Omega g(-\Phi_c)\,dx = P$. We set

$$V_\lambda := -\lambda \Phi_c + \frac{\lambda}{\mathrm{meas}(\Omega} \int_\Omega \Phi_c\,dx,$$

$$\alpha_\lambda = c - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \Phi_c\,dx, \quad \beta_\lambda = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \Phi_c\,dx,$$

and

$$n_\lambda = n_c = g\left(\alpha_\lambda - \lambda^{-1} V_\lambda\right), \quad p_\lambda = p_c = g\left(\beta_\lambda + \lambda^{-1} V_\lambda\right).$$

Then it is easy to see that $(n_\lambda, p_\lambda) \in \mathcal{C}$ satisfy the variational inequalities

$$\begin{aligned} h(n_\lambda) + \lambda^{-1} V_\lambda = \alpha_\lambda \quad \text{on} \quad \{n_\lambda > 0\} \\ h(n_\lambda) + \lambda^{-1} V_\lambda \geq \alpha_\lambda \quad \text{on} \quad \{n_\lambda = 0\} \end{aligned},$$

$$\begin{aligned} h(p_\lambda) - \lambda^{-1} V_\lambda = \beta_\lambda \quad \text{on} \quad \{p_\lambda > 0\} \\ h(p_\lambda) - \lambda^{-1} V_\lambda \geq \beta_\lambda \quad \text{on} \quad \{p_\lambda = 0\} \end{aligned},$$

where $V_\lambda = V[n_\lambda - p_\lambda - C]$. It is left to the reader to verify that the validity of these variational inequalities ensures that (n_λ, p_λ) is the unique minimizer of E_λ in $\mathcal{C}$ and to prove that n_λ, p_λ have the properties as specified in the Theorem. $\square$

2.2 $\lambda \to 0$

The main result of this section is

Theorem 5. *Assume* **A.1-A.5**. *Let $N, P \in \mathbb{R}^+$ with*

$$N - P = \int_\Omega C \, dx.$$

For $\lambda \in \mathbb{R}^+$ let (n_λ, p_λ) be the unique minimizer of E_λ in $\mathcal{C}$, see theorem 4, and $V_\lambda = V[n_\lambda - p_\lambda - C]$. Then:

1. *There is $(n_0, p_0) \in \mathcal{C} \cap (L^\infty(\Omega))^2$ such that*

$$n_\lambda \rightharpoonup n_0, p_\lambda \rightharpoonup p_0 \quad \text{weak* in } L^\infty(\Omega) \text{ as } \lambda \to 0,$$

2. *$V_\lambda \to V_0 = V[n_0 - p_0 - C]$ strongly in $H^1(\Omega)$ as $\lambda \to 0$.*

Furthermore,

3. *If $N > \int_\Omega C^+ \, dx$, $P > \int_\Omega C^- \, dx$, then $V_0 = 0$ and (n_0, p_0) is the unique minimizer of*

$$E : \mathcal{C} \to \mathbb{R} \cup \{\infty\}, \quad E(n,p) = \int_\Omega H(n) \, dx + \int_\Omega H(p) \, dx$$

in

$$\mathcal{C}^0 = \{(n,p) \in \mathcal{C} : n - p = C\} \neq \emptyset,$$

i.e. there is a unique $\gamma \in \mathbb{R}$ and a unique $\rho \in L^\infty(\Omega)$ with

$$\max\{-C^+, -C^-\} \leq \rho, \quad \int_\Omega \rho \, dx = N - \int_\Omega C^+ \, dx = P - \int_\Omega C^- \, dx,$$

$$h(C^+ + \rho) + h(C^- + \rho) = \gamma \quad on \quad \{\rho > -C^+\} \cap \{\rho > -C^-\}$$

$$h(C^+ + \rho) + h(C^- + \rho) \geq \gamma \quad on \quad \{\rho = -C^+\} \cup \{\rho = -C^-\}$$

determining n_0, p_0 via

$$n_0 = C^+ + \rho, \quad p_0 = C^- + \rho.$$

4. *If $N = \int_\Omega C^+ \, dx$, $P = \int_\Omega C^- \, dx$, then $V_0 = 0$ and*

$$n_0 = C^+, \quad p_0 = C^-.$$

5. *If $N < \int_\Omega C^+ \, dx$, $P < \int_\Omega C^- \, dx$ then there is a unique pair (Ω_n, Ω_p) of measurable subsets of Ω such that*

$$\mathrm{meas}(\Omega_n \cap \Omega_p) = 0, \quad \Omega_n \subset \{C \geq 0\},\ \Omega_p \subset \{C \leq 0\},$$

$$\int_{\Omega_n} C^+ \, dx = N, \quad \int_{\Omega_p} C^- \, dx = P,$$

$$\begin{array}{lll} n_0 = C^+ & on & \Omega_n \\ n_0 = 0 & on & \Omega \setminus \Omega_n \end{array}, \qquad \begin{array}{lll} p_0 = C^- & on & \Omega_p \\ p_0 = 0 & on & \Omega \setminus \Omega_p \end{array},$$

$$\begin{array}{lll} V_0 = \operatorname{ess\,inf}_\Omega V_0 & on & \Omega_n \\ \Delta V_0 = C & on & \Omega \setminus (\Omega_n \cup \Omega_p) \\ V_0 = \operatorname{ess\,sup}_\Omega V_0 & on & \Omega_p \end{array}.$$

Proof. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ with $\lim_{k\to\infty} \lambda_k = 0$. By the uniform estimates on $\|n_\lambda\|_{L^\infty(\Omega)}$, $\|p_\lambda\|_{L^\infty(\Omega)}$ there is a subsequence $(\lambda_{k_1(k)})_{k\in\mathbb{N}}$ of $(\lambda_k)_{k\in\mathbb{N}}$ and there is $(n_0, p_0) \in \mathcal{C}$ such that

$$n_{\lambda_{k_1(k)}} \rightharpoonup n_0, p_{\lambda_{k_1(k)}} \rightharpoonup p_0 \quad \text{weak* in } L^\infty(\Omega) \text{ as } k \to \infty.$$

Hence

$$V_{\lambda_{k_1(k)}} \to V_0 = V[n_0 - p_0 - C] \quad \text{strongly in } H^1(\Omega) \text{ as } k \to \infty,$$

hence after passing to a subsequence $(k_2(k))_{k\in\mathbb{N}}$ of $(k_1(k))_{k\in\mathbb{N}}$,

$$V_{\lambda_{k_2(k)}} \to V_0 \quad \text{for almost all } x \in \Omega \text{ as } k \to \infty.$$

We have to prove that (n_0, p_0) are actually independent of the sequence $(\lambda_k)_{k\in\mathbb{N}}$. We proceed by a case-distinction and set

$$F : \mathcal{C} \to \mathbb{R}_0^+ \cup \{\infty\}, \quad F(n,p) = \int_\Omega |\nabla V[n-p-C]|^2 \, dx.$$

The main ingredient is the application of theorem 3.

Case I: $N > \int_\Omega C^+$ and $P > \int_\Omega C^-$.
We have $\mathcal{C}^0 \neq \emptyset$. By theorem 3, the pair (n_0, p_0) minimizes E in the set of all minimizers of F for which E is bounded, i.e. in $\{(n,p) \in \mathcal{C}^0 : E(n,p) < \infty\}$. By strict convexity, the functional E has exactly one minimizer in that set. It is left to the reader to deduce the variational inequalities as specified in 3.

Case II: $N = \int_\Omega C^+ \, dx$ and $P = \int_\Omega C^- \, dx$.
With the notations of **Case I** we have $\mathcal{C}^0 = \{(C^+, C^-)\}$. The statement follows from theorem 3.

Case III: $N < \int_\Omega C^+ \, dx$ and $P < \int_\Omega C^- \, dx$. In this case the set $\mathcal{C}^0$ is void. However, we deduce from theorem 3 that (n_0, p_0) - we note that $E(n_0, p_0) < \infty$ because $n_0, p_0 \in L^\infty(\Omega)$ - is a minimizer of F in $\mathcal{C}^* = \{(n,p) \in \mathcal{C} : E(n,p) < \infty\}$.

We observe: V_0 does not vanish identically. (Otherwise $n_0 - p_0 = C$ implying $N \geq \int_\Omega C^+ \, dx$, $P \geq \int_\Omega C^- \, dx$). Hence due to $\int_\Omega V_0 \, dx = 0$,

$$0 < \operatorname{meas}(\{V_0 < 0\}), \operatorname{meas}(\{V_0 > 0\}) < \operatorname{meas}(\Omega),$$

and therefore

$$\operatorname*{ess\,inf}_\Omega V_0 < \operatorname*{ess\,sup}_\Omega V_0.$$

The sequence $(\alpha_{\lambda_{k_2(k)}} - \lambda_{k_2(k)}^{-1} V_{\lambda_{k_2(k)}}(x))_{k\in\mathbb{N}}$ is, for almost all $x \in \{V_0 < 0\}$, bounded above as $k \to \infty$. Hence $\lim_{k\to\infty} \alpha_{\lambda_{k_2(k)}} = -\infty$. By the well-known identification of weak* limits in L^∞ and pointwise almost-everywhere limits of sequences of functions we obtain for almost all $x \in \Omega$: If $(\alpha_{\lambda_{k_2(k)}} - \lambda_{k_2(k)}^{-1} V_{\lambda_{k_2(k)}}(x))_{k\in\mathbb{N}}$ converges in $\mathbb{R} \cup \{\pm\infty\}$, then $n_0(x) = \lim_{k\to\infty} g(\alpha_{\lambda_{k_2(k)}} - \lambda_{k_2(k)}^{-1} V_{\lambda_{k_2(k)}}(x))$. Hence, up to a set of measure zero,

$$\{V_0 > 0\} \subseteq \{n_0 = 0\}.$$

Now take $x \in \Omega$ such that $\lim_{k\to\infty} V_{\lambda_{k_2(k)}}(x) = V_0(x)$, that the sequence $(\alpha_{k_2(k)} - \lambda_{k_2(k)}^{-1} V_{\lambda_{k_2(k)}}(x))_{k\in\mathbb{N}}$ is bounded above and $V_0(x) > \operatorname{ess\,inf}_\Omega V_0$. Then there is $z \in \Omega$ with $\lim_{k\to\infty} V_{\lambda_{k_2(k)}}(z) = V_0(z) = V_0(x) - \eta$ with some $\eta \in \mathbb{R}^+$ and the sequence $(\alpha_{k_2(k)} - \lambda_{k_2(k)}^{-1} V_{\lambda_{k_2(k)}}(z))_{k\in\mathbb{N}}$ is bounded above, too. Since $\lim_{k\to\infty}(V_{\lambda_{k_2(k)}}(x) - V_{\lambda_{k_2(k)}}(z))/\lambda_{k_2(k)} = \infty$, we obtain up to a set of measure zero:

$$\text{If } \lim_{k\to\infty} V_{\lambda_{k_2(k)}}(x) > \operatorname{ess\,inf}_\Omega V_0, \text{ then } \lim_{k\to\infty} \Big(\alpha_{k_2(k)} - \lambda_{k_2(k)}^{-1} V_{\lambda_{k_2(k)}}(x)\Big) = -\infty, \\ \text{and } n_0(x) = 0 \text{ follows.}$$

Hence

$$\Omega_n = \{n_0 > 0\} \subseteq \{V_0 = \operatorname*{ess\,inf}_{\Omega} V_0\}$$

and in analogy

$$\{V_0 < 0\} \subseteq \{p_0 = 0\}, \quad \Omega_p = \{p_0 > 0\} \subseteq \{V_0 = \operatorname*{ess\,sup}_{\Omega} V_0\},$$

in particular $\{n_0 > 0\} \cap \{p_0 > 0\} = \emptyset$. Now it is clear that V_0 satisfies the double-obstacle problem as specified in 5. of the theorem. Concerning the pointwise behaviour of n_0 on Ω_n we observe that V is constant on Ω_n. Since $V \in H^1(\Omega)$ we deduce $\nabla V = 0$ almost everywhere on Ω_n, see [Zie89]. Furthermore, due to the standard regularity theory of elliptic PDEs we have $V \in H^2_{loc}(\Omega)$. Following [Zie89] once again we deduce $\partial_j\partial_i V = 0$, $i, j = 1, \dots, d$, almost everywhere on each subset of Ω where $\partial_i V$ is constant, in particular on Ω_n. Hence $\Delta V = 0$ almost everywhere on Ω_n which implies $n - C = 0$ almost everywhere on Ω_n, because $p = 0$ almost everywhere on Ω_n. Hence $n = C^+$ almost everywhere on Ω_n. In analogy we deduce $p = C^-$ almost everywhere on Ω_p.

As a consequence, the triple (n_0, p_0, V_0) has the properties as specified in 5. of the theorem.

It remains to be shown that there is at most one minimizer of F in $\mathcal{C}^*$. Assume that (n_*, p_*) is a minimizer of F in $\mathcal{C}^*$. If $n_0 - p_0 \neq n_* - p_*$, then by strict convexity we would obtain

$$\begin{aligned}
&F((n_0+n_*)/2, (p_0+p_*)/2) \\
&\qquad = \int_\Omega |\nabla V[((n_0 - p_0 - C)/2) + ((n_* - p_* - C)/2)]|^2 \, dx \\
&\qquad < \frac{1}{2}\int_\Omega |\nabla V[n_0 - p_0 - C]|^2 \, dx + \frac{1}{2}\int_\Omega |\nabla V[n_* - p_* - C]|^2 \, dx \\
&\qquad\qquad = \frac{F(n_0,p_0) + F(n_*,p_*)}{2}.
\end{aligned}$$

Hence $n_* - p_* = n_0 - p_0$ and therefore $n_* = n_0 + \rho$, $p_* = p_0 + \rho$ for some $\rho \in L^1(\Omega)$ with $\int_\Omega \rho \, dx = 0$. We have on $\Omega \setminus \Omega_n$ $n_0 = 0$, hence $\rho \geq 0$, and we have in analogy $\rho \geq 0$ on $\Omega \setminus \Omega_p$. Since Ω_n and Ω_p are disjoint we obtain $\rho \geq 0$, hence $\rho = 0$. □

Remark 3. *a) Point 3 of Theorem 5 represents the case, where the limit satisfies local charge-neutrality everywhere in Ω.*
b) The limit V_0 in point 5 of Theorem 5 is the unique solution of a double obstacle problem. The coincidence set Ω_n of the smaller obstacle $\operatorname{ess\,inf}_\Omega V_0$ is the set where the electron density n equals the doping profile, the coincidence set Ω_p of the larger obstacle $\operatorname{ess\,sup}_\Omega V_0$ is the set where the hole density p equals the negative doping profile and the noncoincidence set $\Omega \setminus (\Omega_n \cup \Omega_p)$ is the depletion (vacuum) region where $n = p = 0$. Note that the double obstacle problem for V_0

is somewhat nonstandard since the obstacles are not a-priorily given but determined by the constraints $\int_{\Omega_n} C^+ \, dx = N$ and $\int_{\Omega_p} C^- \, dx = P$ on the coincidence sets. However, the a-posteriori regularity theory for the free boundaries is by now standard [Caf00]. They are (locally) $C^{1,\alpha}$-surfaces if the inhomogeneity $C(x)$ is C^α.

Appendix A: Analysis of J_λ

We shall explain in a simple case the connection between the functionals J_λ and E_λ, thus introducing for simplicity several technical assumptions that can be removed with a more detailed analysis.

Consider as in the introduction the functional

$$\begin{aligned} J_\lambda[\Phi] = \frac{\lambda}{2} \int_\Omega |\nabla \Phi|^2 \, dx - \int_\Omega C(x)\Phi \, dx \\ + \int_\Omega G(\alpha[\Phi] + \Phi) \, dx + \int_\Omega G(\beta[\Phi] - \Phi) \, dx \\ - N\alpha[\Phi] - P\beta[\Phi], \end{aligned}$$

where $\alpha = \alpha[\Phi]$ and $\beta = \beta[\Phi]$ are determined by the condition

$$\int_\Omega g(\alpha[\Phi] + \Phi) \, dx = N, \tag{29}$$

$$\int_\Omega g(\beta[\Phi] - \Phi) \, dx = P, \tag{30}$$

with $g = G' \geq 0$, $g \not\equiv 0$, $\lim_{s\to-\infty} g(s) = 0$ and bounded $\Omega \subseteq \mathbb{R}^d$. Note that for $\Phi \in L^\infty(\Omega)$, the map $\alpha \mapsto \int_\Omega g(\alpha + \Phi) \, dx$ is well defined, continuous by Lebesgue's theorem of dominated convergence and converges to 0 and $+\infty$ as α tends to $-\infty$ and $+\infty$, respectively. Analogous properties of course hold for the map $\beta \mapsto \int_\Omega g(\beta - \Phi) \, dx$. Note also that if g is nondecreasing (which is the case if G is convex), then g' is a positive measure on $\mathbb{R}$. If we further assume that $g' \in L^\infty_{loc}(\mathbb{R})$ and $g' > 0$ on $\mathbb{R}$, then the functionals $\alpha[.]$ and $\beta[.]$ are actually C^1 on $L^\infty(\Omega)$, and

$$d\alpha[\phi] \cdot \chi = -\frac{\int_\Omega g'(\alpha[\phi] + \phi)\chi \, dx}{\int_\Omega g'(\alpha[\phi] + \phi) \, dx}$$

$$d\beta[\phi] \cdot \chi = \frac{\int_\Omega g'(\beta[\phi] - \phi)\chi \, dx}{\int_\Omega g'(\beta[\phi] - \phi) \, dx}$$

Proposition 6. *Assume* **A1** *and* **A5***. If $G \in C^2(\mathbb{R})$, $G'' = g' \in L^\infty_{loc}(\mathbb{R})$, $g' > 0$ on $\mathbb{R}$ and $\lim_{s\to+\infty} g(s) = +\infty$, $\lim_{s\to-\infty} g(s) = 0$, then Φ_λ is a weak solution of*

$$\lambda \Delta \Phi_\lambda = g(\alpha[\Phi_\lambda] + \Phi_\lambda) - g(\beta[\Phi_\lambda] - \Phi_\lambda) - C(x) \tag{31}$$

in $H^1 \cap L^\infty(\Omega)$ subject to homogeneous Neumann boundary conditions if and only if it is a critical point of $J_\lambda[\Phi]$ in $H^1 \cap L^\infty(\Omega)$. The functional $J_\lambda[\Phi]$ is strictly convex, so Φ_λ is unique.

Proof. A straightforward calculation shows that for any $\phi,\ \chi \in H^1 \cap L^\infty(\Omega)$,

$$
\begin{aligned}
dJ_\lambda[\phi]\cdot\chi = \int_\Omega \Big[\lambda\nabla\phi\cdot\nabla\chi + \Big(g(\alpha[\phi]+\phi) - g(\beta[\phi]-\phi) - C(x)\Big)\chi\Big]\ dx \\
+ \Big[\int_\Omega (g(\alpha[\phi]+\phi)\ dx - N\Big]\ d\alpha[\phi]\cdot\chi \\
+ \Big[\int_\Omega (g(\beta[\phi]-\phi)\ dx - P\Big]\ d\beta[\phi]\cdot\chi\ .
\end{aligned}
$$

Using the constraints (29) and (30), we find that $dJ_\lambda[\Phi_\lambda]\cdot\chi = 0$ exactly means that Φ_λ is a weak solution of (31).

To prove the convexity of J_λ, we shall consider ϕ_1 and ϕ_2 in $H^1 \cap L^\infty(\Omega)$. Denoting $\psi = \phi_2 - \phi_1$, $\phi^t = t\phi_1 + (1-t)\phi_2$ and $j(t) = J_\lambda[\phi^t]$ for any $t \in (0,1)$, a direct computation shows that

$$
j'(t) = \int_\Omega \Big[\lambda\nabla\phi^t\cdot\nabla\psi + \Big(g(\alpha[\phi^t]+\phi^t) + g(\beta[\phi^t]-\phi^t) - C(x)\Big)\psi\Big]\ dx\ ,
$$

and

$$
\begin{aligned}
j''(t) = \lambda\int_\Omega |\nabla\psi|^2\ dx + \int_\Omega \big[g'(\alpha[\phi^t]+\phi^t) - g'(\beta[\phi^t]-\phi^t)\big]\ |\psi|^2\ dx \\
- \frac{(\int_\Omega g'(\alpha[\phi^t]+\phi^t)\psi\ dx)^2}{\int_\Omega g'(\alpha[\phi^t]+\phi^t)\ dx} - \frac{(\int_\Omega g'(\beta[\phi^t]-\phi^t)\psi\ dx)^2}{\int_\Omega g'(\beta[\phi^t]-\phi^t)\ dx} \\
\geq \lambda\int_\Omega |\nabla\psi|^2\ dx
\end{aligned}
$$

by the Cauchy-Schwarz inequality. □

To prove the boundedness from below of J_λ, it is actually simpler to consider the functional

$$
E_\lambda(n,p) = \int_\Omega H(n)\ dx + \int_\Omega H(p)\ dx + \frac{1}{2\lambda}\int_\Omega |\nabla V[n-p-C]|^2\ dx
$$

with $H' = h$ and g related according to **A4**. Note that **A2** and **A3** are consequences of the assumptions of Proposition 6. This functional has (according to Theorem 3) a unique minimizer (n_λ, p_λ) in $\mathcal{C}$ such that

$$
\begin{gathered}
\Phi_\lambda = \lambda^{-1}V_\lambda \\
\lambda\Delta\Phi_\lambda = n_\lambda - p_\lambda - C \\
\text{with } n_\lambda = g(\alpha[\Phi_\lambda]+\Phi_\lambda) \text{ and } p_\lambda = g(\beta[\Phi_\lambda]-\Phi_\lambda)\ ,
\end{gathered}
$$

which is a critical point of J_λ, so (by convexity) J_λ is bounded from below by $J_\lambda[\Phi_\lambda]$.

Note also that the constraints (29) and (30) can be rewritten as

$$\begin{array}{c} \alpha[\Phi_\lambda] + \Phi_\lambda = h(n_\lambda) \\ \beta[\Phi_\lambda] - \Phi_\lambda = h(p_\lambda) \end{array},$$

where we make use of $n_\lambda, p_\lambda > 0$ on Ω due to $g > 0$ on $\mathbb{R}$. This is why

$$\alpha[\Phi_\lambda]\, N = \textstyle\int_\Omega (h(n_\lambda) - \Phi_\lambda) n_\lambda \; dx$$

$$\beta[\Phi_\lambda]\, P = \textstyle\int_\Omega (h(p_\lambda) + \Phi_\lambda) p_\lambda \; dx$$

and therefore

$$\begin{aligned} J_\lambda[\Phi_\lambda] \;&=\; -\frac{\lambda}{2}\int_\Omega \Phi_\lambda \, \Delta\Phi_\lambda \; dx + \int_\Omega G(\alpha[\Phi_\lambda]{+}\Phi_\lambda)dx + \int_\Omega G(\beta[\Phi_\lambda]{-}\Phi_\lambda)dx \\ &\qquad - \int_\Omega C\Phi_\lambda \; dx - N\,\alpha[\Phi_\lambda] - P\,\beta[\Phi_\lambda] \\ &= -\frac{1}{2}\int_\Omega (n_\lambda - p_\lambda - C)\Phi_\lambda \; dx + \int_\Omega (Goh(n_\lambda) - n_\lambda h(n_\lambda)) \; dx \\ &\qquad + \int_\Omega (Goh(p_\lambda) - p_\lambda h(p_\lambda)) \; dx + \int_\Omega \Phi_\lambda (n_\lambda - p_\lambda - C) \; dx \\ &= -\int_\Omega H(n_\lambda) \; dx - \int_\Omega H(p_\lambda) \; dx - \frac{1}{2\lambda}\int_\Omega |\nabla V_\lambda[n_\lambda - p_\lambda - C]|^2 \; dx \\ &= -E_\lambda[n_\lambda, p_\lambda] \end{aligned}$$

using the fact that $\frac{d}{dt}[th(t) - Goh(t)] = h(t)$, and $H(t) = th(t) - Goh(t)$ (up to a constant which is chosen here equal to 0).

Appendix B: Unbounded domains

In this section we shall investigate (19)-(24) on unbounded domains. In this case the analysis has to overcome several additional difficulties and further assumptions are needed.

We put

$$C_c^\infty(\overline{\Omega}) = \{\phi \restriction \Omega : \phi \in C_c^\infty(\mathbb{R}^d)\},$$

and we assume

V.1 There is a linear operator $V : L^1_{loc}(\overline{\Omega}) \supset \mathsf{dom}(V) \to L^1_{loc}(\overline{\Omega})$, such that for all $f \in \mathsf{dom}(V)$, the function $V[f]$ is a weak solution of

$$-\Delta V[f] = f, \quad + \text{ homogeneous Neumann boundary conditions,} \qquad (32)$$

i.e.

$$-\int_\Omega V[f]\Delta\phi\, dx = \int_\Omega f\phi\, dx, \quad \forall \phi \in C_c^\infty(\overline{\Omega}).$$

V.2 The set $\mathsf{dom}_2(V) := \{f \in \mathsf{dom}(V) : \nabla V[f] \in L^2(\Omega : \mathbb{R}^d)\}$ contains

$$\mathtt{T} := \left\{\rho \in L^\infty(\Omega) : (\mathsf{supp}(\rho) \subset\subset \overline{\Omega}) \wedge \left(\int_\Omega \rho\, dx = 0\right)\right\},$$

and for all $f \in \mathsf{dom}_2(V)$,

$$\int_\Omega \nabla V[f] \cdot \nabla V[\rho]\, dx = \int_\Omega V[f]\rho\, dx, \quad \forall \rho \in \mathtt{T}.$$

V.3 If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathsf{dom}_2(V)$ with

$$f_n \rightharpoonup f, \quad \text{weakly in } L^1(\Omega) \text{ as } n \to \infty,$$

and if $(\nabla V[f_n])_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega : \mathbb{R}^d)$, then $f \in \mathsf{dom}(V)$ and

$$\lim_{n\to\infty} \int_\Omega V[f_n]\, \phi\, dx = \int_\Omega V[f]\, \phi\, dx, \quad \forall \phi \in C_c^\infty(\overline{\Omega}).$$

Remark 4. *a) It is easy to see that* **V.1**, **V.2**, **V.3** *hold for bounded domains Ω where $\mathsf{dom}(V)$ is the set of all $f \in L^1(\Omega)$ with $\int_\Omega f\, dx = 0$ for which a function $Z \in H^1(\Omega)$ exists such that $\int_\Omega Z\, dx = 0$ and $\int \nabla Z \cdot \nabla \phi\, dx = \int_\Omega f\phi\, dx$ for all $\phi \in H^1(\Omega)$, compare the definition of "$V[f]$" in the previous sections.*
b) Assumptions **V.1**-**V.3** *apply in particular to $\Omega = \mathbb{R}^d$, $d \in \mathbb{N}$, and to respective half-space problems, see the discussion below.*
c) The verification of **V.1**-**V.3** *of the examples of b) rely on the knowledge of a Green's function. Whenever such a function is available, then one may proceed similiar as in b) to investigate the validity of* **V.1**, **V.2**, **V.3**.
d) We note that (as e.g. in case of $\Omega = \mathbb{R}^d$, $d \geq 3$) the domain $\mathsf{dom}(V)$ of the operator $V[.]$ may consist of functions which do not *satisfy the global electroneutrality condition $\int_\Omega f\, dx = 0$.*

Example 1 $\Omega = \mathbb{R}$. We set $\mathsf{dom}(V) = L^1(\mathbb{R})$ and introduce for $x \in \mathbb{R}$ and for $f \in \mathsf{dom}(V)$,

$$E(f)(x) := -\int_{-\infty}^x f(s)\, ds, \quad V(x) := \begin{cases} \displaystyle\int_0^x E(f)(s)\, ds & , \ x \geq 0 \\ \displaystyle -\int_x^0 E(f)(s)\, ds & , \ x < 0 \end{cases}.$$

It is easy to see: $E : \mathsf{dom}(V) \to L^\infty(\mathbb{R})$ is non-expansive, i.e. Lipschitz-continuous with Lipschitz constant 1.
Furthermore, $V' = E(f)$ and $V'' = -f$ in the sense of distributions (see, e.g., [Rud]).
Hence $V : \mathsf{dom}(V) \to C^1(\mathbb{R})$ satisfies **V.1**.
If $\rho \in L^\infty(\mathbb{R})$ has compact support and satisfies $\int_{\mathbb{R}} \rho \, dx = 0$, then $E(\rho)$ has compact support, too. Since $E(\rho) \in L^\infty(\mathbb{R})$ we deduce $V'[\rho] = E(\rho) \in L^2(\mathbb{R})$, thus $\mathtt{T} = \left\{\rho \in L^\infty(\mathbb{R}) : (\mathsf{supp}(\rho) \subset\subset \mathbb{R}) \wedge \left(\int_{\mathbb{R}} \rho \, dx = 0\right)\right\} \subseteq \mathsf{dom}_2(V)$.

Now let $f \in \mathsf{dom}_2(V)$ and let $\rho \in \mathtt{T}$. We take a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R})$ with $f_n \to f$ strongly in $L^1(\Omega)$ as $n \to \infty$ (see, e.g. [Ada]) and $\int_{\mathbb{R}} f_n \, dx = 0$ for all $n \in \mathbb{N}$. Then $V[f_n] \in C_c^\infty(\mathbb{R})$ for all $n \in \mathbb{N}$ and therefore

$$\int_{\mathbb{R}} \rho V[f_n] \, dx = -\int_{\mathbb{R}} V[\rho] V''[f_n], \quad n \in \mathbb{N}.$$

Since $V[\rho]$ and $V''[f_n]$ are continuously differentiable and since $V[\rho]$ is compactly supported, we calculate by means of an integration by parts,

$$-\int_{\mathbb{R}} V[\rho] V''[f_n] = \int_{\mathbb{R}} V'[f_n] V'[\rho] \, dx = \int_{\mathbb{R}} E(f_n) E(\rho) \, dx, \quad n \in \mathbb{N}.$$

Since $E(\rho) \in L^\infty(\mathbb{R})$ is compactly supported and since $E(f_n) \to E(f)$ strongly in $L^\infty(\mathbb{R})$, we deduce $\int_{\mathbb{R}} E(f_n)E(\rho) \, dx \to \int_{\mathbb{R}} E(f)E(\rho) \, dx$ as $n \to \infty$. On the other hand it is easy to see that $E(f_n) \to E(f)$ strongly in $L^\infty(\mathbb{R})$ implies $V[f_n] \to V[f]$ in $L^1_{loc}(\mathbb{R})$ as $n \to \infty$. This settles $\int_{\mathbb{R}} \rho V[f_n] \, dx \to \int_{\mathbb{R}} \rho V[f] \, dx$ as $n \to \infty$ and therefore

$$\begin{aligned}
&\int_{\mathbb{R}} V'[f] V'[\rho] \, dx \\
&\qquad = \int_{\mathbb{R}} E(f)E(\rho) \, dx = \lim_{n\to\infty} \int_{\mathbb{R}} E(f_n)E(\rho) \, dx = \lim_{n\to\infty} \int_{\mathbb{R}} V[f_n] \rho \, dx \\
&\qquad = \int_{\mathbb{R}} V[f] \rho \, dx.
\end{aligned}$$

It remains to verify **V.3**. If $f_n \rightharpoonup f$ weakly in $L^1(\mathbb{R})$, then $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\mathbb{R})$. Hence $(V'[f_n])_{n\in\mathbb{N}} = (E(f_n))_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R})$. Furthermore, since the indicator function of $(-\infty, x)$ is in $L^\infty(\mathbb{R})$ for each $x \in \mathbb{R}$, we deduce $E(f_n)(x) \to E(f)(x)$ as $n \to \infty$ for all $x \in \mathbb{R}$. As a consequence, $V[f_n](x) \to V[f](x)$ for all $x \in \mathbb{R}$. Due to uniform boundedness of $(E(f_n))_{n\in\mathbb{N}}$ in $L^\infty(\mathbb{R})$ we also have: $V[f_n] \to V[f]$ in $L^1_{loc}(\mathbb{R})$ as $n \to \infty$. Hence $V[f_n] \to V[f]$ in the sense of distributions as well. $\square$

Example 2 $\Omega = \mathbb{R}^2$. We set $\mathsf{dom}(V) := L^1(\mathbb{R}^2)$. The Green's function of the Laplace operator is

$$K(x) := -\frac{1}{2\pi} \log |x|, \quad x \in \mathbb{R}^2, x \neq 0.$$

Since $K(x) \in L^1_{loc}(\mathbb{R}^2)$ we have for each $f \in L^1(\mathbb{R}^2)$,

$$K \star f := \int_{\mathbb{R}^2} K(. - y) f(y)\, dy \in L^1_{loc}(\mathbb{R}^2).$$

For $f \in L^1(\mathbb{R}^2)$ we set

$$V[f] := -K \star f.$$

$V[.]$ certainly satisfies **V.1**.
We observe for $i = 1, 2$,

$$\partial_i V[f] = -(\partial_i K) \star f, \quad \partial_i K(x) = -\frac{1}{2\pi} \frac{1}{|x|} \frac{x_i}{|x|},$$

in the sense of distributions.

In the sequel we shall use several estimates on $V[f]$ and on $\nabla V[f]$. These results can be found in [ArNi],

Proposition 7. *(Follows from* **Lemma 3.3**c) in [ArNi]*) There is a positive real number K_8 such that: If*

$$f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \quad \textit{and} \quad \int_{\mathbb{R}^2} f\, dx = 0, \quad \textit{and} \quad \int_{\mathbb{R}^2} |x|\, |f(x)|\, dx < \infty,$$

then $\nabla V[f] \in L^2(\mathbb{R}^2 : \mathbb{R}^2)$ and

$$\|\nabla V[f]\|_{L^2(\mathbb{R}^2:\mathbb{R}^2)} \leq K_8 \left(\|f\|_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} |x|\, |f(x)|\, dx \right).$$

Proposition 8. *(Follows from* **Lemma 3.1** in [ArNi]*) There is a positive real number K_9 such that: If*

$$f \in L^1(\mathbb{R}^2), \quad \textit{and} \quad \int_{\mathbb{R}^2} f\, dx = 0,$$

then for all $x \in \mathbb{R}^2$,

$$|V[f](x)| \leq K_9 \left(\int_{\mathbb{R}^2} (1 + |y|)\, |f(y)|\, dy + \left(\int_{\mathbb{R}^2} (1 + |y|)^2\, |f(y)|^2\, dy \right)^{\frac{1}{2}} \right).$$

Proposition 9. *(Follows from* **Lemma 3.1** in [ArNi]*) There is a positive real number K_{10} such that: If*

$$f \in L^1(\mathbb{R}^2), \quad \textit{and} \quad \int_{\mathbb{R}^2} f\, dx = 0,$$

then $V[f] \in L^6(\mathbb{R}^2)$ with

$$\|V[f]\|_{L^6(\mathbb{R}^2)} \leq K_{10}. \left(\int_{\mathbb{R}^2} (1 + |y|)\, |f(y)|\, dy + \left(\int_{\mathbb{R}^2} (1 + |y|)^{\frac{3}{2}}\, |f(y)|^{\frac{3}{2}}\, dy \right)^{\frac{2}{3}} \right).$$

Now we shall verify **V.2**. If $\rho \in L^\infty(\mathbb{R}^2)$ is compactly supported with $\int_{\mathbb{R}^2} \rho \, dx = 0$, then ρ fullfills the requirements of Proposition 7. Hence $\nabla V[\rho] \in L^2(\mathbb{R}^2 : \mathbb{R}^2)$ and therefore $\rho \in \mathsf{dom}_2(V)$.

Let $f \in \mathsf{dom}_2(V)$. Then the function $(x, y) \to K(x-y)f(y)\rho(x)$ belongs to $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$. Hence due to the Fubini-Tonelli Theorem,

$$\int_{\mathbb{R}^2} V[f]\rho \, dx = -\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^d} K(x-y) \, f(y) \, dy \right) \rho(x) \, dx$$
$$= -\int_{\mathbb{R}^2 \times \mathbb{R}^2} K(x-y) \, f(y) \, \rho(x) \, d(x,y) = -\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} K(x-y) \, \rho(x) \, dx \right) \, f(y) \, dy$$
$$= -\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} K(y-x) \, \rho(x) \, dx \right) \, f(y) \, dy = \int_{\mathbb{R}^2} V[\rho] f \, dx,$$

where we made use of $K(-x) = K(x)$. Thus, in order to verify **V.2**, it remains to prove

$$\int_{\mathbb{R}^2} \nabla V[f] \cdot \nabla V[\rho] \, dx = \int_{\mathbb{R}^2} V[\rho] f \, dx.$$

We take a sequence $(\rho_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^2)$ with $\lim_{n\to\infty} \|\rho - \rho_n\|_{L^2(\mathbb{R}^2)} = 0$, see [Ada]. We can assume: There is $R \in (0, \infty)$ such that $\mathsf{supp}(\rho) \subseteq \{|y| \le R\}$ and $\mathsf{supp}(\rho_n) \subseteq \{|y| \le R\}$ for all $n \in \mathbb{N}$. We deduce from Proposition 7:

$$\nabla V[\rho_n] \to \nabla V[\rho] \quad \text{strongly in } L^2(\mathbb{R}^2 : \mathbb{R}^2) \text{ as } n \to \infty. \tag{33}$$

Furthermore, since $K \in L^1_{loc}(\mathbb{R}^2)$ and since $\rho_n \in C_c^\infty(\mathbb{R}^2)$, we have $V[\rho_n] \in C^\infty(\mathbb{R}^2)$ for all $n \in \mathbb{N}$, see e.g. [Ada].

Let us take $\Theta \in C^\infty(\mathbb{R})$ with $\Theta = 1$ on $(-\infty, 0]$ and $\Theta = 0$ on $[1, \infty)$. For $k \in \mathbb{N}$ we introduce

$$\Theta_k : \mathbb{R}^2 \to \mathbb{R}, \quad \Theta_k(x) = \Theta\left(\frac{|x|}{k} - 1 \right).$$

Then $\Theta_k \in C_c^\infty(\mathbb{R}^2)$ for all $k \in \mathbb{N}$, hence $\Theta_k V[\rho_n] \in C_c^\infty(\mathbb{R}^2)$ for all $n, k \in \mathbb{N}$. We calculate for all $n, k \in \mathbb{N}$,

$$\int_{\mathbb{R}^2} f \, (\Theta_k V[\rho_n]) \, dx = -\int_{\mathbb{R}^2} V[f] \, \Delta(\Theta_k V[\rho_n]) \, dx = \int_{\mathbb{R}^2} \nabla V[f] \cdot \nabla(\Theta_k V[\rho_n]) \, dx$$
$$= \int_{\mathbb{R}^2} (\nabla V[f] \cdot \nabla \Theta_k) \, V[\rho_n] \, dx + \int_{\mathbb{R}^2} (\nabla V[f] \cdot \nabla V[\rho_n]) \, \Theta_k \, dx,$$

where made use of $V[\rho_n] \in L^\infty(\mathbb{R}^2)$ (which follows from Proposition 8) and $\nabla V[\rho_n] \in L^2(\mathbb{R}^2 : \mathbb{R}^2)$ for all $n \in \mathbb{N}$.

We consider the limit $n \to \infty$ now.

Due to Proposition 8 the sequence $(V[\rho_n])_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^2)$. By Proposition 9 the sequence $(V[\rho_n])_{n\in\mathbb{N}}$ converges strongly in $L^6(\mathbb{R}^2)$, to $V[\rho]$ as $n\to\infty$. Hence for all $k\in\mathbb{N}$,

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^2} f\left(\Theta_k V[\rho_n]\right)\,dx &= \lim_{n\to\infty}\int_{\{k\le|x|\le 2k\}} f\left(\Theta_k V[\rho_n]\right)\,dx\\ &= \int_{\{k\le|x|\le 2k\}} f\left(\Theta_k V[\rho]\right)\,dx = \int_{\mathbb{R}^d} f\left(\Theta_k V[\rho]\right)\,dx.\end{aligned}$$

Similiar argumentations (in particular exploiting the fact that each integration is in fact an integration over a fixed (i.e. independent of $n\in\mathbb{N}$) bounded domain) yield for all $k\in\mathbb{N}$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(\nabla V[f]\cdot\nabla\Theta_k\right)V[\rho_n]\,dx = \int_{\mathbb{R}^2}\left(\nabla V[f]\cdot\nabla\Theta_k\right)V[\rho]\,dx,$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(\nabla V[f]\cdot\nabla V[\rho_n]\right)\Theta_k\,dx = \int_{\mathbb{R}^2}\left(\nabla V[f]\cdot\nabla V[\rho]\right)\Theta_k\,dx.$$

Hence for all $k\in\mathbb{N}$,

$$\int_{\mathbb{R}^2}\left(f\ V[\rho]\right)\Theta_k\,dx = \int_{\mathbb{R}^2}\left(\nabla V[f]\cdot\nabla\Theta_k\right)V[\rho]\,dx + \int_{\mathbb{R}^2}\left(\nabla V[f]\cdot\nabla V[\rho]\right)\Theta_k\,dx. \tag{34}$$

Now we consider the limit $k\to\infty$. Since $f\ V[\rho], (\nabla V[f]\cdot\nabla V[\rho_n])\in L^1(\mathbb{R}^d)$ and since $\lim_{k\to\infty}\Theta_k(x)=1$ for all $x\in\mathbb{R}^d$ with $0\le\Theta_k\le 1$ for all $k\in\mathbb{N}$, we have

$$\lim_{k\to\infty}\int_{\mathbb{R}^2}\left(f\ V[\rho]\right)\Theta_k\,dx = \int_{\mathbb{R}^2} f\ V[\rho]\,dx, \tag{35}$$

$$\lim_{k\to\infty}\int_{\mathbb{R}^2}\left(\nabla V[f]\cdot\nabla V[\rho]\right)\Theta_k\,dx = \int_{\mathbb{R}^2}\nabla V[f]\cdot\nabla V[\rho]\,dx. \tag{36}$$

Furthermore, for all $k\in\mathbb{N}$ and for all $x\in\mathbb{R}^d$,

$$|\nabla\Theta_k(x)| \le \frac{\sup_{s\in[0,1]}|\Theta'(s)|}{k} =: \frac{K_{11}}{k}.$$

Now we have for all $k\in\mathbb{N}$,

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\left(\nabla V[f]\cdot\nabla\Theta_k\right)V[\rho]\,dx\right| &= \left|\int_{\{k\le|x|\le 2k\}}\left(\nabla V[f]\cdot\nabla\Theta_k\right)V[\rho]\,dx\right|\\ &\le \int_{\{k\le|x|\le 2k\}}|\nabla V[f]|\,|\nabla\Theta_k|\,|V[\rho]|\,dx\\ &\le \frac{K_{11}}{k}\,\|V[\rho]\|_{L^\infty(\mathbb{R}^2)}\,\sqrt{3k^2\pi}\,\left(\int_{\{k\le|x|\le 2k\}}|\nabla V[f]|^2\,dx\right)^{\frac{1}{2}}\\ &\le 4K_{11}\,\|V[\rho]\|_{L^\infty(\mathbb{R}^2)}\left(\int_{\{k\le|x|\le 2k\}}|\nabla V[f]|^2\,dx\right)^{\frac{1}{2}}.\end{aligned} \tag{37}$$

Due to $|\nabla V[f]| \in L^2(\mathbb{R}^2 : \mathbb{R}^2)$,

$$\lim_{k\to\infty} \int_{\{k\le |x|\le 2k\}} |\nabla V[f]|^2 \, dx = 0.$$

We deduce from (37),

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} (\nabla V[f] \cdot \nabla \Theta_k) \, V[\rho] \, dx = 0. \tag{38}$$

Employing (35), (36), (38) we deduce from (34) by passing to the limit $k \to \infty$,

$$\int_{\mathbb{R}^2} f \, V[\rho] \, dx = \int_{\mathbb{R}^2} \nabla V[f] \cdot \nabla V[\rho] \, dx.$$

Finally, let us verify **V.3**. Let $\phi \in L^\infty(\mathbb{R}^2)$ be compactly supported, let's say $\mathsf{supp}(\phi) \subseteq \{|y| \le R\}$ for some positive R. Then it is easy to deduce for all $x \in \mathbb{R}^2$,

$$\left| \int_{\mathbb{R}^2} K(x-y)\phi(y) \, dy \right| \le \|\phi\|_{L^\infty(\mathbb{R}^2)} \int_{\{|y|\le R\}} |K(y)| \, dy \le K_R \, \|\phi\|_{L^\infty(\mathbb{R}^2)},$$

where K_R is a positive number only depending on R. As a consequence, $K \star \phi \in L^\infty(\mathbb{R}^2)$. Now let $(f_n)_{n\in N}$ be a sequence in $L^1(\mathbb{R}^2)$ with $f_n \rightharpoonup f$ weakly in $L^1(\mathbb{R}^2)$ as $n \to \infty$. Then for all $\phi \in L^\infty(\mathbb{R}^2)$ with compact support,

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f_n(y) \, (K \star \phi)(y) \, dy = \int_{\mathbb{R}^2} f(y) \, (K \star \phi)(y) \, dy,$$

while on the other hand for all $n \in \mathbb{N}$ by the Fubini-Tonelli Theorem,

$$\begin{aligned}\int_{\mathbb{R}^2} V[f_n]\phi \, dx &= -\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} K(x-y) \, f(y) \, dy \right) \phi(x) \, dx \\ &= -\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} K(x-y) \, \phi(x) \, dx \right) f(y) \, dy = -\int_{\mathbb{R}^2} f_n(y) \, (K \star \phi)(y) \, dy,\end{aligned}$$

and

$$\int_{\mathbb{R}^2} V[f]\phi \, dx = -\int_{\mathbb{R}^2} f(y) \, (K \star \phi)(y) \, dy$$

as well. $\square$

Example 3 $\Omega = \mathbb{R}^d$, $d \ge 3$. We set $\mathsf{dom}(V) := L^1(\mathbb{R}^d)$. The Green's function of the Laplace operator is

$$K(x) := \frac{1}{\omega_d (d-2) \, |x|^{d-2}} \, , \quad x \in \mathbb{R}^d, \; x \neq 0,$$

where ω_d is the $(d-1)$-dimensional surface measure of the unit sphere in $\mathbb{R}^d$. Since $K \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ we have for each $f \in L^1(\mathbb{R}^d)$,

$$K \star f = \int_{\mathbb{R}^d} K(. - y) f(y)\, dy \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d).$$

For $f \in L^1(\mathbb{R}^d)$ we set

$$V[f] := -K \star f.$$

$V[.]$ certainly satisfies **V.1**.
We observe for all $i = 1, \dots, d$,

$$\partial_i V[f] = -(\partial_i K) \star f, \quad \partial_i K(x) = -\frac{1}{\omega_d\, |x|^{d-1}}\, \frac{x_i}{|x|},$$

in the sense of distributions. Furthermore, for each $i = 1, \dots, d$, the function $\partial_i K$ belongs to $L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$.

Now let $\rho \in L^\infty(\mathbb{R}^d)$ be compactly supported[1]. We take $R \in (0, \infty)$ with $\mathsf{supp}(\rho) \subseteq \{|y| \le R\}$. Then $\rho \in L^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$ and we calculate for all $i = 1, \dots, d$, for all $x \in \mathbb{R}^d$ and for all $q \in (1, \frac{d}{d-1})$, $p = \frac{q}{q-1}$,

$$\begin{aligned}
|\partial_i V[\rho](x)| &= \frac{1}{\omega_d} \left| \int_{\mathbb{R}^d} |x-y|^{1-d} \frac{x_i - y_i}{|x-y|} \rho(y)\, dy \right| \le \frac{1}{\omega_d} \int_{\mathbb{R}^d} |x-y|^{1-d} |\rho(y)|\, dy \\
&= \frac{1}{\omega_d} \int_{|y| \le R} |x-y|^{1-d} |\rho(y)|\, dy \le \frac{1}{\omega_d} \left(\int_{|y| \le R} |x-y|^{q(1-d)}\, dy \right)^{1/q} \|\rho\|_{L^p(\mathbb{R}^d)} \\
&\le \frac{\|\rho\|_{L^p(\mathbb{R}^d)}}{\omega_d} \left(\int_{|y| \le R} |y|^{q(1-d)}\, dy \right)^{1/q} = \frac{\|\rho\|_{L^p(\mathbb{R}^d)}}{\omega_d^{1/p}} \left(\int_0^R s^{(d-1)(1-q)}\, ds \right)^{1/q} \\
&= \frac{R^{1-d+(d/q)}}{(d+q-dq)^{1/q}\, \omega_d^{1/p}}\, \|\rho\|_{L^p(\mathbb{R}^d)}, \qquad (39)
\end{aligned}$$

and the estimate $|\partial_i V[\rho](x)| \le R\, \|\rho\|_{L^\infty(\mathbb{R}^d)}$ follows in analogy.

On the other hand, we have for all $x \in \mathbb{R}^d$ with $|x| > 2R$,

$$\begin{aligned}
|\partial_i V[\rho](x)| &\le \frac{1}{\omega_d} \left(\int_{|y| \le R} |x-y|^{q(1-d)}\, dy \right)^{1/q} \|\rho\|_{L^p(\mathbb{R}^d)} \\
&\le \frac{(|x| - R)^{1-d}}{\omega_d} \left(\int_{|y| \le R} 1\, dy \right)^{1/q} \|\rho\|_{L^p(\mathbb{R}^d)} = \frac{R^{d/q}}{d^{1/q}\, \omega_d^{1/p}}\, (|x| - R)^{1-d}\, \|\rho\|_{L^p(\mathbb{R}^d)},
\end{aligned} \qquad (40)$$

[1] $\int_{\mathbb{R}^d} \rho\, dx = 0$ is *not* required here.

and the estimate $|\partial_i V[\rho](x)| \leq \frac{R^d}{d}\ (|x|-R)^{1-d}\ \|\rho\|_{L^\infty(\mathbb{R}^d)},\ |x|>R$, follows in analogy.

We set for $p \in (d, \infty)$,

$$K(d,p,R) := \begin{cases} \omega_d^{-1/p}\ \max\left\{ \dfrac{(p-1)^{1-\frac{1}{p}}\ R^{1-\frac{d}{p}}}{(p-d)^{1-\frac{1}{p}}}, \dfrac{R^{(d-\frac{d}{p}}}{d^{1-\frac{1}{p}}} \right\} & ,\quad p \in (d,\infty) \\ \max\left\{ R, \dfrac{R^d}{d} \right\} & ,\quad p = \infty \end{cases}.$$

Furthermore, we put

$$u_{d,R}(x) := \begin{cases} 1 & ,\quad |x| \leq R+1 \\ (|x|-R)^{1-d} & ,\quad |x| > R+1 \end{cases}.$$

Then we deduce from (39), (40):

$$|\partial_i V[\rho]| \leq K(d,p,R)\ \|\rho\|_{L^p(\mathbb{R}^d)}\ u_{d,R}, \quad p \in (d,\infty]. \tag{41}$$

Since $u_{d,R} \in L^r(\mathbb{R}^d)$ for all $r \in (d/(d-1), \infty]$, we deduce from (41):

$\forall p \in (d,\infty], \forall r \in (d/(d-1),\infty]$:

There is a constant $K_4(d,r,p,R) \in (0,\infty)$ such that

$$\|\nabla V[\rho]\|_{L^r(\mathbb{R}^d:\mathbb{R}^d)} \leq K_4\ \|\rho\|_{L^p(\mathbb{R}^d)}. \tag{42}$$

In particular: $\nabla V[\rho] \in L^2(\mathbb{R}^d : \mathbb{R}^d)$. Hence $\rho \in \mathsf{dom}_2(V)$.

Remark 5. *In a similiar way one can prove the estimate*

$$|V[\rho]| \leq K_1(d,p,R)\ \|\rho\|_{L^p(\mathbb{R}^d)}\ v_{d,R}, \tag{43}$$

where

$$p \in (d/2, \infty], \quad \rho \in L^\infty(\mathbb{R}^d),\ \mathsf{supp}(\rho) \subseteq \{|y| \leq R\},$$

and

$$K_1(d,p,R) := \begin{cases} \dfrac{1}{(d-2)\ \omega_d^{1/p}}\ \max\left\{ \dfrac{(p-1)^{1-\frac{1}{p}}\ R^{2-\frac{d}{p}}}{(2p-d)^{1-\frac{1}{p}}}, \dfrac{R^{d-\frac{d}{p}}}{d^{1-\frac{1}{p}}} \right\} & ,\quad p \in (\frac{d}{2},\infty) \\ \dfrac{1}{d-2} \max\left\{ \dfrac{R^2}{2}, \dfrac{R^d}{d} \right\} & ,\quad p = \infty \end{cases},$$

$$v_{d,R}(x) := \begin{cases} 1 & ,\quad |x| \leq R+1 \\ (|x|-R)^{2-d} & ,\quad |x| > R+1 \end{cases}.$$

Then one can proceed as above to derive from (43) *that* $V[\rho] \in L^r(\mathbb{R}^d)$ *for all* $r \in (\frac{d}{d-2}, \infty]$ *and furthermore*

$$\forall p \in (d/2, \infty], \forall r \in (d/(d-2), \infty] : \quad \text{There is a constant } K_5(d, r, p, R) \in (0, \infty) \text{ such that} \quad \|V[\rho]\|_{L^r(\mathbb{R}^d)} \leq K_5 \, \|\rho\|_{L^p(\mathbb{R}^d)}. \tag{44}$$

The estimates (41), (42), (43), (44) allow to proceed similiar as for $d = 2$.

Let $f \in \mathsf{dom}_2(V)$ and let $\rho \in L^\infty(\mathbb{R}^d)$ be compactly supported. Then the function $(x, y) \mapsto K(x - y)\, f(y)\, \rho(x)$ belongs to $L^1(\mathbb{R}^d \times \mathbb{R}^d)$. We proceed as for $d = 2$ with the aid of the Fubini-Tonelli Theorem to conclude

$$\int_{\mathbb{R}^d} V[f]\rho \; dx = \int_{\mathbb{R}^d} V[\rho]f \; dx,$$

and it remains to prove

$$\int_{\mathbb{R}^d} \nabla V[f] \cdot \nabla V[\rho] \; dx = \int_{\mathbb{R}^d} V[\rho]f \; dx.$$

We take a sequence $(\rho_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ with $\lim_{n\to\infty} \|\rho - \rho_n\|_{L^{2d}(\mathbb{R}^d)} = 0$, see [Ada]. We can assume: There is $R \in (0, \infty)$ such that $\mathsf{supp}(\rho) \subseteq \{|y| \leq R\}$ and $\mathsf{supp}(\rho_n) \subseteq \{|y| \leq R\}$ for all $n \in \mathbb{N}$. We deduce from (41):

$$\nabla V[\rho_n] \to \nabla V[\rho] \quad \text{strongly in } L^2(\mathbb{R}^d : \mathbb{R}^d) \text{ as } n \to \infty. \tag{45}$$

Furthermore, since $K \in L^1_{loc}(\mathbb{R}^d)$ and since $\rho_n \in C_c^\infty(\mathbb{R}^d)$, we have $V[\rho_n] \in C^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$, see e.g. [Ada].

Let us take $\Theta \in C^\infty(\mathbb{R})$ with $\Theta = 1$ on $(-\infty, 0]$ and $\Theta = 0$ on $[1, \infty)$. For $k \in \mathbb{N}$ we introduce[2]

$$\Theta_k : \mathbb{R}^d \to \mathbb{R}, \quad \Theta_k(x) = \Theta(|x| - k).$$

Then $\Theta_k \in C_c^\infty(\mathbb{R}^d)$ for all $k \in \mathbb{N}$, hence $\Theta_k V[\rho_n] \in C_c^\infty(\mathbb{R}^d)$ for all $n, k \in \mathbb{N}$. We calculate for all $n, k \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^d} f \; (\Theta_k V[\rho_n]) \; dx &= -\int_{\mathbb{R}^d} V[f] \; \Delta(\Theta_k V[\rho_n]) \; dx = \int_{\mathbb{R}^d} \nabla V[f] \cdot \nabla(\Theta_k V[\rho_n]) \; dx \\ &= \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla \Theta_k) \; V[\rho_n] \; dx + \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla V[\rho_n]) \; \Theta_k \; dx,\end{aligned}$$

where made use of $V[\rho_n] \in L^\infty(\mathbb{R}^d)$ and $\nabla V[\rho_n] \in L^2(\mathbb{R}^d : \mathbb{R}^d)$ for all $n \in \mathbb{N}$.

We consider the limit $n \to \infty$ now.

[2] The function Θ_k is different from the corresponding function for $d = 2$.

Due to (44) the sequence $(V[\rho_n])_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^d)$ and converges strongly in $L^r(\mathbb{R}^d)$, $r \in (\frac{d}{d-2}, \infty)$ to $V[\rho]$ as $n \to \infty$. Hence for all $k \in \mathbb{N}$,

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}^d} f\ (\Theta_k V[\rho_n])\ dx &= \lim_{n\to\infty} \int_{\{k\le|x|\le k+1\}} f\ (\Theta_k V[\rho_n])\ dx \\ &= \int_{\{k\le|x|\le k+1\}} f\ (\Theta_k V[\rho])\ dx = \int_{\mathbb{R}^d} f\ (\Theta_k V[\rho])\ dx.\end{aligned}$$

Similiar argumentations (in particular exploiting the fact that each integration is in fact an integration over a fixed (i.e. independent of $n \in \mathbb{N}$) bounded domain) yield for all $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla\Theta_k)\ V[\rho_n]\ dx = \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla\Theta_k)\ V[\rho]\ dx,$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla V[\rho_n])\ \Theta_k\ dx = \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla V[\rho])\ \Theta_k\ dx.$$

Hence for all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} (f\ V[\rho])\ \Theta_k\ dx = \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla\Theta_k)\ V[\rho]\ dx + \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla V[\rho])\ \Theta_k\ dx. \tag{46}$$

Now we consider the limit $k \to \infty$. Since $f\ V[\rho], (\nabla V[f] \cdot \nabla V[\rho_n]) \in L^1(\mathbb{R}^d)$ and since $\lim_{k\to\infty} \Theta_k(x) = 1$ for all $x \in \mathbb{R}^d$ with $0 \le \Theta_k \le 1$ for all $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} (f\ V[\rho])\ \Theta_k\ dx = \int_{\mathbb{R}^d} f\ V[\rho]\ dx, \tag{47}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla V[\rho])\ \Theta_k\ dx = \int_{\mathbb{R}^d} \nabla V[f] \cdot \nabla V[\rho]\ dx. \tag{48}$$

Furthermore, for all $k \in \mathbb{N}$ and for all $x \in \mathbb{R}^d$,

$$|\nabla\Theta_k(x)| \le \sup_{s\in[0,1]} |\Theta'(s)| =: K_2.$$

Now we have for all $k \in \mathbb{N}$,

$$\begin{aligned}\left|\int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla\Theta_k)\ V[\rho]\ dx\right| &= \left|\int_{\{k\le|x|\le k+1\}} (\nabla V[f] \cdot \nabla\Theta_k)\ V[\rho]\ dx\right| \\ &\le \int_{\{k\le|x|\le k+1\}} |\nabla V[f]|\, |\nabla\Theta_k|\, |V[\rho]|\ dx \\ &\le K_2\ \left(\int_{\{k\le|x|\le k+1\}} |\nabla V[f]|^2\ dx\right)^{\frac{1}{2}}\ \left(\int_{\{k\le|x|\le k+1\}} |V[\rho]|^2\ dx\right)^{\frac{1}{2}}.\end{aligned} \tag{49}$$

If we take $R \in (0,\infty)$ with $\mathsf{supp}(\rho) \subseteq \{|y| \le R\}$, then we have due to (43) for all $k \in \mathbb{N}$ with $k > \max\{2R, R+1\}$,

$$\begin{aligned}\int_{\{k\le|x|\le k+1\}} |V[\rho]|^2 \, dx &\le K_1(d,\infty,R)\, \|\rho\|_\infty \int_{\{k\le|x|\le k+1\}} (|x|-R)^{2-d}\, dx \\ &= \omega_d\, K_1(d,\infty,R)\, \|\rho\|_\infty \int_k^{k+1} (s-R)^{4-2d}\, s^{d-1}\, ds \\ &\le 2^{2d-4}\, \omega_d\, K_1(d,\infty,R)\, \|\rho\|_\infty \int_k^{k+1} s^{3-d}\, ds \\ &\le 2^{2d-4}\, \omega_d\, K_1(d,\infty,R)\, \|\rho\|_\infty, \end{aligned} \tag{50}$$

while on the other hand due to $|\nabla V[f]| \in L^2(\mathbb{R}^d)$,

$$\lim_{k\to\infty} \int_{\{k\le|x|\le k+1\}} |\nabla V[f]|^2 \, dx = 0.$$

We deduce from (49) and from (50),

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} (\nabla V[f] \cdot \nabla \Theta_k)\, V[\rho]\, dx = 0. \tag{51}$$

Employing (47), (48), (51) we deduce from (46) by passing to the limit $k \to \infty$,

$$\int_{\mathbb{R}^d} f\, V[\rho]\, dx = \int_{\mathbb{R}^d} \nabla V[f] \cdot \nabla V[\rho]\, dx.$$

The verification of **V.3** can be performed in analogy to $d = 2$. $\square$

Example 4 $\Omega = \{x \in \mathbb{R}^d : x_1 > 0\}$, $d \in \mathbb{N}$. In this case we consider for $f \in L^1(\Omega)$ the Poisson equation

$$-\Delta V = f^*, \quad \text{in } \mathbb{R}^d, \tag{52}$$

where for $(x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$f^*(x_1,\ldots,x_d) = \begin{cases} f(x_1,\ldots,x_d) & , \ x_1 > 0 \\ f(-x_1, x_2, \ldots, x_d) & , \ x_1 < 0 \\ 0 & , \ \text{else} \end{cases}.$$

Then we set $V[f] := V_{\mathbb{R}^d}[f^*] \downharpoonleft \Omega$, where $V_{\mathbb{R}^d}[.]$ is the solution operator of (52) as discussed in the previous examples. The verification of **V.1**, **V.2**, **V.3** is straight-forward. $\square$

For $\lambda > 0$, the equation

$$\lambda \Delta \Phi = n - p - C$$

with $n = g(\alpha[\Phi] + \Phi)$ and $p = g(\beta[\Phi] - \Phi)$ has no solution with n and p in $L^1(\Omega)$ if $g > 0$ on $\mathbb{R}$ (i.e. if $\underline{h} = -\infty$). In that case it is essential to introduce a confinement by an external potential W, see [BDM99], i.e. one has to replace (19) - (24) by

$$\begin{aligned}
\lambda\Delta\Phi &= n[\Phi] - p[\Phi] - C(x), \quad x \in \Omega, && (53)\\
n[\Phi] &= g(\alpha[\Phi] + W + \Phi), && (54)\\
p[\Phi] &= g(\beta[\Phi] + W - \Phi), && (55)\\
\int_\Omega n[\Phi]\,dx &= \int_\Omega g(\alpha[\Phi] + \Phi - W)\,dx \;=\; N, && (56)\\
\int p[\Phi]\,dx &= \int_\Omega g(\beta[\Phi] - \Phi - W)\,dx \;=\; P, && (57)
\end{aligned}$$

with $W \to \infty$ as $|x| \to \infty$.

Remark 6. *One may ask about the physical interpretation of W, in particular whether W can be viewed as electrostatic potential. The problem with two species of particles with opposite sign charges is that an external electrostatic potential cannot be confining for both species, except if $W = |x|^2$ (by rescaling), but it is then not possible to have a charge density C unless it is concentrated at the origin, see [BDM99]. The physical problem is then the question of the free expansion of two species of particles with opposite charges but same total charge, which has already been studied (for the whole space case) in [BDM99]. We shall therefore keep in mind that if one only wishes to consider electrostatic models of W, then for $\underline{h} = -\infty$, the only realistic cases correspond to $W = |x|^2$ when Ω is a cone and $C \equiv 0$, or either $N = 0$ or $P = 0$ and W confining.*

We proceed as in the previous sections to analyze (19)-(22), (56), (57) by means of the functional E_λ, formally defined as

$$\begin{aligned}
&E_\lambda(n,p) = E^a(n,p) + \frac{1}{\lambda}\,E^{el}(n,p),\\
&\qquad E^a(n,p) := \int_\Omega H(n)\,dx + \int_\Omega H(p)\,dx + \int_\Omega (n+p)W\,dx,\\
&\qquad\qquad E^{el}(n,p) := \frac{1}{2}\int_\Omega |\nabla V[n-p-C]|^2\,dx, \quad (58)
\end{aligned}$$

where $(n,p) \in \mathcal{C}$, i.e. $n,p \in L^1_+(\Omega)$ with $\|n\|_{L^1(\Omega)} = N$ and $\|p\|_{L^1(\Omega)} = P$. As in the previous sections we set the electrostatic energy $E^{el}(n,p)$ equal to ∞ whenever $n - p - C \notin \mathsf{dom}_2(V)$. The confining potential W of (58) is assumed to be bounded below. Hence the last integral of $E^a(n,p)$ has a well-defined value in

$\mathbb{R} \cup \{\infty\}$. A bit more delicate is the integrability of $H(n)$, $H(p)$. In contrast to the situation for bounded domains, the convexity of H is not sufficient to assign to each of the first two integrals of $E^a(n,p)$ a value in $\mathbb{R} \cup \{\infty\}$. In fact, we have to impose an additional assumption included in the following

Proposition 10. *Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be a nonvoid domain. Let $N, P \geq 0$ and let $W \in L^1_{loc}(\Omega)$ be bounded below. Furthermore, assume* **A.2 - A.5, V.1 - V.3** *and*

A.6 *There are $\mu, \nu \in \mathbb{R}$ with $n_N := g(\mu - W), p_P := g(\nu - W) \in L^1(\Omega)$,*

$$\int_\Omega n_N \, dx = N, \quad \int_\Omega p_P \, dx = P,$$

and $H^-(n_N), H^-(p_P) \in L^1(\mathbb{R})$.

Then E_λ is bounded below on $\mathcal{C}$.

Proof. We exploit the convexity of H. We have for each $n \in L^1_+(\Omega)$ with $\int_\Omega n \, dx = N$,

$$\begin{aligned} ((H(n) + n\,W) - (H(n_N) + n_N\,W)) &= H(n) - H(n_N) + W\,(n - n_N) \\ &\geq H'(n_N)\,(n - n_N) + W\,(n - n_N) \\ &= (h(n_N) + W)\,(n - n_N) \geq \mu\,(n - n_N), \end{aligned}$$

because $h(n_N) = h(g(\mu - W)) = \mu - W$ whenever $n_N > 0$ and $h(n_N)\,(n - n_N) = h(n_N)\,n \geq (\mu - W)\,n = (\mu - W)\,(n - n_N)$ whenever $n_N = 0$. Hence,

$$\begin{aligned} &\int_\Omega \Big(H(n) + nW\Big)\,dx - \int_\Omega \Big(H(n_N) + n_N W\Big)\,dx \\ &\qquad \geq \int_\Omega \mu\,(n - n_N)\,dx = \mu\,\left(\int_\Omega n\,dx - \int_\Omega n_N\,dx\right) = \mu\,(M - M) = 0, \end{aligned}$$

and the inequality

$$\int_\Omega \Big(H(p) + pW\Big)\,dx - \int_\Omega \Big(H(p_P) + p_P W\Big)\,dx \geq 0$$

with $p \in L^1_+(\Omega)$ with $\int_\Omega p \, dx = P$ follows analogously. Due to the assumuption $H^-(n_N), H^-(p_P) \in L^1(\Omega)$ and due to the assumed boundedness of W from below we deduce that $\int_\Omega (H(n_N) + n_N W)\,dx$, $\int_\Omega (H(p_P) + p_P W)\,dx$ have values in $\mathbb{R} \cup \{\infty\}$. $\square$

Remark 7. *a) Since g is strictly increasing there is at most one pair $(\mu, \nu) \in \mathbb{R}^2$ satisfying* **A.6**.
b) If $N = 0$ and if $g > 0$ on $\mathbb{R}$ (i.e. if and only if $\underline{h} = -\infty$), then there is no

$\mu \in \mathbb{R}$ with $\int_\Omega g(\mu - W)\,dx = N = 0$. Hence, if one wishes to consider the case $N = 0$ with $\underline{h} = -\infty$, then one has to modify the functional E_λ by cancelling (or setting to zero, respectively) all terms which involve n. We leave the details of the corresponding analysis to the reader.
c) Proposition 10 does not exclude *$E_\lambda \equiv \infty$ on $\mathcal{C}$.*

Now let us turn our attention to the existence of minimizers of E_λ in $\mathcal{C}$. As an example, we state the following result in case of $\overline{h} = +\infty$ for which a purely variational argument provides an immediate answer.

Theorem 11. *Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be a nonvoid domain. Let $N, P > 0$ and let $W \in L^1_{loc}(\Omega)$. Furthermore, assume* **A.2 - A.6, V.1 - V.3** *and*

1. *$C \in L^1(\Omega)$.*
2. *W is bounded below.* $\liminf_{x \in \Omega,\, |x| \to \infty} W(x) = \infty$.
3. *$\overline{h} = \infty$.*
4. *$E^a + E_{el} \not\equiv \infty$ on $\mathcal{C}$.*
5. *$E^a(n_N, p_P) < \infty$ (that is, $H(n_N) + H(p_P) + (n_N + p_P)W \in L^1(\Omega)$), where n_N, p_P are as in* **A.6**.

Then E_λ has a unique minimizer (n_0, p_0) in $\mathcal{C}$ and the triple $(\Phi_0, n_0, p_0) = (-\lambda^{-1}V_0, n_0, p_0)$ with $V_0 = V[n_0 - p_0 - C]$ is a solution (the unique "equilibrium solution") of (53)-(57).

Remark 8. *a) Assumption 4. $E_\lambda \not\equiv \infty$ on $\mathcal{C}$ implies in case of $\Omega = \mathbb{R}^1$, $\Omega = \mathbb{R}^2$ or in case of $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with* $\mathrm{meas}(\Omega) < \infty$ *global charge neutrality $N - P = \int_\Omega C$ which is* not the case *for $\Omega = \mathbb{R}^d$, $d \geq 3$*
b) By straight-forward modifications one can also include $N = 0, P > 0$ or $N > 0, P = 0$.

Proof. E_λ is bounded below by Proposition 10. Furthermore, $E_\lambda \not\equiv \infty$ on $\mathcal{C}$. Hence $\inf_{\mathcal{C}} E_\lambda > -\infty$. We apply a standard minimization argument. Since E_λ is strictly convex, E_λ has at most one minimizer. This minizer is shown to exist by taking the limit of a minimizing sequence $(n_k, p_k)_{k \in \mathbb{N}}$. The limit (n_0, p_0) of this sequence belongs to $\mathcal{C}$ because of the weak-L^1 compactness of any minimizing sequence according to the Dunford-Pettis criterion: there is no concentration because H is superlinear at ∞ ($\overline{h} = \infty$) and no vanishing (because of the growth of W at ∞) in the language of concentration-compactness theory.

As shown in the proof of Proposition 10, we have $H(n) + H(p) + (n + p)W \geq H(n_N) + H(p_P) + (n_N + p_P)W \in L^1(\Omega)$ for all $(n, p) \in \mathcal{C}$. Hence the functional

E^a_λ is lower semicontinuous with respect to weak convergence in $L^1(\Omega)$ and we have

$$\int_\Omega H(n_0)\,dx + \int_\Omega H(p_0)\,dx + \int_\Omega W(n_0+p_0)\,dx \\ \leq \liminf_{k\to\infty} \int_\Omega H(n_k)\,dx + \int_\Omega H(p_k)\,dx + \int_\Omega W(n_k+p_k)\,dx. \quad (59)$$

Furthermore, since $(\nabla V_k)_{k\in\mathbb{N}}$, $V_k = V[n_k - p_k - C]$, is bounded in $L^2(\Omega : \mathbb{R}^d)$, we have - maybe after extracting a subsequence but without changing notations - for all $i = 1, \ldots, d$,

$$\partial_i V_k \rightharpoonup Z_i, \quad \text{weakly in } L^2(\Omega) \text{ as } k \to \infty,$$

hence

$$\int_\Omega (Z_1^2 + \ldots + Z_d^2)\,dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla V_k|^2\,dx. \quad (60)$$

Since $C \in L^1(\Omega)$ we have $n_k - p_k - C \rightharpoonup n_0 - p_0 - C$ weakly in $L^1(\Omega)$ as $k \to \infty$. We deduce from **V.3**, $V_k \to V_0 = V[n_0 - p_0 - C]$ in the sense of distributions. Hence $Z_i = \partial_i V_0$, $i = 1, \ldots, d$ and we deduce $E_\lambda(n_0, p_0) \leq \liminf_{k\to\infty} E_\lambda(n_k, p_k) = \inf_{\mathcal{C}} E_\lambda$ from (59) and (60).

Now we shall derive the associated variational inequalities. With the aid of **V.2** we deduce from standard arguments,

$$h(n_0) + W + \lambda^{-1}V_0 = \alpha \quad \text{on } \{n_0 > 0\}, \quad h(p_0) + W - \lambda^{-1}V_0 = \beta \quad \text{on } \{p_0 > 0\},$$

for some $\alpha, \beta \in \mathbb{R}$. In case of $\underline{h} > -\infty$ the function H is differentiable on each compact subset of $[0, \infty)$ and in this case we obtain by standard arguments

$$\underline{h} + W + \lambda^{-1}V_0 \geq \alpha \quad \text{on } \{n_0 = 0\}, \quad \underline{h} + W - \lambda^{-1}V_0 \geq \beta \quad \text{on } \{p_0 = 0\},$$

and we conclude $n_0 = g\,(\alpha - W - \lambda^{-1}V_0)$, $p_0 = g\,(\beta - W + \lambda^{-1}V_0)$,
i.e. $(-\lambda^{-1}V_0, n_0, p_0)$ is a solution of (53)-(57). It remains to consider the case $\underline{h} = -\infty$. It sufficies to prove: $n_0 > 0$ and $p_0 > 0$ almost everywhere on Ω. This is shown in an indirect way. If $n_0 = 0$ on a subset Ω_0 of Ω with $\text{meas}(\Omega_0) > 0$, then there is a compact subset K_0 of Ω_0 with $\text{meas}(K_0) > 0$ as well and one can take a test function $\phi \in L^\infty(\Omega)$ which is compactly supported in $\overline{\Omega}$, $\mathsf{ran}(\phi) = \{c_0, -c_1\}$ (where c_0, c_1 are positive real numbers), $\phi(x) = c_0$ if and only if $x \in K_0$, $\phi^{-1}(-c_1) \subseteq \{\varepsilon \leq n_0 \leq \varepsilon^{-1}\}$ (where $\varepsilon \in (0,1)$ is apropriately chosen), and $\int_\Omega \phi\,dx = 0$. We observe: $(n_0 + \delta\phi, p_0) \in \mathcal{C}$ for all sufficiently small δ. Since (n_0, p_0) minimizes E_λ in $\mathcal{C}$ one has for all sufficiently small $\delta \in (0, \infty)$,

$$E_\lambda(n_0 + \delta, p_0) - E_\lambda(n_0, p_0) \geq 0,$$

while it is not difficult to deduce from $\underline{h} = -\infty$ via **V.2** that $\lim_{\delta\to 0}(E_\lambda(n_0 + \delta, p_0) - E_\lambda(n_0, p_0))/\delta = -\infty$. Hence $n_0 > 0$ almost everywhere on Ω. $p_0 > 0$ almost everywhere on Ω follows in analogy. $\square$

Remark 9. *a) Propositon 11 highlights the role of the confining potential W which ensures weak compactness of minimizing sequences of E_λ.*
b) One may ask whether each solution (Φ_0, n_0, p_0) of (19)-(22), (56), (57) *is actually a minimizer of E_λ in $\mathcal{C}$. The discussion of this (very natural) question, however, requires rather technical additional assumptions (and is therefore omitted here): In contrast to the situation for bounded domains there is no a priori estimate which ensures $E_\lambda(n_0, p_0) < \infty$.*

The introduction of the confining potential W is inevitable whenever $\underline{h} = -\infty$. Let us turn our attention to a discussion of (19)-(24) (i.e. the model equations without W) in the case $\underline{h} > -\infty$ now.

In this situation it is possible to build solutions with compact support as follows (we recall: without any external confinement potential !).

Take a $\Phi \in C^2(\Omega)$ with compact support in Ω and consider $n_\lambda = g(\alpha + \Phi)$ and $p_\lambda = g(\beta - \Phi)$ for α and β such that $\underline{h} - \alpha > 0 > \beta - \underline{h}$. The density n_λ (respectively p_λ) is supported in the set corresponding to $\Phi > \underline{h} - \alpha$ (respectively $\Phi < \beta - \underline{h}$). We may then define

$$C = -\lambda \Delta \Phi + g(\alpha + \Phi) - g(\beta - \Phi) \ ,$$

and compute N and P corresponding to α, β and Φ. Then the triple $(\Phi, n_\lambda, p_\lambda)$ is certainly a solution of (19)-(24). In particular, we observe that global neutrality holds due to the fact that Φ has a compact support. $N \neq 0$ (respectively $P \neq 0$) holds if and only if $\operatorname{ess\,sup}_\Omega \Phi > \underline{h} - \alpha$ (respectively $\operatorname{ess\,inf}_\Omega \Phi < \beta - \underline{h}$).

We note that this construction actually does not depend on whether Ω is bounded or not, but only on the condition $\operatorname{supp}(\Phi) \subset\subset \Omega$. The condition that Φ has a compact support can be replaced by the condition that Φ is equal to a constant Φ_∞ outside a compact subset of Ω, with $\underline{h} - \alpha > \Phi_\infty > \beta - \underline{h}$. In unbounded domains, it is actually sufficient to assume that $\Phi = \Phi_\infty$ in a neighborhood of $\partial\Omega$ and

$$\beta - \underline{h} < \liminf_{|x| \to \infty,\, x \in \Omega} \Phi(x) \leq \limsup_{|x| \to \infty,\, x \in \Omega} \Phi(x) < \underline{h} - \alpha \ .$$

If there exists a solution Φ with $\Phi - \Phi_\infty$ compactly supported in Ω for a constant $\Phi_\infty \in (\beta - \underline{h}, \underline{h} - \alpha)$, then the global neutrality condition

$$N - P - \int_\Omega C \, dx = 0 \tag{61}$$

holds, but this is not necessarily the case if Ω is unbounded. Consider for instance a radial potential Φ which is asymptotically periodic, oscillating between two values in $(\beta - \underline{h}, \underline{h} - \alpha)$, with $C = -\Delta\Phi$ for $|x|$ large enough (case $\underline{h} > -\infty$): C is clearly not in $L^1(\Omega)$.

The global electroneutrality does not either hold in the presence of an external potential W. We can for instance (see [Dol91]) consider in $\Omega = \mathbb{R}^d$ ($d \geq 3$) the equation

$$\Delta\Phi = N \, \frac{e^{\Phi - W(x)}}{\int_\Omega e^{\Phi - W(x)} \, dx} - P \, \frac{e^{-\Phi - W(x)}}{\int_\Omega e^{-\Phi - W(x)} \, dx}$$

with $\lim_{|x|\to+\infty} \Phi(x) = 0$, $W(x) = \frac{1}{2}|x|^2$, $C \equiv 0$. If $N \neq P$, the global electroneutrality condition (61) is not satisfied. This example is easy to generalize to any potential W such that $\liminf_{|x|\to+\infty} |x|^{d-2} W(x) = +\infty$ and $e^{-W} \in L^1(\mathbb{R}^d)$ and $C \in L^1(\mathbb{R}^d)$ (with compact support for example). More striking is the fact that we can state an existence result without global electronuetrality even in the case without confining potential ($W \equiv 0$) provided $d \geq 3$ (and $\underline{h} > -\infty$).

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