The Wess–Zumino term and quantum tunneling

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Abstract

The significance of the Wess–Zumino term in spin tunneling is explored, and a formula is established for the splitting of energy levels of a particle with large fermionic spin as an applied magnetic field is switched on.

The socalled Wess–Zumino term in an action integral has increasingly been realised to be a quantity of considerable significance and with far–reaching physical consequences. The original work usually cited, ref.[1], is not very illuminating in this respect. The truly deep content of the quantity has, however, been uncovered particularly in ref.[2] in connection with the quantisation of the Skyrme model (and the identification of the integer contained in the coefficient of the Wess–Zumino term with the number of colours), and in ref.[3], for instance, in the very different area of macroscopic quantum tunneling. Here we consider the latter case. We show why the term – which arises from the quasiclassical consideration which here implies the use of the coherent state representation – is a Wess–Zumino term, we explore some of its characteristics and establish a formula for the level splitting of a particle with large spin S (integral or half–integral) in an applied magnetic field. The fermionic suppression of the level splitting in special cases has been established earlier (cf. refs.[3, 4, 5]). Our aim here is to extend to the half–integral case the formula obtained earlier for integral S by path integral methods [6] and from Schrödinger theory [7, 8] and in the presence of an applied magnetic field. In many one–dimensional cases Schrödinger theory is not only useful for comparison purposes, but is in particular simpler than the path integral method for transitions at the level of excited states, as a comparison shows (cf. e.g. [8] and [9, 10, 11]).

The consideration of spin systems is basically a discrete problem but in a quasiclassical treatment its mapping into a continuous system by first replacing spin

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operators $\hat{S}$ by classical vectors of length S has turned out to be an exceedingly useful procedure for the construction of models. Thus the spin Hamiltonian with two different anisotropy axes and without application of an external magnetic field,

$$\hat{\mathcal{H}} = -k_x\hat{S}_x^2 + k_z\hat{S}_z^2 \tag{1}$$

($k_x, k_z > 0$) is related to the classical Hamiltonian ($0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$)

$$\mathcal{H} = -k_x S^2 \sin^2\theta\cos^2\phi + k_z S^2\cos^2\theta \tag{2}$$

with two degenerate minima at $\theta = \frac{\pi}{2}, \phi = 0, \pi$. Quantum mechanically the spin can tunnel between these minima. The quasiclassical formulation implies the use of coherent (i.e. minimal uncertainty) states in the evaluation of the path integral. We impose the condition that these states be single–valued under the transformation $\phi \rightarrow \phi + 2\pi$ in order to force phase effects into the action (thus leading to the Wess–Zumino term). The coherent states can be shown to be only asymptotically (i.e. for $S \rightarrow \infty$) orthogonal, and it is their overlap at neighbouring Euclidean time steps which gives rise to a phase, the Wess–Zumino term in the Euclidean action S_E[3].

If we consider a single spin, the action is an integral over Euclidean time τ (and one obtains a level splitting). If we consider a large number of individual spins which align at low temperature, we can view the system as a single large spin, and the action is in addition to the integral over τ a sum over lattice sites (with separation a) or an integral over a spatial coordinate x (resulting in energy bands due to spatial translational invariance). In this case the Hamiltonian has in addition a Heisenberg term, i.e.

$$\hat{\mathcal{H}} = \frac{1}{aS^2}\left[-J\sum_{i,\rho}\hat{S}_i\cdot\hat{S}_{i+\rho} - K_x\sum_i(\hat{S}_i^x)^2 + K_z\sum_i(\hat{S}_i^z)^2\right] \tag{3}$$

In either case the equivalent effective quasiclassical continuum action S_E in Euclidean time obtained with the use of coherent states assumes the general form [3]

$$S_E = S_{WZ} + \int_0^\beta d\tau L, \quad S_{WZ} = i\gamma\int_0^\beta d\tau\dot{\psi}, \;\; L = \frac{1}{2}M(\psi)\dot{\psi}^2 + V(\psi) \tag{4}$$

where $\dot{\psi} = d\psi/d\tau$, and the constant γ, the effective mass $M(\psi)$ and the periodic potential $V(\psi)$ are quantities with model dependent parameters. In the single spin case referred to above ψ is simply the azimuthal angle ϕ with variation from one potential minimum at 0 or π to the other at π or 0 respectively as τ varies from 0 to β, and $\gamma = S$ (the polar angle θ can conveniently be chosen to be $\frac{\pi}{2}$). The Wess–Zumino action S_{WZ} does not only depend on the boundary values $\phi(0), \phi(\beta)$ but also on the path. The endpoint $\phi(\beta)$ can be reached on the circle of unit radius along the right hand semicircular arc a_r in the positive direction of $\mathbf{e}_\phi$ or along the left hand arc a_l in the opposite direction. This direction–dependence implies a handedness or chirality $C_\pm$ in S_{WZ}, i.e.

$$S_{WZ} = iS\pi C_\pm, \quad C_\pm = \frac{1}{\pi}\left[\int_{a_{r,l}} d\tau\dot{\phi}\right]_0^\beta = \frac{1}{\pi}\left[a_{r,l}\right]_0^\beta \tag{5}$$

which in the present case (without applied magnetic field) implies $C_\pm = \pm 1$. The effective mass $M(\phi)$ and potential $V(\phi)$ are found to be (with the conventions of eq.(1)) [6, 7, 8]

$$M(\phi) = \frac{1}{2k_z\left(1 + \frac{k_x}{k_z}\cos^2\phi\right)}, \quad V(\phi) = -k_x S^2 \cos^2\phi \tag{6}$$

The Euclidean time equation of motion can be shown to possess periodic instanton solutions [5, 10] which reduce in the limit $\frac{k_x}{k_z} \to 0$ to the periodic and vacuum instanton solutions of sine–Gordon theory. It is only the vacuum configuration which saturates the Bogomol'nyi inequality which can be constructed from the expression of its (positive) energy. Since in most cases in the context of spin tunneling the relevant pseudoparticle configurations are not the vacuum ones, this inequality is not of primary importance here. In the second, multiple spin, case referred to above the periodic instanton configurations appear as static (i.e. τ–independent) solutions of the 1–dimensional equation of motion. The translational invariance of the equation then implies the existence of a dynamical collective coordinate $\chi(\tau)$. Reexpressing the effective action entirely in terms of this collective coordinate, one again arrives at an expression like that of eq.(4) with ψ replaced by χ [3]. In this case $S_{WZ} = iS\pi C_\pm$ where now $S = Ns|\chi(\beta) - \chi(0)|/a$, N being the number of spins and s the spinvalue of one of them. It is the action integral S_E of eq.(4) which defines the basic model theory of the ferromagnetic sample we consider.

That S_{WZ} is the action of a Wess–Zumino term as in field theory follows from its properties and the fact that we can rewrite it in the defining form of ref. [2]. We first observe that as a total derivative it does not contribute to the equation of motion. Clearly we can rewrite $C_\pm$ as

$$C_\pm = -\frac{i}{\pi}\int_{a_{r,l}} d\tau [g^\dagger \frac{\partial}{\partial\tau} g], \quad g = e^{i\psi(\tau)} \tag{7}$$

thus demonstrating its appearance as a Wess–Zumino term (the higher dimensional form contains more factors like the τ–dependent one here). We observe that without the Wess–Zumino term the Lagrangian density is separately invariant under the global replacements $\tau \to -\tau$ and $\psi \to -\psi$, but with it only under the combination of both. Thus the Wess–Zumino term restricts the symmetry of the Lagrangian. As illustrated with the help of a simple model in ref. [2] it is precisely such a condition which gives rise to a Wess–Zumino term. The chirality associated with this term leads to a gauge potential, $\mathbf{A}$, which in the present case is simply a unit vector $\mathbf{e}_\phi$ around the unit circle $S^1 = a_l \cup a_r$ in the xy–plane, travelling either in the clockwise sense or in the anticlockwise sense through an angle 2π. Thus

$$\int_a d\tau \dot{\phi}(\tau) = \int_a \mathbf{e}_\phi \cdot \boldsymbol{d\phi} \equiv \int_a \mathbf{A} \cdot \boldsymbol{d\phi} = \int_a \boldsymbol{A} \cdot \dot{\boldsymbol{\phi}}\, d\tau \tag{8}$$

and

$$e^{S_{WZ}(a_r \cup a_l)} = 1, \quad S_{WZ}(a_r \cup a_l) = 2iN\pi$$

(N an integer), which is the condition corresponding to the Dirac charge quantisation condition (and, in fact, in ref. [2] the number of colours appears in the coefficient of the Wess–Zumino term in a similar way). Thus also

$$e^{S_{WZ}(a_r)} = e^{-S_{WZ}(a_l)}$$

One should note, that even in the 2–dimensional case of the multi–spin case above, the Wess-Zumino term is the same one-dimensional one as above. A 2-dimensional generalisation of the above, say to τ and x, would not only be zero (in view of the necessary $\epsilon_{\tau x}$ in front, but in addition would not satisfy such a symmetry requirement). Another way to see that the Wess–Zumino term does not affect the *mechanical* energy is to write down the Hamiltonian. With the definition $p_\phi = \frac{\partial(L+iS\dot{\phi})}{\partial\dot{\phi}} = M(\phi)\dot{\phi} + iS$ of the conjugate momentum (the first part being the mechanical momentum), one finds immediately that in the Hamiltonian $H = p_\phi\dot{\phi} - (L + iS\dot{\phi})$ the Wess–Zumino contribution drops out and one obtains

$$H = \frac{1}{2}M(\phi)\dot{\phi}^2 - V(\phi)$$

as expected. Expressed in terms of canonical variables the Hamiltonian, of course, assumes the form as in gauge theory, i.e.

$$H = \frac{(p_\phi - iS)^2}{2M(\phi)} - V(\phi)$$

This also shows that the Wess–Zumino term does not affect the classical energy of a pseudoparticle solution of the equations of motion.

The problem becomes more complicated if we also allow an externally applied magnetic field $\mathcal{B}$, e.g. perpendicular to the easy axis x, which can be taken into account by adding to the Hamiltonian of eq.(1) the contribution $-h\hat{S}_y$, where $h = g\mu_B\mathcal{B}$. Then

$$\hat{\mathcal{H}} = -k_x\hat{S}_x^2 + k_z\hat{S}_z^2 - h\hat{S}_y \tag{9}$$

In the corresponding quasiclassical continuum representation the new contribution implies that M and V (with $\psi \to \phi$) change to (cf. e.g. ref.[6])

$$\begin{aligned} M(\phi) &= \left[2k_z + 2k_x\cos^2\phi + \frac{h\sin\phi}{S+\frac{1}{2}}\right]^{-1}, \\ V(\phi) &= -\left[k_xS(S+1)\cos^2\phi + h(S+\frac{1}{2})\sin\phi\right] \end{aligned} \tag{10}$$

For integral values of S this case has been considered earlier in refs.[6, 8, 12]. Our interest here concerns the general case of S integral or half–integral and the establishment of a formula for the level splitting particularly in the latter case. We therefore follow ref.[8]. Instead of using the path integral method as in refs. [6, 12], we consider the equivalent Schrödinger theory and proceed as in refs.[7, 8]. The procedure employed there exploits the known level splitting for a periodic potential with identical barriers on either side of one of the wells of the potential. The appropriate one–dimensional Schrödinger equation with mass $\frac{1}{2}$ and potential $V(\phi) = 2h_m^2\cos 2\phi$ with h_m^2 assumed to be large (high barriers) leads to a splitting of the n–th asymptotically–single oscillator level given by the eigenvalue difference (cf. [7, 8] and references therein)

$$\triangle_{q_0=2n+1} = \frac{2(16h_m)^{q_0/2+1}e^{-4h_m}}{(8\pi)^{1/2}[\frac{1}{2}(q_0-1)]!}\Big(1+O(\frac{1}{h_m})\Big) \tag{11}$$

This result has also been obtained by path integral methods [11]. Identifying parameters with the present case, one finds (cf.[8])

$$h_m^2 = \frac{k_x}{4k_z}S(S+1)(1-a^2), \;\; a = \frac{h}{2k_x(S+\frac{1}{2})}, \;\; b = \frac{k_z}{k_x} \tag{12}$$

The effect of the applied magnetic field and so h is to shift the minima of the periodic potential slightly away from an integral value of π, and also to make successive barriers of the potential asymmetric, i.e. the barrier on one side of a minimum differs in size from that on the other side. Choosing the potential minimum at $\phi = \pi - \arcsin a$ as our reference minimum as in refs. [8, 12], the adjacent minima to the left and to the right are located angles

$$\phi_l = \arcsin a, \;\; \phi_r = 2\pi + \arcsin a$$

implying arc lengths

$$a_l = -\pi + 2\arcsin a, \quad a_r = \pi + 2\arcsin a$$

respectively, so that the distance from the minimum to the left of the reference minimum to that on the right is -2π. Correspondingly the periodic instantons in these barriers also differ (the explicit expressions are given in refs. [8, 12]). The actions $\mathcal{A}_{l,r}$ of these instantons enter the argument of the exponential contained in the level splitting whereas the factor in front is determined by the matching of different branches of the wave function in domains of overlap. Our present consideration differs from that in ref. [8] in that we now also wish to take into account the Wess–Zumino term. Thus we have to add to the action of either periodic instanton a contribution $iS\pi C_\pm$. This means that the tunneling exponential, which in the fieldless case ($a = 0$) is

$$e^{-\frac{2S}{\sqrt{b}}}$$

must be replaced by half the sum of two exponentials with arguments $-\mathcal{A}_r - iS\pi C_+$ and $-\mathcal{A}_l - iS\pi C_-$ respectively, i.e.

$$\begin{aligned} \frac{1}{2}e^{-\frac{2S}{\sqrt{b}}\left(\sqrt{1-a^2}+a\arcsin a\right)} & \Big[\;\; e^{-i(\pi+2\arcsin a)S}.e^{a(s+\frac{1}{2})\frac{\pi}{\sqrt{b}}} \\ & + \;\; e^{i(\pi-2\arcsin a)S}.e^{-a(s+\frac{1}{2})\frac{\pi}{\sqrt{b}}}\Big] \\ = e^{-\frac{2S}{\sqrt{b}}\left(\sqrt{1-a^2}+a\arcsin a\right)} & \; . \;\; e^{-i2S\arcsin a}.\cos\left(S\pi - \frac{ia(S+\frac{1}{2})\pi}{\sqrt{b}}\right) \end{aligned} \tag{13}$$

The a–dependent parts of the phase appearing here are sometimes called "Aharonov–Bohm" contributions [13]. With the appropriate replacement in eq. (12) the level splitting in this general case becomes

$$\begin{aligned}\triangle_{q_0=2n+1} &= \frac{2(16h_m)^{q_0/2+1}}{(8\pi)^{1/2}[\frac{1}{2}(q_0-1)]!}.e^{-\frac{2S}{\sqrt{b}}(\sqrt{1-a^2}+a\arcsin a)} \\ &. \left|e^{-i2S\arcsin a}\cos\left(S\pi-\frac{ia(S+\frac{1}{2})\pi}{\sqrt{b}}\right)\right| \qquad (14)\end{aligned}$$

We take the modulus of the phase factor not only because the energy must be positive, but also in view of the way the factor has to be handled in the path integral method (cf. ref. [3]). For integral values of S this formula reduces to

$$\triangle_{q_0=2n+1} = \frac{2(16h_m)^{q_0/2+1}}{(8\pi)^{1/2}[\frac{1}{2}(q_0-1)]!}.e^{-\frac{2S}{\sqrt{b}}(\sqrt{1-a^2}+a\arcsin a)}\cosh\left(\frac{a(S+\frac{1}{2})\pi}{\sqrt{b}}\right) \qquad (15)$$

This is the formula obtained in [8] in agreement with the results in [6]. For half–integral values of S and applied field zero, i.e. $a=0$, the splitting vanishes – in agreement with the socalled Kramer's degeneracy. The really interesting case is that of half–integral values of S and magnetic field unequal zero. In this case we obtain

$$\triangle_{q_0=2n+1} = \frac{2(16h_m)^{q_0/2+1}}{(8\pi)^{1/2}[\frac{1}{2}(q_0-1)]!}.e^{-\frac{2S}{\sqrt{b}}(\sqrt{1-a^2}+a\arcsin a)}\sinh\left(\frac{a(S+\frac{1}{2})\pi}{\sqrt{b}}\right) \qquad (16)$$

which is a plausible result because it vanishes completely for vanishing magnetic field. Comparing eq.(15) with eq.(16) we see that a very weak but nonzero magnetic field for which the level splitting increases linearly with the magnetic field is indicative of a macroscopic fermionic state. It would be interesting to see this observed.

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