

# TOWARDS $SO(2,10)$ -INVARIANT $M$ -THEORY: MULTILAGRANGIAN FIELDS

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## Abstract

The  $SO(2,10)$  covariant extension of M-theory superalgebra is considered, with the aim to construct a correspondingly generalized  $M$ -theory, or  $11d$  supergravity. For the orbit, corresponding to the  $11d$  supergravity multiplet, the simplest unitary representations of the bosonic part of this algebra, with sixth-rank tensor excluded, are constructed on a language of field theory in  $66d$  space-time. The main peculiarities are the presence of more than one equation of motion and corresponding Lagrangians for a given field and that the gauge and SUSY invariances of the theory mean that the sum of variations of these Lagrangians (with different variations of the same field) is equal to zero.

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# 1 Introduction

Recent progress in the investigation of theories with maximal supersymmetry leads to the notions of  $M$  and  $F$  theories[1], unifying many previously disconnected ones, such as 11d supergravity, superstrings, branes, etc. One of the lessons of this progress is that all features of the supersymmetry algebra appear to be important and contain interesting information on the physics behind them. Here we make an attempt to develop a systematic approach, aimed at using the well known property of maximally extended 11d supersymmetry algebra, namely its invariance with respect to  $SO(2, 10)$  rotations [2]. This feature has been discussed in a number of papers [3] - [11], where, in particular, twelve-dimensional theories were constructed, the particles and branes models, etc. What is the motivation for an  $SO(2, 10)$  approach to the  $M$ - theory superalgebra?

The 11-dimensional supersymmetry algebra with maximal number of “central” charges [12] (which are not central but rather are tensors with respect to the space-time rotations) has the following anticommutator of supercharges

$$\begin{aligned} \{\bar{Q}, Q\} &= \Gamma^i P_i + \Gamma^{ij} Z_{ij} + \Gamma^{ijklm} Z_{ijklm}, \\ i, j, \dots &= 0, 1, 2, \dots, 10 \end{aligned} \quad (1)$$

Here the  $Q$ 's are 11d Majorana spinors,  $P_i$  are the usual momenta, and tensors  $Z_{ij}, Z_{ijklm}$  the abovementioned central charges. The interesting feature of this relation is its  $SO(2, 10)$  invariance, which means that it is invariant with respect to the  $SO(2, 10)$  group of rotations of space-time with two “times” and 10 space coordinates. Thus 11 dimensional Majorana spinors  $Q$  can be identified as 12d Majorana-Weyl spinors, and generators on the r.h.s. combine into a 12d tensor  $P_{\mu\nu}$  and a selfdual sixth-rank tensor  $Z_{\mu\nu\lambda\rho\sigma\delta}^+$ . Instead of (1) we have now in 12d notation

$$\begin{aligned} \{\bar{Q}, Q\} &= \Gamma^{\mu\nu} P_{\mu\nu} + \Gamma^{\mu\nu\lambda\rho\sigma\delta} Z_{\mu\nu\lambda\rho\sigma\delta}^+ \\ \mu\nu, \dots &= 0', 0, 1, \dots, 10 \end{aligned} \quad (2)$$

Reducing the right hand side from 12d to 11d notation we obtain again relation (1). The important difference between these two relations is the absence in the second one of the vector object, momentum  $P_i$ . We ignore first

supercharges  $Q$ , and consider the pure ‘‘Poincare’’ algebra, implied by (2), which now consists of the usual generators of  $SO(2, 10)$  rotations and tensors  $P_{\mu\nu}$  with evident commutation relations: components of  $P_{\mu\nu}$  are mutually commutative, and transform as second rank tensors under rotations. We shall obtain features, which differ considerably from the usual ones. The main difference originates from the presence of many invariants (Casimir’s operators), constructed from  $P_{\mu\nu}$ :

$$\begin{aligned} Tr P^2 &= P_{\mu\nu} P^{\nu\mu}, \\ Tr P^4 &= P_{\mu\nu} P^{\nu\lambda} P_{\lambda\rho} P^{\rho\mu}, \\ &\dots \\ Tr P^{12} &= P_{\mu\mu_1} P^{\mu_1\mu_2} \dots P^{\mu_{11}\mu}. \end{aligned} \tag{3}$$

instead of the single one in  $11d$ :  $P^2 = P_i P^i$ . Of course, in  $11d$ , one can also obtain a lot of invariants if one considers those constructed from  $P_i$  and tensors  $Z_{ij}, Z_{ijklm}$ . But it is not necessary, and the minimal theory (i.e.  $11d$  supergravity) can be constructed starting from  $P_i$  only. From the  $SO(2, 10)$  point of view we are forced to switch on tensorial charges from the beginning. Here we shall consider the problem of construction of field theories with tensor  $P_{\mu\nu}$  instead of the usual momenta  $P_i$ . The final goal is the construction of a theory with  $SO(2, 10)$  invariance, which has to reduce to  $11d$  theories under some conditions. In particular, although the algebra (2) has many different BPS representations [3], there is one, which corresponds to the usual BPS representation of  $11d$  supergravity - massless superparticle multiplet. This is the multiplet, corresponding to the orbit of a particular tensor

$$P_{\mu\nu} = (P_{0i}, P_{ij} = 0), \tag{4}$$

$$P^2 = P_{0i} P^{0i} = 0, \tag{5}$$

$$P^{0i} = (1, 1, 0, \dots, 0) \tag{6}$$

From the  $SO(2, 10)$  point of view, the (compact) little group of this orbit is  $SO(9)$ , exactly as in  $11d$  superalgebra, and the representations are the same, in particular the smallest representation of the corresponding superalgebra includes one spin-vector (128 degrees of freedom), a traceless second-rank tensor (44), and an antisymmetric third rank tensor (84). One should note also that the natural space on which the algebra with  $SO(2, 10)$  and  $P_{\mu\nu}$

generators is realized is not  $12d$  space-time, but  $66d$  “space-time” with coordinates  $X^{\mu\nu}$  - antisymmetric tensor of second rank. The orbit referred to has the property, that all polynomial invariants (3) are equal to zero. The entire little group of this orbit can be found from its Lie algebra: Consider all elements of  $so(2,10)$  algebra, which leave the tensor (4) unchanged. Actually (4) itself, with raised second index (we use “mostly plus” signature), is an element of  $so(2,10)$ , so we are seeking its stabilizer in this algebra. It is easy to show, that the following matrices of  $so(2,10)$  are exactly all matrices, commuting with (4):

$$\begin{pmatrix} 0 & a & a & 0 & 0 & \dots & 0 \\ -a & 0 & 0 & b & c & \dots & d \\ a & 0 & 0 & -b & -c & \dots & -d \\ 0 & b & b & 0 & e & \dots & f \\ 0 & c & c & -e & 0 & \dots & g \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & d & d & -f & -g & \dots & 0 \end{pmatrix} \quad (7)$$

The algebra of matrixes (7) is a direct sum of the  $so(1,1)$  algebra of matrixes (7) with only non-zero entry  $a$ , and an algebra, which is a semidirect sum of  $so(9)$  (represented by matrixes (7) with  $a = b = c = \dots = d = 0$ ) and an Abelian algebra of matrixes (7) with non-zero elements  $b, c, \dots, d$  only. The unitary finite-dimensional representations of this little group algebra are those of  $so(9)$  subalgebra, with other generators represented by zero, which is possible due to the structure of the algebra.

The main feature of theories of this kind (i.e. those having  $P_{\mu\nu}$  instead of  $P_i$  in the algebra of symmetries) is the appearance of many Lagrangians - even in the case of one scalar field - which are simultaneously necessary for the description of the theory. Or, equivalently, many equations of motion for the same field are necessary, which is immediately connected with the existence of many invariants of  $P_{\mu\nu}$ . Their role is to bring the general function of the momenta  $P_{\mu\nu}$  to a function on the orbit of  $SO(2,10)$ . The notion of a symmetry generalizes in this case, as we shall see below, to the single equation, which implies that a sum of variations of these actions, with different infinitesimal variation of the same field in different actions, is equal to zero.

In the following we shall discuss mainly the theories without supersymmetry generators. Supersymmetry will appear in Section 6. Also, an algebra

of symmetry will be generalized to the semidirect product of  $so(2, q)$  and that of  $P_{\mu\nu}$  which are tensors under  $SO(2, q)$  Lorenz rotation, and components of  $P_{\mu\nu}$  commute with themselves. We are mainly interested in  $q = 10$ , but some examples will be considered for  $q = 2$ . In Sections 3, 4, 5 scalar, spinor and vector fields, respectively, are considered. Section 2 contains the discussion of orbits in the case of  $q=2$ . The conclusion is devoted to the discussion of results and prospects.

## 2 Classification of orbits: Example of $q=2$

The method of induced representations requires the classification of all orbits of the group  $SO(2, 10)$  on the space of tensors  $P_{\mu\nu}$ . This classification can be presented in a form of a list of representatives, one for each orbit, so called standard forms of  $P_{\mu\nu}$ . It is also convenient to have an identification of orbits with values of invariants, constructed from  $P_{\mu\nu}$ , i.e. (3). This correspondence can be easily obtained if standard forms are known. The classification for the  $(2 + 10)d$  case is not known to us. It is known in a simpler case, e.g. for Maxwell field strength in  $(1 + 3)d$  (see, e.g. [14]). Here we shall present this for the case of  $SO(2, 2)$ . In this case the space of tensors  $P_{\mu\nu}$  can be invariantly divided into two subspaces of self-dual and antiself-dual tensors, which have, correspondingly, the forms (with second index raised)

$$P_{\mu}^{+\nu} = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ b & -c & 0 & -a \\ c & b & a & 0 \end{pmatrix} \quad (8)$$

and

$$P_{\mu}^{-\nu} = \begin{pmatrix} 0 & k & l & n \\ -k & 0 & n & -l \\ l & n & 0 & k \\ n & -l & -k & 0 \end{pmatrix} \quad (9)$$

The triplets  $(a, b, c)$  and  $(k, l, n)$  transform under algebra  $so(2, 2)$  as vectors with respect to the two  $so(1, 2)$  subalgebras of  $so(2, 2) = so(1, 2) + so(1, 2)$ . Correspondingly (neglecting some subtleties concerning factorization over discrete subgroups) these vectors can be brought to the different forms depending on the values of their squares, namely to the forms  $(m, 0, 0), (0, 0, m)$

and  $(1, 0, 1)$ . Substitution of these forms into (8), (9) gives the standard forms of  $P_{\mu\nu}^{\pm}$ . The invariants, which define to which orbit a given  $P_{\mu\nu}$  belongs are traces of squares of self-dual and antiself-dual parts of  $P_{\mu\nu}$ :

$$P_{\mu}^{+\nu} P_{\nu}^{+\mu} = 4(-a^2 + b^2 + c^2) \quad (10)$$

$$P_{\mu}^{-\nu} P_{\nu}^{-\mu} = 4(-k^2 + l^2 + n^2)$$

The cases of negative, positive, and zero values of these invariants correspond to the three abovementioned forms of tensors (8) and (9). In particular, it is easy to see, that the case of both invariants equal to zero gives the matrix of the form (4), (5) and (6).

### 3 Scalar field

According to the little group method of construction of the unitary representations of semidirect product groups, we have to choose a particular value of  $P_{\mu\nu}$ , take a particular unitary representation of the corresponding little group, which is, by definition, the subgroup of  $SO(2, q)$ , which leaves that particular  $P_{\mu\nu}$  unchanged, and induce this representation on the whole group. In field theories with the usual Poincare symmetry it is known that the same representations can be described also in the language of fields and their equations of motion. In the simplest case of scalar (trivial) representation of the little group it is necessary to use the space of the usual functions on the orbit, which can be described in this case as the space of functions  $\Phi(P_{\mu\nu})$ , satisfying the equations of motion

$$\begin{aligned} (Tr P^2 - 2m_1^2)\Phi(P_{\mu\nu}) &= 0, \\ (Tr P^4 - 2m_2^4)\Phi(P_{\mu\nu}) &= 0, \\ &\dots \\ (Tr P^{12} - 2m_6^{12})\Phi(P_{\mu\nu}) &= 0 \end{aligned} \quad (11)$$

The first is the equation of the usual Klein-Gordon type, the others are on the same footing as the first one, and altogether they define functional  $\Phi(P_{\mu\nu})$  on the orbit, which is characterized by the numbers  $m_1, m_2, \dots, m_6$ . It is easy to check that on the orbit (4) this set of equations of motion reduces to the

usual Klein-Gordon equation in 11d:

$$\begin{aligned} (P_{0'i}P^{0'i} - m_1^2)\Phi(P_{0i}) &= 0 \\ m_2 = m_3 = m_4 = m_5 = m_6 &= m_1 \end{aligned} \quad (12)$$

Actions, giving (11), are, in a coordinate representations,

$$\begin{aligned} S_i &= \frac{1}{2} \int [dX^{\mu\nu}] (\Phi(X) (\frac{\partial}{\partial X^{\mu_1\mu_2}} \dots \frac{\partial}{\partial X^{\mu_{2i-1}\mu_{2i}}} - m_i^{2i}) \Phi(X)) \\ i &= 1, 2, \dots, 6. \end{aligned} \quad (13)$$

The question, whether it is possible to replace this set of Lagrangians by a single one, giving all necessary equations (11), is open, even in the simplest case of the scalar field. For the other hand, field theory can be considered as a second-quantized version of the theory of particles. This particle Hamiltonian, giving the same equations (11) as an equation for the wave function of the quantized particle, can be represented in a standard form:

$$H = \lambda_1(Tr P^2 - 2m_1^2) + \dots + \lambda_6(Tr P^{12} - 2m_6^{12}) \quad (14)$$

where  $\lambda_i$  are arbitrary gauge functions (consequently, functions, multiplying them, are constraints to be imposed on a wave function, cf. Eqs. (11)). We can define the Lagrangian after exclusion of momenta in the usual relation:

$$L = P_{\mu\nu} \dot{X}^{\mu\nu} - H \quad (15)$$

It may be worth mentioning, that a very similar problem was discussed many years ago by Fierz and Pauli [13]<sup>1</sup> They considered, particularly, the problem of construction of the Lagrangian for higher spin massive theories. These theories are described by a few equations of the same field, as in the present situation, and Fierz and Pauli developed the method of introduction of auxiliary fields, leading to a single Lagrangian, with an equivalent set of equations of motion.

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<sup>1</sup>We are indebted to M. Vasiliev for bringing this article to our attention.

## 4 Dirac equation(s)

We now consider the spinor  $\Psi$  as a function of tensorial momenta  $P_{\mu\nu}$  with the following set of equations of motion:

$$\Gamma^{\mu_1\mu_2} P_{\mu_1\mu_2} \Psi(P_{\mu\nu}) = 0 \quad (16)$$

$$\Gamma^{\mu_1\mu_2\mu_3\mu_4} P_{\mu_1\mu_2} P_{\mu_3\mu_4} \Psi(P_{\mu\nu}) = 0 \quad (17)$$

$$\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} P_{\mu_1\mu_2} P_{\mu_3\mu_4} P_{\mu_5\mu_6} \Psi(P_{\mu\nu}) = 0 \quad (18)$$

$$\Gamma^{\mu_1\mu_2\cdots\mu_8} P_{\mu_1\mu_2} P_{\mu_3\mu_4} P_{\mu_5\mu_6} P_{\mu_7\mu_8} \Psi(P_{\mu\nu}) = 0 \quad (19)$$

$$\Gamma^{\mu_1\mu_2\cdots\mu_{10}} P_{\mu_1\mu_2} P_{\mu_3\mu_4} \cdots P_{\mu_9\mu_{10}} \Psi(P_{\mu\nu}) = 0 \quad (20)$$

$$\Gamma^{\mu_1\mu_2\cdots\mu_{12}} P_{\mu_1\mu_2} \cdots P_{\mu_{11}\mu_{12}} \Psi(P_{\mu\nu}) = 0 \quad (21)$$

and actions:

$$S_i = \frac{1}{2} \int [dX^{\mu\nu}] (\bar{\Psi}(X) (\Gamma^{\mu_1\mu_2\cdots\mu_{2i-1}\mu_{2i}} \frac{\partial}{\partial X^{\mu_1\mu_2}} \cdots \frac{\partial}{\partial X^{\mu_{2i-1}\mu_{2i}}}) \Psi(X)) \quad (22)$$

$$i = 1, 2, \dots, 6, \quad \bar{\Psi} = \Psi^T \Gamma^0 \Gamma^{0'}$$

Again we can check that on the orbit(4) this set is equivalent to the 11d Dirac equation:

$$\Gamma^{0'i} P_{0'i} \Psi(P_{0'i}) = 0 \quad (23)$$

Then one can check that on-shell scalar invariants are zero (3) :

$$Tr P^2 \Psi(P_{\mu\nu}) = 0, \quad (24)$$

$$Tr P^4 \Psi(P_{\mu\nu}) = 0, \quad (25)$$

...

$$Tr P^{12} \Psi(P_{\mu\nu}) = 0. \quad (26)$$

To prove this, one has to multiply equations (16) to (21) on different matrices like:

$$\Gamma^{\mu\nu} P_{\mu\nu}, \quad \Gamma^{\mu\nu\lambda\rho} P_{\mu\nu} P_{\lambda\rho}, \quad \dots$$

and make some  $\Gamma$ -algebra calculations. For example, squaring (16), we have:

$$\begin{aligned} \Gamma^{\mu_1\mu_2} P_{\mu_1\mu_2} \Gamma^{\mu_3\mu_4} P_{\mu_3\mu_4} \Psi(P_{\mu\nu}) &= \\ \left( \Gamma^{\mu_1\mu_2\mu_3\mu_4} P_{\mu_1\mu_2} P_{\mu_3\mu_4} + 2Tr P^2 \right) \Psi(P_{\mu\nu}) &= 0 \end{aligned} \quad (27)$$



i.e. Eq.. (24). Multiplying (16) on  $\Gamma^{\mu_1\mu_2} P^3_{\mu_1\mu_2}$  we obtain

$$\begin{aligned} \Gamma^{\mu_1\mu_2} P^3_{\mu_1\mu_2} \Gamma^{\mu_3\mu_4} P_{\mu_3\mu_4} \Psi(P_{\mu\nu}) &= \\ \left( \Gamma^{\mu_1\mu_2\mu_3\mu_4} P^3_{\mu_1\mu_2} P_{\mu_3\mu_4} + 2Tr P^4 \right) \Psi(P_{\mu\nu}) &= 0 \end{aligned} \quad (28)$$

Squaring (17) and using (24) and higher equations from (16) to (21), we obtain

$$\begin{aligned} \left( \Gamma^{\mu_1\mu_2\mu_3\mu_4} P_{\mu_1\mu_2} P_{\mu_3\mu_4} \right)^2 \Psi(P_{\mu\nu}) &= \\ \left( 32\Gamma^{\mu_1\mu_2\mu_3\mu_4} P^3_{\mu_1\mu_2} P_{\mu_3\mu_4} + 16Tr P^4 \right) \Psi(P_{\mu\nu}) &= 0 \end{aligned} \quad (29)$$

From (28), (29) and (24) follows (25) and so on. Equivalently, it is possible to use instead of (16) to (21) equations (16) and

$$G_i(P)\Psi(P_{\mu\nu}) = 0, \quad i = 2, 3, 4, 5, 6 \quad (30)$$

$$(31)$$

Here  $G_i$  are combinations of invariants defined below in (44). Equations (30) express higher invariants through  $Tr P^2$  and (16) gives the on-shell condition  $Tr P^2 = 0$  and fixes the spinor structure of  $\Psi$ .

## 5 Vector field and gauge invariance

We consider the Abelian vector field satisfying the usual Maxwell equation:

$$P_j F^{ji} = \left( -P^2 \eta^{ij} + P^i P^j \right) A_j = 0 \quad (32)$$

This equation has a U(1) gauge invariance:

$$\delta A_i = P_i \alpha(P_i) \quad (33)$$

which allows one to impose a gauge fixing condition off-shell and remove the longitudinal component:

$$P^i A_i = 0 \quad (34)$$

Assuming  $P^2 \neq 0$  we can rewrite the Maxwell equation (32) as:

$$P^j P_j A_i^{tr} = P^2 \left( \delta_i^j - \frac{P_i P^j}{P^2} \right) A_j = 0 \quad (35)$$

This equation means that  $A_i^{tr} = 0$  which implies the absence of physical degrees of freedom. Therefore on-shell we have to put  $P^2 = 0$  and  $P^i A_i = 0$  which means in an appropriate frame:  $P_i = (1, 1, 0, 0, \dots)$  and  $A_0 = A_1$ . These relations are equivalent to well-known result that an on-shell vector field has  $d - 2$  physical components with the massless condition  $P^2 = 0$ .

We now turn to the construction of the vector field theory in an  $SO(2, 10)$  invariant way. Using the vector  $A_\mu(P_{\lambda\nu})$  and the ‘‘field strength’’

$$F_{\mu\nu\lambda} = P_{\mu\nu} A_\lambda + P_{\nu\lambda} A_\mu + P_{\lambda\mu} A_\nu \quad (36)$$

we can define the following set of equations of motion:

$$P^{\mu\nu} F_{\mu\nu\lambda} = 0 \quad (37)$$

$$(P^3)^{\mu\nu} F_{\mu\nu\lambda} = 0 \quad (38)$$

$$\dots \quad (39)$$

$$(P^{11})^{\mu\nu} F_{\mu\nu\lambda} = 0 \quad (40)$$

An equivalent set of equations is:

$$P^{\mu\nu} F_{\mu\nu\lambda} = 2 \left( P_\lambda^{2\mu} - \frac{1}{2} Tr P^2 \delta_\lambda^\mu \right) A_\mu = 0 \quad (41)$$

$$G_i(P) A_\mu = 0 \quad i = 2, 3, 4, 5, 6 \quad (42)$$

$$(43)$$

Here  $G_i$  with  $G_1 = -\frac{1}{2} Tr P^2$  are combinations of invariants (3) coinciding with the coefficients in the expansion of the characteristic polynomial for matrix  $P_\mu{}^\nu$ :

$$\begin{aligned} f(x) &= \det(P - x) \\ &= x^{12} + x^{10} G_1 + x^8 G_2 + \dots + G_6 \end{aligned} \quad (44)$$

Again this set is equivalent to the ordinary Maxwell system under the condition  $P_{ij} = 0$  and therefore reproduces the masslessness condition  $P^2 = 0$ .

We can now define the set of actions corresponding to equations (41) to (42) in coordinate space:

$$S_1 = \int [dX^{\mu\nu}] \left( -\frac{1}{6} F^{\mu\nu\lambda} F_{\mu\nu\lambda} \right), \quad (45)$$

$$S_i = \int [dX^{\mu\nu}] \left( \frac{1}{2} A_\lambda G_i (\partial_{X^{\mu\nu}}) A^\lambda \right), \quad (46)$$

$$i = 2, 3, \dots, 6.$$

Actions, corresponding to the other form of equations of motion (37) to (40) are:

$$K_1 = \int [dX^{\mu\nu}] \left( -\frac{1}{6} F^{\mu\nu\lambda} F_{\mu\nu\lambda} \right), \quad (47)$$

$$K_i = \int [dX^{\mu\nu}] \left( \frac{1}{2} A^\mu (\delta_{\mu\nu} Tr(P^{2i-2}) - 2(P^{2i-2})_{\mu\lambda}) A^\lambda \right), \quad (48)$$

$$i = 1, 2, \dots, 6. \quad (49)$$

Another possible set of actions is

$$N_1 = \int [dX^{\mu\nu}] \left( -\frac{1}{6} F^{\mu\nu\lambda} F_{\mu\nu\lambda} \right), \quad (50)$$

$$N_i = \int [dX^{\mu\nu}] \left( -\frac{1}{6} F^{\mu\nu\lambda} (P^{2i-2})^\sigma_\lambda F_{\mu\nu\sigma} \right), \quad (51)$$

$$i = 1, 2, \dots, 6. \quad (52)$$

What about gauge invariance? We can define the following set of variations corresponding to each equation of the set

$$\begin{aligned} \delta_1 A_\mu(P_{\lambda\rho}) &= P_{\mu\nu}^{10} \alpha^\nu(P_{\lambda\rho}) \\ \delta_2 A_\mu(P_{\lambda\rho}) &= P_{\mu\nu}^8 \alpha^\nu(P_{\lambda\rho}) \\ &\vdots \\ \delta_6 A_\mu(P_{\lambda\rho}) &= \alpha_\mu(P_{\lambda\rho}) \end{aligned} \quad (53)$$

Then it is easy to see that the following equation is satisfied:

$$\delta_1 S_1 + \delta_2 S_2 + \dots + \delta_6 S_6 = 0 \quad (54)$$

due to the well-known Hamilton-Cayley identity for characteristic polynomials:

$$f(P) = 0 \quad (55)$$

where

$$f(x) = \det(P - x) \quad (56)$$

The equation similar to (54) is valid for other sets of actions, with different variations  $\delta_i A_\mu$ . Under the condition  $P_{ij} = 0$  the remaining symmetry is:

$$\delta A_{0'i} = P_{0'\nu}^{10} \alpha^\nu(P_{0'i}) = \beta \quad (57)$$

$$\delta A_i(P_{0'i}) = P_{i\nu}^{10} \alpha^\nu(P_{0'i}) = P_{i0'} \gamma \quad (58)$$

The first of these can be used for gauging away the additional twelfth component  $A_{0'}$ , the second gives the usual gauge transformation for the remaining  $11d$  Abelian gauge field. The subtlety is that: if we consider the on-shell condition  $P^2 = 0$  from the beginning, it is not possible to gauge away  $A_{0'}$ , as is seen from (57). It is not clear, whether this subtlety spoils the equivalence of the space of solutions of (37) (or (41), (42)) with the corresponding unitary representation of the group considered.

## 6 Supersymmetry

In this section we will consider  $(2+2)d$  supersymmetric theories realizing the following algebra of supersymmetry:

$$\begin{aligned} \{\bar{Q}, Q\} &= \Gamma^{\mu\nu} P_{\mu\nu} \\ \mu\nu, \dots &= 0', 0, 1, 2. \end{aligned} \quad (59)$$

We first define actions and the supersymmetry transformation for the supersymmetry multiplets in the  $(2+2)d$  theory, containing Majorana spinor and scalar and spinor and vector fields. For the scalar (Wess-Zumino) multiplet we can define the actions :

$$S_1 = \frac{1}{2} \int [dX^{\mu\nu}] \left( \Phi \left( \frac{\partial}{\partial X^{\mu_1 \mu_2}} \frac{\partial}{\partial X_{\mu_1 \mu_2}} \right) \Phi + \bar{\Psi} \Gamma^{\mu_1 \mu_2} \frac{\partial}{\partial X^{\mu_1 \mu_2}} \Psi \right) \quad (60)$$

$$S_2 = \frac{1}{2} \int [dX^{\mu\nu}] \left( \Phi G_2 \Phi + \bar{\Psi} \Gamma^{\mu_1 \mu_2 \mu_3 \mu_4} \frac{\partial}{\partial X^{\mu_1 \mu_2}} \frac{\partial}{\partial X^{\mu_3 \mu_4}} \Psi \right) \quad (61)$$

and variations of fields:

$$\delta_1 \Phi = \bar{\varepsilon} \Psi, \quad \delta_1 \Psi = \frac{1}{2} \Gamma^{\mu_1 \mu_2} \frac{\partial}{\partial X^{\mu_1 \mu_2}} \Phi \varepsilon \quad (62)$$

$$\delta_2 \Psi = -\Phi \varepsilon, \quad \delta_2 \Phi = 0 \quad (63)$$

The commutator of transformation (62) is  $\Gamma^{\mu\nu} P_{\mu\nu}$ , according to algebra (59), the commutator of (63) is zero.

It is easy to check the following supersymmetry relation:

$$\delta_1 S_1 + \delta_2 S_2 = 0 \quad (64)$$

Again, as in the case of gauge invariance, the statement of supersymmetry is of the kind of (54): the sum of variations of actions of different dimensionality is equal to zero.

Similar equations for the vector multiplet are:

$$S_1 = \int [dX^{\mu\nu}] \left( -\frac{1}{6} F^{\mu\nu\lambda} F_{\mu\nu\lambda} + \frac{1}{2} \bar{\Psi} \Gamma^{\mu_1\mu_2} \frac{\partial}{\partial X_{\mu_1\mu_2}} \Psi \right), \quad (65)$$

$$S_2 = \int [dX^{\mu\nu}] \left( \frac{1}{2} A_\lambda G_2(\partial_{X^{\mu\nu}}) A^\lambda + \frac{1}{2} \bar{\Psi} \Gamma^{\mu_1\mu_2\mu_3\mu_4} \frac{\partial}{\partial X_{\mu_1\mu_2}} \frac{\partial}{\partial X_{\mu_3\mu_4}} \Psi \right), \quad (66)$$

$$\delta_1 A_\mu = \bar{\varepsilon} \Gamma_\mu \Psi \quad (67)$$

$$\delta_1 \Psi = F_{\mu\nu\lambda} \Gamma^{\mu\nu\lambda} \varepsilon \quad (68)$$

$$\delta_2 \Psi = -\frac{1}{2} A_\mu \Gamma^\mu \varepsilon \quad (69)$$

$$\delta_2 A_\mu = 0 \quad (70)$$

$$\delta_1 S_1 + \delta_2 S_2 = 0 \quad (71)$$

It is possible to find additional symmetries in the above actions which, however, may not be important since all these theories are free.

## 7 Conclusion and Outlook

In the above we considered the construction of field theories, the space-time algebra of symmetries of which is a semidirect sum of  $so(2, q)$  and the Abelian algebra of second-rank tensors  $P_{\lambda\rho}$ . The reason is that the bosonic part of the  $M$ -theory algebra has this form, with  $q = 10$ , modulo a sixth-rank central charge, and the aim of the present investigation is the development of an  $SO(2, 10)$ -invariant approach to  $M$ -theory. This algebra differs from the usual Poincare one in two respects: the momentum  $P_i$  is replaced by  $P_{\mu\nu}$ , and second time-like direction appears in the metric. The free theories of scalar, spinor, vector fields are constructed, as well as the supersymmetric free theory for  $q = 2$ . One of the main features, which is different in the present situation from the usual ones, is the necessity of having simultaneously many Lagrangians for the same field. This follows from the existence of many invariants, constructed from  $P_{\mu\nu}$ , which all have to be fixed for a given unitary

representation of algebra of symmetry. For a scalar field, e.g., these equations are Klein-Gordon ones and their higher derivative analogs. Another peculiarity is the form of the gauge and susy invariance. The Lagrangians mentioned are not separately invariant with respect to these transformations, but only the sum of their variations - with different variations of the same field in different Lagrangians - is equal to zero. It is well-known, that free theories can possess very different kinds of algebras of symmetries, and only interaction selects the very few, such as susy algebra. So, it is important to have an example of an interacting theory. This can be constructed for  $q = 2$ . The similar “example” for  $q = 10$  would be an explicitly  $SO(2, 10)$ -invariant formalism for  $11d$  supergravity, unknown so far. It is possible to construct that on a linearized level. On this stage we can propose the first level equations of motion for graviton  $h_{\mu\nu}$ , gravitino  $\Psi_\mu$ , and third rank antisymmetric tensor  $A_{\mu\nu\lambda}$ :

$$-\frac{1}{2}\partial_{\alpha\beta}\partial^{\beta\alpha}h_{\mu\nu} - \partial_\mu^\alpha\partial_\nu^\beta h_{\alpha\beta} + \partial_\mu^2 h_{\alpha\nu} + \partial_\nu^2 h_{\alpha\mu} - \partial_{\mu\nu}^2 h = 0 \quad (72)$$

$$\gamma^{\mu\nu\lambda\rho}\partial_{\nu\lambda}\Psi_\rho = 0 \quad (73)$$

$$\partial^{\mu\nu}F_{\mu\nu\lambda\rho\sigma} = 0, \quad F_{\mu\nu\lambda\rho\sigma} = \partial_{[\mu\nu}A_{\lambda\rho\sigma]} \quad (74)$$

It is easy to see that these equations go to usual  $11d$  linearized supergravity equations on the orbit (4). The main problem here is the formulation of high level equations of motion with complete set of symmetries (global SUSY and gauge), which will be discussed separately. The other interesting question is the construction of extended objects with explicit  $SO(2, 10)$  invariance, appropriate for this hypothetical form of  $11d$  supergravity. We refer to [11],[8] and [9],[10] for some suggestions.

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