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A note on the analyticity of AdS scalar exchange graphs in the crossed channel

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Abstract

We discuss the analytic properties of AdS scalar exchange graphs in the crossed channel. We show that the possible non-analytic terms drop out by virtue of non-trivial properties of generalized hypergeometric functions. The absence of non-analytic terms is a necessary condition for the existence of an operator product expansion for CFT amplitudes obtained from AdS/CFT correspondence.

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Scalar exchange conformal graphs are basic ingredients in a calculation of conformally invariant n-point functions for $n \geq 4$. Such graphs have been extensively studied in earlier works on conformal field theory (CFT) in $d > 2$ [1–3]. Consider a conformal four-point function [4] $\mathcal{G}(x_1,..,x_4) = \langle \mathcal{O}(x_1)..\mathcal{O}(x_4)\rangle$ where the scalar field $\mathcal{O}(x)$ has dimension $\tilde{\Delta}$. One contribution to this four-point function is given by the skeleton graphs in which a scalar field $\sigma(x)$ of dimension Δ is exchanged between external legs corresponding to $\mathcal{O}(x)$. For such skeleton graphs to exist it suffices that the three-point function $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\sigma(x_3)\rangle$ is non-zero. [5] The relevant graph is depicted in Fig.1. Then, bose symmetry requires that $\mathcal{G}(x_1,..,x_4)$ receives contributions from all different graphs which can be obtained from Fig.1 by suitable relabeling of the external points.

$B(x_1, x_3; x_2, x_4) =$

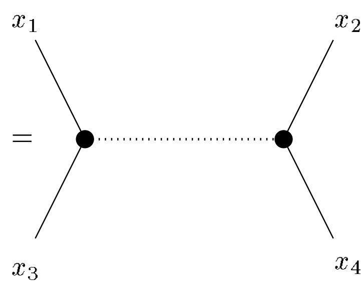

Figure 1: *The standard CFT_d scalar exchange graph. The solid lines represent the full propagator of $\mathcal{O}(x)$, the dotted line represents the full propagator of $\sigma(x)$ and the dark blobs stand for the full vertex functions* [2, 3].

One of the basic implications of conformal invariance is the existence of an operator product expansion (OPE) [5] which is essentially equivalent to a partial-wave decomposition of conformal n-point functions. Such a property finds an explicit application in the study of the four-point function $\mathcal{G}(x_1,..,x_4)$, as it implies that its calculation in terms of skeleton graphs should be compatible with a conformal OPE. By conformal invariance $\mathcal{G}(x_1,..,x_4)$ is determined up to an arbitrary analytic function of the two biharmonic ratios

$$u = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \quad , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} , \quad x_{ij} = x_i - x_j \, . \tag{1}$$

Then, calculating the relevant skeleton graphs obtained from Fig.1 one finds a general expansion of the form

$$\begin{aligned} \mathcal{G}^{(1)}(x_1, x_2, x_3, x_4) \quad &= \quad \frac{1}{(x_{12}^2 x_{34}^2)^{\tilde{\Delta}}} \sum_{n,m=0}^{\infty} \frac{u^n (1-v)^m}{n!m!} \Bigg[-A_{nm}(\tilde{\Delta}, \Delta) \ln u + B_{nm}(\tilde{\Delta}, \Delta) \\ &\quad + u^{\frac{1}{2}\Delta - \tilde{\Delta}} C_{nm}(\tilde{\Delta}, \Delta) + u^{\frac{1}{2}(d-\Delta) - \tilde{\Delta}} C_{nm}(\tilde{\Delta}, d-\Delta) \Bigg] , \end{aligned} \tag{2}$$

[4]For clarity we choose all fields in the four-point function to have equal dimensions. The generalization of our results in the case of different dimensions is straightforward.

[5]Conformal invariance determines the form of three-point functions. Then, using the D'EPP formula [4] for amputation one can obtain the full three-point vertex function which is used in the skeleton graphs.

where the coefficients A_{nm}, B_{nm} and C_{nm} have been given in a number of works [2,3]. This expression is compatible with a conformal OPE when the latter is inserted in the *direct channel* $x_1 \to x_3$, $x_2 \to x_4$ or equivalently $u \to 0$, $v \to 1$. The first two terms on the r.h.s. of (2) together with the contributions from the zeroth order terms (disconnected graphs) [6]

$$\mathcal{G}^{(0)}(x_1, x_2, x_3, x_4) = \frac{1}{(x_{13}^2 x_{24}^2)^{\tilde{\Delta}}} + \frac{1}{(x_{12}^2 x_{34}^2)^{\tilde{\Delta}}} \left[1 + v^{-\tilde{\Delta}}\right] , \tag{3}$$

represent the contributions of infinitely many conformal tensor fields with definite dimension and rank. The last two terms on the r.h.s. of (3) determine the canonical dimensions of these tensor fields, while the logarithmic terms on the r.h.s. of (2) determine their anomalous dimension. A general method to identify and study the contributions of conformal tensor fields in four-point functions is described in [6].

In order that a four-point function admits a conformal OPE it is necessary that it does not contain non-analytic terms e.g. the double summation in (2) is restricted over $n, m \in \mathbf{N}_0$. The analyticity of conformally invariant amplitudes is a non-trivial property of CFT_d and should be checked in all explicit calculations. In this letter we show that general AdS_{d+1} scalar exchange graphs are analytic in the crossed channel expansion and therefore admit a conformal OPE. More technical details of our calculation as well as a thorough OPE analysis of AdS_{d+1} scalar exchange graphs are contained in [6].

Our main motivation for studying the analyticity properties of AdS_{d+1} scalar exchange graphs comes from similar studies of conformal graphs such as the one depicted in Fig.1. Using the D'EPP formula and Symanzik's technique [7] the exchange graph in Fig.1 has been calculated in the direct channel as

$$\begin{aligned} B(x_1, x_3; x_2, x_4) &= \frac{\pi^d u^{-\tilde{\Delta}}}{(x_{12}^2 x_{34}^2)^{\tilde{\Delta}}} \sum_{n,m=0}^{\infty} \frac{u^n (1-v)^m}{n!m!} \left[u^{\frac{1}{2}\Delta} c_{nm}(d,\Delta) + u^{\frac{1}{2}d - \frac{1}{2}\Delta} c_{nm}(d, d-\Delta)\right], & (4) \\ c_{nm}(d,\Delta) &= \alpha(\Delta)\alpha^2(\tfrac{1}{2}d - \tfrac{1}{2}\Delta) \frac{(\frac{1}{2}\Delta)_n^2 (\frac{1}{2}\Delta)_{n+m}^2}{(\Delta)_{2n+m}(\Delta - \frac{1}{2}d + 1)_n} , & (5) \\ \alpha(b) &= \frac{\Gamma(\frac{1}{2}d - b)}{\Gamma(b)} , \quad (b)_n = \frac{\Gamma(b+n)}{\Gamma(b)} . & (6) \end{aligned}$$

This result is explicitly compatible with a conformally invariant OPE when the latter is inserted in the direct channel. Namely, the first term on the r.h.s. of (4) is the full contribution of the scalar field $\sigma(x)$ in the four-point function, while the second term on the r.h.s. represents the so-called *shadow symmetric* singularities of the first series. [7] From (4) we can obtain the results for both the *crossed channels*. For clarity we consider here one of two crossed channels, namely

[6] Here and in the following we set to 1 the normalization of two-point functions.

[7] The term *shadow symmetry* was introduced for the first time in [8]. It corresponds to an intertwiner [9] of the conformal group in $d > 2$ that maps the equivalent representations with dimensions η and $d - \eta$ onto each other. Shadow symmetric singularities may correspond to physical *shadow fields* if the dimensions of the latter satisfy the unitarity bound e.g. $d - \eta \geq d/2 - 1$. See [2,3] and also [11].

when we set $x_3 \leftrightarrow x_4$ in Fig.1. This is obtained from (4) by the interchange $u \leftrightarrow v$. However, since we are interested in comparing the result with the OPE inserted in the direct channel, the above crossing transformation requires an analytic continuation of the result in (4). Indeed, the hypergeometric m-sum on the r.h.s. of (4) gives, by virtue of the crossing transformation, a hypergeometric series in the variable $(1-u)$. In order to compare this with the OPE inserted in the direct channel we use a degenerate Kummer transformation [10] to obtain a series in the variable u. Nevertheless, the remaining series is now a series in the variable v and it is not suitable for a comparison with the direct channel OPE which requires $v \to 1$. Eventually, we need to transform this series into a power series in $(1-v)$ and this step requires an analytic continuation which may not always be possible. For the explicit case of (4) we obtain after some algebra,

$$B(x_1,x_4;x_2,x_3) = K\frac{1}{(x_{12}^2x_{34}^2)^{\tilde{\Delta}}}\sum_{n,m=0}^{\infty}\frac{u^m}{(m!)^2}\mathcal{D}_m(\frac{\partial}{\partial\xi})\Big[(1-v)^{s-2t}\mathcal{A}_m(\xi)+\mathcal{B}_m(\xi)\Big]_{\xi=0}, \quad (7)$$

$$\mathcal{A}_m(\xi) = v^{\frac{1}{2}\Delta}\Gamma(2t-s)\,{}_2F_1\Big(s-t,s-t;1+s-2t;1-v\Big) - v^{\frac{1}{2}d-\frac{1}{2}\Delta}\Gamma(2t-s)\,{}_2F_1\Big(1-t,1-t;1+s-2t;1-v\Big), \quad (8)$$

$$\mathcal{B}_m(\xi) = v^{\frac{1}{2}\Delta}\,\Gamma(s-2t)\,{}_2F_1\Big(t,t;2t-s+1;1-v\Big)\left[\frac{\Gamma^2(t)}{\Gamma^2(s-t)}-\frac{\Gamma^2(1+t-s)}{\Gamma^2(1-t)}\right], \quad (9)$$

where

$$s = \Delta-\frac{1}{2}d+1\,,\; t = \frac{1}{2}\Delta+m+\xi\,, \quad (10)$$

$$\mathcal{D}_m(\frac{\partial}{\partial\xi}) = -\ln u+2\psi(m+1)-\frac{\partial}{\partial\xi}\,, \quad (11)$$

and K is an overall constant whose explicit value is not important here [6]. Notice that the second term on the r.h.s. of (7) is analytic, while the first is not. Nevertheless, the non-analytic term is exactly zero as can be easily seen using Euler's identity [10] for either of the two hypergeometric functions in (8). The remaining analytic term can then be easily computed from (7) and (9) and the result was given in [2, 3].

In the above calculation we observe that the existence of the shadow singularities in the direct channel (4) was essential for the cancellation of the non-analytic terms in the crossed channel. Let us turn now to the calculation of the standard AdS_{d+1} scalar exchange graph [12, 13] depicted in Fig.2. This, together with the two crossed symmetric graphs and the three disconnected terms (3) contribute to the four-point function of $\mathcal{O}(x)$ in the boundary CFT_d in the context of AdS/CFT correspondence [14]. Our starting point is the following Mellin-Barnes representation for the graph in Fig.2 which is obtained using either Symanzik's method [6] or by other techniques [13]

$$C(x_1,x_3;x_2,x_4) = \frac{k}{(x_{12}^2x_{34}^2)^{\tilde{\Delta}}}\int_{\mathcal{C}}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\Gamma^2(-s)\left[\frac{\Gamma^4(\tilde{\Delta}+s)\Gamma^2(\frac{1}{2}\Delta+\tilde{\Delta}-\frac{1}{2}d)\Gamma(\frac{1}{2}\Delta-\tilde{\Delta}-s)}{\Gamma(2\tilde{\Delta}+2s)\Gamma(\Delta-\frac{1}{2}d+1)\Gamma(\frac{1}{2}\Delta+\tilde{\Delta}-\frac{1}{2}d-s)}\right.$$

$$C(x_1, x_3; x_2, x_4) =$$

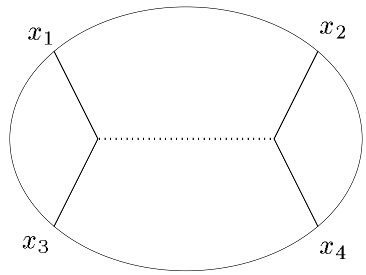

Figure 2: *The AdS_{d+1} scalar exchange graph. The solid lines represent the "bulk-to-boundary" propagator of $\mathcal{O}(x)$ and the dotted line represents the standard "bulk-to-bulk" propagator of $\sigma(x)$* [12].

$$\times {}_3F_2\left(\tfrac{1}{2}\Delta + \tilde{\Delta} - \tfrac{1}{2}d, \tfrac{1}{2}\Delta + \tilde{\Delta} - \tfrac{1}{2}d, \tfrac{1}{2}\Delta - \tilde{\Delta} - s; \Delta - \tfrac{1}{2}d + 1, \tilde{\Delta} + \tfrac{1}{2}\Delta - \tfrac{1}{2}d - s; 1\right)$$
$$\times u^s\, {}_2F_1\left(\tilde{\Delta} + s, \tilde{\Delta} + s; 2\tilde{\Delta} + 2s; 1 - v\right)\Bigg], \qquad (12)$$

where k is an unimportant for what follows constant [6]. The contour $\mathcal{C}$ is chosen such that it separates the "right" from the "left" poles of the integrand [13, 15]. The existence of such a contour is necessary for the convergence of (12) which can then be shown to give a well-defined expansion for the scalar exchange graph of Fig.2 in the direct channel. This expansion can be matched with a conformally invariant OPE [17, 6]. However, one finds that in the explicit result for (12) terms similar to the second term on the r.h.s. of (4) are missing [12, 13, 17, 6]. In view of the role played by such terms (shadow singularities), for the analyticity of the standard CFT_d exchange graphs it is important to check the analyticity of the AdS_{d+1} graph in Fig.2 in the crossed channel. We choose for clarity to study the crossed channel which is obtained from (12) by the interchange $u \leftrightarrow v$. In this case, the argument of the ${}_2F_1$ hypergeometric function becomes $(1-u)$ and we again need to use degenerate Kummer's relations to analytically continue it into a series in the variable u. Furthermore, the generalized hypergeometric ${}_3F_2$ function in (12) is *Saalschutzian* [16] and we can use the non-terminating form of Saalschutz's theorem (Eq. 4.3.4.2 of [16]) to write it as a sum of a ratio of Γ-functions and another ${}_3F_2$ function. Then, after some algebra which involves calculating the Mellin-Barnes integral we obtain

$$C(x_1, x_4; x_2, x_3) = \frac{k_1}{(x_{12}^2 x_{34}^2)^{\tilde{\Delta}}}\left[\sum_{m=0}^{\infty} \frac{u^m}{(m!)^2}\mathcal{D}_m\left(\frac{\partial}{\partial \xi}\right)\left(v^{\frac{1}{2}\Delta - \tilde{\Delta}} f_1(v,\xi) - f_2(v,\xi) - f_3(v,\xi)\right)\right]_{\xi=0} \qquad (13)$$

$$f_1(v,\xi) = \frac{\Gamma^2(\frac{1}{2}\Delta + m + \xi)}{\Gamma(\Delta - \frac{1}{2}d + 1)}\, {}_2F_1\left(\tfrac{1}{2}\Delta + m + \xi, \tfrac{1}{2}\Delta + m + \xi; \Delta - \tfrac{1}{2}d + 1; v\right), \qquad (14)$$

$$f_2(v,\xi) = \frac{\Gamma^2(\tilde{\Delta} + m + \xi)}{\Gamma(\frac{1}{2}\Delta + \tilde{\Delta} - \frac{1}{2}d + 1)\Gamma(\tilde{\Delta} - \frac{1}{2}\Delta + 1)}$$
$$\times {}_3F_2\left(\tilde{\Delta} + m + \xi, \tilde{\Delta} + m + \xi, 1; \tfrac{1}{2}\Delta + \tilde{\Delta} - \tfrac{1}{2}d + 1, \tilde{\Delta} - \tfrac{1}{2}\Delta + 1; v\right), \qquad (15)$$

$$f_3(v,\xi) \;=\; \frac{\Gamma(2\tilde{\Delta}-\frac{1}{2}d)}{\Gamma^2(\tilde{\Delta}-\frac{1}{2}\Delta)\Gamma^2(\tilde{\Delta}-\frac{1}{2}\Delta-\frac{1}{2}d)}\int_{\mathcal{C}}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\frac{\Gamma^2(-s)\Gamma^2(\tilde{\Delta}+m+s+\xi)}{(s+1)(\tilde{\Delta}+\frac{1}{2}\Delta-\frac{1}{2}d)}v^s$$
$$\times {}_3F_2\left(\frac{1}{2}\Delta-\tilde{\Delta}+1,\frac{1}{2}\Delta+\tilde{\Delta}-\frac{1}{2}d+1+s,1;s+2,\frac{1}{2}\Delta+\tilde{\Delta}-\frac{1}{2}d+1;1\right), \quad (16)$$

where again the constant k_1 is unimportant here. First we show that $f_3(v,\xi)$ does not contain non-analytic terms as $v\to 1$. The argument, whose details can be found in [6], is as follows. The $f_3(v,\xi)$ term has the form

$$f_3(v,\xi)=\int_{\mathcal{C}}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\,\Gamma^2(-s)\,g(s,\xi)\,v^s\,, \quad (17)$$

for some function $g(s,\xi)$. The analyticity of $f_3(v,\xi)$ at $v=0$ allows us to write the result of the Mellin-Barnes integration in (17) as

$$f_3(v,\xi)=\frac{\partial}{\partial\epsilon}\left[\sum_{n=0}^{\infty}\frac{v^{n-\epsilon}g(n-\epsilon,\xi)}{\Gamma^2(n+1-\epsilon)}\right]_{\epsilon=0}. \quad (18)$$

One way to obtain (18) is to shift by an infinitesimal parameter ϵ one of the two Γ-functions in (17) in order to regularize the double poles. The possible non-analytic terms as $v\to 1$ in (18) are determined by the large-n asymptotics of the ratio $g(n-\epsilon,\xi)/\Gamma^2(n+1-\epsilon)$. This in turn can be found using, for example, the Stirling formula (24) below and an asymptotic expansion for ${}_3F_2$ obtained from Eq. 4.3.4.2 of [16] by setting (in Slater's notation) $c=\frac{1}{2}\Delta+\tilde{\Delta}-\frac{1}{2}d+1+s$, $e=s+2$. In this way we obtain

$$\left.\frac{g(n-\epsilon,\xi)}{\Gamma^2(n+1-\epsilon)}\right|_{n\to\infty}\approx\sum_{i=1}^{2}\sum_{\lambda}^{\infty}\tilde{\sigma}_{\lambda,i}\frac{\Gamma(A_i+n+1-\epsilon-\lambda)}{\Gamma(n+1-\epsilon)}\,, \quad (19)$$

for some parameters A_i, which depend among others on m and ξ. The coefficients $\tilde{\sigma}_{\lambda,i}$ can in principle be explicitly determined in terms of the Bernoulli numbers (see below), but in this case we only require their existence. Then, by virtue of (19) we obtain the non-analytic terms

$$\left.\sum_{n=0}^{\infty}\frac{v^{n-\epsilon}g(n-\epsilon,\xi)}{\Gamma^2(n+1-\epsilon)}\right|_{n.a.} \approx \left.\sum_{i=1}^{2}\sum_{\lambda}^{\infty}\tilde{\sigma}_{\lambda,i}\,v^{-\epsilon}\frac{\Gamma(A+1-\epsilon-\lambda)}{\Gamma(1-\epsilon)}{}_2F_1(A_i+1-\epsilon-\lambda,1;1-\epsilon;v)\right|_{n.a.}$$
$$\approx \sum_{i=0}^{2}\sum_{\lambda}^{\infty}\tilde{\sigma}_{\lambda,i}\,\Gamma(A_i+1-\lambda)(1-v)^{-A_i-1+\lambda}\,, \quad (20)$$

where to get the second line of (20) we used a Kummer transformation [10]. The crucial point is now that the non-analytic terms are independent of the parameter ϵ and therefore drop out when we substitute (20) into (18).

Next, by yet another Kummer transformation it is easy to see that the $f_1(v,\xi)$ term gives the following non-analytic contribution to (13)

$$v^{\frac{1}{2}\Delta-\tilde{\Delta}}(1-v)^{1-\frac{1}{2}d-2m-2\xi}\Gamma(\frac{1}{2}d-1+2m+2\xi)\sum_{n=0}^{\infty}\frac{(1-v)^n}{n!}\frac{(\frac{1}{2}\Delta-\frac{1}{2}d+1-m-\xi)_n^2}{(2-\frac{1}{2}d-2m-2\xi)_n}\,, \quad (21)$$

which is similar to the one found in (7). It therefore remains to see whether the $f_2(v,\xi)$ term involves a non-analytic part which cancels the above contribution. To this end, we would like to exploit some kind of "Kummer-like" transformations for the generalized hypergeometric function ${}_3F_2$ in (15). However, it seems that such transformations do not exist in general. For this reason we use again an asymptotic expansion argument as follows. The relevant term in (15) reads

$$\sum_{n=0}^{\infty} \frac{v^n}{n!} \frac{\Gamma^2(\tilde{\Delta}+m+\xi+n)\Gamma(n+1)}{\Gamma(\frac{1}{2}\Delta+\tilde{\Delta}-\frac{1}{2}d+1+n)\Gamma(\tilde{\Delta}-\frac{1}{2}\Delta+1+n)}\,, \tag{22}$$

and we have extracted an inessential common constant factor from (21) and (22). In order to study the existence of non-analytic terms as $v \to 1$ in (22) we can use the identity

$$\sum_{n=1}^{\infty} \frac{\Gamma(a+n+1)}{\Gamma(n+1)}\, v^n = \Gamma(a+1)\,(1-v)^{-a-1}\,. \tag{23}$$

To do this, we first need to transform (22) into the form (23). This can be accomplished with the help of the Stirling formula [10] which gives the asymptotics of the ratio

$$\left.\frac{\Gamma(a+r+1)}{\Gamma(r+1)}\right|_{r\to\infty} \approx \exp\left[a\ln(r+1)+\sum_{k=1}^{\infty}\frac{\mathcal{P}_{k+1}(a)}{(r+1)^k}\right]\,, \tag{24}$$

$$\mathcal{P}_{k+1}(a) = \frac{(-1)^{k+1}}{k}\sum_{l=0}^{a-1} l^k\,. \tag{25}$$

Using (24) we can then write

$$\left.\frac{\Gamma^2(\tilde{\Delta}+m+\xi+n)}{\Gamma(\frac{1}{2}\Delta+\tilde{\Delta}-\frac{1}{2}d+1+n)\Gamma(\tilde{\Delta}-\frac{1}{2}\Delta+1+n)}\right|_{n\to\infty} \approx \sum_{\lambda=0}^{\infty}\sigma_\lambda\frac{\Gamma(\beta-\lambda+n+1)}{\Gamma(n+1)}\,, \tag{26}$$

where the coefficients σ_λ are recursively obtained from

$$\sum_{\lambda=0}^{\infty}\frac{\sigma_\lambda}{(n+1)^\lambda}\exp\left[\sum_{k=1}^{\infty}\frac{\mathcal{P}_{k+1}(\beta-\lambda)}{(n+1)^k}\right] = \exp\left[\sum_{k=1}^{\infty}\frac{2\mathcal{P}_{k+1}(t_1)-\mathcal{P}_{k+1}(t_2)-\mathcal{P}_{k+1}(t_3)}{(n+1)^k}\right], \tag{27}$$

$$t_1 = \tilde{\Delta}+m+\xi-1\,,\ \ t_2 = \tilde{\Delta}-\frac{1}{2}\Delta\,,\ \ t_3 = \frac{1}{2}\Delta+\tilde{\Delta}-\frac{1}{2}d\,, \tag{28}$$

$$\beta = 2t_1-t_2-t_3 = \frac{1}{2}d-2+2m+2\xi\,, \tag{29}$$

by matching the powers of $1/(n+1)$ on both sides of (27). Then, using (26) and (27) we are able to extract the possible non-analytic behavior of (22), which is essentially dictated by its large-n behavior, to obtain

$$\left.\sum_{n=0}^{\infty}\frac{v^n}{n!}\frac{\Gamma^2(\tilde{\Delta}+m+\xi+n)\Gamma(n+1)}{\Gamma(\frac{1}{2}\Delta+\tilde{\Delta}-\frac{1}{2}d+1+n)\Gamma(\tilde{\Delta}-\frac{1}{2}\Delta+1+n)}\right|_{n.a.} \approx \sum_{\lambda}^{\infty}\sigma_\lambda\left[\sum_{n=0}^{\infty}\frac{\Gamma(\beta-\lambda+n+1)}{\Gamma(n+1)}v^n\right]$$

$$\approx \sum_{\lambda=0}^{\infty}\sigma_\lambda\,\Gamma(\frac{1}{2}d-1+2m+2\xi-\lambda)(1-v)^{1-\frac{1}{2}d-2m-2\xi+\lambda}\,. \tag{30}$$

Notice that our approach does not determine the analytic part of (22). Therefore, from (13), (21) and (30) the cancellation of non-analytic terms requires that the equal powers of $1-v$ coincide to all orders in both sides of the following identity

$$\sum_{\lambda=0}^{\infty}\sigma_\lambda\frac{(-1)^\lambda(1-v)^\lambda}{(2-\frac{1}{2}d-2m-2\xi)_\lambda}=\sum_{k,l=0}^{\infty}\frac{(1-v)^{k+l}}{k!l!}\frac{(\tilde{\Delta}-\frac{1}{2}\Delta)_l(\frac{1}{2}\Delta-\frac{1}{2}d+1-m-\xi)_k^2}{(2-\frac{1}{2}d-2m-2\xi)_k}\,. \qquad (31)$$

This highly non-trivial identity, which involves on the l.h.s. among others the Bernoulli numbers through the polynomials (25), is analytically proven in [6] and it corresponds to "Kummer-like" identities for the generalized hypergeometric ${}_3F_2$ function. Its first few lower orders can be also shown to be satisfied with the help of e.g. a simple MAPLE algorithm [6].

Concluding, we outlined our proof that the possible non-analytic terms in AdS$_{d+1}$ scalar exchange graphs drop out by virtue of some highly non-trivial identities. This is in contrast with the case of scalar exchange graphs in standard CFT where the cancellation of the non-analytic terms was due to the appearance of the shadow singularities in the direct channel. Our result lends support to the idea that the amplitudes obtained from AdS/CFT correspondence can be analyzed using conformal OPE techniques [17]. Such an analysis, which also includes more technical details of our proof here, is presented in [6].

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References

[1] E. S. Fradkin and M. Ya. Palchik, Phys. Rep. **44** (1978) 249.

[2] K. Lang and W. Rühl, Nucl. Phys. **B377** (1992) 371.

[3] A. C. Petkou, "Conserved currents, consistency relations and operator product expansions in the conformally invariant $O(N)$ vector model for $2<d<4$", Ann. Phys. **249** (1996) 180, hep-th/9410093; "Operator product expansions and consistency relations in a $O(N)$ invariant fermionic CFT for $2<d<4$", Phys. Lett. **B389** (1996) 18, hep-th/9602054.

[4] M. D'Eramo, G. Parisi and L. Peliti, Lett. Nuovo Cim. **2** (1971), 878.

[5] G. Mack and I. T. Todorov, Phys. Rev. **D8** (1973) 1764;
V. K. Dobrev, V. B. Petkova, S. G. Petrova and I. T. Todorov, Phys. Rev. **D13** (1976) 887.

[6] L. Hoffmann, A. C. Petkou and W. Rühl, "*Aspects of the conformal operator product expansion in AdS/CFT correspondence*", to appear.

[7] K. Symanzik, Lett. Nuovo Cim. **3** (1972), 734.

[8] S. Ferrara, A. F. Grillo and G. Parisi, Lett. Nuovo Cim. **5** (1972) 147; S. Ferrara, A. F. Grillo, R. Gatto and G. Parisi, Lett. Nuovo Cim. **4** (1972) 115; S. Ferrara and G. Parisi, Nucl. Phys. **B42** (1972) 281.

[9] K. Koller, Commun. Math. Phys. **40** (1975) 15.

[10] I. S. Gradsteyn and I. M. Ryshik, *Table of Integrals, Series and Products*, 5th Ed. Academic Press (1994).

[11] I. Klebanov and E. Witten, "AdS/CFT correspondence and symmetry breaking", Nucl. Phys. **B556** (1999) 89, hep-th/9905104.

[12] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, "Comments on 4-point functions in the CFT/AdS correspondence," Phys. Lett. **B452** (1999) 61, hep-th/9808006; E. D'Hoker and D. Z. Freedman, 'General scalar exchange in AdS_{d+1}', Nucl. Phys. **B550** (1999) 261, hep–th/9811257; E. D'Hoker, D.Z. Freedman, S.D. Mathur, A. Matusis, L. Rastelli, "Graviton and gauge boson propagators in AdS_{d+1}," Nucl. Phys. **B562** (1999) 330, hep-th/9902042; E. D'Hoker, D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, "Graviton exchange and complete 4-point functions in the AdS/CFT correspondence," Nucl. Phys. **B562** (1999) 353, hep-th/9903196; E. D'Hoker, D.Z. Freedman and L. Rastelli, "AdS/CFT 4-point functions: How to succeed at z-integrals without really trying," Nucl. Phys. **B562** (1999) 395, hep-th/9905049.

[13] H. Liu, 'Scattering in Anti–de Sitter space and operator product expansion', Phys. Rev. **D60** 106005 (1999), hep–th/9811152.

[14] J. Maldacena, "The large N limit of superconformal field theories and supergravity", Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "Gauge theory correlators from noncritical string theory", Phys. Lett. B428 (1998) 105, hep-th/9802109; E. Witten, "Anti-de Sitter space and holography", Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150; O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity", hep-th/9905111.

[15] E. C. Titchmarsh, *The theory of Functions*, 2nd Ed. Oxford University Press (1939). (Reprinted 1975).

[16] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press (1966).

[17] E. D'Hoker, S.D. Mathur, A. Matusis and L. Rastelli, "The operator product expansion of $\mathcal{N}$=4 SYM and the 4-point functions of supergravity", hep-th/9911222.