THE CONSTRUCTION OF TRIGONOMETRIC INVARIANTS FOR WEYL GROUPS AND THE DERIVATION OF CORRESPONDING EXACTLY SOLVABLE SUTHERLAND MODELS

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Abstract

Trigonometric invariants are defined for each Weyl group orbit on the root lattice. They are real and periodic on the coroot lattice. Their polynomial algebra is spanned by a basis which is calculated by means of an algorithm. The invariants of the basis can be used as coordinates in any cell of the coroot space and lead to an exactly solvable model of Sutherland type. We apply this construction to the F_4 case.

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1 Introduction

Integrable models of the Calogero-Moser class and their trigonometric and rational limit models are conventionally described by the simple Lie algebras, i.e. those contained in the classical sequences A_n, B_n, C_n, D_n or the exceptional set G_2, F_4, E_6, E_7, E_8. We have shown [1, 2] that the Weyl groups underlying these algebras are the essential ingredients in the construction of the trigonometric or rational models. Exact solvability is easily proven by expressing the Schrödinger equation in terms of the Weyl group invariants as coordinates. For the rational models the crystallographic property of the Weyl groups, which guarantees the existence of a root lattice, can be abandoned. For the proper Coxeter groups H_3 and H_4 and the infinite sequence of dihedral groups [2, 3] Calogero type models can be constructed as well.

For the classical Lie algebras and G_2 the Weyl group invariants are not derived but guessed by intuition [4]. But this method failed for the exceptional algebras F_4 and E_6, E_7, E_8. Therefore a concept supplying us with all trigonometric polynomial invariants and their algebraic basis is highly desirable. In the case of the rational polynomial invariants the existence of the algebraic basis is guaranteed by Chevalley's theorem [5]. The Jacobian for the transition from cartesian coordinates in root space to the basic invariants as new coordinates can be factorized ("factorization theorem"). Both the Chevalley theorem and the factorization theorem are valid also in the trigonometric case as we shall show here for F_4 (for the classical Lie algebras and G_2 they are known to be valid, too).

Our construction of invariants proceeds as follows. We decompose the root lattice Λ into orbits Ω (infinitely many, they can be ordered by the length of vectors which is constant over the orbit). For each orbit we define an invariant trigonometric polynomial

$$T_\Omega(x) = \sum_{\beta \in \Omega} \exp i(\beta, x) \tag{1.1}$$

These functions obey fusion rules

$$T_{\Omega_1}(x) T_{\Omega_2}(x) = \sum_{\Omega_3} C^{\Omega_3}_{\Omega_1 \Omega_2} T_{\Omega_3}(x) \tag{1.2}$$

with "fusion coefficients" $C^{\Omega_3}_{\Omega_1 \Omega_2}$ that are nonnegative integers. In (1.2) the null-orbit consisting only of the null-vector in Λ must be included

$$T_{\Omega_0}(x) = 1 \tag{1.3}$$

The system of equations (1.2) is of triangular shape and can be solved trivially for $T_{\Omega_{\max}}(x)$. Each pair Ω_1, Ω_2 defines in fact a unique $\Omega_{\max}(\Omega_1, \Omega_2)$ with

$$C^{\Omega_{\max}(\Omega_1, \Omega_2)}_{\Omega_1 \Omega_2} = 1 \tag{1.4}$$

By recursive substitutions we isolate then an algebraic basis

$$\{T_{\Omega_1}, T_{\Omega_2}, \ldots T_{\Omega_n}\} \qquad n = \text{rank (Weyl group)} \tag{1.5}$$

and for all other orbits we obtain explicitly

$$T_\Omega(x) = \text{pol}\{T_{\Omega_1}(x), \ldots T_{\Omega_n}(x), T_{\Omega_0}(x)\} \tag{1.6}$$

This is an explicit and constructive version of Chevalley's theorem. It is obtained only in a case-by-case study (e.g. for F_4). In Section 2 we do this in great detail for F_4.

In Section 3 we apply this technique to construct the F_4 Sutherland model, using the approach developed in [1, 2]. By the way some minor theorems are proven by explicit calculation (e.g. existence of the $r_i^{(a)}$-coefficients as polynomials in the Chevalley basis).

2 Trigonometric invariants of Weyl groups.

Weyl groups are generated by reflections along roots $\alpha \in \mathbb{R}_n$ [5]

$$x \in \mathbb{R}_n : s_\alpha x = x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha \tag{2.1}$$

Each root α is an integer linear combination of simple roots

$$\{\alpha_1, \alpha_2, ... \alpha_n\} \tag{2.2}$$

These simple roots span an integral lattice $\Lambda \in \mathbb{R}_n$

$$\beta \in \Lambda : \; \beta = \sum_{i=1}^{n} m_i \alpha_i, \quad m_i \in \mathbb{Z} \tag{2.3}$$

If the Weyl group W acts on a vector $\beta \in \Lambda$ it produces an orbit Ω

$$\Omega = \{w\beta, \; w \in W\} \tag{2.4}$$

How can such orbit be characterized?

Since W is a discrete subgroup of $O(n)$ acting on $\mathbb{R}_n$ we obtain

$$\begin{aligned} \|w\beta\|^2 &= (w\beta, w\beta) \\ &= \|\beta\|^2 \end{aligned} \tag{2.5}$$

so that an orbit appears as a discrete set on a sphere of radius $\|\beta\|$ and (see (2.3))

$$\|\beta\|^2 = \sum_{i,j} (\alpha_i, \alpha_j) m_i m_j \tag{2.6}$$

Here (α_i, α_j) is contained in the Cartan matrix as

$$A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{\|\alpha_i\|^2} \tag{2.7}$$

Only for simply laced Weyl groups the lengths of the simple roots are all equal.

For the non-simply laced Weyl group $W = F_4$ we use as basis in $\mathbb{R}_4$

$$\{e_i, \; i \in \{1,2,3,4\}, \quad (e_i, e_j) = \delta_{ij}\}$$

Then $W = F_4$ can be generated from the reflections along e_i and f_i (all i) where

$$\begin{aligned} f_1 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \\ f_2 &= \frac{1}{2}(e_1 + e_2 - e_3 - e_4) \\ f_3 &= \frac{1}{2}(e_1 - e_2 + e_3 - e_4) \\ f_4 &= \frac{1}{2}(e_1 - e_2 - e_3 + e_4) \end{aligned} \tag{2.8}$$

$$(f_i, f_j) = \delta_{ij} \tag{2.9}$$

and the permutation group S_4 of the basis $\{e_i\}$. The set of roots decomposes into two orbits

$$\Omega_1 = \{\pm e_i,\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\} \tag{2.10}$$

$$\Omega_2 = \{\pm e_i \pm e_j,\ i < j\} \tag{2.11}$$

which are characterized by a subscript denoting the length squared of the roots. In addition we need the null-orbit

$$\Omega_0 = \{\text{null-vector}\} \tag{2.12}$$

All other orbits can be characterized by an integral radius squared and a further "degeneracy" label. A list of orbits up to $\|\beta\|^2 = 24$ is given in Table 1.

Trigonometric invariants are defined for each orbit by

$$T_{n,a}(x) = \sum_{\beta \in \Omega_{n,a}} \exp i(\beta, x) \tag{2.13}$$

(i.e. $\|\beta\|^2 = n$), so that

$$T_{n,a}(0) = \#\Omega_{n,a} \tag{2.14}$$

Now we consider pairs of such trigonometric invariants and expand their product as

$$T_{n,a}(x)T_{m,b}(x) = \sum_{k,c} C^{(k,c)}_{(n,a)(m,b)} T_{k,c}(x) \tag{2.15}$$

Applying permutations of S_4, reflections along the coordinate axis e_i, $i \in \{1, 2, 3, 4\}$ and reflection along f_1 to any generating vector yields the whole orbit.

We expand $x \in \mathbb{R}_n$ in a co-root basis

$$x = \sum_i \xi_i \tilde{\alpha}_i \tag{2.16}$$

Then by inversion of (2.15) we obtain

$$(2\pi)^{-n} \int_{\text{cell}} d^n\xi T_{m,a}(x) T_{n,b}(x) T_{k,c}(x) = \#\Omega_{k,c} \cdot C^{(k,c)}_{(m,a)(n,b)} \tag{2.17}$$

Note that the r.h.s. is symmetric in the three orbits. We denote (2.15), (2.17) the "fusion rules" for trigonometric invariants. The "fusion coefficients" $C^{(k,c)}_{(m,a),(n,b)}$ are nonnegative integers.

We expect that a Chevalley theorem of the following type is valid:
Any polynomial in the $\{T_{n,a}(x)\}$ can be expressed as a polynomial in an algebraic basis of invariants including $T_{\Omega_0} = 1$. The number of nontrivial basis elements is rank W.

In the case of $W = F_4$ this algebraic basis is constructed by inversion of (2.15) and consists of

$$T_1, T_2, T_3, T_6{}^{*)} \tag{2.18}$$

This inversion is possible by the triangular shape of the fusion rules. Namely inserting (2.13) into (2.15) and using the triangular inequality we obtain

$$\begin{aligned} C^{(k,c)}_{(n,a)(m,b)} = 0 \qquad & \text{except possibly for} \\ & |\sqrt{n} - \sqrt{m}| \leq \sqrt{k} \leq \sqrt{n} + \sqrt{m} \end{aligned} \tag{2.19}$$

Thus there is a maximal k for each n, m

$$k_{\max} \leqq (\sqrt{n} + \sqrt{m})^2 \tag{2.20}$$

and we solve (2.15) for $T_{k_{\max},c}(x)$ where c is such that the fusion coefficient is one. The result for

$$T_{k,c}, \quad k \notin \{1,2,3,6\}, \quad k \leq 24 \tag{2.21}$$

expressed as a polynomial in T_1, T_2, T_3, T_6 is given in Table 2.

The task to introduce the trigonometric invariants as coordinates in each cell of the space $\mathbb{R}_n$ leads to the study of the Jacobian matrix

$$\left\{ \frac{\partial T_m}{\partial x_i} \right\}_{m \in \text{basic set of invariants}} \tag{2.22}$$

In the program of constructing exactly solvable models we have to compute the (inverse) Riemannian

$$\begin{aligned} g^{-1}_{mn} &= \sum_i \frac{\partial T_m}{\partial x_i} \frac{\partial T_n}{\partial x_i}(x) \\ &= -\sum_{\beta \in \Omega_m} \sum_{\beta \in \Omega_n} (\beta, \beta') \exp i(\beta + \beta', x) \\ &= -\frac{1}{2} \sum_{k,a} (k - m - n) C^{(k,a)}_{(m)(n)} T_{k,a}(x) \end{aligned} \tag{2.23}$$

which can obviously be expressed as a polynomial in the basic invariants. For F_4 the largest orbit appearing is Ω_{24} in g^{-1}_{66}.

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*) We could replace T_6 by one of the following invariants: $\{T_7, T_8, T_{9,1}, T_{9,2}, T_{10}\}$ or constant linear combinations thereof.

3 The F_4 Sutherland model

From (2.23) a first version of the Riemannian is obtained by insertion of the fusion coefficients

$$
\begin{aligned}
g^{-1}_{11} &= -T_4 - T_3 + 4T_1 + 24 \\
g^{-1}_{12} &= -T_5 + 6T_1 \\
g^{-1}_{13} &= -\frac{3}{2}T_7 - 3T_6 - 2T_5 + 3T_3 + 12T_2 + 12T_1 \\
g^{-1}_{16} &= -2T_{11} - 2T_{9,1} + 4T_5 + 6T_3 \\
g^{-1}_{22} &= -2T_8 - 2T_6 + 8T_2 + 48 \\
g^{-1}_{23} &= -2T_{9,1} - 3T_7 + 6T_3 + 24T_1 \\
g^{-1}_{26} &= -3T_{14} - 6T_{12} - 4T_{10} + 6T_6 + 24T_4 + 24T_2 \\
g^{-1}_{33} &= -3T_{12} - 5T_{11} - 8T_{10} - 12T_{9,2} - 6T_{9,1} - 12T_8 - 3T_7 + 4T_5 \\
&\quad +12T_4 + 21T_3 + 48T_2 + 60T_1 + 288 \\
g^{-1}_{36} &= -4T_{17,1} - 6T_{15} - 6T_{13,2} - 8T_{13,1} - 4T_{11} \\
&\quad +6T_7 + 16T_5 + 36T_3 + 48T_1 \\
g^{-1}_{66} &= -6T_{24} - 10T_{22} - 16T_{20} - 12T_{18,2} - 24T_{18,1} \\
&\quad -24T_{16} - 6T_{14} + 8T_{10} + 24T_8 + 42T_6 + 96T_4 + 120T_2 + 576
\end{aligned}
\tag{3.1}
$$

A second form is obtained by substitution of the algebraic basis T_1, T_2, T_3, T_6 by means of Table 2.

$$
\begin{aligned}
g^{-1}_{11} &= -T_1^2 + T_3 + 6T_2 + 12T_1 + 48 \\
g^{-1}_{12} &= -T_1T_2 + 3T_3 + 12T_1 \\
g^{-1}_{13} &= -\frac{3}{2}T_1T_3 + 4T_1T_2 + 12T_1^2 + \frac{3}{2}T_6 - 24T_3 - 42T_2 - 96T_1 - 288 \\
g^{-1}_{16} &= -2T_1T_6 + 2T_2T_3 + 4T_1T_2 - 12T_3 - 48T_1 \\
g^{-1}_{22} &= -2T_2^2 + 12T_1^2 + 2T_6 - 24T_3 - 48T_2 - 96T_1 - 192 \\
g^{-1}_{23} &= -2T_2T_3 + 3T_1T_3 - 4T_1T_2 - 24T_1^2 - 9T_6 + 60T_3 \\
&\quad +108T_2 + 240T_1 + 576
\end{aligned}
$$

$$
\begin{aligned}
g^{-1}_{26} &= -4T_1^2T_2 - 24T_1^3 - 3T_2T_6 + 3T_3^2 + 8T_3T_2 - 6T_6T_1 \\
&\quad +72T_3T_1 + 104T_2T_1 + 12T_2^2 + 96T_1^2 - 48T_6 + 288T_3 \\
&\quad +480T_2 + 1536T_1 + 2304 \\
g^{-1}_{33} &= 4T_2T_1^2 + 12T_1^3 - 4T_2T_3 - 3T_3^2 + T_6T_1 - 36T_3T_1 \\
&\quad -60T_1T_2 - 96T_1^2 + 12T_6 - 48T_3 - 48T_2 - 384T_1 \\
g^{-1}_{36} &= 2T_1T_2T_3 - 16T_1^2T_2 - 4T_3T_6 - 6T_2T_6 - 8T_1T_6 + 72T_2^2 \\
&\quad +32T_2T_3 - 12T_1T_3 + 144T_1T_2 + 96T_1^2 + 36T_6 \\
&\quad -240T_3 - 48T_2 - 960T_1 - 2304 \\
g^{-1}_{66} &= -16T_1^3T_2 - 4T_1T_2T_6 - 8T_1^2T_6 + 2T_2T_3^2 \\
&\quad +48T_1T_2T_3 - 96T_1^2T_2 + 48T_1T_2^2 - 192T_1^3 \\
&\quad -6T_6^2 + 16T_3T_6 - 24T_2T_6 + 16T_1T_6 + 24T_3^2 \\
&\quad +512T_2T_3 + 576T_1T_3 + 864T_2^2 + 2880T_1T_2 + 1344T_1^2 \\
&\quad +96T_6 + 1152T_3 + 6144T_2 + 7680T_1 + 9216
\end{aligned}
\tag{3.2}
$$

Since F_4 has two orbits in the roots we have according to the factorization theorem

$$\det g^{-1} = \frac{1}{4} P_1 \cdot P_2 \tag{3.3}$$

where P_i corresponds to Ω_i (2.10, 2.11). We find explicitely

$$\begin{aligned}
P_1 &= 110592T_1 + 41472T_2 + 27648T_3 + 110592 - 3456T_6 - 1728T_1T_6 \\
&+192T_1T_3^2 - 432T_3T_6 - 112T_1^3T_3 - 384T_1^2T_3 \\
&+5184T_2T_3 + 20736T_1T_2 - 1728T_1^2T_2 + 144T_1^2T_6 \\
&-48T_1^3T_2 - 648T_2T_6 + 14976T_1T_3 + 4T_6T_1^3 \\
&-T_3^2T_1^2 + 216T_1T_2T_3 - 18T_6T_1T_3 - 4608T_1^3 \\
&+1728T_3^2 + 18432T_1^2 + 3888T_2^2 + 27T_6^2 + 16T_1^5 \\
&+4T_3^3 \qquad (3.4) \\
P_2 &= 10616832T_1 + 4423680T_2 + 1769472T_3 + 7077888 - 221184T_6 \\
&-221184T_1T_6 + 103680T_1T_3^2 - 27648T_3T_6 \\
&+34560T_1^3T_3 + 663552T_1^2T_3 + 774144T_2T_3 \\
&+4866048T_1T_2 + 1465344T_1^2T_2 - 62208T_1^2T_6 \\
&-6912T_1^3T_2 - 78336T_2T_6 + 1990656T_1T_3 \\
&-1728T_6T_1^3 + 18144T_3^2T_1^2 - T_2^2T_6^2 \\
&+566784T_1T_2T_3 - 17280T_6T_1T_3 - 8T_6T_1T_2^3 \\
&+79488T_1^2T_3T_2 - 48384T_6T_1T_2 - 2592T_6T_1T_2^2 \\
&+43200T_1T_2^2T_3 + 36T_6^2T_1T_2 - 2592T_3T_1^2T_6 \\
&-8640T_3T_1^3T_2 - 4608T_2T_3T_6 - 5184T_2T_1^2T_6 \\
&+576T_6T_1^3T_2 - 18T_6T_2T_3^2 + 13392T_1T_2T_3^2 \\
&-144T_2^2T_3T_6 - 1728T_2^2T_3T_1^2 + 96T_2^3T_3T_1 \\
&-216T_3^2T_1^2T_2 - 108T_1T_6T_3^2 + 72T_1^2T_6T_2^2 \\
&+774144T_1^3 + 138240T_3^2 + 5308416T_1^2 + 1096704T_2^2 \\
&+1728T_6^2 - 20736T_1^5 + 3456T_3^3 - 103680T_1^4 + 129024T_2^3 \\
&+6384T_2^4 + 1728T_1^6 + 27T_3^4 + 64T_2^5 + 4T_6^3 \\
&+119808T_3T_2^2 + 787968T_1T_2^2 + 36288T_2T_3^2 \\
&+88128T_1T_2 + 108T_1T_6 + 45888T_1T_2 \\
&-18432T_1^3T_2^2 - 9024T_2^2T_6 + 2520T_2^2T_3^2 \\
&+192T_2T_6^2 + 864T_1T_6^2 - 432T_6T_3^2 - 328T_2^3T_6 \\
&-10368T_1^4T_3 - 32T_1^3T_2^3 + 6592T_2^3T_3 - 2976T_1^2T_2^3 \\
&+432T_1^4T_2^2 + 1728T_1^5T_2 - 432T_3^2T_1^3 + 1296T_3^3T_1 \\
&+864T_1^4T_6 + 432T_2T_3^3 - 34560T_1^4T_2 + 32T_2^4T_3 \\
&-16T_1^2T_2^4 + 224T_1T_2^4 + 4T_2^3T_3^2 \\
&-1296T_6T_1T_2T_3 \qquad (3.5)
\end{aligned}$$

If we reexpress these polynomials in the Chevalley basis T_1, T_2, T_3, T_6 as functions of Cartesian coordinates we find

$$P_1 = -2^{24} \prod_{\alpha \in \Omega_1^+} [\sin \frac{1}{2}(\alpha, x)]^2 \tag{3.6}$$

$$P_2 = -2^{24} \prod_{\alpha \in \Omega_2^+} [\sin \frac{1}{2}(\alpha, x)]^2 \tag{3.7}$$

Here we made use of the fact that each orbit $\Omega_{k,a}$ can be decomposed into a positive and a negative semiorbit

$$\Omega_{k,a} = \Omega_{k,a}^+ \cup \Omega_{k,a}^- \tag{3.8}$$

by a hypersurface

$$(\alpha, \xi) = 0 \tag{3.9}$$

so that

$$\Omega_{k,a}^\pm = \{\alpha \in \Omega_{k,a}; (\alpha, \xi) \gtrless 0\} \tag{3.10}$$

For example, we define

$$\Omega_{k,a}^+ = \{\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4 \in \Omega_{k,a}; \tag{3.11}$$
$$\mu_1 > 0 \text{ or } \mu_1 = 0, \mu_2 > 0 \tag{3.12}$$
$$\text{or } \mu_1 = \mu_2 = 0, \mu_3 > 0 \tag{3.13}$$
$$\text{or } \mu_1 = \mu_2 = \mu_3 = 0, \mu_4 > 0\} \tag{3.14}$$

Then the coefficients ξ_n of ξ

$$\xi = \sum_n \xi_n e_n \tag{3.15}$$

must satisfy inequalities, i.e. from Ω_1 and Ω_2

$$\xi_1 > \xi_2 > \xi_3 > \xi_4 > 0 \tag{3.16}$$
$$\xi_1 > \xi_2 + \xi_3 + \xi_4 \tag{3.17}$$

Ω_3 implies new inequalities, etc., so that a vector ξ exists for any finite set of orbits. We assume (3.16),(3.17) to hold throughout.

We consider the asymptotic behaviour along an imaginary direction in x–space

$$x = -i\lambda\xi, \quad \lambda \to \infty \tag{3.18}$$

In each orbit there exist maximal vectors $\alpha_{\max}(k, a)$ so that for fixed ξ

$$(\alpha_{\max}(k, a), \xi) = \max_{\alpha \in \Omega_{k,a}} (\alpha, \xi) \tag{3.19}$$

Then (for unique $\alpha_{\max}(k, a)$)

$$T_{k,a} \sim e^{\lambda(\alpha_{\max}(k,a),\xi)} \tag{3.20}$$
$$\text{for } \lambda \to \infty \tag{3.21}$$

In particular, we have

$$\alpha_{\max}(1) = e_1 \tag{3.22}$$
$$\alpha_{\max}(2) = e_1 + e_2 \tag{3.23}$$
$$\alpha_{\max}(3) = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4) \tag{3.24}$$
$$\alpha_{\max}(4) = 2e_1 + e_2 + e_3 \tag{3.25}$$

From (3.18),(3.19) we find as leading term in $P_1 (P_2)$ the unique monomial $-T_1^2 T_3^2 (-T_2^2 T_6^2)$ with asymptotic behaviour

$$P_{1,2} \sim -e^{\lambda(\rho_{1,2},\xi)} \tag{3.26}$$

where

$$\begin{aligned} \rho_1 &= 5e_1 + e_2 + e_3 + e_4 \\ \rho_2 &= 6e_1 + 4e_2 + 2e_3 \end{aligned} \tag{3.27}$$

These are special cases of Weyl type vectors

$$\rho_{k,a} = \sum_{\alpha \in \Omega^+_{k,a}} \alpha \tag{3.28}$$

The same leading term (3.26) results from (3.6), (3.7). In this fashion the constants in the factorization formulas (3.6), (3.7) can be controlled.

From (2.23) follows that

$$\det\, g^{-1} = \frac{1}{4} P_1 P_2 = (\det\, J)^2 \tag{3.29}$$

where J is the Jacobian matrix (2.22). Due to (3.6),(3.7) $\det J$ vanishes in first order along the hyperplanes

$$(\alpha, x) = 2n\pi, \quad n \in \mathbb{Z}, \alpha \in \Omega_{1,2} \tag{3.30}$$

The root space $\mathbb{R}_4$ is therefore divided into cells bounded by the walls (3.30) where (see (3.42)) repulsive and impenetrable potentials are positioned.

Next we derive the r–coefficients from

$$r_m^{(a)} = \sum_n g^{-1}_{mn} \frac{\partial \log P_a}{\partial T_n} \tag{3.31}$$

that ought to be polynomials in the $\{T_n\}_{n\in\{1,2,3,6\}}$. In fact, we find

$$r^{(1)} = (-5T_1 - 24, -6T_2 - 6T_1, -9T_3 - 12T_2 - 24T_1, -4T_1T_2 - 12T_6 - 24T_2 + 24T_1) \tag{3.32}$$

$$r^{(2)} = (-6T_1, -10T_2 - 48, -12T_3 - 24T_1, -24T_1^2 - 18T_6 + 48T_3 + 96T_2 + 192T_1 + 576) \tag{3.33}$$

From (3.2) and (3.32),(3.33) we obtain the algebraic differential operator

$$D = -\sum_{m,n} \frac{\partial}{\partial T_m} g^{-1}_{mn} \frac{\partial}{\partial T_n} + \sum_{a=1,2} \gamma_a \sum_m r_m^{(a)} \frac{\partial}{\partial T_m} \tag{3.34}$$

containing two real coupling constants γ_a as free parameters. This operator is exactly solvable in terms of polynomial eigenfunctions

$$Dp_\lambda(T_1, T_2, T_3, T_6) = \lambda p_\lambda(T_1, T_2, T_3, T_6) \tag{3.35}$$

One can show that D possesses a flag of invariant polynomial spaces $\{V_N\}$

$$\begin{aligned} & DV_N \subset V_N && (3.36)\\ V_N \;=\; & \mathrm{span}\{T_1^{n_1} T_2^{n_2} T_3^{n_3} T_6^{n_6}, n_i \in \mathbb{Z}_\geq, && (3.37)\\ & n_1 + 2n_2 + 3n_3 + 4n_6 \leq N\} && (3.38) \end{aligned}$$

Thus the eigenfunctions $\{p_\lambda\}$ can be calculated by linear algebra.

On the other hand we know from the general scheme that D corresponds by a "gauge transformation" [1] to the Schrödinger operator in $\mathbb{R}_4$

$$\mathcal{H} = -\Delta + W \tag{3.39}$$

with a Laplacian Δ and a potential W

$$W = \frac{1}{4} \sum_{a,b} (\gamma_a \gamma_b - \frac{1}{4}) R_{ab} \tag{3.40}$$

$$\begin{aligned} R_{ab} &= \sum_{m,n} g^{-1}_{mn} \frac{\partial \log P_a}{\partial T_m} \frac{\partial \log P_b}{\partial T_n} \\ &= \sum_n r_n^{(a)} \frac{\partial}{\partial T_n} \log P_b \end{aligned} \tag{3.41}$$

In Cartesian coordinates we can for F_4 express

$$R_{ab} = \rho_{ab} \sum_{\alpha \in \Omega_a^+} [\sin \frac{1}{2}(\alpha, x)]^{-2} + C_{ab} \tag{3.42}$$

where

$$\begin{array}{rr} \rho_{11} = +1, & C_{11} = -28 \\ \rho_{22} = +2, & C_{22} = -56 \\ \rho_{12} = \rho_{21} = 0, & C_{12} = C_{21} = -36 \end{array} \tag{3.43}$$

can be determined by an asymptotic analysis.

In this fashion we have derived the Sutherland model for F_4.

References

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A Tables

Table 1: Orbits, generating vectors and cardinals for F_4

Ω	$\sum n_i e_i$	$\#\Omega$
Ω_1	e_1	24
Ω_2	e_1+e_2	24
Ω_3	$e_1+e_2+e_3$	96
Ω_4	$2e_1, e_1+e_2+e_3+e_4$	24
Ω_5	$2e_1+e_2$	144
Ω_6	$2e_1+e_2+e_3$	96
Ω_7	$2e_1+e_2+e_3+e_4$	192
Ω_8	$2e_1+2e_2$	24
$\Omega_{9,1}$	$2e_1+2e_2+e_3$	288
$\Omega_{9,2}$	$3e_1$	24
Ω_{10}	$3e_1+e_2,\ 2e_1+2e_2+e_3+e_4$	144
Ω_{11}	$3e_1+e_2+e_3$	288
Ω_{12}	$2e_1+2e_2+2e_3,\ 3e_1+e_2+e_3+e_4$	96
$\Omega_{13,1}$	$3e_1+2e_2$	144
$\Omega_{13,2}$	$2e_1+2e_2+2e_3+e_4$	192
Ω_{14}	$3e_1+2e_2+e_3$	192
Ω_{15}	$3e_1+2e_2+e_3+e_4$	576
Ω_{16}	$2e_1+2e_2+2e_3+2e_4,\ 4e_1$	24
$\Omega_{17,1}$	$3e_1+2e_2+2e_3$	288
$\Omega_{17,2}$	$4e_1+e_2$	144
$\Omega_{18,1}$	$3e_1+3e_2$	24
$\Omega_{18,2}$	$4e_1+e_2+e_3,\ 3e_1+2e_2+2e_3+e_4$	288
$\Omega_{19,1}$	$3e_1+3e_2+e_3$	288
$\Omega_{19,2}$	$4e_1+e_2+e_3+e_4$	192
Ω_{20}	$4e_1+2e_2,\ 3e_1+3e_2+e_3+e_4$	144
$\Omega_{21,1}$	$4e_1+2e_2+e_3$	576
$\Omega_{21,2}$	$3e_1+2e_2+2e_3+2e_4$	192
Ω_{22}	$3e_1+3e_2+2e_3,\ 4e_1+2e_2+e_3+e_4$	288
Ω_{23}	$3e_1+3e_2+2e_3+e_4$	576
Ω_{24}	$4e_1+2e_2+2e_3$	96

Table 2: $T_{n,a}$ expressed as polynomial in T_1, T_2, T_3, T_6 for F_4

$$
\begin{aligned}
T_4 &= T_1^2 - 2T_3 - 6T_2 - 8T_1 - 24 \\
T_5 &= T_2T_1 - 3T_3 - 6T_1 \\
T_7 &= T_3T_1 - 4T_2T_1 - 8T_1^2 - 3T_6 + 22T_3 + 36T_2 + 80T_1 + 192 \\
T_8 &= T_2^2 - 6T_1^2 - 2T_6 + 12T_3 + 28T_2 + 48T_1 + 120 \\
T_{9,1} &= T_3T_2 - 3T_3T_1 + 8T_2T_1 + 24T_1^2 + 9T_6 - 60T_3 - 108T_2 - 228T_1 - 576 \\
T_{9,2} &= T_1^3 - 3T_3T_1 - 3T_2T_1 + 3T_6 - 21T_3 - 36T_2 - 99T_1 - 192 \\
T_{10} &= T_2T_1^2 - 2T_3T_2 - 8T_2T_1 - 6T_2^2 - 3T_6 - 30T_2 \\
T_{11} &= T_6T_1 - 2T_3T_2 + 3T_3T_1 - 8T_2T_1 - 24T_1^2 - 9T_6 + 63T_3 + 108T_2 + 240T_1 + 576 \\
T_{12} &= -4T_2T_1^2 - 8T_1^3 - 2T_6T_1 + 8T_3T_2 + 12T_2^2 + 24T_3T_1 + 56T_2T_1 + T_3^2 + 60T_1^2 \\
&\quad +40T_3 + 120T_2 + 288T_1 + 288 \\
T_{13,1} &= T_2^2T_1 - 6T_1^3 - 2T_6T_1 - T_3T_2 + 15T_3T_1 + 19T_2T_1 + 24T_1^2 - 9T_6 \\
&\quad +63T_3 + 108T_2 + 354T_1 + 576 \\
T_{13,2} &= T_3T_1^2 - 2T_3^2 - T_6T_1 - 6T_3T_2 - 6T_3T_1 - 8T_2T_1 - 16T_1^2 - 6T_6 \\
&\quad +20T_3 + 72T_2 + 152T_1 + 384 \\
T_{14} &= 8T_2T_1^2 + 24T_1^3 + T_6T_2 + 6T_6T_1 - 72T_3T_1 - 136T_2T_1 - 20T_2^2 - 16T_3T_2 - 3T_3^2 \\
&\quad -144T_1^2 + 22T_6 - 192T_3 - 400T_2 - 1152T_1 - 1536 \\
T_{15} &= T_3T_2T_1 - 3T_3T_1^2 - 4T_2^2T_1 - 8T_2T_1^2 - 3T_6T_2 + T_6T_1 + 40T_3T_2 + 21T_3T_1 \\
&\quad 96T_2T_1 + 6T_3^2 + 36T_2^2 + 24T_1^2 + 9T_6 - 6T_3 + 84T_2 - 240T_1 - 576 \\
T_{16} &= T_1^4 - 4T_3T_1^2 - 4T_2T_1^2 + 4T_6T_1 + 2T_3^2 + 8T_3T_2 + 6T_2^2 - 16T_3T_1 \\
&\quad -16T_2T_1 - 76T_1^2 + 12T_6 - 40T_3 - 72T_2 - 416T_1 - 552 \\
T_{17,1} &= -2T_3T_2T_1 + 3T_3T_1^2 + 4T_2^2T_1 + 16T_2T_1^2 + 12T_1^3 + T_6T_3 + 6T_6T_2 + 5T_6T_1 \\
&\quad -6T_3^2 - 55T_3T_2 - 72T_2^2 - 51T_3T_1 - 200T_2T_1 - 72T_1^2 + 9T_6 - 120T_3 \\
&\quad -492T_2 - 468T_1 - 576 \\
T_{17,2} &= -3T_3T_2T_1 + T_2T_1^3 - 3T_2^2T_1 + 3T_6T_2 - T_6T_1 - 36T_2^2 - 19T_3T_2 \\
&\quad -3T_3T_1 - 92T_2T_1 + 24T_1^2 + 9T_6 - 60T_3 - 300T_2 - 234T_1 - 576 \\
T_{18,1} &= T_2^3 - 24T_1^3 - 15T_2T_1^2 - 3T_6T_2 - 6T_6T_1 + 3T_3^2 + 30T_3T_2 + 72T_3T_1 + 54T_2^2 \\
&\quad +192T_2T_1 + 144T_1^2 - 21T_6 + 192T_3 + 549T_2 + 1152T_1 + 1536 \\
T_{18,2} &= T_6T_1^2 - 28T_2T_1^2 - 72T_1^3 - 2T_6T_3 - 9T_6T_2 - 26T_6T_1 + 56T_3T_2 + 216T_3T_1 + 84T_2^2 \\
&\quad +440T_2T_1 + 9T_3^2 + 432T_1^2 - 84T_6 + 576T_3 + 1308T_2 + 3456T_1 + 4608 \\
T_{19,1} &= T_3T_2^2 - T_3T_2T_1 + 8T_2T_1^2 + 4T_2^2T_1 - 3T_3T_1^2 - 2T_6T_3 + 3T_6T_2 - 3T_6T_1 + 6T_3^2 \\
&\quad -8T_3T_2 - 36T_2^2 + 18T_3T_1 - 68T_2T_1 + 48T_1^2 + 18T_6 - 69T_3 - 408T_2 \\
&\quad -480T_1 - 1152 \\
T_{19,2} &= -8T_1^4 - 4T_2T_1^3 + T_3^2T_1 + 9T_3T_2T_1 + 23T_3T_1^2 + 12T_2^2T_1 - 2T_6T_1^2 + 48T_2T_1^2 \\
&\quad +48T_1^3 - T_6T_3 - 3T_6T_2 - 7T_6T_1 + 2T_3^2 + 24T_3T_2 + 36T_2^2 + 72T_3T_1 + 244T_2T_1 \\
&\quad +384T_1^2 + 18T_3 + 192T_2 + 512T_1 \\
T_{20} &= -1716T_2 + 48T_6 - 2T_3T_2^2 - 8T_1T_2^2 - 2T_1^2T_6 - 2928T_1 - 636T_3 \\
&\quad +76T_1^2T_2 + T_1^2T_2^2 - 228T_2^2 - 152T_2T_3 + 22T_1T_6 - 264T_1T_3 - 306T_1^2 \\
&\quad -680T_1T_2 + 12T_2T_6 - 27T_3^2 + 24T_1^2T_3 + 120T_1^3 + 4T_3T_6 - 6T_2^3 - 3600 - 6T_1^4
\end{aligned}
$$

$$
\begin{aligned}
T_{22} &= -4608 - 1008T_2 + 111T_6 - 576T_3 - 3456T_1 + 34T_1T_6 + 4T_3T_6 + 8T_3T_2^2 - 16T_2T_3 \\
&\quad -216T_1T_3 - 152T_1T_2 + 88T_1^2T_2 + 56T_1T_2^2 - 2T_1^2T_6 - 8T_1^3T_2 + 15T_2T_6 + T_2T_3^2 \\
&\quad -4T_1^2T_2^2 - 2T_6T_1T_2 + 24T_1T_2T_3 - 432T_1^2 - 9T_3^2 + 36T_2^2 + 72T_1^3 + 12T_2^3 \\
T_{24} &= 4320 + 1104T_2 - 152T_6 + 912T_3 + 3648T_1 - 64T_1T_6 \\
&\quad -16T_3T_6 - 8T_3T_2^2 + 96T_2T_3 + 384T_1T_3 \\
&\quad +128T_1T_2 - 208T_1^2T_2 - 80T_1T_2^2 + 8T_1^2T_6 \\
&\quad +16T_1^3T_2 - 24T_2T_6 - 2T_2T_3^2 + 4T_1^2T_2^2 \\
&\quad -48T_1^2T_3 + 4T_6T_1T_2 - 48T_1T_2T_3 + 312T_1^2 \\
&\quad +48T_3^2 + 12T_2^2 - 192T_1^3 - 8T_2^3 + 12T_1^4 + T_6^2
\end{aligned}
$$