KL–TH 97/4

THE CONTINUOUS SERIES OF CRITICAL POINTS OF THE TWO-MATRIX MODEL AT N $\to \infty$ IN THE DOUBLE SCALING LIMIT

S. Balaska, J. Maeder and W. Rühl

Department of Physics, University of Kaiserslautern, P.O.Box 3049
67653 Kaiserslautern, Germany
E-mail: ruehl@physik.uni-kl.de

Abstract

The critical points of the continuous series are characterized by two complex numbers $l_{1,2}(Re(l_{1,2}) < 0)$, and a natural number $n(n \geq 3)$ which enters the string susceptibility constant through $\gamma = \frac{-2}{n-1}$. The critical potentials are analytic functions with a convergence radius depending on l_1 (or l_2). We use the orthogonal polynomial method and solve the Schwinger-Dyson equations with a technique borrowed from conformal field theory.

March 1997

1 Introduction

We study matrix models whose action depends on hermitean $N \times N$ matrices as dynamical variables. They are coupled to a chain of r vertices and $r-1$ connecting links

$$S(M^{(1)}, M^{(2)}, M^{(3)} \ldots M^{(r)}) =$$
$$= Tr\left\{\sum_{\alpha=1}^{r} V_\alpha(M^{(\alpha)}) - \sum_{\alpha=1}^{r-1} c_\alpha M^{(\alpha)} M^{(\alpha+1)}\right\} \tag{1.1}$$

Little is known about models with $r \geq 3$, but the two-matrix models ($r = 2$) seem to exhibit the full richness of critical structures. It turns out that they possess two series of critical points: the well-known "discrete series" for which the potentials V_α are polynomials, and a "continuous series" which we will describe in this work.

Statistical ensembles of matrices appeared first in connection with problems of nuclear physics [1]. As generalizations of vector sigma models they served as objects for the study of phase transitions and renormalization theory [2]. Recently they attracted interest as models for the coupling of conformal field matter with the gravitational field [3]. In this case they are analyzed in their critical domain defined by the "double scaling limit". We shall also apply this limiting procedure in this work.

All investigations of the matrix models in the double scaling limit are based on the orthogonal polynomial method which will be outlined at the end of this introduction. If the critical potentials are polynomials, the matrix models can be solved perturbatively in the double scaling domain. This method has been applied to study all types of critical behaviour of the polynomial two-matrix models in [3, 4]. The final result can be described as follows. Let the polynomial degrees of the potentials be l_1 and l_2, $l_1 \leq l_2$. If l_2 does not divide l_1, the universality class of the maximal critical point of this model is

$$[p, q] = [l_1, l_2] \tag{1.2}$$

where p and q denote the degrees of differential operators of the generalized KdV hierarchy. If, however, l_2 divides l_1, but differs from two, then

$$[p, q] = [l_1 + 1, l_2] \tag{1.3}$$

The string susceptibility exponent γ is

$$\gamma = \frac{-2}{p + q - 1} \tag{1.4}$$

The continuous series of critical points necessitates nonpolynomial critical potentials that are holomorphic inside a circle of finite radius of convergence.

Each depends on a parameter l_1 (respectively l_2). A third parameter, a natural number n, is connected with the perturbative order at which the equation

$$[B_2, B_1] = 1 \tag{1.5}$$

can be fulfilled and enters the string susceptibility exponent γ as

$$\gamma = \frac{-2}{n-1} \tag{1.6}$$

Whereas the discrete series is intimately connected with the theory of the Korteweg-deVries equations [5] and positive integer powers of quasi-differential operators, we will use complex powers of such operators (as described in [6]) only marginally. The differential equations arising at the end are trivial and have polynomial solutions.

The partition function Z for the two-matrix model with action S (1.1) is

$$Z = \int \prod_{\alpha=1,2} \prod_{i\leq j} d(ReM_{ij}^{(\alpha)}) \prod_{k<l} d(ImM_{kl}^{(\alpha)}) e^{-S} \tag{1.7}$$

The matrices $M^{(\alpha)}$ are diagonalized

$$M^{(\alpha)} = U^{(\alpha)}\Lambda^{(\alpha)}U^{(\alpha),-1} \tag{1.8}$$

with unitary $N \times N$ matrices $U^{(\alpha)}$. After the integration over these unitary matrices we have

$$Z = C(N) \int \prod_{\alpha=1,2} \prod_{i} d\lambda_i^{(\alpha)} \Delta(\lambda^{(1)})\Delta(\lambda^{(2)}) \cdot e^{S(\Lambda^{(1)},\Lambda^{(2)})} \tag{1.9}$$

$(\Lambda^{(\alpha)} = \mathrm{diag}\{\lambda_i^{(\alpha)}\})$
where $\Delta(\lambda)$ is the Vandermonde determinant

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) \tag{1.10}$$

The method of orthogonal polynomials [7, 8] applied to (1.9) uses a biorthogonal system

$$\{\Pi_m(\lambda),\ \tilde{\Pi}_m(\mu)\}_{m=0}^{\infty} \tag{1.11}$$

$$\deg \Pi_m = \deg \tilde{\Pi}_m = m \tag{1.12}$$

so that

$$\int_{\mathbb{R}_2} d\lambda d\mu \Pi_m(\lambda)\tilde{\Pi}_n(\nu) \times \exp\{-V_1(\lambda) - V_2(\mu) + c\lambda\mu\} = \delta_{nm} \tag{1.13}$$

Differentiation operators A_1, A_2 and multiplication operators B_1, B_2 are introduced by

$$\begin{aligned} \Pi'_m(\lambda) &= \sum_n (A_1)_{mn} \Pi_n(\lambda) \\ \tilde{\Pi}'_m(\mu) &= \sum_n (A_2)_{nm} \tilde{\Pi}_n(\mu) \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} \lambda \Pi_m(\lambda) &= \sum_n (B_1)_{mn} \Pi_n(\lambda) \\ \mu \tilde{\Pi}_m(\mu) &= \sum_n (B_2)_{nm} \tilde{\Pi}_n(\mu) \end{aligned} \tag{1.15}$$

so that

$$[B_1, A_1] = [A_2, B_2] = 1 \tag{1.16}$$

We normalize c in (1.13) to one and derive Schwinger-Dyson equations in the usual way

$$\begin{aligned} A_1 + B_2 &= V'_1(B_1) \\ A_2 + B_1 &= V'_2(B_2) \end{aligned} \tag{1.17}$$

Then (1.16) implies with (1.17)

$$[B_2, B_1] = 1 \tag{1.18}$$

From the definitions (1.14), (1.15) we deduce that

$$\begin{array}{lllll} (A_1)_{mn} = 0 & \text{except possibly for} & n-m & \leq & -1 \\ (A_2)_{mn} = 0 & \text{except possibly for} & n-m & \geq & +1 \\ (B_1)_{mn} = 0 & \text{except possibly for} & n-m & \leq & +1 \\ (B_1)_{mn} = 0 & \text{except possibly for} & n-m & \geq & -1 \end{array} \tag{1.19}$$

If (1.17) and (1.19) are fulfilled, then the commutation (1.18) is diagonal. We call this assertion "the lemma". Contrary to our procedure in [4] we exploit (1.17) only partially in this work and therefore we impose (1.18) as a strong additional constraint.

The Schwinger-Dyson equations (1.17) are evaluated in the "double scaling limit". All our notations are standard, in particular identical with those in [4].

The critical potentials are expanded as

$$\begin{aligned} V'_1(t)^c &= \sum_{k=0}^{\infty} f^c_k t^k \\ V'_2(t)^c &= \sum_{k=0}^{\infty} g^c_k t^k \end{aligned} \tag{1.20}$$

where $f_k^c(g_k^c)$ are certain analytic functions of l_1 and l_2. We do not tune the coupling constants to these critical values but multiply the whole critical action by a new parameter

$$\frac{N}{g} \tag{1.21}$$

Then we tune

$$N \to \infty, \quad g \to g_c \tag{1.22}$$

The matrix labels n, m become continuous in this limit

$$\frac{n}{N} = \xi, \quad 0 \le \xi \le 1 \tag{1.23}$$

We replace the label N by the string coupling constant a

$$\frac{1}{N} = a^{2-\gamma}, \quad (\gamma < 0) \tag{1.24}$$

so that $a \to 0$ if $N \to \infty$. Moreover

$$\xi = \frac{g_c}{g}(1 - a^2 x) \tag{1.25}$$

We introduce a variable

$$z = e^{i\varphi} \tag{1.26}$$

dual (in the Fourier series sense) to the discrete matrix label n. Then we scale φ as

$$\varphi = a^{-\gamma} p \tag{1.27}$$

so that due to (1.24), (1.25)

$$p = i\frac{d}{dx} \tag{1.28}$$

is the quantum mechanical momentum operator corresponding to x.

New in this work is that the multiplication operator $B_1(B_2)$ decomposes into "blocks" corresponding to a nondegenerate $\mathbb{N}^2$-lattice [1]

$$B_1 = \sum_{[n_1,n_2] \in \mathbb{N}^2} B_1^{[n_1,n_2]} \tag{1.29}$$

where each block possesses a double scaling expansion

$$B_1^{[n_1,n_2]} \cong \sum_{n=0}^{\infty} a^{-\gamma[n+l_2-(n_1 l_1+n_2 l_2)]} \cdot Q_n^{[n_1,n_2]}(x;p) \tag{1.30}$$

Obviously it is necessary that

$$Re\, l_{1,2} < 0 \tag{1.31}$$

[1] In this paper $\mathbb{N}$ includes zero.

in order to render the expansion (1.29) perturbative. Analogously we expand B_2 in terms of $P_n^{[n_1,n_2]}(x;p)$. Both Q_n and P_n are given as asymptotic expansions for $p \to +\infty$ (or $p \to -\infty$) and with p ordered to the right of x. They are quasidifferential operators involving complex powers of the differential symbol p. For the block $[0,0]$ which is "basic" in some sense, we found simple expressions

$$Q_0^{[0,0]}(x;p) = (e^{i\frac{\pi}{2}} L(x;p))^{l_2} \tag{1.32}$$

$$P_0^{[0,0]}(x;p) = (e^{-i\frac{\pi}{2}} L(x;p))^{l_1} \tag{1.33}$$

where L has the form

$$L(x;p) = p + \sum_{n=1}^{\infty} u_n(x) p^{-n} \tag{1.34}$$

The meaning of the complex labels l_1 and l_2 can be fixed by (1.32), (1.33). The obvious commutativity of these operators can be extended to the whole perturbative series (1.30)

$$[B_2^{[0,0]}, B_1^{[0,0]}] \cong 0 \tag{1.35}$$

2 The critical potentials

Our aim is now to solve the Schwinger-Dyson equations (1.17), (1.19) with analytic methods. We introduce the functions

$$b_1(z) = \sum_{k=-\infty}^{+1} (B_1)_{n,n+k} z^k \tag{2.1}$$

$$b_2(z) = \sum_{k=-1}^{+\infty} (B_2)_{n,n+k} z^k \tag{2.2}$$

where the n-dependence is not made explicit on the l.h.s. $b_1(z)$ ($b_2(z)$) possesses a first order pole at infinity (zero) but is otherwise assumed to be analytic in a punctured disc around infinity (zero). We fix an irrelevant scale and assume that this disc extends to $z = 1$. At $z = 1$ either function will get a logarithmic branch point.

In polar coordinates we define the Fourier series boundary values

$$\lim_{r \searrow 1} b_1(re^{i\varphi}) = b_1(e^{i\varphi})_\downarrow \tag{2.3}$$

$$\lim_{r \nearrow 1} b_2(re^{i\varphi}) = b_2(e^{i\varphi})_\uparrow \tag{2.4}$$

Projection of the Fourier series $f(e^{i\varphi})$ on its non-negative (non-positive) frequency part is denoted $f(e^{i\varphi})_+$ ($f(e^{i\varphi})_-$). Then the Schwinger-Dyson equations (1.17) with (1.19) take the form

$$(b_1)_{\downarrow -} = V_2'(b_2)_{\uparrow -} \tag{2.5}$$

$$(b_2)_{\uparrow +} = V_1'(b_1)_{\downarrow +} \tag{2.6}$$

In the double-scaling asymptotic region we expand the functions $b_{1,2}(z)$ (2.1), (2.2) as

$$b_1(z) = r(z) + \sum_{m=1}^{\infty} a^{-(m+1)\gamma} U_m(x;z) \tag{2.7}$$

$$b_2(z) = s(z) + \sum_{m=1}^{\infty} a^{-(m+1)\gamma} V_m(x;z) \tag{2.8}$$

where the functions U_m (V_m) are later expanded in the neighborhood of $z = 1$ into an asymptotic power series in $(1-\frac{1}{z})$ (or $(1-z)$). For $r(z)$ and $s(z)$ we make the ansatz ($l_1, l_2 \in \mathbb{C}$, (1.31))

$$\begin{aligned} r(z) \quad &= \quad z(1-\frac{1}{z})^{l_2} = \sum_{k=-1}^{\infty} \rho_{-k} z^{-k} \\ &\quad (|z| > 1) \end{aligned} \tag{2.9}$$

$$\begin{aligned} s(z) \quad &= \quad \frac{1}{z}(1-z)^{l_1} = \sum_{k=-1}^{\infty} \sigma_k z^k \\ &\quad (|z| < 1) \end{aligned} \tag{2.10}$$

At leading order (2.6) goes into

$$k \geq 0: \ \sigma_k = \sum_{s=0}^{\infty} M(l_2)_{ks} f_s^c \tag{2.11}$$

where $\{f_s^c\}_0^\infty$ are the critical coupling constants of the potential V_1 (see (1.20)) and from (2.10)

$$M(l)_{ks} = (-1)^{s-k} \binom{ls}{s-k} \tag{2.12}$$

This matrix $M(l)$ is upper triangular with ones on the diagonal. So its inverse exists, but we must know whether $\{\sigma_k\}$ is in the domain of $M(l)^{-1}$ when we invert (2.11).

In order to invert $M(l)$ we invert the function

$$t = r(z) \tag{2.13}$$

to

$$z = \beta(t) \tag{2.14}$$

so that for $z \to \infty$

$$\beta(t) = t + l_2 - \binom{l_2}{2}\frac{1}{t} + O\left(\frac{1}{t^2}\right) \tag{2.15}$$

This function $\beta(t)$ is holomorphic for

$$
\begin{aligned}
|t| > |t_c| : \\
t_c = (-l_2)^{l_2}(1-l_2)^{1-l_2}
\end{aligned}
\tag{2.16}
$$

where the principal branch of the logarithm with cut on the negative real axis is used. When t approaches t_c

$$
\beta(t) \sim A\left(1-\frac{t_c}{t}\right)^{\frac{1}{2}} \tag{2.17}
$$

This singularity arises from a stationary point of $r(z)$, and (2.16) can be obtained this way.

If we define

$$
\beta(t)^n = \sum_{k=-\infty}^{n} a_{kn} t^k \tag{2.18}
$$

we obtain

$$
a_{kn} = \begin{cases} (-1)^{n-k}\binom{-kl_2}{n-k}\frac{n}{k} & (k \neq 0) \\ \delta_{n0} + l_2(1-\delta_{n0}) & (k = 0) \end{cases} \tag{2.19}
$$

The expansion coefficients in (2.15) are a_{k1}, and we will see in a moment that

$$
(M(l_2)^{-1})_{mn} = a_{mn} \tag{2.20}
$$

To prove this we note that for $z \to \infty$ (2.13), (2.15) imply

$$
(r(z)^k)_+ = t^k + O\left(\frac{1}{t}\right) \tag{2.21}
$$

and consequently

$$
V_1'(r(z))_+ = V_1'(t) + O\left(\frac{1}{t}\right) \tag{2.22}
$$

We will see in a moment that $V_1'(t)$ converges for (with (2.16))

$$
|t| < |t_c|
$$

so that for $t \to t_c$

$$
V_1'(t) \sim C\left(1-\frac{t}{t_c}\right)^{\frac{1}{2}} \tag{2.23}
$$

since the exponents in (2.17), (2.23) are bigger than the limit of integrability -1, (see our discussion in the Appendix) the Fourier series obtained in the limit

$$
t = |t_c|e^{i\Theta} \tag{2.24}
$$

are equal

$$
V_1'(|t_c|e^{i\Theta})_{\uparrow,+} = \sum_{k=0}^{\infty} \sigma_k(\beta(|t_c|e^{i\Theta})^k)_{\downarrow,+} \tag{2.25}
$$

provided the sum on the r.h.s. converges appropriately. From (2.25) follows (2.20) immediately.

The critical coupling constants can be obtained from (2.11), (2.20) and (2.19)

$$\begin{aligned} f_k^c &= \sum_{m=0}^{\infty} a_{km}\sigma_m \\ &= (l_1 l_2 - l_1 - l_2)\frac{(-1)^k\Gamma(l_1 - l_2 k)}{\Gamma(l_1 - k)\Gamma(2 + (1 - l_2)k)} \end{aligned} \tag{2.26}$$

Since the summation in (2.26) is hypergeometric, absolute convergence follows from

$$Re(l_1 - l_2 k) > 0 \tag{2.27}$$

Due to (1.3) this may be violated for a finite number of k. In these cases we postulate (2.26) to be true by "analytic continuation". Of course this amounts to a renormalization by subtraction of infinite counter terms. Using (2.26) we can by applying Stirling's formula immediately prove (2.23).

3 Evaluation of the Schwinger-Dyson equations

The Schwinger-Dyson equations (1.17) are evaluated in the neighborhood of $z = 1$ by an asymptotic expansion in powers of $1 - z$ or $1 - \frac{1}{z}$. The functions $U_m(x; z), V_m(x; z)$ in (2.7), (2.8) are decomposed into contributions of blocks, too

$$\begin{aligned} U_m(x;z) &= \sum_{[n_1,n_2]\in\mathbb{N}^2} \sum_{r=0}^{\infty} U_{mr}^{[n_1,n_2]}(x) \\ &\quad \times z\left(1 - \frac{1}{z}\right)^{l_2 - (n_1 l_1 + n_2 l_2) - (m+1) + r} \end{aligned} \tag{3.1}$$

$$\begin{aligned} V_m(x;z) &= \sum_{[n_1,n_2]\in\mathbb{N}^2} \sum_{r=0}^{\infty} V_{mr}^{[n_1,n_2]}(x) \\ &\quad \times \frac{1}{z}(1 - z)^{l_1 - (n_1 l_1 + n_2 l_2) - (m+1) + r} \end{aligned} \tag{3.2}$$

The appearance of these blocks is a necessary consequence of the recursion relations that we will derive next. Otherwise the form of the expansion is an intuitive generalisation of what has been found for the discrete series [4].

These recursion relations are derived from the identities

$$\left(\frac{1}{z}(1 - z)^{l_1}\right)_{\uparrow,+} = \sum_{k=0}^{\infty} f_k^c \left(z^k(1 - \frac{1}{z})^{k l_2}\right)_{\downarrow,+} \tag{3.3}$$

$$\left(z(1-\frac{1}{z})^{l_2}\right)_{\downarrow,-} = \sum_{k=0}^{\infty} g_k^c \left(z^{-k}(1-z)^{kl_1}\right)_{\uparrow,-} \tag{3.4}$$

from which we derived the critical coupling constants (2.26). We shall denote (3.4) "dual" to (3.3) and extend this notation also on the recursion relations derived from (3.4). This "duality" is connected with the replacements

$$l_1 \longleftrightarrow l_2,\ z \longleftrightarrow \frac{1}{z} \tag{3.5}$$

On the r.h.s. of (3.3) we have terms (see (2.9))

$$f_k^c r(z)^k$$

If we replace one factor $r(z)$ by an order-m term from (2.7) and use (3.1) we obtain

$$U_{ms}^{[n_1,n_2]}(x) \sum_{k=1}^{\infty} f_k^c k z^k \left(1-\frac{1}{z}\right)^{kl_2-\lambda-(m+1)+s}$$
$$(\lambda = n_1 l_1 + n_2 l_2) \tag{3.6}$$

We make an ansatz

$$\left(\sum_{k=1}^{\infty} k f_k^c z^k \left(1-\frac{1}{z}\right)^{kl_2}\right)_{\downarrow,+} =$$
$$= \left(\sum_{n=0}^{\infty} c_n (1-z)^{l_1+n}\right)_{\uparrow,+}$$
$$+ \text{ holomorphic function at } z=1 \tag{3.7}$$

Taking derivatives $z\frac{d}{dz}$ of (3.3) (taking derivatives is compatible with the operations $\downarrow, +$ etc.) gives

$$c_n = +\frac{l_1 + \sum_{k=1}^{n} l_2^k}{l_2^{n+1}} \tag{3.8}$$

The holomorphic part in (3.7) is unavoidable and is essential for the sequel, remember that in the case of the discrete series there are <u>only</u> holomorphic parts.

Inserting (3.7) into (3.6) we obtain

$$\left((1-\frac{1}{z})^{-\lambda-(m+1)+s}\right)_{\downarrow} \left((1-z)^{l_1+n}\right)_{\uparrow}$$

which we evaluate with the help of the identity ($\epsilon \searrow 0$)

$$((1-e^{-i\varphi-\epsilon})^a(1-e^{+i\varphi-\epsilon})^b)_+$$
$$= \frac{\sin \pi b}{\sin \pi(a+b)}(1-e^{i\varphi-\epsilon})^{a+b}(1-(1-e^{i\varphi-\epsilon}))^{-a}$$
$$+\binom{a+b-1}{b} {}_2F_1(1,-b;1-a-b;1-e^{i\varphi-\epsilon}) \tag{3.9}$$

which is valid for $\varphi \to 0, a+b \notin \mathbb{Z}$. We have made the holomorphic part in (3.9) explicit. If $b \in \mathbb{Z}, a+b \notin \mathbb{Z}$, only this holomorphic part of (3.9) survives. Therefore the holomorphic function in (3.7) contributes only to the holomorphic part of (3.9).

The recursion relations that can be derived in this fashion express functions

$$V_{mr}^{[n_1,n_2]}(x), \quad [n_1,n_2] \neq [1,0]$$

in terms of the functions $U_{m's}^{[n_1',n_2']}(x)$ and their derivatives. First there is a linear contribution ($\lambda = n_1 l_1 + n_2 l_2$)

$$\begin{aligned} + \quad & \frac{\sin \pi l_1}{\sin \pi (l_1 - \lambda)} \sum_{n=0}^{r} \sum_{s=0}^{r-n} (-1)^{m+n+r+1} \\ & \times \binom{\lambda + m + 2 - s}{r - n - s} c_n U_{ms}^{[n_1,n_2]}(x) \end{aligned} \tag{3.10}$$

In order to obtain expressions which are quadratic in U or linear in its derivatives we need sets of coefficients generalizing the $\{c_n\}$ in (3.7)

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} k^2 f_k^c z^k \left(1 - \frac{1}{z}\right)^{kl_2} \right)_{\downarrow,+} \\ & = \left(\sum_{n=0}^{\infty} d_n (1-z)^{l_1+n} \right)_{\uparrow,+} \\ & + \text{holomorphic function at } z = 1 \end{aligned} \tag{3.11}$$

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} k^3 f_k^c z^k \left(1 - \frac{1}{z}\right)^{kl_2} \right)_{\downarrow,+} \\ & = \left(\sum_{n=0}^{\infty} e_n (1-z)^{l_1+n} \right)_{\uparrow,+} \\ & + \text{holomorphic function at } z = 1 \end{aligned} \tag{3.12}$$

By differentiating (3.3) twice or three times we obtain the recursions

$$\begin{aligned} & l_2^2 d_n - 2 l_2 d_{n-1} + d_{n-2} = +l_2(c_n - c_{n-1}) + 1 \\ & +(l_1^2 - l_1 - 1)\delta_{n0} - (l_1 - 1)^2 \delta_{n1} \\ & (d_n = 0 \text{ for } n < 0) \end{aligned} \tag{3.13}$$

$$\begin{aligned} & -l_2^3 e_n + 3 l_2^2 e_{n-1} - 3 l_2 e_{n-2} + e_{n-3} \\ & +3 l_2 (l_2 d_n - (l_2 + 1) d_{n-1} + d_{n-2}) \\ & -l_2 (2c_n - 3c_{n-1} + c_{n-2}) = \\ & -1 - [l_1(l_1 - 1)(l_1 - 2) - 1]\delta_{n0} \\ & +[2 l_1 (l_1 - 1)(l_1 - 2) + 1]\delta_{n1} - (l_1 - 1)^3 \delta_{n2} \\ & (e_n = 0 \text{ for } n < 0) \end{aligned} \tag{3.14}$$

from which we derive that

$$d_0 = +\frac{l_1^2}{l_2^2} \tag{3.15}$$

$$d_1 = \frac{l_1(2l_1+1)}{l_2^3} - \frac{l_1^2 - l_1 - 1}{l_2^2} \tag{3.16}$$

$$e_0 = +\frac{l_1^3}{l_2^3} \tag{3.17}$$

$$e_1 = +\frac{l_1}{l_2^4}(3l_1^2 + 3l_1 + 1) - \frac{1}{l_2^3}(2l_1^3 - 2l_1 - 1) \tag{3.18}$$

With the help of these sets of coefficients we obtain as further contributions to the function $V_{mr}^{[n_1,n_2]}(x)$

$$\begin{aligned}
&+\frac{\sin \pi l_1}{\sin \pi(l_1-\lambda)} \sum_n \{ \sum_{s_1,s_2} (-1)^{m+n+r+1} \binom{\lambda+m+2-s_1-s_2}{r-n-s_1-s_2} \frac{1}{2}(d_n - c_n) \\
&\times \sum_{n_1',n_2',m_1} U_{m_1 s_1}^{[n_1',n_2']}(x) U_{m-m_1-1,s_2}^{[n_1-n_1',n_2-n_2']}(x) \\
&+\sum_{s=0}^{r-n} (-1)^{m+n+r} \binom{\lambda+m+1-s}{r-n-s} \frac{1}{2}[l_2(d_n-c_n) - (d_{n-1}-c_{n-1})] \frac{d}{dx} U_{m-1,s}^{[n_1,n_2]}(x) \\
&+\sum_{s=0}^{r-n} (-1)^{m+n+r+1} \binom{\lambda+m-s}{r-n-s} \frac{1}{12}[2(l_2^2 e_n - 2l_2 e_{n-1} + e_{n-2}) \\
&-3(l_2(l_2+1)d_n - 3l_2 d_{n-1} + d_{n-2}) \\
&+l_2(l_2+3)c_n - 5l_2 c_{n-1} + c_{n-2}] \frac{d^2}{dx^2} U_{m-2,s}^{[n_1,n_2]}(x) \\
&+O(U^3, UU', UU'', (U')^2, U''', ...)\}
\end{aligned} \tag{3.19}$$

The corresponding dual relations are obtained from (3.4) and express

$$U_{mr}^{[n_1,n_2]}(x)$$

through $V_{m',s}^{[n_1',n_2']}(x)$ by means of coefficients $\{\tilde{c}_n\}, \{\tilde{d}_n\}, \{\tilde{e}_n\} \ldots$ which are derived from $\{c_n\}, \{d_n\}, \{e_n\} \ldots$ by the replacement $l_1 \longleftrightarrow l_2$. Before we are able to show that the full set of relations is "recursive" in the proper sense, we have to eliminate "circles".

First we mention that there are two types of gradings which are "conserved" by these relations.

(α) We define the "block-grade" of

$$\left(\frac{d}{dx}\right)^k U_{mr}^{[n_1,n_2]}(x)$$

to be

$$[n_1, n_2]$$

Then $V_{m,s}^{[n_1,n_2]}(x)$ obtains contributions of only those monomials in $\left(\frac{d}{dx}\right)^{k_i} U_{m_i,r_i}^{[n_{1i},n_{2i}]}(x)$ so that

$$\begin{aligned} n_1 &= \sum_i n_{1i} \\ n_2 &= \sum_i n_{2i} \end{aligned} \tag{3.20}$$

(β) We define the "m-grade" of

$$\left(\frac{d}{dx}\right)^k U_{mr}^{[n_1,n_2]}(x)$$

to be

$$\mu = m + 1 + k \tag{3.21}$$

Then $V_{m,s}^{[n_1,n_2]}(x)$ obtains contributions of only those monomials in $\left(\frac{d}{dx}\right)^{k_i} U_{m_i,r_i}^{[n_{1i},n_{2i}]}(x)$ so that

$$m + 1 = \sum_i \mu_i \tag{3.22}$$

The recursion of (3.10), (3.19) and their duals is in both grades.

We denote the blocks

$$[0,0],\ [0,1],\ [1,0],\ [1,1]$$

the "elementary blocks". The blocks $[n_1, n_2]$ with $n_1 > 1$ or $n_2 > 1$ are called "nonelementary". We show first that we can express recursively

$$U_{mr}^{[n_1,n_2]}(x),\ V_{m',s}^{[n_1',n_2']}(x)$$

from nonelementary blocks by the functions of the elementary blocks. Namely reinsert the dual recursion relation for $U_{ms}^{[n_1,n_2]}$ into the recursion relation for $V_{mr}^{[n_1,n_2]}$

$$\begin{aligned} V_{mr}^{[n_1,n_2]}(x) &= \frac{\sin \pi l_1 \sin \pi l_2}{\sin \pi (l_1 - \lambda) \sin \pi (l_2 - \lambda)} \sum_{s=0}^{r} A_{rs} V_{ms}^{[n_1,n_2]}(x) \\ &+ \text{ derivative and nonlinear terms} \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} A_{rs} &= \sum_{s'=s}^{r} \sum_{n=0}^{r-s'} \sum_{n'=0}^{s'-s} (-1)^{n+n'+r+s'} \\ & \binom{\lambda + m + 2 - s'}{r - n - s'} \binom{\lambda + m + 2 - s}{s' - n' - s} c_n \tilde{c}_{n'} \end{aligned} \tag{3.24}$$

This matrix can be shown to be the unit matrix. For

$$\lambda = n_1 l_1 + n_2 l_2$$

not belonging to an elementary block we have

$$\frac{\sin \pi l_1 \sin \pi l_2}{\sin \pi (l_1 - \lambda) \sin \pi (l_2 - \lambda)} \neq 1 \tag{3.25}$$

Thus we cast the first term of the r.h.s. of (3.23) on the l.h.s. and divide by

$$1 - \frac{\sin \pi l_1 \sin \pi l_2}{\sin \pi (l_1 - \lambda) \sin \pi (l_2 - \lambda)} \tag{3.26}$$

Next we come to the elementary blocks which we deal with one after the other. The basic block $[0,0]$ is also the simplest one. The recursion relations for $U_{mr}^{[0,0]}(x), V_{mr}^{[0,0]}(x)$ involve only functions of the same block $[0,0]$ and are therefore recursive only in the m-grade. The connection of the recursion relations with the dual recursion relations is quite simple: one set is the inverse of the other set. So every function $V_{mr}^{[0,0]}(x)$ can be expressed as a polynomial in $U_{m's}^{[0,0]}(x)$ and its derivatives, and these $U_{m's}^{[0,0]}(x)$ can be considered as being free.

Next we study the block $[1,0]$. From the ansatz (3.2) we see that the corresponding parts of the functions $V_m(x;z)$ are single-valued in a neighborhood of $z = 1$, i.e. they have only zeros and poles. From (3.9) we recognize that poles are in fact not needed. Therefore we make the "no-pole-assumption"

$$V_{mr}^{[1,0]}(x) = 0 \text{ for } m \geq r \tag{3.27}$$

One can also show easily that if the Schwinger-Dyson equations are evaluated asymptotically for $z \to 1$, pole terms are unconstrained.

By means of the dual recursion relations we can express the $U_{mr}^{[1,0]}(x)$ as polynomials in $V_{mr}^{[1,0]}(x)$, $V_{mr}^{[0,0]}(x)$ and their derivatives. Application of the direct recursion makes no sense, because the holomorphic part of the functions $V_m(x;z)$ is produced by a completely different mechanism. They stem from an infinite number of sources and cannot be computed. We shall therefore consider these functions as freely eligible.

The block $[0,1]$ behaves very similar as the block $[1,0]$. The "no-pole-assumption" is

$$U_{mr}^{[0,1]}(x) = 0 \text{ for } m \geq r \tag{3.28}$$

The functions $V_{mr}^{[0,1]}(x)$ are expressed as polynomials in $U_{mr}^{[0,1]}(x)$, $U_{mr}^{[0,0]}(x)$ and their derivatives by the recursion relations. The dual recursion relations are suppressed.

Finally we come to the block $[1,1]$. Of course (3.26) is zero. But the matrix A in (3.24) is also the unit matrix. Thus the l.h.s. of (3.23) cancels the first term of the r.h.s. What is the remainder? Taking into account all recursion relations for the blocks $[0,0]$, $[0,1]$, $[1,0]$, we can show that the remainder vanishes, too. Therefore the recursion relations and dual recursion relations are inverses of each other modulo the recursion relations of the other elementary blocks.

Thus we have obtained an algorithm by which all functions

$$U_{mr}^{[n_1,n_2]}(x),\ V_{mr}^{[n_1,n_2]}(x)$$

can be expressed as polynomials in

$$\begin{array}{l} U_{mr}^{[0,0]}(x),\ V_{mr}^{[1,0]}(x)\ (r \geq m+1), \\ U_{mr}^{[0,1]}(x)\ (r \geq m+1),\ U_{mr}^{[1,1]}(x) \\ (m \geq +1 \text{ in all cases}) \end{array} \tag{3.29}$$

and their derivatives. It is not possible to restrict the blocks from $\mathbb{N}^2$ to the single block $[0,0]$. The $\mathbb{N}^2$ lattice is generated by the functions $V_{mr}^{[1,0]}(x), U_{mr}^{[0,1]}(x)$ that can always be present on the r.h.s. of the recursion relations and their duals and correspond to holomorphic behaviour at $z=1$.

4 The double scaling limits of B_1 and B_2

In the double scaling limit the $a^{-\gamma}$-expansions (2.7), (2.8) are combined with the $(1-z)$-expansions (3.1), (3.2), and (1.26), (1.27) are inserted so that a simple expansion in powers of $a^{-\gamma}$ results:

$$B_1 = \sum_{[n_1,n_2]\in\mathbb{N}^2} B_1^{[n_1,n_2]} \tag{4.1}$$

$$B_2 = \sum_{[n_1,n_2]\in\mathbb{N}^2} B_2^{[n_1,n_2]} \tag{4.2}$$

and with

$$\lambda = n_1 l_1 + n_2 l_2 \tag{4.3}$$

$$B_1^{[n_1,n_2]} \simeq \sum_{n=0}^{\infty} a^{-(l_2-\lambda+n)\gamma} Q_n^{[n_1,n_2]}(x;p) \tag{4.4}$$

$$B_2^{[n_1,n_2]} \simeq \sum_{n=0}^{\infty} a^{-(l_1-\lambda+n)\gamma} P_n^{[n_1,n_2]}(x;p) \tag{4.5}$$

The $P_n^{[n_1,n_2]}$, $Q_n^{[n_1,n_2]}$ are quasidifferential operators. The contributions of $r(z)$ (2.7) and $s(z)$ (2.8) are attributed to the block $[0,0]$.

We define expansion coefficients

$$e^x(1-e^{-x})^l = \sum_{k=0}^{\infty} t_k(l) x^{k+l} \tag{4.6}$$

so that the first few are

$$
\begin{aligned}
t_0(l) &= 1 \\
t_1(l) &= 1 - \frac{1}{2}\binom{l}{1} \\
t_2(l) &= \frac{1}{2} - \frac{1}{3}\binom{l}{1} + \frac{1}{4}\binom{l}{2}
\end{aligned}
\tag{4.7}
$$

From the U's and V's we go over to new functions ($\lambda = n_1 l_1 + n_2 l_2$)

$$\Phi_{mn}^{[n_1,n_2]}(x) = \sum_{r=0}^{n} t_{n-r}(l_2 - \lambda - (m+1) + r) U_{mr}^{[n_1,n_2]}(x) \tag{4.8}$$

$$\Psi_{mn}^{[n_1,n_2]}(x) = \sum_{r=0}^{n} t_{n-r}(l_1 - \lambda - (m+1) + r) V_{mr}^{[n_1,n_2]}(x) \tag{4.9}$$

$(1 \leq m < \infty)$
and

$$\Phi_{-1,n}^{[0,0]} = t_n(l_2) \tag{4.10}$$

$$\Psi_{-1,n}^{[0,0]} = t_n(l_1) \tag{4.11}$$

The quasidifferential operators $Q_n^{[n_1,n_2]}$, $P_n^{[n_1,n_2]}$ are then given by

$$Q_n^{[n_1,n_2]}(x;p) = \sum_m \Phi_{mn}^{[n_1,n_2]}(x) \left(e^{i\frac{\pi}{2}} p\right)^{-n_1 l_1 - (n_2-1) l_2 - (m+1) + n} \tag{4.12}$$

$$P_n^{[n_1,n_2]}(x;p) = \sum_m \Psi_{mn}^{[n_1,n_2]}(x) \left(e^{-i\frac{\pi}{2}} p\right)^{-(n_1-1) l_1 - n_2 l_2 - (m+1) + n} \tag{4.13}$$

Here the summation over m extends from $\min m$ to infinity where

$$\min(m+1) = 2\max(n_1, n_2) \tag{4.14}$$

as a consequence of the recursion relations (3.10), (3.19).

Now we face the problem to solve all the constraints following from the commutator (1.5), (1.18). Since

$$
\begin{aligned}
[B_2^{[n_1,n_2]}, B_1^{[n_1',n_2']}] \cong \sum_{n=0}^{\infty} & a^{-\gamma[n-(n_1+n_1'-1)l_1-(n_2+n_2'-1)l_2]} \\
& \times \sum_{s=0}^{n} [P_{n-s}^{[n_1,n_2]}, Q_s^{[n_1',n_2']}]
\end{aligned}
\tag{4.15}
$$

and

$$
\begin{aligned}
&[P_{n-s}^{[n_1,n_2]}, Q_s^{[n_1',n_2']}] = \\
&= \sum_{k=1}^{\infty} \sum_{m_1,m_2} \exp i\frac{\pi}{2}[2s - n + (n_1 - n_1' - 1)l_1 + (n_2 - n_2' + 1)l_2 + m_1 - m_2] \\
&\qquad \times \Big\{ \binom{l_1 - \lambda + n - s - m_1 - 1}{k} \Psi_{m_1,n-s}^{[n_1,n_2]}(x) \left(i\frac{d}{dx}\right)^k \Phi_{m_2,s}^{[n_1',n_2']}(x) \\
&\qquad - \binom{l_2 - \lambda' + s - m_2 - 1}{k} \Phi_{m_2,s}^{[n_1',n_2']}(x) \left(i\frac{d}{dx}\right)^k \Psi_{m_1,n-s}^{[n_1,n_2]}(x) \Big\} \\
&\qquad \times p^{n-(n_1+n_1'-1)l_1-(n_2+n_2'-1)l_2-m_1-m_2-k-2}
\end{aligned}
\tag{4.16}
$$

Integer powers of p result only if (for generic l_1 and l_2)

$$
\begin{aligned}
n_1 + n_1' &= 1 \\
n_2 + n_2' &= 1
\end{aligned}
\tag{4.17}
$$

All commutators (4.15) not satisfying (4.17) must vanish, whereas

$$
\sum_{\substack{n_1, n_1', n_2, n_2' \\ n_1 + n_1' = 1 \\ n_2 + n_2' = 1}} [B_2^{[n_1,n_2]}, B_1^{[n_1',n_2']}] = 1 \tag{4.18}
$$

We shall now demonstrate that this problem can be solved with the sole free functions (3.30).

We assume first that for all m, r

$$
\frac{d}{dx} U_{mr}^{[0,0]} = \frac{d}{dx} U_{mr}^{[0,1]} = \frac{d}{dx} V_{mr}^{[1,0]} = 0 \tag{4.19}
$$

Then of all the commutators (4.16) those with

$$
n_1 \cdot n_2 = 0 \text{ and } n_1' \cdot n_2' = 0 \tag{4.20}
$$

vanish trivially whereas (4.18) reduces to

$$
[B_2^{[0,0]}, B_1^{[1,1]}] + [B_2^{[1,1]}, B_1^{[0,0]}] = 1 \tag{4.21}
$$

Next assume that $n_0 \geq 3$ exists to that

$$
\frac{d}{dx} U_{mn}^{[1,1]} = 0 \, \text{for all} \, m \, \text{and} \, n < n_0 \tag{4.22}
$$

Then for the leading order

$$
a^{-\gamma n_0} \tag{4.23}
$$

in (4.15) we obtain from (4.21) using (4.16) and $k=1$

$$\sum_{s=0}^{n_0}\sum_{m_1,m_2}\Big\{e^{i\frac{\pi}{2}(2s-n_0-2l_1+m_1-m_2)} \times(l_1+n_0-s-m_1-1)\Psi^{[0,0]}_{m_1,n_0-s}(x)i\frac{d}{dx}\Phi^{[1,1]}_{m_2,s} -e^{i\frac{\pi}{2}(2s-n_0+2l_2+m_1-m_2)} \times(l_2+s-m_2-1)\Phi^{[0,0]}_{m_2,s}(x)i\frac{d}{dx}\Psi^{[1,1]}_{m_1,n_0-s}(x)\Big\} \times p^{n_0-m_1-m_2-3} \tag{4.24}$$

so

$$n_0=3 \tag{4.25}$$

is possible for

$$\begin{aligned} m_1=-1, \quad m_2=+1, \quad &\text{(first term)}\\ m_1=+1, \quad m_2=-1, \quad &\text{(second term)} \end{aligned} \tag{4.26}$$

In this case we obtain

$$\left[-l_1e^{-i\pi l_1}-l_1e^{+i\pi l_2}\cdot\frac{\sin\pi l_1}{\sin\pi l_2}\right]\frac{d}{dx}U^{[1,1]}_{13}(x)=1 \tag{4.27}$$

The bracket is

$$-l_1\frac{\sin\pi(l_1+l_2)}{\sin\pi l_2}\neq 0 \tag{4.28}$$

and after normalization of x by translation we obtain

$$U^{[1,1]}_{13}(x)=\alpha_{13}x \tag{4.29}$$

Letting m_1 and m_2 grow beyond the values (4.26) we obtain

$$U^{[1,1]}_{m3}(x)=\alpha_{m3}x+\beta_{m3} \tag{4.30}$$

where all α_{m3} are determined (e.g. $\alpha_{23}=0$) but all β_{m3} are free integration constants (except $\beta_{13}=0$).

Choosing $n_0>3$ we can show in the same way that

$$U^{[1,1]}_{mn_0}(x)=\alpha_{mn_0}x+\beta_{mn_0} \tag{4.31}$$

Given any $n_0\geq 3$ γ is fixed by the usual argument to

$$\gamma=\frac{-2}{n_0-1} \tag{4.32}$$

Still all commutator constraints must be fulfilled where

$$\max(n_1 + n_1' - 1,\ n_2 + n_2' - 1) > 0 \tag{4.33}$$

But this is necessary only up to (and including) the order (4.23), namely whenever (see (4.25))

$$n - (n_1 + n_1' - 1)Rel_1 - (n_2 + n_2' - 1)Rel_2 \leq n_0 \tag{4.34}$$

Now among the two functions

$$\Phi^{[n_1', n_2']}_{m_2, s}(x), \quad \Psi^{[n_1, n_2]}_{m_1, n-s}(x) \tag{4.35}$$

at least one must contain a function

$$U^{[1,1]}_{m,r},\ r \geq n_0$$

which is nonconstant if (4.16) is nonzero. Let us assume that this is the first in (4.35), so that

$$n_1' \geq 1,\ n_2' \geq 1$$

Then from (4.34) together with (4.33) we obtain

$$n < n_0 \tag{4.36}$$

On the other hand from (4.35)

$$s \geq r \geq n_0 \tag{4.37}$$

and from (4.16)

$$n \geq s \tag{4.38}$$

But (4.37), (4.38) contradict (4.36). Therefore all commutator constraints are satisfied.

5 Expectation values and concluding remarks

We can evaluate the partition function (1.7) in the scaling domain with standard methods (see e.g. [4], eqs. (181), (182))

$$\begin{aligned} F(\zeta) &= \log Z \\ &= \text{const} + a^{-2\gamma}(1 - a^2\zeta)^{-2} \int_{a^{-2}}^{\zeta} dx(\zeta - x) \\ &+ \left\{ \log\left[1 + \sum_{m=1}^{\infty} a^{-(m+1)\gamma} \operatorname*{Res}_{z=\infty} U_m(x; z)\right] \right. \\ &+ \left. \log\left[1 + \sum_{m=1}^{\infty} a^{-(m+1)\gamma} \operatorname*{Res}_{z=0} V_m(x; z)\right] \right\} \end{aligned} \tag{5.1}$$

where ζ is defined so that (see (1.25))

$$\zeta = x(\xi)|_{\xi=1} \tag{5.2}$$

In analogy with $D = 2$ conformal field theory we may assume maximal holomorphy for the functions $U_m(x;z)$ and $V_m(x;z)$ in z. $V_m(x;z)$ is holomorphic for $|z| < 1$ with exception of a simple pole at $z = 0$, and $U_m(x;z)$ behaves analogously. Moreover, the expansions (3.1), (3.2) are assumed to converge in these domains. Then

$$\underset{z=\infty}{\mathrm{Res}} \quad U_m(x;z) = \sum_{r=0}^{\infty} \sum_{[n_1,n_2]\in\mathbb{N}^2} U_{mr}^{[n_1,n_2]}(x) \tag{5.3}$$

$$\underset{z=0}{\mathrm{Res}} \quad V_m(x;z) = \sum_{r=0}^{\infty} \sum_{[n_1,n_2]\in\mathbb{N}^2} V_{mr}^{[n_1,n_2]}(x) \tag{5.4}$$

Due to (4.14) for fixed m, the sum over the blocks $[n_1, n_2]$ is finite. Moreover we can cut off the sum over r at n_0 as follows from the arguments in the preceding section. Then it follows

$$\underset{z=\infty}{\mathrm{Res}} \quad U_1(x;z) = \alpha_{1n_0} x + \mathrm{const} \tag{5.5}$$

A second order polynomial in x appears first in

$$\underset{z=\infty}{\mathrm{Res}} \quad U_3(x;z)$$

The critical exponents l_1 and l_2 that define the structure of the quasi-differential operators (see (1.32), (1.33)) are only implicitly contained in (5.3), (5.4), e.g. in the coefficient α_{1n_0} in (5.5).

Finally we want to remark that what we have presented here is not a mathematical construction, in particular not an existence proof for the continuous series. This follows from the "lemma" mentioned after (1.19). Obviously the commutator (1.18) is not diagonal automatically. This must be interpreted as an incomplete evaluation of the Schwinger-Dyson equations. Instead we have imposed the commutator (1.18) as a constraint. In the case of the discrete series we proved that the latter procedure gives equivalent results as a complete evaluation of the Schwinger-Dyson equations. In the case of the continuous series this is at most plausible but not yet proved.

Appendix: Technical considerations

The method applied in sections 2 and 3 uses projection on positive (or negative) frequency parts of a Fourier series and a study of asymptotic expansions in the limit $\varphi \to 0$.

Consider a function $g(z)$ which is holomorphic in the whole complex plane except a cut from 1 to ∞ along the real axis. Then the Taylor expansion

$$g(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n e^{in\varphi} \qquad (z = re^{i\varphi}) \tag{A.1}$$

for $r < 1$ fixed, is a Fourier series with positive frequencies only and converges exponentially. On the other hand for $r > 1$ fixed

$$g(z) = \sum_{n=-\infty}^{+\infty} b_n(r) e^{in\varphi} \tag{A.2}$$

provided the discontinuity along the cut is integrable

$$\int_1^M d\zeta |g(\zeta + i\epsilon) - g(\zeta - i\epsilon)| < \infty \qquad (M > r) \tag{A.3}$$

Then by deformation of the contour

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \cdot \oint \frac{dz}{z^{n+1}} g(z) \\ &= \frac{b_n(r)}{r^n} + \frac{1}{2\pi i} \int_1^r \frac{d\zeta}{\zeta^{n+1}} [g(\zeta + i\epsilon) - g(\zeta - i\epsilon)] \end{aligned} \tag{A.4}$$

and

$$a_n = \lim_{r \searrow 1} b_n(r) \tag{A.5}$$

In particular this limit vanishes for negative n since

$$a_n = 0 \text{ for } n < 0. \tag{A.6}$$

Consider the typical function

$$g(z) = (1 - z)^\lambda, \; \lambda \in \mathbb{C} \tag{A.7}$$

Then

$$a_n = (-1)^n \binom{\lambda}{n} \tag{A.8}$$

$$b_n(r) = \frac{\sin \pi\lambda}{\pi} \frac{r^\lambda}{\lambda - n} {}_2F_1(-\lambda, -\lambda + n; -\lambda + n + 1; \frac{1}{r}) \tag{A.9}$$

Provided

$$Re\lambda + 1 > 0 \tag{A.10}$$

we can verify (A.5).

Now consider $\lambda = -1$. Then

$$a_n = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{A.11}$$

$$\lim_{r \searrow 1} b_n(r) = \begin{cases} -1 & n < 0 \\ 0 & n \geq 0 \end{cases} \tag{A.12}$$

so that

$$\sum_{n=-\infty}^{+\infty} (a_n - b_n(1+))e^{in\varphi} = 2\pi\delta(\varphi) \tag{A.13}$$

In the asymptotic expansion at $\varphi \to 0$ the r.h.s. of (A.13) is unobservable. A similar situation arises if λ is equal any negative integer.

If

$$Re\lambda + 1 \leq 0, \ -\lambda \notin \mathbb{N} \tag{A.14}$$

then we can perform the limit $r \searrow 1$ on (A.9) after subtraction of a diverging expression which renders the result analytic in λ ("analytic regularization"). This diverging subtraction term consists of local distributions

$$\frac{d^s}{d\varphi^s}\delta(\varphi)$$

multiplied with diverging coefficients

$$\sim \left(1 - \frac{1}{r}\right)^{\lambda+1-s}$$

In any case instead of (A.5) we have

$$a_n = \lim_{r \searrow 0} \{b_n(r) - \text{ asymptotically unobservable terms}\} \tag{A.15}$$

As an example we prove (3.8). Derivation of (3.3) gives

$$\left(-\frac{1}{z}(1-z)^{l_1} - l_1(1-z)^{l_1-1)}\right)_{\uparrow,+} \tag{A.16}$$

on the l.h.s. and

$$\left(\left(1 + \frac{l_2}{z(1-\frac{1}{z})}\right) \sum_{k=1}^{\infty} k f_k^c z^k (1 - \frac{1}{z})^{kl_2}\right)_{\downarrow,+} \tag{A.17}$$

on the r.h.s. The factor in front of the sum of (A.17) produces only nonpositive frequencies and can therefore be written as

$$\left(\left(1+\frac{l_2}{z(1-\frac{1}{z})}\right)_{\downarrow}\left(\sum_{k=1}^{\infty} k f_k^c z^k (1-\frac{1}{z})^{kl_2}\right)_{\downarrow,+}\right)_+$$
$$=\left(\left(1+\frac{l_2}{z(1-\frac{1}{z})}\right)_{\downarrow}\left(\sum_{k=1}^{\infty} c_n (1-z)^{l_1+n}\right)_{\uparrow}\right)_+$$
$$+ \text{ irrelevant terms} \tag{A.18}$$

Now (A.13) implies

$$\left(\frac{1}{z(1-\frac{1}{z})}\right)_{\downarrow} = -\left(\frac{1}{1-z}\right)_{\uparrow} + 2\pi\delta(\varphi) \tag{A.19}$$

so that we can continue (A.18) to

$$\left(\sum_{n=0}^{\infty} c_n \left[(1-z)^{l_1+n} - l_2(1-z)^{l_1+n-1}\right]\right)_{\uparrow,+}$$
$$+ \text{ irrelevant terms} \tag{A.20}$$

where the latter include a holomorphic part at $z=1$ with an infinite constant (if $Re\, l_1 < 0$) and eventually a first order pole at $z=1$. Equating (A.16) and (A.18) we obtain (3.8).

Acknowledgement

S.B. would like to thank the German Academic Exchange Service (DAAD) for financial support.

References

[1] M.L. Metha, Random matrices and the statistical theory of energy levels, Acad. Press, New York 1967.

[2] C. Itzykson, J.-M. Drouffe, Statistical field theory, 2 Vols, Camb. Univ. Press, Cambridge 1989, Section 10.3

[3] E. Brzin, V.A. Kazakov, Phys. Lett. B236 (1990) 144;
D.J. Gross, A.A. Migdal, Phys. Rev. Lett. 64 (1990) 127;
M.R. Douglas, S.H. Shenker, Nucl. Phys. B335 (1990) 635;
Proceedings of 1990 Cargse workshop on "Random surfaces and quantum gravity", Eds. O. Alvarez, E. Marinari, P. Windey. NATO ASI Series B: Physics Vol. 262 (1992);
P. di Francesco, P. Ginsparg, J. Zinn-Justin, Physics Reports 254 (1995) 1-133.

[4] S. Balaska, J. Maeder, W. Rühl, Nucl. Phys. B 486 (1997) 673.

[5] M.R. Douglas, Phys. Lett B238 (1990) 176;
P. Ginsparg, M. Goulian, M.R. Plesser, J. Zinn-Justin, Nucl. Phys. B342 (1990) 539.

[6] I.M. Gelfand, L.A. Dikii, Funks. Anal. Prilozhen., 10, 4 (1976) 13.

[7] D. Bessis, Commun. Math. Phys. 69 (1979) 147;
D. Bessis, C. Itzykson, J.-B. Zuber, Adv. Appl. Math. 1 (1980) 109.

[8] M.L. Metha, Commun. Math. Phys. 79 (1981) 327;
S. Chadha, G. Mahoux, M.L. Mehta, J. Phys. A 14 (1981) 579.