arXiv:hep-th/9511085 v3 16 Nov 1995

GREENSITE-HALPERN STABILIZATION OF A_k SINGULARITIES IN THE $N \to \infty$ LIMIT

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ABSTRACT

The Greensite-Halpern method of stabilizing bottomless Euclidean actions is applied to zerodimensional O(N) sigma models with unstable A_k singularities in the $N = \infty$ limit.

Dedicated to J. Lukierski to his 60th birthday.

1. Classical actions which are unbounded from below do not define Euclidean quantum field theories because the partition functions diverge. A method to modify the classical actions in such a fashion that convergence is guaranteed on the one hand whereas the classical actions are only minimally changed on the other hand has been proposed by Greensite and Halpern [1]. We refer to this method as "Greensite-Halpern stabilization". Modifications of a theory are considered minimal if the stabilized and the original "bottomless" theory have the same

1. classical limit;
2. perturbative series;
3. $N \to \infty$ limit.

In [1] it has been proved for typical models that these requirements are indeed fulfilled. The Greensite-Halpern stabilization applied to a stable theory leaves it unchanged.

A famous example of a classical bottomless theory is Euclidean gravity. The same problem of instability arises in matrix models of pure gravity. Applications of Greensite-Halpern stabilization to these models can be found in [2,3].

The most popular method of stabilization is analytic continuation of the classical action in a coupling constant. Expectation values are then not necessarily analytic [4] but it seems that the perturbative series is always invariant under continuation. So the three axioms of minimality formulated by Greensite and Halpern may also be fulfilled. It is, however, known that both stabilization methods are inequivalent.

We want to apply the Greensite-Halpern stabilization method to zero dimensional sigma models that exhibit A_k singularities with $k > 1$, ($k = 1$ appears in [1]). In these cases we have to perform double scaling limits, where N goes to infinity and coupling constants $\{f_r\}$ tend to their critical values $\{f_r^c\}$. There arise scale invariant variables $\{\zeta_r\}_1^{k-1}$ and the singular factor in the partition function is a generalized Airy function depending on these variables (see [5] for the details). The cases A_k with $k = 2n$ are unstable. If $k = 2n + 1$ there are two signs $A^{\pm}_{2n+1}$ one of which (the "wrong sign" A^{-}_{2n+1}) is also unstable.

The generalized Airy functions are given by integral representations. In the stable cases the integral contours are the real axis. Mathematical textbooks [6] teach us that we have to choose complex contours in the unstable cases. Though this leads to well-defined Airy functions, it is not clear whether they are suited for a probabilistic interpretation in at least a subdomain of the variables $\{\zeta_r\}$. At the end of this article we will make a clarifying comment on this problem. On the other hand the Greensite-Halpern stabilized theories have an obvious probabilistic interpretation for all $\{\zeta_r\} \in \mathbb{R}_{k-1}$.

2. We consider zerodimensional sigma models

$$Z = \int \prod_{a=1}^{N} d\phi_a e^{-S}, \quad \phi_a \in \mathbb{R} \tag{1}$$

$$S = \frac{1}{2}\phi \cdot \phi + \sum_{r=2}^{k} \frac{f_r}{2r} N^{-r+1} (\phi \cdot \phi)^r \tag{2}$$

$$\phi \cdot \phi = Nz \tag{3}$$

$$\tilde{S}(z) = \frac{1}{N} S = \frac{1}{2} z + \sum_{r=2}^{k} \frac{f_r}{2r} z^r. \tag{4}$$

Angular integration gives

$$Z = \frac{(\pi N)^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_0^\infty dz \frac{1}{z} \exp N(\frac{1}{2} \log z - \tilde{S}(z)). \tag{5}$$

The exponent in (5) may exhibit a singularity $A_n (n \leq k)$ which in the limit $N \to \infty$ allows us to expand Z and any expectation value in a series of fractional negative powers of N. In the present context we will deal with only the leading term which for Z gives a generalized Airy function.

The starting point of the Greensite-Halpern stabilization is the Schrˇödinger-equation

$$\left[-\frac{1}{2}\Delta_\phi + \frac{1}{8} \sum_a \left(\frac{\partial S}{\partial \phi_a} \right)^2 - \frac{1}{4} \Delta_\phi S \right] \psi_0(\phi) = E_0 \psi_0(\phi) \tag{6}$$

with normalized ground state wave function $\psi_0(\phi)$ and eigenvalue E_0. The ill-defined probability density

$$\frac{1}{Z}e^{-S}$$

is replaced by $|\psi_0(\phi)|^2$. Change of the coordinates (3) and action (4) gives

$$\left[-\frac{2z}{N}\frac{\partial^2}{\partial z^2}-\frac{\partial}{\partial z}+NV(Z)\right]\tilde{\psi}_0(z)=E_0\tilde{\psi}_0(z) \tag{7}$$

$$\psi_0(\phi)=\tilde{\psi}_0(z(\phi)) \tag{8}$$

$$V(z)=\frac{1}{2}z(\tilde{S}')^2-\frac{1}{2}\tilde{S}'-\frac{z}{N}\tilde{S}''. \tag{9}$$

Next we apply the $N\to\infty$ limit to the equation (7) [1,7]: We factorize

$$\tilde{\psi}_0(z)=z^{-\frac{N}{4}}\varphi_0(z) \tag{10}$$

and rescale the equation

$$\left[-\frac{2z}{N}\frac{\partial^2}{\partial z^2}+N\left(\frac{1}{8z}+V(z)\right)+O(1)\right]\varphi_0(z)=E_0\varphi_0(z) \tag{11}$$

in the neighborhood of the singularity.

If this singularity is A_1, its location z_0 is determined from

$$-\frac{1}{8z_0^2}+V'(z_0)=0. \tag{12}$$

It has been shown in [1] that the left hand side factorizes

$$-\frac{1}{8z^2}+V'(z)=F_1(z)F_2(z) \tag{13}$$

with

$$F_1(z)=\tilde{S}'(z)-\frac{1}{2z} \tag{14}$$

$$F_2(z)=z\tilde{S}''(z)+\frac{1}{2}\tilde{S}'(z)+\frac{1}{4z}. \tag{15}$$

If

$$F_1(z_0)=0 \tag{16}$$

we have an A_1 singularity in the action (5) as well. An additional branch of A_1 singularities in the potential of the Schrödinger equation (11) arises at

$$F_2(z_0)=0. \tag{17}$$

We will not consider it here (see, however, [1]). The ground state energy is in this approximation

$$\begin{aligned} E_0 &= N\left(\frac{1}{8z_0} + V(z_0)\right) \\ &= \frac{1}{2} N z_0 \left(\tilde{S}'(z_0)\right)^2 \geq 0 \end{aligned} \tag{18}$$

which remains valid in the $A_k, k > 1$, case.

To complete the discussion of the A_1 case we prove stability (i.e. A_1 is A_1^+). We expand the potential to next order

$$\frac{1}{8z} + V(z) = \frac{1}{8z_0} + V(z_0) + \frac{1}{2}(z - z_0)^2\omega^2 + O((z - z_0)^3) \tag{19}$$

and

$$\begin{aligned} \omega^2 &= F_1'(z_0)F_2(z_0) + F_1(z_0)F_2'(z_0) \\ &= F_1'(z_0)F_2(z_0) \end{aligned} \tag{20}$$

if (16) holds.

Now from (14), (15) we obtain

$$F_2(z) = zF_1'(z) + \frac{1}{2}F_1(z) \tag{21}$$

so that once again from (16)

$$\omega^2 = z_0(F_1'(z_0))^2 > 0 \tag{22}$$

and we have (local) stability. We will later see that any A_{n+1} singularity in the action (5) implies a (stable) A_{n+1} singularity in the potential of the Schrödinger equation. Other singularities in the potential (such as A_1 (17)) are not automatically stable.

The ground state energy E_0 is to next order

$$E_0 = N\left(\frac{1}{8z_0} + V(z_0)\right) + \epsilon_1 \tag{23}$$

where to leading order now

$$\left[-\frac{2z_0}{N}\frac{\partial^2}{\partial z^2} + \frac{1}{2}N\omega^2(z - z_0)^2\right]\varphi_0(z) = \epsilon_1\varphi_0(z). \tag{24}$$

This equation is rescaled by

$$x = N^{\frac{1}{2}}(z - z_0) \tag{25}$$

so that the oscillator equation

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\frac{\omega^2}{4z_0}x^2\right]\varphi_0(z(x))=\frac{\epsilon_1}{4z_0}\varphi_0(z(x)) \tag{26}$$

results. It follows

$$\epsilon_1 = z_0^{\frac{1}{2}}\omega \tag{27}$$

and

$$\varphi_0(z(x)) = A\cdot e^{-\frac{1}{2}\frac{\omega}{\sqrt{4z_0}}x^2}. \tag{28}$$

3. An A_{n+1} singularity in (5) shows up at z_0 if

$$\begin{aligned} F_1^{(m)}(z_0)=0, \quad 0\le m\le n \\ F_1^{(n+1)}(z_0)\neq 0. \end{aligned} \tag{29}$$

If in (2) and (4) we choose $k=n+1$ (the "minimal set" of coupling constants) there is exactly one such singularity and corresponding critical coupling constants $\{f_r^c\}_2^{n+1}$ (see [5]). Since from (21)

$$F_2^{(m)} = zF_1^{(m+1)} + (m+\frac{1}{2})F_1^{(m)} \tag{30}$$

(29) implies

$$\begin{aligned} F_2^{(m)}(z_0)=0, \quad 0\le m\le n-1 \\ F_2^{(n)}(z_0)\neq 0. \end{aligned} \tag{31}$$

At such point z_0

$$(F_1(z)F_2(z))^{(2n+1)}|_{z_0} = \binom{2n+1}{n+1} z_0 \left(F_1^{(n+1)}(z_0)\right)^2 > 0 \tag{32}$$

whereas

$$(F_1(z)F_2(z))^{(m)}|_{z_0}=0, \quad m\le 2n. \tag{33}$$

It follows that at leading order in $z-z_0$ the potential in the Schrödinger equation is

$$+N\cdot\frac{g_{2n+2}}{2n+2}(z-z_0)^{2n+2} \tag{34}$$

with

$$g_{2n+2}=\frac{z_0(F_1^{(n+1)}(z_0))^2}{n!(n+1)!}>0. \tag{35}$$

So the Greensite-Halpern program produces a stable potential in the Schrödinger equation for each A_{n+1}.

If the Schrödinger equation is rescaled at $N \to \infty$ in analogy to (24), (25) we obtain

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\frac{1}{n+1}x^{2n+2}\right)\varphi_0(z(x)) = \\ = \frac{N}{\lambda^2}\cdot\frac{\epsilon_1}{4z_0}\cdot\varphi_0(z(x)) \qquad (36)$$

where

$$x=\lambda(z-z_0) \qquad (37)$$

and

$$\lambda=\left(\frac{N^2 g_{2n+2}}{4z_0}\right)^{\frac{1}{2n+4}}. \qquad (38)$$

So $\varphi_0(z(x))$ is a universal function of x and

$$\epsilon_1=\frac{4z_0\lambda^2}{N}\tilde{\epsilon}_1 \qquad (39)$$

where $\tilde{\epsilon}_1$ is a universal number (depending on n). The function $\varphi_0(z(x))=\chi_0(x)$ is symmetric in x and for $x\to\infty$ behaves as $(n>0)$

$$\chi_0(x)=A\exp\Big\{ -\frac{x^{n+2}}{(n+1)^{\frac{1}{2}}(n+2)}+\frac{1}{2}(n+1)\log x \\ -\frac{(n+1)^{\frac{1}{2}}}{n}\tilde{\epsilon}_1 x^{-n}+O(x^{-n-2})\Big\}. \qquad (40)$$

Squaring this function and substituting (37), (38) we obtain the Greensite-Halpern probability distribution over the real z-axis for large $|z|$

$$|\varphi_0(z)|^2=A^2\exp\left\{-N\frac{|F_1^{(n+1)}(z_0)|}{(n+2)!}|z-z_0|^{n+2}+O(\log z)\right\} \qquad (41)$$

(and analogously for large N). Here the effect of the stabilization can be clearly seen: all "wrong signs"are eliminated.

Now we consider a deformed A_{n+1} singularity: the coupling constants $\{f_r\}$ are different from the critical ones $\{f_r^c\}$

$$f_r-f_r^c=\Theta_r \qquad (42)$$

but with $N\to\infty$ these Θ_r go to zero in such a fashion that

$$G(x;\{\zeta\}) = \lim_{N\to\infty} N\left\{\tilde{S}(z)-\tilde{S}(z_0)-\frac{1}{2}\log\frac{z}{z_0}\right\} \\ = \sum_{r=1}^{n}\frac{\zeta_r}{r!}x^r+\epsilon\frac{x^{n+2}}{(n+2)!} \qquad (43)$$

($\epsilon = \pm 1$ for even n).
Thus in terms of λ (37) (the normalization in (38) is marginally changed)

$$\zeta_r = \lim_{N\to\infty} N\lambda^{-r} F_1^{(r-1)}(z_0) \tag{44}$$

and the point z_0 is kept fixed by the requirement that the power of order $n+1$ in (43) vanishes. In this case the saddle point integration of (5) gives

$$Z_{\text{sing}} = \frac{(\pi N)^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{1}{\lambda z_0} \int_C dx \exp\{-G(x;\{\zeta\})\} \tag{45}$$

where C is a chain running from infinity to infinity along which the integral converges exponentially. The integral is a generalized Airy function.

Now we apply the analogous procedure in the Greensite-Halpern stabilization program, which results in a measure

$$\frac{d\mu_{GH}}{dx} = \exp\{-\tilde{G}(x;\{\zeta\})\} \tag{46}$$

$$\int d\mu_{GH} = 1. \tag{47}$$

In order to calculate $\tilde{G}$ we repeat the rescaling of the Schrödinger equation using (21), (44) and (37) and get

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{8}\left(\frac{\partial}{\partial x} G(x;\{\zeta\})\right)^2\right] \chi_0(x) = \tilde{\epsilon}_1 \chi_0(x) \tag{48}$$

and

$$\chi_0(x) = \exp\{-\frac{1}{2}\tilde{G}(x;\{\zeta\})\}. \tag{49}$$

It follows for $x \to +\infty$

$$\begin{aligned} \tilde{G}(x;\{\zeta\}) &= \epsilon G(x;\{\zeta\}) \\ &\quad + \log(\epsilon \frac{\partial}{\partial x} G(x;\{\zeta\}) \\ &\quad + O(1). \end{aligned} \tag{50}$$

4. Now we consider an example: the singularity A_2. The Airy function is the proper one

$$\begin{aligned} \pi Bi(\zeta) &= \int_0^\infty dx \left\{\exp(-\frac{1}{3}x^3 + \zeta x) + \sin(+\frac{1}{3}x^3 + \zeta x)\right\} \\ &= \int_C dx \exp(-\frac{1}{3}x^3 + \zeta x) \end{aligned} \tag{51}$$

where C is defined as follows. Let C_q, $q \in \mathbb{Q}$, denote the contour along the ray $\{re^{2\pi iq},\ 0 \le r < \infty\}$ oriented from zero to infinity. Then

$$C = C_0 - \frac{1}{2}(C_{\frac{2}{3}} + C_{\frac{4}{3}}). \tag{52}$$

How can this Airy function be used to calculate expectation values? Consider a polynomial

$$P_M(x) = \sum_{r=0}^{M} a_r x^r. \tag{53}$$

It is natural to define then

$$\langle P_M(x)\rangle = Bi(\zeta)^{-1} P_M\left(\frac{d}{d\zeta}\right) Bi(\zeta). \tag{54}$$

In order that a probabilistic interpretation is possible, the matrix $\mathcal{P}_M(\zeta)$

$$\mathcal{P}_M(\zeta) = \begin{pmatrix} 1 & \langle x\rangle & \langle x^2\rangle & \dots & \langle x^M\rangle \\ \langle x\rangle & \langle x^2\rangle & \langle x^3\rangle & \dots & \langle x^{M+1}\rangle \\ & & & & \\ \langle x^M\rangle & & & & \langle x^{2M}\rangle \end{pmatrix} \tag{55}$$

must be positive (for some ζ_M and all M) at least for

$$\zeta > \zeta_M. \tag{56}$$

From the asymptotic expansion of the Airy function ([8], equ. 10.4.63) follows

$$\langle (P_M(x))^2\rangle \underset{\zeta\to\infty}{=} (P_M(\zeta^{\frac{1}{2}}))^2 + \text{lower order terms} \tag{57}$$

so that $\mathcal{P}_M(\zeta)$ has one positive eigenvalue for large ζ. It can be shown that the other eigenvalues are positive for large ζ, too. With this knowledge it suffices to calculate

$$D_M(\zeta) = \det \mathcal{P}_M(\zeta). \tag{58}$$

For low M we find (for $\zeta \to \infty$) e.g.

$$D_1(\zeta) = \frac{1}{2\zeta^{\frac{1}{2}}} + O(\zeta^{-2}) \tag{59}$$

$$D_2(\zeta) = \frac{1}{4\zeta^{\frac{3}{2}}} + O(\zeta^{-3}). \tag{60}$$

Assume that $D_M(\zeta) > 0$ for $\zeta \to \infty$ has been shown. Then ζ_M is the largest zero of D_M. For $M = 1$ we obtain (using the tables in [8])

$$\zeta_1 = 0.4003. \tag{61}$$

Finally we have to prove

$$\zeta_c = \sup_M \zeta_M < \infty \tag{62}$$

which is so far only wishful thinking.

In the Greensite-Halpern approach we have to solve

$$\left[-\frac{1}{2} \frac{\partial}{\partial x^2} + \frac{1}{8}(x^2 - \zeta)^2 \right] \chi_0(x) = \tilde{\epsilon}_1 \chi_0(x). \tag{63}$$

By symmetry we have

$$\langle x^{2n+1} \rangle_{GH} = 0, \text{ all n}. \tag{64}$$

Moreover we find e.g.

$$\langle x^2 \rangle_{GH} = \begin{cases} \zeta + \text{ lower order terms for } \zeta \to \infty, \\ O(|\zeta|^{\frac{1}{2}}) \text{ for } \zeta \to -\infty \end{cases} \tag{65}$$

as compared with

$$\langle x^2 \rangle = \zeta \tag{66}$$

from Airy's differential equation ([8], eqn. 10.4.1). The difference between the two approaches becomes more striking if we compare the dispersions

$$\langle x^2 \rangle - \langle x \rangle^2 = D_1(\zeta) \tag{67}$$

$$\langle x^2 \rangle_{GH} - (\langle x \rangle_{GH})^2 = \langle x^2 \rangle_{GH}. \tag{68}$$

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