DOUBLE SCALING LIMITS, AIRY FUNCTIONS AND MULTICRITICAL BEHAVIOUR IN O(N) VECTOR SIGMA MODELS

J. MAEDER and W. RÜHL
Department of Physics, University of Kaiserslautern, P.O.Box 3049
67653 Kaiserslautern, Germany
E-mail: ruehl@gypsy.physik.uni-kl.de

ABSTRACT

O(N) vector sigma models possessing catastrophes in their action are studied. Coupling the limit $N \to \infty$ with an appropriate scaling behaviour of the coupling constants, the partition function develops a singular factor. This is a generalized Airy function in the case of spacetime dimension zero and the partition function of a scalar field theory for positive spacetime dimension.

1. Introduction

Matrix models have been understood as representing stochastic triangulated surfaces and thus interpreted as quantum gravity theories. They are treated in the "double scaling limit" $N \to \infty$, $g \to g_c$ [1]. The same kind of limits can be applied to O(N) vector models [2,3,4] which are connected with statistical ensembles of polymers. These models have been studied with the usual $\frac{1}{N}$ expansion and renormalization group techniques [2,3,4,5]. This approach involves a certain amount of guess work.

If an action possesses a singularity (synonymous: a catastrophe) the double scaling limit is naturally defined by letting N tend to infinity and the deformation of the singularity go to zero. The partition function develops then a singular factor which is a generalized Airy function if spacetime has dimension zero or a partition function of a scalar field theory if this dimension is nonzero (positive). The Airy function depends in general on ν scale invariant parameters, where ν is the codimension of the singularity.

Singularities can be classified [6] and form s-dimensional families. The s parameters of these families are called "moduli". If $s = 0$ the families are discrete and are grouped into A, D, and E series. They are related with the corresponding Lie algebras by their symmetry. The A series can be realized in one-vector models, D and E series need two O(N) vectors at least. The class of quasihomogeneous singularities is the biggest class we have studied so far and shows some hitherto unobserved structures.

By application of diffeomorphisms singularities can be brought to a canonical form. These diffeomorphisms "reparametrize" the coupling constants. The canonical form contains the full information of the singularity and defines the universality class of the multicritical behaviour. The notation of universality class thus obtains mathematical rigour. But the problem of ill-defined (unstable) partition functions can also be approached by the theory of singularites. Stability is either obtained by using complex contours (as used for the Airy function $\mathrm{Bi}(\zeta)$) or by an appropriate infinitesimal deformation. Moreover, renormalization group equations and beta functions can be derived a posteriori.

In a zerodimensional sigma model not all field degrees of freedom participate in the singularity. The residual degrees behave Gaussian at the saddle point and are integrated over. They give rise to the regular part of the partition function and possess a standard $\frac{1}{N}$ expansion. In sigma models with nonzero spacetime dimension small momenta are also treated as deformation parameters. Large momenta are included in the Gaussian variables. The elimination of the Gaussian variables is a nontrivial procedure. It ought to be correct in the singular variables at least to the order of the singularity (the method used by Di Vecchia et. al. in [3,4] is wrong). In section 4 we describe this elimination in a nontrivial case.

2. Elementary catastrophes in zerodimensional spacetime

We outline here the results of [7]. Let Φ be an $O(N)$ vector field and $d\Phi$ be the Lebesgue measure on $\mathbb{R}_N$. Then we define the partition function

$$Z_N(g) = \int d\Phi \exp[-Ng(\Phi)] \tag{1}$$

where $g(\Phi)$ possesses the expansion

$$g(\Phi) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{g_k}{k}(\Phi^2)^k. \tag{2}$$

Formal, analytic or finite power series are permitted. After integration of the angles we obtain

$$Z_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_0^\infty \frac{dx}{x} e^{\frac{N}{2}f(x)} \tag{3}$$

$$f(x) = \log x - \sum_{k=1}^{\infty} \frac{g_k}{k} x^k. \tag{4}$$

We normalize g_1 to one.

If we require

$$g_k = 0,\ k \geq m+1 \tag{5}$$

we have

$$f^{(m+1)}(x) = (-1)^m \frac{m!}{x^{m+1}} \neq 0 \tag{6}$$

and a catastrophe of the type A_m can be produced at x_0 if

$$f^{(k)}(x_0) = 0 \text{ for all } 1 \leq k \leq m. \tag{7}$$

This is achieved if the coupling constants and x_0 are chosen as the critical values

$$g_k^c = (-1)^{k-1}\binom{m}{k}m^{-k} \tag{8}$$

$$x_0^c = m. \tag{9}$$

From (8) we have $g_m^c > 0$ only if m is odd. Thus the partition function (1) at the critical point is ill-defined. However, if instead of (5) we have

$$g_{m+1} = \epsilon > 0, \quad g_k = 0,\ k \geq m+2 \tag{10}$$

convergence is guaranteed.

The critical values (8), (9) define the canonical form of the A_m catastrophe. We define deformations by

$$g_k = g_k^c + \Theta_k \quad (2 \leq k \leq m) \tag{11}$$

$$x_0 = x_0^c + \Theta_0 \tag{12}$$

so that instead of (7) we have (for $1 \leq k \leq m-1$)

$$f^{(k)}(x_0^c + \Theta_0) = -(k-1)!\sum_{l=k}^{m}\binom{l-1}{k-1}m^{l-k}\Theta_l + \mathrm{O}_2(\Theta). \tag{13}$$

In fact, translation invariance permits us to submit the deformations (11), (12) to the constraint

$$f^{(m)}(x_0^c + \Theta_0) = 0 \tag{14}$$

which fixes Θ_0 in terms of $\{\Theta_k\}_2^m$.

Now we scale x as

$$x - x_0 = \lambda\eta \tag{15}$$

so that $N \to \infty$ and $\lambda \to 0$ combine to render the $(m+1)$st order term finite and normalized

$$\frac{N}{2}\frac{\lambda^{m+1}}{(m+1)!}|f^{(m+1)}(m)| = \frac{1}{m+1}. \tag{16}$$

This implies

$$\lambda = m\left(\frac{2}{N}\right)^{\frac{1}{m+1}}. \tag{17}$$

If each linear combination (13) scales as

$$f^{(k)}(m + \Theta_0) = a_k\left(\frac{2}{N}\right)^{\sigma_k} \tag{18}$$

for $N \to \infty$ then for $1 \leq k \leq m-1$

$$\lim_{N\to\infty}\frac{N}{2}\frac{\lambda^k}{k!}f^{(k)}(m + \Theta_0) = \zeta_k \tag{19}$$

is finite, provided

$$\sigma_k = 1 - \frac{k}{m+1} \tag{20}$$

and

$$\zeta_k = \frac{m^k}{k!} a_k. \tag{21}$$

The resulting partition function is to leading order in $\frac{1}{N}$

$$Z_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} e^{\frac{N}{2} f(m)} \left(\frac{2}{N}\right)^{\frac{1}{m+1}} \cdot Y(\zeta_1, \zeta_2, \ldots, \zeta_{m-1}) \tag{22}$$

where Y is the A_m-type generalized Airy function

$$Y(\zeta_1, \zeta_2, \ldots, \zeta_{m-1}) = \int_{C^{(m)}} d\eta \exp\left\{ \sum_{k=1}^{m-1} \zeta_k \eta^k + (-1)^m \frac{\eta^{m+1}}{m+1} \right\}. \tag{23}$$

If m is odd, $C^{(m)}$ is the real axis with positive orientation. If m is even, $C^{(m)}$ is chosen as a linear combination of complex contours each one running from infinity to infinity. The singular part of the free energy is

$$F_s(\zeta) = \log Y(\zeta_1, \zeta_2, \ldots, \zeta_{m-1}). \tag{24}$$

It satisfies Airy function differential equations and a renormalization group equation

$$\left\{ N \frac{\partial}{\partial N} - \sum_{k=2}^{m} \beta_k(\Theta) \frac{\partial}{\partial \Theta_k} \right\} F_s(\zeta(N, \Theta)) = 0 \tag{25}$$

where (13), (19) has been substituted for ζ_k. The $\{\zeta_k\}$ are the "double scale invariant" variables. The beta functions $\{\beta_k\}$ in (25) can be calculated to the first order in the $\{\Theta_k\}$.

3. Quasihomogeneous singularities

The singularities of Section 2 involve only one variable. If we want singularities with r variables we have to define a corresponding class of more complex models. We shall see that models with r vector fields have the desired properties.

Let

$$g(\Phi_1, \Phi_2, \ldots, \Phi_r) = \frac{1}{2} \left\{ \sum_{1 \le i \le r} g_i(\Phi_i^2) + \sum_{1 \le i < j \le r} s_{ij}((\Phi_i \Phi_j)^2) \right\} \tag{26}$$

and

$$Z_N^{(r)} = \int \prod_{i=1}^{r} d\Phi_i \exp\{-N g(\Phi_1, \Phi_2, \ldots, \Phi_r)\}. \tag{27}$$

We introduce the shorthands

$$x_i = \Phi_i^2 \tag{28}$$

$$e_i = \frac{\Phi_i}{\sqrt{x_i}} \tag{29}$$

$$t_{ij} = \frac{(\Phi_i \Phi_j)^2}{x_i x_j}. \tag{30}$$

With the uniform measure $d\Omega$ on S_{N-1} define the angular integrals

$$F(t) = \int \prod_{i=1}^{r} d\Omega(e_i) \prod_{j<k} \delta((e_j \cdot e_k)^2 - t_{jk}) \tag{31}$$

where only those (j,k) are included in the product for which s_{jk} is nonzero. If all s_{jk} are nonzero and N is big enough, $F(t)$ (31) contains the factor

$$\Gamma^{(r)}(e_1, e_2, \dots, e_r)^{\frac{N}{2}} \tag{32}$$

where $\Gamma^{(r)}$ is the Gram determinant of the arguments listed. It is a polynomial of $\{t_{jk}^{\frac{1}{2}}\}$ of degree r. For the partition function (27) we obtain

$$\begin{aligned} Z_N^{(r)} &= C_N^{(r)} \int_{\mathbb{D}} \prod_{j<k} dt_{jk}\, h(t) \\ &\cdot \int_{(\mathbb{R}_+)^r} \prod_{i=1}^{r} \frac{dx_i}{x_i} \exp\left[\frac{N}{2}\Psi(x_i, t_{jk})\right] \end{aligned} \tag{33}$$

where h is independent of N, $\mathbb{D}$ takes account of the Riemann sheets necessary, and the phase function (reduced action) Ψ is

$$\begin{aligned} \Psi &= \log\left\{(\prod_{i=1}^{r} x_i)\Gamma^{(r)}(e_1, e_2, \dots, e_r)\right\} \\ &- \sum_{1\le i\le r} g_i(x_i) - \sum_{1\le i<j\le r} s_{ij}(x_i x_j t_{ij}). \end{aligned} \tag{34}$$

The singular saddle point has to be found in $\mathbb{D} \times (\mathbb{R}_+)^r$.

Notation is simplified if we use

$$t_{ij} = t_{ji}, \quad s_{ij} = s_{ji}, \quad \zeta_{ij} = x_i x_j t_{ij}. \tag{35}$$

Then the saddle point S has to satisfy

$$x_i {g_i}'(x_i) + \sum_{j(\neq i)} \zeta_{ij} {s_{ij}}'(\zeta_{ij}) = 1, \ \text{(for all } i\text{)} \tag{36}$$

$$\frac{t_{jk}}{\Gamma^{(r)}} \frac{\partial \Gamma^{(r)}}{\partial t_{jk}} - \zeta_{jk} {s_{jk}}'(\zeta_{jk}) = 0, \ \text{(for all } (j,k),\ j<k\text{)}. \tag{37}$$

In the generic case this saddle point is Gaussian but we are interested in the case in which the Hessian has nonzero corank which is achieved by an adjustment of coupling constants. With

$$
\begin{aligned}
-x_i^2\frac{\partial^2\Psi}{\partial x_i^2} &= 1+x_i^2{g_i}''(x_i)+\sum_{j(\neq i)}{\zeta_{ij}}^2{s_{ij}}''(\zeta_{ij}) \\
&:= U_i \qquad (38)\\
-x_ix_j\frac{\partial^2\Psi}{\partial x_i\partial x_j} &= \zeta_{ij}{s_{ij}}'(\zeta_{ij})+\zeta_{ij}^2{s_{ij}}''(\zeta_{ij}) \\
&:= \sigma_{ij} \quad (i\neq j) \qquad (39)
\end{aligned}
$$

we obtain for $i \neq j$

$$
-x_it_{ij}\frac{\partial^2\Psi}{\partial x_i\partial t_{ij}}=\sigma_{ij} \qquad (40)
$$

and for i, j, k pairwise different

$$
-x_it_{jk}\frac{\partial^2\Psi}{\partial x_i\partial t_{jk}}=0. \qquad (41)
$$

Thus a corank r of the Hessian can be achieved if in addition to (36), (37)

$$
\begin{aligned}
U_i &= 0 \quad (\text{all } i) \qquad (42)\\
\sigma_{jk} &= 0 \quad (\text{all } (j,k),\ j<k). \qquad (43)
\end{aligned}
$$

The location of the singularity S is at $\{x_i^{(0)}, t_{jk}^{(0)}\}$ and we use normalized variables

$$
\begin{aligned}
\xi_i &= \frac{x_i-x_i^{(0)}}{x_i^{(0)}} \qquad (44)\\
\tau_{jk} &= \frac{t_{jk}-t_{jk}^{(0)}}{t_{jk}^{(0)}}. \qquad (45)
\end{aligned}
$$

For $r = 2$ we have performed the calculation explicitly [7].

After integration of the Gaussian degrees of freedom $\{\tau_{jk}\}$ we have an r variable action

$$
\frac{N}{2}f(\xi) \qquad (46)
$$

and the singular partition function

$$
Z_{N,s}^{(r)}=\int d^r\xi\, h(\xi)\exp\left[\frac{N}{2}f(\xi)\right]. \qquad (47)
$$

Assume $f(\xi)$ is a deformation of a critical function

$$
f_c(\xi)=f_\delta(\xi)+f_{>\delta}(\xi) \qquad (48)
$$

where f_δ is quasihomogeneous of degree δ and type α and $f_{>\delta}$ has higher degree than δ. To explain these concepts consider a monomial of ξ

$$\xi^{\vec{k}} = \xi_1^{k_1} \xi_2^{k_2} \dots \xi_r^{k_r} \tag{49}$$

and apply the multiplicative group $\mathbb{R}_+$ to ξ by

$$\xi_i \to \lambda^{\alpha_i} \xi_i, \; \alpha_i \in Q \tag{50}$$

so that

$$\xi^{\vec{k}} \to \lambda^{\vec{k} \cdot \vec{\alpha}} \xi^{\vec{k}}. \tag{51}$$

Then $\xi^{\vec{k}}$ is called "of degree δ and of type $\vec{\alpha}$" if

$$\vec{k} \cdot \vec{\alpha} = \delta. \tag{52}$$

A polynomial is quasihomogeneous if each monomial contained in it satisfies (52).

Consider as an example the singularity W_{10} ($r = s = 2$). Its canonical form is [6]

$$\xi_1^4 + (a_0 + a_1 \xi_2) \xi_1^2 \xi_2^3 + \xi_2^6 \tag{53}$$

with a_0, a_1 as moduli ($a_0 \neq \pm 2$) and the principal quasihomogeneous part

$$f_\delta(\xi) = \xi_1^4 + a_0 \xi_1^2 \xi_2^3 + \xi_2^6 \tag{54}$$

with

$$\vec{\alpha} = (\frac{1}{4}, \frac{1}{6}), \; \delta = 1 \tag{55}$$

$$f_{>\delta}(\xi) = a_1 \xi_1^2 \xi_2^4. \tag{56}$$

We will always adjust $\vec{\alpha}$ so that $\delta = 1$.

Now we consider a deformation of $f_c(\xi)$

$$f(\xi) = \sum_{\delta=0}^{1} f_\delta(\xi) + f_{>1}(\xi) \tag{57}$$

obtained by a change of coupling constants

$$\{\gamma_n^c\} \to \{\gamma_n^c + \Theta_n = \gamma_n\} \tag{58}$$

and expansion points

$$\{\xi_i = 0\}_1^r \to \{\Delta_i\}_1^r. \tag{59}$$

We can impose r constraints from translation invariance so that the $\{\Delta_i\}_1^r$ are expressed linearly in terms of the $\{\Theta_n\}$. The scaling is done such that the degree $\delta = 1$ terms absorb the factor N

$$\frac{N}{2} \cdot \lambda = 1. \tag{60}$$

Each term of lower degree $0 < \delta < 1$

$$f_\delta(\xi) = \sum_{\substack{\vec{k} \\ (\vec{\alpha}\cdot\vec{k}=\delta)}} t_{\delta,\vec{k}} \xi^{\vec{k}} \tag{61}$$

can be expanded in the $\{\Theta_n\}$

$$t_{\delta,\vec{k}} = \sum_n \mathcal{N}_{\delta,\vec{k};n} \Theta_n + \mathrm{O}_2(\Theta) \tag{62}$$

if it is not kept equal to zero by a translational invariance constraint. Then we perform the double scaling

$$Z_{\delta,\vec{k}} = \lim_{\substack{N\to\infty \\ \forall\Theta_n\to 0}} \left(\frac{N}{2}\right)^{1-\delta} t_{\delta,\vec{k}}. \tag{63}$$

In the singular partition function appears the generalized Airy function

$$\Phi[Z] = \int_{\tilde{C}} d^r\xi \exp[-Z(\xi)] \tag{64}$$

$$Z(\xi) = \sum_{0<\delta\le 1} \sum_{\substack{\vec{k} \\ (\vec{\alpha}\cdot\vec{k}=\delta)}} Z_{\delta,\vec{k}} \xi^{\vec{k}}. \tag{65}$$

Of course the $\delta = 1$ part of $Z(\xi)$ can be normalized to the canonical form. The number of arguments of Φ is thus reduced to the codimension of the singularity. If the sigma model studied is too "narrow", some of these arguments may be missing (see X_9 singularity in [7]). The contour $\tilde{C}$ belongs to an equivalence class which runs from infinity to infinity and is thus invariant under translations and dilations (51).

In [8] we derive string equations for these models. The singular partition function serves as a ground state functional for the W-algebra and is invariant under a "triangular subalgebra" depending only on the type $\vec{\alpha}$. If the singularity is homogeneous and

$$\vec{\alpha} = \alpha_0(\frac{1}{r}, \frac{1}{r}, \dots, \frac{1}{r}) \tag{66}$$

this subalgebra is the inhomogeneous $gl(r)$ (r translations in coupling constant space as invariant subalgebra).

4. Nonzero spacetime dimensions

In the case of spacetime dimensions $D > 0$ we obtain partition functions for r scalar fields instead of Airy functions. We will sketch here only the case of the A_m singularities ($r = 1$). A partly incomplete, partly wrong discussion of these cases was given by Di Vecchia et. al. in [3,4].

Consider the action

$$S = \int d^Dx \left\{ \frac{1}{2}\partial_\mu\Phi\partial_\mu\Phi(x) + \frac{1}{2}\beta^2\Phi^2(x) + U(\Phi^2)(x) \right\} \tag{67}$$

with the potential

$$U(\sigma) = \sum_{n=2}^{\infty} \frac{f_n}{n} \sigma^n. \tag{68}$$

Then a standard trick gives us the partition function in terms of two scalar fields σ and ρ

$$Z = \int D\sigma D\rho \exp[-N S_{eff}(\sigma, \rho)] \tag{69}$$

with

$$\begin{aligned} S_{eff} &= \int d^D x [U(\sigma)(x) - i\sigma(x)\rho(x)] \\ &+ \frac{1}{2} \text{Tr} \log [-\Delta + \beta^2 + 2i\rho]. \end{aligned} \tag{70}$$

The saddle point is assumed at constant values σ_0, ρ_0

$$U'(\sigma_0) = i\rho_0 \tag{71}$$

$$\sigma_0 = \int \frac{d^D p}{(2\pi)^D} (p^2 + m^2)^{-1} \tag{72}$$

$$m^2 = \beta^2 + 2i\rho_0. \tag{73}$$

The dimensions D are interpolated and firstly restricted to the interval $0 < D < 2$, so that renormalizations are not necessary.

The fields σ and ρ fluctuate around their stationary values (71)-(73)

$$\sigma(x) = \sigma_0(1 + \alpha(x)) \tag{74}$$

$$\rho(x) = \rho_0(1 + \beta(x)). \tag{75}$$

In terms of the Fourier transforms of α and β the Hessian is

$$\begin{aligned} S^{(2)}_{eff} &= \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} (\hat{\alpha}(-k), \hat{\beta}(-k)) \\ & \begin{pmatrix} \sigma_0^2 U''(\sigma_0) & -i\sigma_0\rho_0 \\ -i\sigma_0\rho_0 & 2\rho_0^2 \Sigma(k) \end{pmatrix} \begin{pmatrix} \hat{\alpha}(k) \\ \hat{\beta}(k) \end{pmatrix} \end{aligned} \tag{76}$$

where

$$\Sigma(k) = \int \frac{d^D p}{(2\pi)^D} [(p^2 + m^2)((p-k)^2 + m^2)]^{-1}. \tag{77}$$

The Hessian is diagonalized by

$$\begin{pmatrix} \hat{\alpha}(k) \\ \hat{\beta}(k) \end{pmatrix} = \begin{pmatrix} a(k) \\ i \end{pmatrix} \hat{\xi}(k) + \begin{pmatrix} -ib(k) \\ 1 \end{pmatrix} \hat{\eta}(k) \tag{78}$$

$$S_{eff}^{(2)} = \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\left\{\lambda_+(k)N_+(k)\hat{\xi}(-k)\hat{\xi}(k) + \lambda_-(k)N_-(k)\hat{\eta}(-k)\hat{\eta}(k)\right\} \tag{79}$$

where $N_\pm(k)$ are the squares of the eigenvectors

$$N_+(k) = a(k)^2 - 1, \quad N_-(k) = 1 - b(k)^2 \tag{80}$$

and it turns out that

$$a(k)b(k) = 1. \tag{81}$$

The eigenvalues $\lambda_\pm(k)$ are

$$\lambda_\pm(k) = \frac{1}{2}\left\{U''(\sigma_0)\sigma_0^2 + 2\rho_0^2\Sigma(k) \pm [(U''(\sigma_0)\sigma_0^2 - 2\rho_0^2\Sigma(k))^2 - 4\sigma_0^2\rho_0^2]^{\frac{1}{2}}\right\}. \tag{82}$$

We will make $\lambda_-(k)$ zero at $k=0$. Then $\lambda_+(k)$ is positive and $\hat{\xi}(k)$ a Gaussian degree of freedom. If we put the system into a box with periodic boundary conditions, then the spectrum of k is discrete and only $\hat{\eta}(0)$ is non-Gaussian, whereas $\hat{\eta}(k)$, $k \neq 0$, are Gaussian. In this case the saddle point for the Gaussian integration is determined by

$$\frac{\partial S_{eff}}{\partial \hat{\xi}(k)} = 0, \text{ all } k \tag{83}$$

$$\frac{\partial S_{eff}}{\partial \hat{\eta}(k)} = 0, \text{ all } k \neq 0 \tag{84}$$

and its location depends on $\hat{\eta}(0)$. Because of translational invariance these equations are solved by

$$\hat{\xi}(k) = \hat{\eta}(k) = 0 \text{ for } k \neq 0 \tag{85}$$

and with

$$S_{eff}|_{\hat{\xi}(k)=\hat{\eta}(k)=0,\, k\neq 0} = \tilde{S}_{red}(\hat{\xi}(0), \hat{\eta}(0)) \tag{86}$$

by additionally

$$\frac{\partial}{\partial \hat{\xi}(0)}\tilde{S}_{red}(\hat{\xi}(0), \hat{\eta}(0)) = 0 \tag{87}$$

which means

$$-\lambda_+(0)N_+(0)\hat{\xi}(0) = \sum_{n=3}^{\infty} \frac{\partial}{\partial \hat{\xi}(0)}\tilde{S}_{red}^{n}(\hat{\xi}(0), \hat{\eta}(0)). \tag{88}$$

Here the superscript n denotes the order in both arguments.

An iterative solution of (88) gives

$$\hat{\xi}(0) = H(\hat{\eta}(0)) = \sum_{l=2}^{\infty} a_l \hat{\eta}(0)^l \tag{89}$$

and

$$\begin{aligned} S_{red}(\hat{\eta}(0)) &= \tilde{S}_{red}(H(\hat{\eta}(0)), \hat{\eta}(0)) \\ &= \sum_{n=2}^{\infty} \frac{g_n}{n} \hat{\eta}(0)^n. \end{aligned} \tag{90}$$

Thus after the Gaussian integration we end up with a zerodimensional case as treated in Section 2. First one calculates the critical coupling constants g_n^c in (90) and in turn the critical constants f_n^c in the potential $U(\sigma)$.

Now we turn to the case of physical interest, namely infinite volume and a continuous momentum spectrum. In the case of the singularity A_m in (90) the coupling constants are deformed as usual

$$f_n = f_n^c + \Theta_n \; (2 \leq n \leq m). \tag{91}$$

In addition the domain of small momenta

$$|k| < \Lambda \tag{92}$$

is considered as deformation, whereas large momenta

$$|k| > \Lambda \tag{93}$$

belong to Gaussian degrees of freedom $\hat{\xi}(k), \hat{\eta}(k)$ and are integrated over. Our aim is a partition function of a scalar field ϕ in conventional form

$$\begin{aligned} \int D\phi \exp \Big\{ - \Big[\frac{1}{2} \int d^D x \, \phi(x)(-\Delta + M^2)\phi(x) \\ + \sum_{l=3}^{m} \zeta_l \int d^D x \, \phi(x)^l + \frac{F_{m+1}}{m+1} \int d^D x \, \phi(x)^{m+1} \Big] \Big\} \end{aligned} \tag{94}$$

which replaces the Airy function (23). Contrary to the Airy function there is no linear term in ϕ, there is a term ϕ^m, and normalization is applied to the kinetic energy and not to ϕ^{m+1}. Moreover the mass term $M^2\phi^2$ must be positive. This can be achieved by approaching the canonical form of the singularity in the neighbourhood of deformations on a specific path, namely on a surface of stable A_1 singularities contingent to A_m.

In the first step we reduce S_{eff} to $S_{red}(\hat{\eta}(k))$, $|k| < \Lambda$, that possesses a power series expansion analogous to (90) in terms of convolution powers $(\hat{\eta})_*^n$ for $n \geq 3$. We expand $\lambda_-(k)$ to first order in k^2 and the coupling constant deformations $\{\Theta_n\}$ (91). Then we scale k by

$$k = N^{-\lambda} k' \tag{95}$$

$$x = N^{\lambda} x'. \tag{96}$$

The exponent λ must be positive to map the domain of small k on $\mathbb{R}_D$ for large N. As in Section 2 wee firstly examine the term ϕ^{m+1} which entails

$$\phi(x') = C^{(m)} N^{\frac{1+D\lambda}{m+1}} \eta(x). \tag{97}$$

With this ansatz we enter the kinetic energy term and find

$$\lambda = \frac{m-1}{2(m+1) - D(m-1)} \tag{98}$$

and

$$C^{(m)} = \sqrt{2\Pi_2 (i\rho_0)^2 \frac{1}{6}(2 - \frac{D}{2})\frac{1}{m^2}} \tag{99}$$

with

$$\Pi_n = \int \frac{d^D k}{(2\pi)^D} (k^2 + m^2)^{-n}. \tag{100}$$

The mass M^2 is obtained from a double scaling constraint on the coupling constants

$$\lim_{\substack{N\to\infty \\ \forall\Theta\to 0}} \left(\sum_{n=2}^{m} (n-1)\Theta_n \sigma_0^{n-2} \right) = \frac{1}{2\Pi_2}\frac{1}{6}(2 - \frac{D}{2})\frac{M^2}{m^2}. \tag{101}$$

This constraint fixes the limiting contour on the stable A_1 surface.

From (91) we obtain for the intermediate coupling constant g_n in (90)

$$g_n - g_n^c = n \sum_{l=2}^{m} \alpha_{nl}^{(m)} \Theta_l. \tag{102}$$

Then the double scaling limit defines the scale invariant variables ζ_n

$$\zeta_n = \lim_{\substack{N\to\infty \\ \forall\Theta\to 0}} (C^{(m)})^{-n} N^{\chi_n^{(m)}} \sum_{l=2}^{m} \alpha_{nl}^{(m)} \Theta_l \tag{103}$$

where the critical indices are

$$\chi_n^{(m)} = \frac{2(m+1-n)}{2(m+1) - D(m-1)} \tag{104}$$

$$\chi_2^{(m)} = 2\lambda \quad \text{(see (98))}. \tag{105}$$

The coupling constant F_{m+1} in (94) is a critical value and thereby fixed

$$F_{m+1} = g_{m+1}^c (C^{(m)})^{-m-1}. \tag{106}$$

However, this coupling constant varies with the mass of the fundamental field Φ.

Now we consider the interval of dimensions $2 < D < 4$. Then Π_1 but no Π_n, $n \geq 2$ (100), diverges. This entails that σ_0, ρ_0 and the bare coupling constants f_n are infinite and must be renormalized. First we continue Π_1 analytically from $0 < D < 2$ to $2 < D < 4$ and call this function Π_1^{an}

$$\Pi_1^{an} = \int \frac{d^D p}{(2\pi)^D} [(p^2 + m^2)^{-1} - (p^2)^{-1}]. \tag{107}$$

Then we replace the saddle point equation (72) by the renormalized equation

$$\sigma_0^{ren} = \Pi_1^{an} = \sigma_0 - \sigma_\infty \tag{108}$$

$$\sigma_\infty = \int \frac{d^D p}{(2\pi)^D} (p^2)^{-1} \text{ (divergent)}. \tag{109}$$

We insert σ_0 (108) into the potential (68) and reorder to obtain

$$U'(\sigma_0) = \frac{1}{2}\Delta m^2 + \sum_{n=2}^{m} f_n^{ren} (\sigma_0^{ren})^{n-1} \tag{110}$$

$$= \frac{1}{2}\Delta m^2 + U^{ren\prime}(\sigma_0^{ren}). \tag{111}$$

The divergent term Δm^2 is absorbed in $i\rho_0^{ren}$ and finally in the renormalized mass.

After these renormalizations all arguments go through as long as λ (98) is positive, i.e. for

$$D < D_\infty = 2\,\frac{m+1}{m-1}. \tag{112}$$

But this is the classical condition for the partition function (94) to be superrenormalizable.

For the renormalizable case $D = D_\infty$ the scaling procedure is not defined due to the divergence of the critical indices $\chi_n^{(m)}$ (104) and λ (98). We can however renormalize the parameter N by introducing

$$N'(D) = N^{\frac{1}{2(m+1) - D(m-1)}} \tag{113}$$

so that the scaling procedure defined in terms of N' is analytic in D. Therefore we propose to treat the renormalizable case for all $2 < D < 4$ by analytic continuation in D as well.

Details about how the results presented in this section are derived and more details can be found in [9].

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