Double Scaling Limits and Catastrophes of the zerodimensional O(N) Vector Sigma Model: The A-Series

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Abstract

We evaluate the partition functions in the neighbourhood of catastrophes by saddle point integration and express them in terms of generalized Airy functions.

1 Introduction

The vector models have attracted interest as one-dimensional quantum gravity theories just as the matrix models are interpreted as representing two dimensional quantum gravity theories. A connection with polymer models has also been emphasized from the outset [2]. Their double scaling limit has been studied with the usual renormalization group and $1/N$ expansion methods. In the zero dimensional case the beta function and the free energy have been calculated for large N exactly.

We will show that the double scaling limits for this most elementary model can be calculated exactly, i.e. asymptotically to any order in a $N^{-\frac{1}{m+1}}$-expansion. Catastrophes are singularities of differentiable maps [6, 7] and by diffeomorphisms can be transformed to canonical forms. We will study such canonical forms only. It does not make sense to reshape these canonical forms by application of diffeomorphisms.

There are elementary and nonelementary catastrophes. The elementary ones are ordered into A, D and E cases [6]. Whereas vector models with one vector field can exhibit only A-series catastrophes (A_m, $m \in \mathbb{N}$), models with two vector fields can also posses D or E series catastrophes . The nonelementary catastrophes show up also first in two-field models.

Application of catastrophe theory to zero dimensional O(N) sigma models leads to a wealth of useful information which can be used as a guideline for studies of more complicated models. The diffeomorphisms which are basic in catastrophe theory replace the 'reparametrisations in

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coupling constant space' ; those properties of catastrophes which hold true independently of diffeomorphisms (characterize cosets of the diffeomorphism group) map onto universal features of phase transitions. This is of course not new. Nevertheless our analysis will lead to some new insights.

We define the zero dimensional O(N)vector models by

$$\mathcal{Z}_N(g) \;=\; \int d\Phi\,\exp\Big\{-N\,g(\Phi)\Big\}, \qquad d\Phi\;:\;\text{Lesbeque measure},\quad \Phi\in I\!R_{I\!N} \tag{1.1}$$

where $g(\Phi)$ is O(N) invariant and has the asymptotic expansion

$$g(\Phi) \;\simeq\; \frac{1}{2}\sum_{k=1}^{\infty}\frac{g_k}{k}\left(\Phi^2\right)^k \tag{1.2}$$

Constraints on the $\{g_k\}$ lead to catastrophes that dominate the large N behaviour of the partition function (1.1) through saddle point expansions. Main ingredients in these expansions are (real nonoscillating) generalized Airy functions.

The free energy is defined by

$$F_N(g) \;=\; -\frac{1}{N}\log\mathcal{Z}_N(g)\;-\;\frac{1}{2}\log\frac{Ng_1}{2\pi} \tag{1.3}$$

where this normalization is such that

$$F_N(g)\Big|_{g_k=0\ \forall k\geq 2} \;=\; 0 \tag{1.4}$$

The simplest catastrophe is of A_1 type (or Morse or Gaussian). We set

$$g_1\;=\;1,\quad g_2\;\geq\;0,\quad g_k\;=\;0\ \forall k\geq 3 \tag{1.5}$$

then $\mathcal{Z}_N$ can be expanded into a $1/N$ expansion around a Gaussian saddle point

$$F_N(g_2)\Big|_{N=\infty} \;=\; \frac{1}{2}\log\frac{1+\sqrt{\Delta}}{2}\;+\;\frac{1}{2}\frac{1}{1+\sqrt{\Delta}}\;-\;\frac{1}{4} \tag{1.6}$$

with

$$\Delta \;=\; 1\;+\;4g_2 \tag{1.7}$$

This function satisfies the (large N) Callan-Symanzik equation.

$$\left(N\frac{\partial}{\partial N}-\beta(g_2)\frac{\partial}{\partial g_2}+\gamma(g_2)\right)\,F_N(g_2) \;=\; R_N(g_2) \tag{1.8}$$

Assume we know that

$$\gamma(g_2) \;=\; 1 \tag{1.9}$$

for the free energy. We can continue $F_N(g_2)\Big|_{N=\infty}$ analytically off the positive real axis till the neighbourhood of

$$g_* \;=\; -\frac{1}{4} \tag{1.10}$$

where $F_N(g_2)\Big|_{N=\infty}$ has a branch point in the variable Δ

$$\begin{aligned} F_N(g_2)\Big|_{N=\infty} &= g(\Delta) + \Delta^{\frac{3}{2}} h(\Delta) \\ \Big(h(0) &= -\frac{1}{3}\Big) \end{aligned} \tag{1.11}$$

both functions g, h being analytic.

Now the singular part is defined to satisfy the homogeneous Callan-Symanzik (renormalization group or RG) equation. It follows

$$\beta(g_2)^{-1} \;=\; \frac{\partial}{\partial g_2} \log\Big(\Delta^{\frac{3}{2}} h(\Delta)\Big) \tag{1.12}$$

$$\beta(g_2) \;=\; \frac{2}{3}(g_2 - g_*) + \mathcal{O}\Big((g_2 - g_*)^2\Big) \tag{1.13}$$

and

$$R_N(g_2) \;=\; \Big(-\beta(g_2)\frac{\partial}{\partial g_2} + 1\Big)\, g(\Delta) \tag{1.14}$$

The results on β, F_n, R_N given in [1] are thus reproduced.

In the neighbourhood $g_2 \simeq g_*$ it is guessed that the singular part of $F_N(g_2)_{\rm sing}$ (which at $N=\infty$ is equal $\Delta^{\frac{3}{2}} h(\Delta)$) possesses the series expansion

$$F_N(g_2)_{\rm sing} \;=\; \sum_{h=0}^{\infty} a_h\, N^{-h}\, (g_2 - g_*)^{2-\gamma_0-\gamma_1 h} \tag{1.15}$$

This is based on the fact that each term of (1.15) satisfies the RG equation when

$$\gamma_1 \;=\; \tfrac{1}{\beta'(g_*)} \;=\; \tfrac{3}{2}, \qquad \gamma_0 \;=\; 2 - \tfrac{\gamma(g_*)}{\beta'(g_*)} \;=\; \tfrac{1}{2} \tag{1.16}$$

However: this argument is too simple, since the RG equation does not determine h to be an integer. If we apply derivation of h to any term in (1.15) we obtain also a permitted contribution

$$a'_h\, N^{-h}\, (g_2 - g_*)^{2-\gamma_0-\gamma_1 h}\, \log\Big[N(g_2 - g_*)^{\gamma_1}\Big] \tag{1.17}$$

In fact such term for $h = 1$ exists.

The g_2 parameter defines a curve of A_1 catastrophes which at $g_2 = g_*$ ends in a A_2 catastrophe. But at this catastrophe and in a neighbourhood we can derive $F_N(g_2)_{\rm sing}$ directly (all of $F_N(g_2)$ in fact). The result (Section 3) is

$$F_N(g_2)_{\rm sing} \;=\; -\frac{1}{N}\log \mathrm{Bi}(\zeta) \tag{1.18}$$

$$\zeta \;=\; \Big(\frac{N}{2}\Big)^{\frac{2}{3}} (g_2 - g_*) \tag{1.19}$$

and Bi is the usual Airy function. Moreover the expansion (1.15) is asymptotic only and

$$a'_1 \;=\; +\frac{1}{6} \tag{1.20}$$

In section 4 we deform A_2 catastrophes by g_3. The aim is to render the partition function for canonical A_2 convergent. A similiar problem with convergence appears for all A_m, m even, and can be solved the same way. In section 5 we discuss the canonical A_m catastrophes for general m and in section 6 we derive the corresponding β-functions.

2 A pedagogical example of mathematical interest

The confluent hypergeometric function

$${}_1F_1(\alpha;\gamma;z) \;=\; \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n}\frac{z^n}{n!}, \qquad (\alpha)_n \;=\; \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{2.1}$$

is a well studied transcendental function known in mathematical physics since more than hundred years. Nevertheless the asymptotic behaviour in the limit

$$\alpha \to \infty, \qquad z \to \infty, \qquad \gamma \quad \text{fixed} \tag{2.2}$$

$$\frac{\alpha}{z} \;=\; \xi, \tag{2.3}$$

$$\xi \to -\frac{1}{4}, \quad \text{so that} \quad (1+4\xi)z^{\frac{2}{3}} \quad \text{is again fixed} \tag{2.4}$$

is not covered in the textbook literature (see e.g. Luke's otherwise extremly useful treatise [3]). This is a typical 'double scaling limit'. The standard approach is a saddle point integration technique.

The integral representation for ${}_1F_1$ is ([4], 9.211.2)

$${}_1F_1(\alpha;\gamma;z) \;=\; \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 dt t^{-1}(1-t)^{\gamma-1}\exp\left\{zt+\alpha\log t-\alpha\log(1-t)\right\} \tag{2.5}$$

for $\Re(\alpha)>0$, $\Re(\gamma-\alpha)>0$. Though the asymptotic region (2.2) lies outside the convergence domain of (2.5), the saddle point expansion we shall derive remains valid. We consider only the contribution of the saddle point and not those of the boundaries. In physical applications one must be more careful. The relevant contribution is always the one which dominates the asymptotic behaviour.

Define using (2.3)

$$f(t) \;=\; t+\xi\log\frac{t}{1-t} \tag{2.6}$$

as 'phase function'. There are two extrema of $f(t)$ at $t_\pm$ if $\xi > -1/4$ and none if $\xi < -1/4$

$$t_\pm \;=\; \frac{1}{2}\left(1\pm\sqrt{\Delta}\right) \tag{2.7}$$

$$\Delta \;=\; 1+4\xi \tag{2.8}$$

For $\Delta = 0$ we obtain a point of inflexion at

$$t_0 \;=\; \frac{1}{2} \tag{2.9}$$

We expand $f(t)$ around t_0

$$f(t) \;=\; f(t_0) + (t - t_0)f'(t_0) + \frac{1}{6}(t - t_0)^3 f'''(t_0) + \mathcal{O}\Big((t - t_0)^4\Big) \tag{2.10}$$

with

$$f(t_0) \;=\; \frac{1}{2}, \qquad f'(t_0) \;=\; \Delta, \qquad f'''(t_0) \;=\; 32\xi \tag{2.11}$$

where we may approximate

$$f'''(t_0) \;=\; -8 \tag{2.12}$$

Now we scale the integration variable so that

$$t - t_0 \;=\; \lambda\,\eta \tag{2.13}$$

$$z\,\frac{1}{3!}\,f'''(t_0)\lambda^3 \;=\; -\frac{1}{3} \tag{2.14}$$

$$\lambda \;=\; (4z)^{-\frac{\lambda}{3}} \tag{2.15}$$

Thus the leading part of ${}_1F_1$ coming from this saddle point is

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} e^{zf(t_0)} t_0^{-1}(1 - t_0)^{\gamma - 1}(4z)^{-\frac{1}{3}}\,\Phi(\zeta) \tag{2.16}$$

where $\Phi(\zeta)$ is of greatest interest for us

$$\Phi(\zeta) \;=\; \int_C d\eta e^{\zeta\eta - \frac{1}{3}\eta^3} \tag{2.17}$$

$$\zeta \;=\; 4^{-\frac{1}{3}} z^{\frac{2}{3}} \Delta \tag{2.18}$$

As a rule we use the average over two 'least deformed real axis' contours on which (2.17) converges. Define

$$r \in Q: \qquad C_r \;=\; \text{contour from zero to } e^{2\pi i r}\infty \text{ along a ray} \tag{2.19}$$

Then in (2.17) we set

$$C \;=\; C_0 - \frac{1}{2}(C_{\frac{1}{3}} + C_{-\frac{1}{3}}) \tag{2.20}$$

With the ray integrals

$$R_r^{(3)}(\zeta) \;=\; \int_{C_r} d\eta e^{\zeta\eta - \frac{1}{3}\eta^3} \tag{2.21}$$

we get

$$\Phi(\zeta) \;=\; R_0^{(3)}(\zeta) - \frac{1}{2}\Big(R_{\frac{1}{3}}^{(3)}(\zeta) + R_{-\frac{1}{3}}^{(3)}(\zeta)\Big) \tag{2.22}$$

Using small ζ expansions to identify functions we find ([5], 10.4.3)

$$\Phi(\zeta) \;=\; \pi\,\mathrm{Bi}(\zeta) \tag{2.23}$$

$$\frac{-i}{2}\Big(R_{\frac{1}{3}}^{(3)}(\zeta) - R_{-\frac{1}{3}}^{(3)}(\zeta)\Big) \;=\; \pi\,\mathrm{Ai}(\zeta) \tag{2.24}$$

From ([5], figs 10.6 and 10.7) we see that

$$\mathrm{Bi}(\zeta_0) \,=\, 0, \qquad \zeta_0 \,=\, -1.173 \tag{2.25}$$

and $\mathrm{Bi}(\zeta)$ oscillates for $\zeta < \zeta_0$ and is positive for $\zeta > \zeta_0$. The function $\mathrm{Ai}(\zeta)$ oscillates everywhere. This justifies the choice of of the contour C (2.20) for $\zeta > \zeta_0$.

The Airy functions possesses a large ζ asymptotic expansion ([5], 10.4.18 and 10.4.63) which is obtained by keeping $\xi \neq -\frac{1}{4}$ fixed and by expanding $f(t)$ around the extrema $t_\pm$ (2.7). This is a saddle point expansion of A_1 type and can be used to obtain information on the domain where Φ is real non oscillating. Moreover we need it to recover the expansion (1.15) and to correct it (Section 3).

The residue $R(t)$ of $f(t)$ in (2.10) which is $\mathcal{O}((t-t_0)^4)$ and has been neglected still can be expanded systematically

$$\exp\Big\{zR(t)\Big\} \;\simeq\; 1 + \sum_{n=1}^{\infty}\sum_{k=2n}^{\infty} a_{n,k}\, z^n\, (t-t_0)^{2k} \tag{2.26}$$

with $a_{n,k}$ polynomials in the derivatives at t_0. Moreover the function

$$B(t) \;=\; t^{-1}\,(1-t)^{\gamma-1} \tag{2.27}$$

can be expanded

$$B(t) \;=\; B(t_0)\,\Big(1 + \sum_{r=1}^{\infty} b_r (t-t_0)^r\Big) \tag{2.28}$$

Inserting both expansions into the partition function and submitting it to the same procedure as before we get (2.16) with $\Phi(\zeta)$ replaced by

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=2n}^{\infty} 2^{-\frac{2}{3}(2k+r)} a_{n,k}\, b_r z^{n-\frac{1}{3}(2k+r)}\Phi^{(2k+r)}(\zeta), \quad (a_{0,0} = b_0 = 0) \tag{2.29}$$

which is an asymptotic expansion in powers of $z^{-\frac{1}{3}}$.

3 Elementary catastrophes , in particular A_2

Families of polynomials

$$\zeta_1 t + \zeta_2 t^2 + \ldots + \zeta_{k-1} t^{k-1} \pm t^{k+1} \tag{3.1}$$

define a catastrophe of type A_k with $\{\zeta_1, \zeta_2, \ldots, \zeta_{k-1}\}$ as deformation parameters. Saddlepoint expansions around deformed catastrophes are dealt with in the encyclopadic treatise [7]. Many questions relevant to our problem remain unanswered. In particular we would like to know where the generalized Airy functions $\Phi(\zeta_1, \zeta_2, \ldots, \zeta_{k-1})$ are real positive. This domain in $\mathbb{R}_{k-1}$ is bounded by a $(k-2)$-dimensional surface which can only be determined numerically. In section 5 we will learn that this question is relevant for even k only.

In this connection the asymptotic behaviour of the functions Φ is of interest. But even for the Pearcy function [8]

$$\Phi(\zeta_1, \zeta_2)$$

which is the last one in this series carrying a name, the above series are unknown.

Let us return to (1.1) now and perform the angle integration

$$\mathcal{Z}_N(g) \;=\; \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_0^\infty \frac{dt}{t} \exp\left\{\frac{N}{2} f(t)\right\} \tag{3.2}$$

with

$$f(t) \;=\; \log t - \sum_{k=1}^{\infty} \frac{g_k}{k} t^k \tag{3.3}$$

We shall always normalize

$$g_1 \;=\; 1 \tag{3.4}$$

We want to evaluate (3.2) in the limit of large N. If

$$f'(t_0) = 0, \qquad f''(t_0) \neq 0 \tag{3.5}$$

we have a Gaussian integral as leading term (A_1 catastrophe or Morse singularity) implying a pure $1/N$ expansion. The A_2 case arises if

$$f'(t_0) = 0, \qquad f''(t_0) = 0, \qquad f'''(t_0) \neq 0 \tag{3.6}$$

For this case to occur it is sufficient to have g_2 as only coupling constant

$$g_k \;=\; 0, \qquad k \geq 3 \tag{3.7}$$

The integral (3.2) can then be evaluated as a sum of two ${}_1F_1$-functions, which can be treated as in the preceding section. But the result thus obtained can be directly derived from the integral (3.2) by a saddle point technique.

Solving (3.6) with (3.3), (3.7) gives

$$t_0 = (-g_2)^{-\frac{1}{2}} = \frac{1}{2} \qquad (g_2 < 0) \tag{3.8}$$

$$f'''(t_0) = 16 \tag{3.9}$$

The deformation of the catastrophe is achieved by one free parameter, say Δ (1.7)

$$\Delta = 1 + 4\, g_2 \tag{3.10}$$

$$f' = (1-\Delta)^{\frac{1}{2}} - 1 \tag{3.11}$$

whereas

$$t_0 = (-g_2)^{-\frac{1}{2}} \tag{3.12}$$

still holds. If we expand $f(t)$ around t_0 at $\Delta \to 0$

$$f(t) = f(t_0) - \frac{1}{2}\Delta(t-t_0) + \frac{8}{3}(t-t_0)^3 + \text{remainder} \tag{3.13}$$

and scale the integration variable

$$t - t_0 = \lambda\eta \tag{3.14}$$

$$\lambda = t_0 \left(\frac{2}{N}\right)^{\frac{1}{3}} \tag{3.15}$$

we obtain as leading part of the partition function

$$\mathcal{Z}_N(g) \simeq \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \exp\left\{\frac{N}{2} f(t_0)\right\} \left(\frac{2}{N}\right)^{\frac{1}{3}} \Phi(\zeta) \tag{3.16}$$

where

$$\zeta = \frac{1}{4}\left(\frac{N}{2}\right)^{\frac{2}{3}} \Delta = \left(\frac{N}{2}\right)^{\frac{2}{3}} (g_2 - g_2^*), \qquad g_2^* = -\frac{1}{4} \tag{3.17}$$

and

$$\Phi(\zeta) = \int_{C'} d\eta\, e^{-\zeta\eta + \frac{1}{3}\eta^3} \tag{3.18}$$

The contour C' must be chosen such that under replacement of

$$\eta \to -\eta$$

$\Phi(\zeta)$ becomes identical with (2.24).

In the case of the Airy functions asymptotic expansions for large ζ need not be obtained from a saddle point expansion but [5], 10.4.63 tells us that

$$\begin{aligned} \mathrm{Bi}(\zeta) &\simeq \pi^{-\frac{1}{2}}\zeta^{-\frac{1}{4}}e^{z}\sum_{k=0}^{\infty} c_k\, z^{-k} &(3.19)\\ z &= \frac{2}{3}\zeta^{\frac{3}{2}} &(3.20)\\ c_k &= \frac{\Gamma(3k+\frac{1}{2})}{54^k k!\Gamma(k+\frac{1}{2})} &(3.21)\end{aligned}$$

which agrees with the saddle point expansion.

If we now take the logarithm we obtain

$$\log\Phi(\zeta) = \frac{1}{2}\log\pi - \frac{1}{4}\log\zeta + \frac{2}{3}\zeta^{\frac{3}{2}} + c_1\frac{3}{2}\zeta^{-\frac{3}{2}} + (c_2 - \frac{1}{2}c_1^2)\frac{9}{4}\zeta^{-3} + \dots \qquad (3.22)$$

Comparison with (1.3)

$$F_N(g_2) = \frac{1}{2} - \frac{1}{2}f(t_0) + \frac{1}{3N}\log\frac{N}{2} - \frac{1}{N}\log\Phi \qquad (3.23)$$

and (1.15) allows us to identify the coefficients a_h, namely

$$\begin{aligned} -a_0 &= +\frac{1}{3}\\ -a_1 &= \text{arbitrary}\\ -a_2 &= 3\,c_1\\ -a_3 &= 9\,(c_2 - \frac{1}{2}c_1^2) &(3.24)\end{aligned}$$

Moreover the singular term

$$+\frac{1}{4N}\log\zeta = \frac{1}{6N}\log\frac{N}{2}(g_2 - g_*)^{\frac{3}{2}} \qquad (3.25)$$

implies (1.20).

The partition function diverges at the critical value $g_c = g_2^* = -\frac{1}{4}$. We shall show in the subsequent section that adding a term to $f(t)$ (3.3), (3.7)

$$-\frac{1}{3}g_3 t^3, \qquad g_3 > 0, \quad \text{small} \qquad (3.26)$$

is the most elegant way to come around this problem.

4 Deformation of an A_3 catastrophe into curves of A_2 catastrophes

Let

$$f(t) = \log t - t - \frac{1}{2}g_2 t - \frac{1}{3}g_3 t^3 \qquad (4.1)$$

where g_3 is a free parameter but we assume still that

$$f'''(t_0) \quad \neq \quad 0 \tag{4.2}$$

then g_3 defines a curve of A_2 catastrophes. This curve has two real analytic branches intersecting at the A_3 catastrophe (Figs 1 and 2), where

$$f'''(t_0) \quad = \quad 0 \tag{4.3}$$

In fact

$$f'(t_0) \quad = \quad f''(t_0) \; = \; 0 \tag{4.4}$$

can be solved in the form (see [1], eqns (78),(79))

$$t_{0,\pm} \quad = \quad -\frac{1}{g_2}\Big[1 \pm (1+3g_2)^{\frac{1}{2}}\Big] \tag{4.5}$$

and

$$-\,27g_{3,\pm} \quad = \quad 2 + 9g_2 \mp 2(1+3g_2)^{\frac{3}{2}} \tag{4.6}$$

These are the curves drawn in Figs. 1 and 2 respectively.

The partition function is convergent with (4.1) if $g_3 > 0$. We mentioned already in Section 3 that an arbitrary small positive g_3 can be used to give a well defined meaning to the t_--branch (at the dot). If we move from this point towards the A_3 catastrophe we have

$$0 \quad \leq \quad 27g_3 \quad < \quad 1 \tag{4.7}$$

$$-\tfrac{1}{3} \quad < g_2 \quad \leq \quad -\tfrac{1}{4} \tag{4.8}$$

and remain continuously connected to the cuspoidal A_2 catastrophe. This makes sense if we can steer the parameters g_2, g_3 at will. What if the system is such that it can adjust the parameter g_3 freely for fixed g_2 ? Then it could jump (first order transition) to the t_+-branch eventually. However, since at $g_2 = -\frac{1}{4}$

$$f(t_{0,+}) - f(t_{0,-})\Big|_{g_2=-\frac{1}{4}} \quad = \quad \log 3 - \frac{4}{3} \quad (= \; -0.2347) \tag{4.9}$$

the t_--branch is stable.

We consider now g_2 and

$$\Delta \quad = \quad -2f'(t_0) \tag{4.10}$$

as deformation parameters. Then

$$t_{0,\pm}^{-1} \quad = \quad \frac{1}{3}\Big[(1-\frac{\Delta}{2}) \mp \Big((1-\frac{\Delta}{2})^2 + 3g_2\Big)^{\frac{1}{2}}\Big] \tag{4.11}$$

Our formulas are applicable in the case

$$g_2 < 0, \qquad \Delta < 2 \tag{4.12}$$

and

$$\left(1 - \frac{\Delta}{2}\right)^2 + 3g_2 \quad > \quad 0 \tag{4.13}$$

Setting (4.13) equal zero gives the A_3 catastrophe since

$$f'''(t_{0,\pm}) \quad = \quad \mp 2t_{0,\pm}^{-2}\left[\left(1 - \frac{\Delta}{2}\right)^2 + 3g_2\right]^{\frac{1}{2}} \tag{4.14}$$

In order to avoid oscillating Airy functions which would lead to complex free energies we must moreover have

$$\mathrm{sign}\Big\{\Delta f'''(t_{0,\pm})\Big\} \quad = \quad +1 \tag{4.15}$$

Now we scale

$$t - t_0 \quad = \quad \lambda\eta \tag{4.16}$$

in

$$f(t) \quad = \quad f(t_0) - \frac{1}{2}\Delta(t - t_0) + \frac{1}{6}f'''(t_0)(t - t_0)^3 + \mathcal{O}\Big((t - t_0)^4\Big) \tag{4.17}$$

so that

$$\frac{N}{2}\frac{1}{6}\lambda^3\Big|f'''(t_0)\Big| \quad = \quad \frac{1}{3} \tag{4.18}$$

or

$$\lambda \quad = \quad \left(\frac{4}{N|f'''(t_0)|}\right)^{\frac{1}{3}} \tag{4.19}$$

The contribution of either branch to the partition function is (leading term only)

$$\mathcal{Z}_N(g_2, g_3)_\pm \quad \simeq \quad \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\exp\left\{\frac{N}{2}f(t_{0,\pm})\right\}t_{0,\pm}^{-1}\left(\frac{4}{N|f'''(t_{0,\pm})|}\right)^{\frac{1}{3}}\Phi_\pm(\zeta_\pm) \tag{4.20}$$

where

$$\zeta_\pm \quad = \quad \frac{1}{4}N^{\frac{2}{3}}|\Delta|\left|\frac{4}{f'''(t_{0,\pm})}\right|^{\frac{1}{3}} \tag{4.21}$$

and

$$\Phi_\pm(\zeta_\pm) \quad = \quad \int_{C_\pm} d\eta\,\exp\left\{\pm\zeta\eta \mp \frac{1}{3}\eta^3\right\} \tag{4.22}$$

$C_\pm$ are the contours obtained by a minimal deformation of the positively orientated real axis that makes the integrals convergent. We obtain

$$\Phi_\pm(\zeta) \quad = \quad \pi\mathrm{Bi}(\zeta) \tag{4.23}$$

in either case. The constraint (4.15) renders ζ positive in either case.

5 The general A_m (cuspoidal) catastrophe

We consider now the case

$$g_1 = 1, \quad g_k = 0, \quad k \geq m+1 \tag{5.1}$$

In this case we have

$$f^{(m+1)}(t) = (-1)^m \frac{m!}{t^{m+1}} \neq 0 \quad \forall t \tag{5.2}$$

The A_m catastrophe occurs if

$$f^{(k)}(t_0) = 0 \quad \forall\, 1 \leq k \leq m \tag{5.3}$$

This leads to the equations

$$\frac{(-1)^{k-1}}{t_0^k} - \sum_{l=k}^{m} g_l \binom{l-1}{k-1} t_0^{l-k} = 0 \tag{5.4}$$

This system can be solved in an elementary fashion for t_0 and $\{g_l\}_2^m$

First we set

$$k = m : \quad \frac{(-1)^{m-1}}{t_0^m} = g_m \tag{5.5}$$

which entails

$$\text{sign } g_m = (-1)^{m-1} \tag{5.6}$$

since t_0 must be positive

$$t_0 = |g_m|^{-\frac{1}{m}} \tag{5.7}$$

Convergence of the partition function necessitates

$$\text{sign } g_m = +1 \tag{5.8}$$

which is compatible with (5.6) only if m is odd. For even m we will employ the procedure discussed in the preceding section: We add a term

$$-\tfrac{1}{m+1}\, g_{m+1}\, t^{m+1}, \quad g_{m+1} \searrow 0 \tag{5.9}$$

to the action $f(t)$.

An intermediary step in solving the system (5.4) is

$$g_k = (-1)^{m-k} \binom{m}{k} t_0^{m-k}\, g_m, \qquad (1 \leq k \leq m) \tag{5.10}$$

which for $k = m$ is trivial. For $k = 1$ we make use of $g_1 = 1$ to obtain $t_0 = m$. In proving (5.10) we need the identity

$$\sum_{l=k}^{m} (-1)^{l-k} \binom{l-1}{k-1} \binom{m}{l} \quad = \quad 1 \tag{5.11}$$

Denote the l.h.s. of (5.11) by $P_k^{(m)}$. Then

$$P_k^{(m)} - P_{k+1}^{(m)} \quad = \quad \sum_l (-1)^{l-k} \binom{m}{l} \Big[\binom{l-1}{k-1} + \binom{l-1}{k} \Big] \tag{5.12}$$

which by Pascal's identity gives

$$\begin{aligned} &= \binom{m}{k} \sum_l (-1)^{l-k} \binom{m-k}{l-k} \\ &= \delta_{mk} \end{aligned} \tag{5.13}$$

Since $P_m^{(m)} = 1$ (5.11) follows. Inserting (5.5) into (5.10) we have finally

$$g_k \quad = \quad (-1)^{k-1} \binom{m}{k} m^{-k} \tag{5.14}$$

Now we deform this catastrophe by

$$g_k \quad = \quad (-1)^{k-1} \binom{m}{k} m^{-k} + \tau_k, \qquad (2 \leq k \leq m) \tag{5.15}$$

$$t_0 \quad = \quad m + \tau_0 \tag{5.16}$$

Referring to translational invariance in t we postulate

$$f^{(m)}(t_0) \quad = \quad 0 \tag{5.17}$$

so that

$$f(t) \quad = \quad \sum_{l=0}^{m-1} \frac{(t-t_0)^l}{l!} f^{(l)}(t_0) \; + \; \frac{(t-t_0)^{m+1}}{(m+1)!} f^{(m+1)}(t_0) \; + \; \mathcal{O}\Big((t-t_0)^{m+2}\Big) \tag{5.18}$$

We expand $f^{(l)}(t_0)$ linearly in all τ_0 and τ_k, $2 \leq k \leq m$.

First we notice that

$$\frac{\partial}{\partial \tau_0} f^{(l)}(m + \tau_0) \Big|_{\tau_0 = \tau_k = 0 \, \forall k} \quad = \quad 0, \quad 0 \leq l \leq m-1 \tag{5.19}$$

since

$$f^{(l+1)}(m) \quad = \quad 0 \tag{5.20}$$

from (5.3). So there remains ($\tau_1 = 0$)

$$f^{(l)}(m+\tau_0) \;=\; -(l-1)!\sum_{k=l}^{m}\binom{k-1}{l-1} m^{k-l}\tau_k + \text{quadratic terms in } \{\tau\} \tag{5.21}$$

Next comes the scaling procedure

$$t - t_0 \;=\; \lambda\eta \tag{5.22}$$

so that

$$\frac{N}{2}\,\frac{\lambda^{m+1}}{(m+1)!}\left|f^{(m+1)}(m)\right| \;=\; \frac{1}{m+1} \tag{5.23}$$

leading to

$$\lambda \;=\; m\Big(\frac{2}{N}\Big)^{\frac{1}{m+1}} \tag{5.24}$$

Then introduce the scaling variables

$$\zeta_l \;=\; \frac{N}{2}\frac{\lambda^l}{l!}\,f^{(l)}(m+\tau_0), \qquad (1 \le l \le m-1) \tag{5.25}$$

which are kept $\mathcal{O}(1)$ at the transition point by definition. It follows

$$f^{(l)}(m+\tau_0) \;=\; A_l\left(\frac{N}{2}\right)^{-\sigma_l} \tag{5.26}$$

where

$$\sigma_l \;=\; 1 - \frac{l}{m+1} \tag{5.27}$$

and

$$A_l \;=\; \frac{l!}{m^l}\,\zeta_l \;=\; \mathcal{O}(1) \tag{5.28}$$

so we have simple scaling of the derivatives implying that the coupling constants

$$\tau_2,\, \tau_3,\, \ldots,\, \tau_{m-1},\, \tau_m \tag{5.29}$$

scale along algebraic curves with $\frac{N}{2}$ as single parameter, whereas τ_0 is coupled to τ_m by

$$\tau_0 \;=\; (-1)^m m^m \tau_m \tag{5.30}$$

as follows from (5.17). In order to solve (5.21) for the τ_k we denote

$$\mathcal{N}_{lk} \;=\; \binom{k-1}{l-1} m^{k-l} \tag{5.31}$$

Then the matrix inverse is

$$\mathcal{N}^{-1}_{kl} \;=\; (-1)^{l-k}\left[\binom{l-1}{k-1}-\binom{m-1}{k-1}\right] m^{l-k}, \qquad \left\{\begin{array}{c} 1\le l\le m-1 \\ 2\le k\le m \end{array}\right\} \tag{5.32}$$

It follows

$$\tau_k \;=\; -\sum_{l=1}^{m-1}\mathcal{N}^{-1}_{kl}\frac{A_l}{(l-1)!}\left(\frac{N}{2}\right)^{-\sigma_l} \tag{5.33}$$

Note that τ_1 vanishes automatically.

The contribution of this saddle point to the partition function is

$$\mathcal{Z}_N(g)\Big|_{A_m} \;=\; \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\exp\left\{\frac{N}{2}f(m)\right\}\left(\frac{2}{N}\right)^{\frac{1}{m+1}}\Phi(\zeta_1,\zeta_2,\ldots,\zeta_{m-1}) \tag{5.34}$$

as leading term where

$$\Phi(\zeta_1,\zeta_2,\ldots,\zeta_{m-1}) \;=\; \int_{C^{(m)}} d\eta\,\exp\left\{\sum_{k=1}^{m-1}\zeta_k\eta^k+(-1)^m\frac{\eta^{m+1}}{m+1}\right\} \tag{5.35}$$

For odd m we identify $C^{(m)}$ with the positive real axis. For even m we define

$$C^{(m)} \;=\; -C_{\frac{1}{2}}+\frac{1}{2}\left\{C_{\frac{1}{2(m+1)}}+C_{\frac{-1}{2(m+1)}}\right\} \tag{5.36}$$

What are the conditions for asymptotic behaviour of Φ to be nonoscillating for large $\{\zeta_k\}$? For m odd there is no problem since

$$m\ \text{odd} \;:\; \Phi(\zeta) \;>\; 0 \tag{5.37}$$

by our definition. For even m only the case $m=2$ has been dealt with already, e.g. by condition (4.15) that leads to the correlation of signs in (4.22). For $m\ge 4$ studying the contributions of all subordinate catastrophes

$$A_n\;,\; n\;<\;m$$

is a complicated algebraic task. An approach giving an insight into the case $m=4$ is presented in the Appendix.

6 Differential equations and the renormalization equation

The function $\Phi(\zeta_1,\zeta_2,\ldots,\zeta_{m-1})$ (5.35) satisfies the following set of linear differential equations

$$\frac{\partial\Phi}{\partial\zeta_k} \;=\; \frac{\partial^k\Phi}{\partial\zeta_1^k}, \qquad k\in\{1,2,\ldots m-1\} \tag{6.1}$$

and

$$(-1)^m \frac{\partial^m}{\partial \zeta_1^m}\Phi + \sum_{k=1}^{m-1} k\zeta_k \frac{\partial}{\partial \zeta_{k-1}}\Phi \quad = \quad 0 \tag{6.2}$$

In turn this system of $m-1$ equations determines Φ to lie in the m-dimensional space of integrals (5.34) with admissable contours $C^{(m)}$.

Let us denote

$$F(\zeta) \quad = \quad \log \Phi(\zeta) \tag{6.3}$$

Then $F(\zeta)$ satisfies a system of nonlinear differential equations, e.g. for $m = 2$

$$F'' + (F')^2 \quad = \quad \zeta \tag{6.4}$$

which is the Airy differential equation in logarithmic camouflage.

Independently of these differential equations F satisfies the renormalization group equation

$$\left(N\frac{\partial}{\partial N} - \sum_{k=2}^{m} \beta_k(\tau)\frac{\partial}{\partial \tau_k}\right) F(\zeta) \quad = \quad 0 \tag{6.5}$$

where we used the deviations τ_k of the coupling constants g_k off their critical values (5.15). Knowledge of the variables $\{\zeta_l\}$ as functions of N and $\{\tau_k\}$ allows us to determine $\beta_k(\tau)$ (insert (5.21) and (5.28) into (5.26))

$$\beta_k(\tau) \quad = \quad \sum_{\tau_l} \frac{\partial \beta_k}{\partial \tau_l}\bigg|_{\tau=0} \tau_l + \mathcal{O}(\tau^2) \tag{6.6}$$

We find

$$N\frac{\partial}{\partial N}F \quad = \quad \sum_l \sigma_l \zeta_l \frac{\partial}{\partial \zeta_l}F \tag{6.7}$$

$$\frac{\partial}{\partial \tau_k}F \quad = \quad \sum_l \frac{\partial \zeta_l}{\partial \tau_k}\frac{\partial}{\partial \zeta_l}F \tag{6.8}$$

and from (6.5), (5.31)

$$\sum_l \left\{ \sigma_l \sum_k \mathcal{N}_{lk}\tau_k \, - \, \sum_k \mathcal{N}_{lk}\beta_k(\tau) + \mathcal{O}(\tau^2)\right\} \frac{\partial F}{\partial \zeta_l} \quad = \quad 0 \tag{6.9}$$

Setting each coefficient of $\frac{\partial F}{\partial \zeta_l}$ equal to zero gives

$$\sigma_l \mathcal{N}_{lk} \quad = \quad \sum_r \mathcal{N}_{lr}\xi_{rk} \tag{6.10}$$

with the susceptibility matrix

$$\xi_{rk} \quad = \quad \frac{\partial \beta_r(\tau)}{\partial \tau_k}\bigg|_{\tau_l = 0\,\forall l} \tag{6.11}$$

So the matrix $\mathcal{N}_{lr}$ diagonalizes this susceptibility matrix and $\{\sigma_l\}$ are its eigenvalues. In our inverse approach we can calculate ξ by

$$\xi_{rk} \;=\; \sum_l \mathcal{N}_{rl}^{-1}\sigma_l \mathcal{N}_{lk} \tag{6.12}$$

The sum can be performed and gives

$$\beta_k(\tau) \;=\; (-1)^k \binom{m-1}{k-1} \frac{m^{2-k}}{m+1}\tau_2 + \frac{m+1-k}{m+1}\tau_k + \frac{mk}{m+1}(1-\delta_{km})\tau_{k+1} + \mathcal{O}(\tau^2) \tag{6.13}$$

$$(2 \leq k \leq m)$$

Appendix: Generalized Airy functions for A_m, m even

The asymptotic expansions of generalized Airy functions for large arguments as defined in (5.35) are treated with the same saddle point techniques which produced them. An Airy function for a catastrophe A_m obtains contributions of all catastrophes A_l, $1 < l < m$.

We restrict the arguments by

$$\zeta_i \;=\; 0, \qquad k < i \leq m-1 \tag{A.1}$$

and consider corresponding reduced phase functions

$$f_k(\eta) \;=\; \sum_{i=1}^{k} \zeta_i \eta^i \;+\; \frac{1}{m+1}\,\eta^{m+1} \tag{A.2}$$

We let

$$\zeta_1 \;=\; \epsilon|\zeta_1|, \qquad |\zeta_1| \to \infty, \qquad \epsilon^2 \;=\; 1 \tag{A.3}$$

and couple the remaining variables to $|\zeta_1|$

$$\zeta_i \;=\; \alpha\,|\zeta_1|^{\frac{m+1-i}{m}}, \qquad 2 \leq i < k \tag{A.4}$$

$$\eta_0 \;=\; \omega\,|\zeta_1|^{\frac{1}{m}} \tag{A.5}$$

where η_0 is the position of the A_l catastrophe, so that

$$\{\alpha_2, \alpha_3, \ldots, \omega\} \;\in\; I\!R^k$$

is fixed in the limiting procedure. Then

$$f_k(\eta_0) \;=\; \Psi_0(\alpha_2, \alpha_3, \ldots, \omega)\,|\zeta_1|^{\frac{m+1}{m}} \tag{A.6}$$

$$\Psi_0(\alpha_2, \alpha_3, \ldots, \omega) \;=\; \epsilon\,\omega + \sum_{l=2}^{k} \alpha_l \omega^l + \frac{1}{m+1}\omega^{m+1} \tag{A.7}$$

We introduce the shorthand

$$\Psi_l(\alpha_2,\alpha_3,\ldots,\omega) \;=\; \frac{\partial^l}{\partial\omega^l}\Psi_0(\alpha_2,\alpha_3,\ldots,\omega) \tag{A.8}$$

Then the A_1 catastrophes appear at

$$\Psi_1(\alpha_2,\alpha_3,\ldots,\omega) \;=\; 0 \tag{A.9}$$

In the case $k=1$ this is

$$\Psi_1(\omega) \;=\; \epsilon+\omega^m \;=\; 0 \tag{A.10}$$

and has m solutions ω. Only that solution is relevant in our context for which

$$\Re\left(\Psi_0\Big|_{\Psi_1=0}\right)$$

is maximal. For a real nonoscillating asymptotic expansion we need moreover

$$\Psi_0 \text{ real}, \qquad \Psi_2<0 \tag{A.11}$$

For $k=1$ this is possible only if

$$\epsilon \;=\; -1, \qquad \omega \;=\; -1 \tag{A.12}$$

Let us consider the case $k=2$ in more detail and concentrate on the case (A.11). Then there are two possibilitues

$$\epsilon \;=\; -1, \qquad -\infty<\omega<0, \qquad \alpha_2 \;=\; \frac{1-\omega^m}{2\omega} \tag{A.13}$$

or

$$\epsilon \;=\; +1, \qquad 0<\omega<(m-1)^{-\frac{1}{m}}, \qquad \alpha_2 \;=\; -\frac{1+\omega^m}{2\omega} \tag{A.14}$$

The case (A.12) is contained in (A.13) but (A.14) is a new branch. At the end of this branch we have

$$\omega \;=\; (m-1)^{-\frac{1}{m}} \tag{A.15}$$

and an A_2 catastrophe. Denoting

$$\chi_k \;=\; \Psi_k\Big|_{\Psi_1=0} \tag{A.16}$$

the A_1 catastrophes contribute as leading term

$$\Phi(\zeta_1,\zeta_2,\ldots,\zeta_k,0,\ldots) \;\simeq\; e^{\chi_0|\zeta_1|^{\frac{m+1}{m}}}\left[\frac{2\pi}{|\chi_2|\,|\zeta_1|^{\frac{m-1}{m}}}\right]^{\frac{1}{2}} \tag{A.17}$$

Now to the A_2 catastrophes. They show up first at $k=2$ as we just found, and are located at

$$\begin{aligned} \Psi_1 &= 0, && \text{(A.18)} \\ \Psi_2 &= 0 && \text{(A.19)} \end{aligned}$$

For $k=2$ these conditions are solved by

$$\begin{aligned} \epsilon &= (m-1)\,\omega^m && \text{(A.20)} \\ \alpha_2 &= -\frac{1}{2} m\,\omega^{m-1} && \text{(A.21)} \end{aligned}$$

We deform the condition (A.18) by a new parameter and use the standard scaling technique

$$\begin{aligned} \Psi_1 &= \Delta_1 && \text{(A.22)} \\ \epsilon &= (m-1)\,\omega^m + \Delta_1 && \text{(A.23)} \end{aligned}$$

but (A.19), (A.21) is maintained. With

$$\lambda = \left[\frac{2}{|\chi_3|\,|\zeta_1|^{\frac{m-2}{m}}}\right]^{\frac{1}{3}} \qquad \text{(A.24)}$$

and

$$z = \operatorname{sign}\chi_3\,\Delta_1\,\lambda\,|\zeta_1| \qquad \text{(A.25)}$$

we obtain the leading term of the asymptotic expansion ($k \geq 2$)

$$\Phi(\zeta_1, \zeta_2, \ldots, \zeta_k, 0, \ldots) \simeq e^{\chi_0 |\zeta_1|^{\frac{m+1}{m}}} \left[\frac{2}{|\chi_3|\,|\zeta_1|^{\frac{m-2}{m}}}\right]^{\frac{1}{3}} \Phi(z) \qquad \text{(A.26)}$$

if Δ_1 approaches zero in such a fashion that z (A.25) remains fixed. Moreover χ_k is now obtained from Ψ_k by restriction to (A.18) and (A.19).

It is obvious that this procedure can be extended to all A_l and k.

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