

**Construction of exactly solvable quantum models of
Calogero and Sutherland type with
translation invariant four-particle interactions**

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Abstract

We construct exactly solvable models for four particles moving on a real line or on a circle with translation invariant two- and four-particle interactions. The Sutherland type model seems to be hitherto unknown.

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1 Introduction

All quantum mechanical models of the infinite sequences A_n, B_n, C_n, BC_n, D_n and in addition some exceptional sequence models which were defined and proved to be completely integrable by Olshanetsky and Perelomov by the "Hamiltonian reduction method" [1] possess limiting versions of Sutherland and Calogero type that can be shown to be exactly solvable by a simple trick. Namely, their Schrödinger operators can be transcribed into quadratic polynomials of related "hidden" Lie algebras. The spectrum and the eigenfunctions can then be determined from representation theory, e.g. using polynomial spaces .

This program was formulated in [2] and successfully applied first to the A_n sequence in [3]. Then it was carried over to the other sequences and even the supersymmetric generalizations in [4, 5].

Our aim is to turn the arguments around and to develop an algorithm which leads to new exactly solvable models. First investigations were presented in [6] (hereafter quoted as (I)). The program contains two major and separate issues: to render a second order differential operator curvature free and to find a first order differential operator satisfying an integrability constraint. If both constraints are fulfilled, the whole operator can be transformed into a Schrödinger operator with a standard Laplacian as kinetic energy and a real potential as potential energy terms.

In this work we apply this method to four-particle models. We start from the second order differential operator for A_3 known from [3] (Calogero case) and [6] (Sutherland case). After a change of the variable τ_3 to $\lambda_3 = \tau_3^2$ we test out all possible first order differential operators. This way we find a new solution leading to a four-particle potential which is automatically translation-invariant. In the Calogero case this model was known before [7], but the Sutherland case seems to be new.

2 The program

We are interested here in the bound state spectrum of Schrödinger operators. The whole analysis is therefore performed in real spaces. Consider a flag of polynomial spaces $V_N, N \in \mathbb{Z}_{\geq}$

$$V_N = \text{span} \{x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} | r_1 p_1 + r_2 p_2 + \dots + r_n p_n \leq N\} \quad (1)$$

$(p_i \in \mathbb{N})$

We consider differential operators of first order

$$D_{[\vec{\alpha}; a]}^{(1)} = x^{[\vec{\alpha}]} \frac{\partial}{\partial x_a} \quad (2)$$

($\vec{\alpha}$ a multi-exponent)
and of second order

$$D_{[\vec{\alpha};a,b]}^{(2)} = x^{[\vec{\alpha}]} \frac{\partial^2}{\partial x_a \partial x_b} \quad (3)$$

that leave each space V_N invariant. If

$$\vec{p} = (1, 1, \dots, 1) \quad (4)$$

then the operators (2) generate the full linear (inhomogeneous) group of \mathbb{R}_n and the operators of second order (3) can be obtained as products from the first order operators, i.e. in (2)

$$\vec{\alpha} = e^{(a)}, \quad e_b^{(a)} = \delta_b^a \quad (5)$$

and in (3)

$$\vec{\alpha} = e^{(a)} + e^{(b)} \quad (6)$$

Now we consider a candidate for a future Schrödinger operator

$$\begin{aligned} D &= - \sum_{\vec{\alpha}, a, b} g_{[\vec{\alpha};a,b]} D_{[\vec{\alpha};a,b]}^{(2)} \\ &+ \sum_{\vec{\beta}, c} h_{[\vec{\beta};c]} D_{[\vec{\beta};c]}^{(1)} \end{aligned} \quad (7)$$

The eigenvectors and values of D in V_N can be calculated easily by finite linear algebra methods (the number of eigenvector in V_N may be smaller than $\dim V_N$, see (I)). If we want completely integrable models we must make sure that a complete set of involutive differential operators exists. For this task Lie algebraic methods may be very helpful.

The first step in transforming D into a Schrödinger operator is to write it symmetrically

$$D = - \sum_{a,b} \frac{\partial}{\partial x_a} g_{ab}^{-1}(x) \frac{\partial}{\partial x_b} + \sum_a r_a(x) \frac{\partial}{\partial x_a} \quad (8)$$

where

$$g_{ab}^{-1} = \sum_{\vec{\alpha}} g_{[\vec{\alpha};a,b]} x^{[\vec{\beta}]} \quad (9)$$

We write g_{ab}^{-1} because this is the inverse of a Riemann tensor. The Riemann tensor g_{ab} is assumed to be curvature free. The task to make it so will not arise in this work. But we mention that we developed a minimal algorithm to solve this issue.

Following the notations of (I) we "gauge" the polynomial eigenfunctions φ of D by

$$\psi(x) = e^{-\chi(x)} \varphi(x) \quad (10)$$

so that

$$e^{-\chi} D e^{+\chi} = -\frac{1}{\sqrt{g}} \sum_{a,b} \frac{\partial}{\partial x_a} (\sqrt{g} g_{ab}^{-1}) \frac{\partial}{\partial x_b} + W(x) \quad (11)$$

($g = (\det g^{-1})^{-1}$).

This is possible if and only if

$$\sum_b g_{ab}^{-1}(x) \frac{\partial}{\partial x_b} [2\chi - \ln \sqrt{g}] = r_a(x) \quad (12)$$

which implies integrability constraints on the functions $\{r_a(x)\}$. If they are fulfilled we obtain a "prepotential"

$$\rho = \ln P \quad (13)$$

so that

$$\rho = 2\chi - \ln \sqrt{g} \quad (14)$$

In all cases studied, we found solutions for ρ as follows. Let

$$\det g^{-1}(x) = \prod_{i=1}^r P_i(x) \quad (15)$$

where $\{P_i(x)\}$ are different real polynomials. Then

$$\rho(x) = \sum_{i=1}^r \gamma_i \ln P_i(x) \quad (16)$$

with free parameters γ_i solves the requirement that $\{r_a(x)\}$ (12) belong to differential operators leaving each V_N invariant. In particular

$$r_a^{(i)}(x) = \frac{1}{P_i(x)} \sum_b g_{ab}^{-1}(x) \frac{\partial P_i}{\partial x_b} \quad (17)$$

are polynomials, and each power $[\vec{\alpha}]$ is bounded by the requirement that the spaces $V_N(1)$ are invariant. Inserting (15), (16) in (14) we obtain finally

$$\chi = \frac{1}{2} \sum_{i=1}^r (\gamma_i - \frac{1}{2}) \ln P_i(x) \quad (18)$$

We will later see that in the case of the models of Calogero type a term

$$\gamma_0 \ln P_0(x) \quad (19)$$

can be added to ρ , where

$$P_0(x) = e^{x^1} \quad (20)$$

is not contained in $\det g^{-1}$ as a factor. This prepotential gives rise to the oscillator potential.

Finally we mention that $e^{-\chi}$ is the ground state wave function of the Schrödinger operator, as follows from (10).

The expression (I, 6.17) for the potential $W(x)$ contains a term linear in χ

$$-\sum_{a,b} \frac{\partial}{\partial x_a} \left(g_{ab}^{-1} \frac{\partial \chi}{\partial x_b} \right) = -\frac{1}{2} \sum_{i=1}^r (\gamma_i - \frac{1}{2}) \sum_a \frac{\partial}{\partial x_a} r_a^{(i)}(x) \quad (21)$$

Each divergence

$$\sum_a \frac{\partial}{\partial x_a} r_a^{(i)}(x) = C^{(i)} \quad (22)$$

ought to be a constant to render the operator (8) formally self-adjoint after subtraction of a constant. From now on we shall dismiss all constant terms in $W(x)$.

We can then write the potential as

$$W(x) = \sum_{i,j} \gamma_{ij} R_{ij} \quad (23)$$

$$R_{ij} = \sum_{a,b} g_{ab}^{-1}(x) \frac{\partial \ln P_i}{\partial x_a} \frac{\partial \ln P_j}{\partial x_b} \quad (24)$$

$$\gamma_{ij} = \frac{1}{4} (\gamma_i \gamma_j - \frac{1}{4}) \quad (i, j \neq 0). \quad (25)$$

In the cases of this article

$$R_{ij} = \text{const if } i \neq j \quad (26)$$

If we then set

$$\gamma_i = -\nu_i + \frac{1}{2} \quad (i \neq 0) \quad (27)$$

we obtain

$$W(x) = \sum_{i=1}^r \gamma_{ii} R_{ii} \quad (28)$$

with

$$\gamma_{ii} = \frac{1}{4} \nu_i (\nu_i - 1) \quad (29)$$

3 A model of Calogero type

In the Schrödinger operator for the A_3 Calogero model

$$H_{\text{cal}} = \frac{1}{2} \sum_{i=1}^4 \left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + g_1 \sum_{i < j} (x_i - x_j)^{-2} \quad (30)$$

we separate the centre-of-mass part by introducing relative coordinates

$$\begin{aligned} X &= x_1 + x_2 + x_3 + x_4 \\ y_i &= x_i - \frac{1}{4}X, \quad i \in \{1, 2, 3\} \end{aligned} \quad (31)$$

Following [3] we introduce elementary symmetric polynomials as new coordinates instead of the $\{\tau_i\}$

$$\begin{aligned} \tau_2 &= \sum_{1 \leq i < j \leq 4} y_i y_j \\ \tau_3 &= \sum_{1 \leq i < j < k \leq 4} y_i y_j y_k \\ \tau_4 &= y_1 y_2 y_3 y_4 \end{aligned} \quad (32)$$

Using the ground state wave function

$$\Psi_0(x) = V(x)^{\nu_1} \exp \left\{ -\frac{1}{2} \omega \sum_{i=1}^4 x_i^2 \right\} \quad (33)$$

with ν_1 any root of

$$g_1 = \nu_1(\nu_1 - 1) \quad (34)$$

and $V(x)$ the Vandermonde determinant, we gauge H_{cal} to

$$\begin{aligned} \tilde{H}_{\text{cal}} &= \Psi_0(x)^{-1} H_{\text{cal}} \Psi_0(x) \\ &= \tilde{H}_{\text{c.o.m.}} + \tilde{H}_{\text{rel}} + \text{const} \end{aligned} \quad (35)$$

Then (see [3] and I (7.1) - (7.6) with $s = 0$, $w_2 = w_3 = w_4 = 1$) we get

$$\begin{aligned} 2\tilde{H}_{\text{rel}} &= - \sum_{a,b=2}^4 g_{ab}^{-1} \frac{\partial}{\partial \tau_a} \frac{\partial}{\partial \tau_b} \\ &\quad + \text{first order differential operator} \end{aligned} \quad (36)$$

with

$$g^{-1} = \begin{pmatrix} -2\tau_2 & -3\tau_3 & -4\tau_4 \\ -3\tau_3 & -4\tau_4 + \tau_2^2 & +\frac{1}{2}\tau_2\tau_3 \\ -4\tau_4 & +\frac{1}{2}\tau_2\tau_3 & -2\tau_2\tau_4 + \frac{3}{4}\tau_3^2 \end{pmatrix} \quad (37)$$

The first order differential operator is discarded.

The relative motion Hamiltonian (36) reflects the symmetry of H_{cal} (30) under

$$x_i \rightarrow -x_i \quad (38)$$

namely it is symmetric under

$$\tau_2 \rightarrow \tau_2, \tau_3 \rightarrow -\tau_3, \tau_4 \rightarrow \tau_4 \quad (39)$$

Thus it leaves polynomial spaces that are even or odd in τ_3 separately invariant. Now we reduce these spaces and consider polynomials only that are even in τ_3 :

$$\tau_2 = \lambda_2, \tau_3^2 = \lambda_3, \tau_4 = \lambda_4 \quad (40)$$

$$\begin{aligned} \frac{\partial}{\partial \tau_3} &= 2\lambda_3^{\frac{1}{2}} \frac{\partial}{\partial \lambda_3} \\ \frac{\partial^2}{\partial \tau_3^2} &= 4\lambda_3 \frac{\partial^2}{\partial \lambda_3^2} + 2 \frac{\partial}{\partial \lambda_3} \end{aligned} \quad (41)$$

(this substitution (40) was invented in [4, 5]). Multiplying the second row and column with $2\lambda_3^{\frac{1}{2}}$ we obtain

$$g_{ab}^{-1} \rightarrow G_{ab}^{-1} \quad (42)$$

Whereas the quadratic differential operator (36) with g_{ab}^{-1} as in (37) and

$$P_1(\tau) = -4 \det g^{-1} \quad (\text{I, (7.22), and appendix (A.1)}) \quad (43)$$

$$P_0(\tau) = e^{\tau_2} \quad (44)$$

$$r^{(1)} = (-12, 0, -2\tau_2) \quad (45)$$

$$r^{(0)} = (-2\tau_2, -3\tau_3, -4\tau_4) \quad (46)$$

allows to reconstruct H_{cal} completely, we have now an additional factor in

$$\det G^{-1} = 4\lambda_3 \left(-\frac{1}{4}\right) P_1(\lambda) \quad (47)$$

($P_1(\lambda)$ is obtained from $P_1(\tau)$ (A.1) by substitution (40)).

The polynomial spaces (1) are now such that

$$n = 3, p_1 = p_3 = 1, p_2 = 2 \quad (48)$$

For the first order differential operators (2) we have

$$[\vec{\alpha}; a] \in \left\{ \begin{aligned} &[\vec{0}; a] \text{ all } a; \\ &[e^{(a)}; 1], [e^{(a)}; 3], a \in \{1, 3\}; \\ &[e^{(a)}; 2], \text{ all } a; \\ &[e^{(a)} + e^{(b)}; 2], a, b \in \{1, 3\} \end{aligned} \right\} \quad (49)$$

If we multiply two of these we obtain possible second order differential operators (3), but in addition we have

$$[\vec{\beta}; a, b] \in \{[e^{(2)}; a, b] \quad a, b \in \{1, 3\}\} \quad (50)$$

Now the program of the preceding section is set in action: From

$$P_1(\lambda) = -\frac{1}{\lambda_3} \det G^{-1} \quad (51)$$

$$P_2(\lambda) = \lambda_3 \quad (52)$$

$$P_0(\lambda) = e^{\lambda_2} \quad (53)$$

we obtain

$$r^{(1)} = (-12, 0, -2\lambda_2) \quad (54)$$

$$r^{(2)} = (-6, 4(\lambda_2^2 - 4\lambda_4), \lambda_2) \quad (55)$$

$$r^{(0)} = (-2\lambda_2, -6\lambda_3, -4\lambda_4) \quad (56)$$

and

$$R_{11} = \frac{16}{P_1} [2\lambda_2^5 + 16\lambda_2^3\lambda_4 - 96\lambda_2\lambda_4^2 + 9\lambda_2^2\lambda_3 + 108\lambda_3\lambda_4] \quad (57)$$

$$R_{22} = \frac{4}{\lambda_3} [\lambda_2^2 - 4\lambda_4] \quad (58)$$

$$R_{00} = -2\lambda_2 \quad (59)$$

R_{11} leads us back to the Calogero and R_{00} to the oscillator potential. The new potential following from R_{22} is obtained as follows. We factorize λ_3 (eliminating $y_4 = -y_1 - y_2 - y_3$)

$$\lambda_3 = \tau_3^2 = (y_1 + y_2)^2 (y_1 + y_3)^2 (y_2 + y_3)^2 \quad (60)$$

and perform a fractional decomposition

$$\frac{Q}{\lambda_3} = \sum_{1 \leq i < j \leq 3} (y_i + y_j)^{-2} \quad (61)$$

We find

$$Q = \lambda_2^2 - 4\lambda_4 \quad (62)$$

so that

$$R_{22} = 16 \sum_{3 \text{ indep. terms}} (x_i + x_j - x_k - x_l)^{-2} \quad (63)$$

Using (29) we obtain finally the new model

$$\begin{aligned} H_{\text{cal}} &= \frac{1}{2} \sum_{i=1}^4 \left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) \\ &+ g_1 \sum_{1 \leq i < j \leq 4} (x_i - x_j)^{-2} \\ &+ g_2 \sum_{3 \text{ indep. terms}} (x_i + x_j - x_k - x_l)^{-2} \end{aligned} \quad (64)$$

with

$$g_2 = 2\nu_2(\nu_2 - 1) \quad (65)$$

In the course of this derivation we have reduced the polynomial spaces of the standard A_3 Calogero model to spaces of polynomials that are even in τ_3 . But afterwards we multiply them with

$$\lambda_3^{\frac{1}{2}\nu_2} = \tau_3^{\nu_2} \quad (\tau_3 > 0) \quad (66)$$

So for $\nu_2 = 0$ we have recovered the even and for $\nu_2 = 1$ the odd spaces. In these cases $g_2 = 0$ as shown by (65). Thus the new potential arises by an interpolation between even and odd spaces.

4 A model of Sutherland type

Having constructed a model of Calogero type the existence of a corresponding model of Sutherland type can be expected. We start from our formulas for the $N = 4$ Sutherland model in (I, Section 7) and put everywhere

$$w_i = 1, \quad i \in \{2, 3, 4\}, \quad s = 2 \quad (67)$$

Again we change variables as in (40), (41) and obtain

$$\det(G^{-1}) = 4\lambda_3\left(-\frac{1}{4}\right)P_1(\lambda) \quad (68)$$

as in (47), and

$$P_2(\lambda) = \lambda_3 \quad (69)$$

as in (52). $P_1(\lambda)$ is given as in the appendix (A.2). There is no place for a $P_0(\lambda)$. The reader is advised to try out what happens with a $P_0(\lambda)$.

Calculations become very lengthy. We find

$$\begin{aligned} r_a^{(1)} &= (2(a_7 - 1)(a_7 - 3)\lambda_4 - 2(a_7 + 5)\lambda_2 - 12, \\ &\quad -24\lambda_3, \\ &\quad +2(a_7 - 9)\lambda_4 - 2\lambda_2) \end{aligned} \quad (70)$$

$$\begin{aligned} r_a^{(2)} &= (-(a_7 - 1)(a_7 - 3)\lambda_4 + (a_7 - 7)\lambda_2 - 8, \\ &\quad +4[(a_7 - 1)\lambda_4 - \lambda_2]^2 - 8\lambda_3 - 16\lambda_4, \\ &\quad -(a_7 + 3)\lambda_4 + \lambda_2) \end{aligned} \quad (71)$$

The parameter a_7 appears only in the Lie algebraic formulation of the same (a_7 independent) Sutherland model.

The potentials R_{ij} come out as

$$R_{11} = \frac{16}{P_1} \{R_1\} \quad (72)$$

with R_1 given in the appendix (A.3)

$$R_{22} = \frac{4}{\lambda_3} \{[(a_7 - 1)\lambda_4 - \lambda_2]^2 - 4\lambda_4\} \\ + \text{irrelevant terms} \quad (73)$$

and R_{12} is constant (irrelevant). We introduce (see (I), eqs. (7.16)-(7.18)) the shorthand for the trigonometric S_4 -symmetric functions

$$\sigma_2 = \sum \sin y_i \sin y_j \cos y_k \cos y_l \\ = -\frac{1}{2} \sum_{1 \leq i < j \leq 3} \sin^2(y_i + y_j) \quad (74)$$

$$\sigma_3 = \sum \sin y_i \sin y_j \sin y_k \cos y_l \\ = - \prod_{1 \leq i < j \leq 3} \sin(y_i + y_j) \quad (75)$$

$$\sigma_4 = \sin y_1 \sin y_2 \sin y_3 \sin y_4 \quad (76)$$

Since

$$\lambda_3 = \sigma_3^2 \quad (77)$$

we study the fractional decomposition

$$\frac{Q}{\lambda_3} = \sum_{1 \leq i < j \leq 3} (\sin(y_i + y_j))^{-2} \quad (78)$$

For Q we find

$$Q = (\sigma_2 - 2\sigma_4)^2 - 4\sigma_4 + \sigma_3^2 \quad (79)$$

On the other hand we get by insertion of λ_2, λ_4 ($\xi = 2\lambda_2, \eta_4 = 4\lambda_4$ in I (7.9) and (7.11))

$$R_{22} = \frac{4Q}{\lambda_3} + \text{const} \quad (80)$$

Thus we end up with a Sutherland model

$$H_{\text{suth}} = \frac{1}{2} \sum_{i=1}^4 \left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) \\ + g_1 \sum_{1 \leq i < j \leq 4} (\sin(x_i - x_j))^{-2} \\ + \frac{1}{4} g_2 \sum_{\substack{\text{3 cases}}} (\sin \frac{1}{2}(x_i + x_j - x_k - x_l))^{-2} \quad (81)$$

with g_2 as in (65). The remarks made at the end of the preceding section remain true in this case.

5 Concluding remarks

Our technique can also be applied to the A_2 model whereas the cases A_{N-1} , $N > 4$, cause hitherto unsolved problems. In the latter cases the form of the trigonometric models in the variables $\{\sigma_n\}_2^N$ analogous to (74) – (76) is also known [8]. The trigonometric variables $\{\eta_n\}_1^{N-1}$ introduced in [3] in which all A_{N-1} models were expressed, are algebraically related with the $\{\sigma_n\}_2^N$ but the explicit form of this relation is not known. The trigonometric variables introduced in [4] eqns. (2.17), (2.18) are $\mathfrak{R}\eta_1, \mathfrak{S}\eta_1$ respectively, those of [4], eqns. (3.4), (3.5) are still different.

The rational A_2 and G_2 models are equivalent if

$$\nu(A_2) = \nu(G_2) + \mu(G_2) \quad (82)$$

where the respective coupling constants are (see [4])

$$g(A_2) = \nu(A_2)(\nu(A_2) - 1) \quad (83)$$

$$g(G_2) = \nu(G_2)(\nu(G_2) - 1) \quad (84)$$

$$g_1(G_2) = 3\mu(G_2)(\mu(G) - 1) \quad (85)$$

Indeed, their Lie algebraic version ([4]), eqns. (2.8), (4.5)) are identical if (82) is satisfied. The trigonometric versions are, however, inequivalent.

Applying our technique to the A_2 model gives the G_2 model, both in the rational and the trigonometric cases. Since the A_2 model is obtained from the G_2 model by specialization of one coupling constant, we should denote both models together as AG_2 . The extension of the A_3 model found by us could to be denoted AG_3 correspondingly. The Lie algebras A_3 and D_3 are identical. But A_3 and D_4 models involve four particles, the center-of-momentum motion of the D_4 model is not separable. Its potential is [1]

$$g \sum_{1 \leq i < j \leq 4} [v(x_i - x_j) + v(x_i + x_j)] \quad (86)$$

whereas our AG_3 model has potential

$$\sum_{1 \leq i < j \leq 4} [g_1 v(x_i - x_j) + \frac{1}{4} g_2 v(y_i + y_j)] \quad (87)$$

They are obviously related, but (87) differs from (86) by independence of the coupling constants and translational invariance.

Finally we want to mention that we have derived the AG_3 model also by the use of the coordinates μ_1, μ_2, μ_3 from

$$\mu_1 = \sum_{i=1}^4 y_i^2 = -2\tau_2 \quad (88)$$

$$\mu_2 = \sum_{1 \leq i < j \leq 4} y_i^2 y_j^2 = +2\tau_4 + \tau_2^2 \quad (89)$$

$$\mu_3 = \sum_{1 \leq i < j < k \leq 4} y_i^2 y_j^2 y_k^2 = -2\tau_2 \tau_4 + \tau_3^2 \quad (90)$$

$$\mu_4 = y_1^2 y_2^2 y_3^2 y_4^2 = \tau_4^2 \quad (91)$$

so that

$$\mu_4 = \frac{1}{4}(\mu_2 - \frac{1}{4}\mu_1^2)^2 \quad (92)$$

This derivation goes along the same lines but is more laborious. It is remarkable that in this case one has to use an infinite ensemble of flags of polynomial spaces

$$V_N = \text{span}\{\mu_1^{n_1} \mu_2^{n_2} \mu_3^{n_3}; \quad n_1 p_1 + n_2 p_2 + n_3 p_3 \leq N\} \quad (93)$$

namely one flag for each admissible triplet $(p_1, p_2, p_3) \in \mathbb{N}^3$. In the following table we give some of these triplets. The minimal admissible triplet is $(2, 3, 5)$.

(1,2,4)	(2,3,5)	(3,5,10)
(1,3,6)	(2,3,6)	(3,7,14)
(1,4,8)	(2,4,5)	(3,8,16)
(1,5,10)	(2,5,6)	(3,10,20)
(1,6,12)	(2,5,10)	(3,11,22)
⋮	(2,6,7)	⋮
(1, p_2 , $2p_2$)	⋮	(3, p_2 , $2p_2$)

Table 1: Table of admissible triplets (p_1, p_2, p_3) .

Criterion for admissibility is that the differential operator D (7),(8) maps V_N in V_N and possesses nontrivial eigenvectors.

6 Note added

Recently we were able to prove that the translation invariant four-particle model AG_3 considered here is equivalent to the translation non-invariant three-particle model belong to B_3 . The arguments rely on a discussion of the underlying Coxeter groups and their invariants.

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7 Appendix: Some formulas

In the case of the Calogero A_3 model we have from (I,(7.22))

$$P_1(\tau) = 27\tau_3^4 - 256\tau_4^3 + 128\tau_2^2\tau_4^2 - 16\tau_2^4\tau_4 + 4\tau_2^3\tau_3^2 - 144\tau_2\tau_3^2\tau_4 \quad (\text{A.1})$$

In the case of the Sutherland A_3 model we have (we give these functions for $a_7 = 3$ only)

$$\begin{aligned} P_1(\lambda) = & -64\lambda_3\lambda_2^3\lambda_4 - 16\lambda_2^5\lambda_4 - 48\lambda_2\lambda_4\lambda_3^2 - 144\lambda_2\lambda_4\lambda_3 + 528\lambda_3\lambda_2\lambda_4^2 + 256\lambda_3\lambda_2^2\lambda_4^2 \\ & - 384\lambda_3\lambda_2\lambda_4^3 + 128\lambda_2^2\lambda_4^2 - 768\lambda_2\lambda_4^3 + 27\lambda_3^2 + 192\lambda_3\lambda_4^2 - 48\lambda_4\lambda_3^2 + 8\lambda_2^2\lambda_3^2 \\ & + 48\lambda_3^2\lambda_4^2 - 384\lambda_3\lambda_4^3 + 192\lambda_3\lambda_4^4 - 256\lambda_4^3 + 768\lambda_4^4 + 4\lambda_3^3 - 208\lambda_2^2\lambda_3\lambda_4 \quad (\text{A.2}) \\ & + 36\lambda_2\lambda_3^2 - 768\lambda_4^5 + 256\lambda_4^6 - 16\lambda_2^4\lambda_4 + 256\lambda_2^3\lambda_4^2 - 1024\lambda_2^2\lambda_4^3 + 144\lambda_2^4\lambda_4^2 \\ & - 512\lambda_2^3\lambda_4^3 + 896\lambda_2^2\lambda_4^4 - 768\lambda_2\lambda_4^5 + 1536\lambda_2\lambda_4^4 + 4\lambda_3\lambda_2^3 + 4\lambda_3\lambda_4^4 \end{aligned}$$

and

$$\begin{aligned} R_1 = & 272\lambda_3\lambda_2^3\lambda_4 + 108\lambda_4\lambda_3 + 56\lambda_2^5\lambda_4 + 216\lambda_2\lambda_4\lambda_3^2 + 9\lambda_2^2\lambda_3 + 732\lambda_2\lambda_4\lambda_3 + 2\lambda_2^5 \\ & - 2352\lambda_3\lambda_2\lambda_4^2 - 1136\lambda_3\lambda_2^2\lambda_4^2 + 1728\lambda_3\lambda_2\lambda_4^3 - 96\lambda_2\lambda_4^2 + 16\lambda_2^3\lambda_4 - 736\lambda_2^2\lambda_4^2 \end{aligned}$$

$$\begin{aligned}
& +3648\lambda_2\lambda_4^3 - 108\lambda_3^2 - 972\lambda_3\lambda_4^2 + 216\lambda_4\lambda_3^2 - 34\lambda_2^2\lambda_3^2 - 216\lambda_3^2\lambda_4^2 & (A.3) \\
& +1728\lambda_3\lambda_4^3 - 864\lambda_3\lambda_4^4 + 1152\lambda_4^3 - 3456\lambda_4^4 - 18\lambda_3^3 + 2\lambda_2^6 + 896\lambda_2^2\lambda_3\lambda_4 \\
& -150\lambda_2\lambda_3^2 + 3456\lambda_4^5 - 1152\lambda_4^6 + 72\lambda_2^4\lambda_4 - 1200\lambda_2^3\lambda_4^2 + 4736\lambda_2^2\lambda_4^3 \\
& -600\lambda_2^4\lambda_4^2 + 2240\lambda_2^3\lambda_4^3 - 4000\lambda_2^2\lambda_4^4 + 3456\lambda_2\lambda_4^5 - 7008\lambda_2\lambda_4^4 - 4\lambda_3\lambda_2^3 \\
& -14\lambda_3\lambda_2^4
\end{aligned}$$