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Sigma models with A_k singularities in Euclidean spacetime of dimension $0 \leq D < 4$ and in the limit $N \to \infty$

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Abstract

For the case of the single-O(N)-vector linear sigma models the critical behaviour following from any A_k singularity in the action is worked out in the double scaling limit $N \to \infty$, $f_r \to f_r^c$, $2 \leq r \leq k$. After an exact elimination of Gaussian degrees of freedom, the critical objects such as coupling constants, indices and susceptibility matrix are derived for all A_k and spacetime dimensions $0 \leq D < 4$. There appear exceptional spacetime dimensions where the degree k of the singularity A_k is more strongly constrained than by the renormalizability requirement.

1 Introduction

Matrix models can be reformulated as representing stochastic triangulated surfaces and are thus interpreted as quantum gravity theories. They are treated in the "double scaling limit" $N \to \infty, g \to g_c$ [1]-[3]. $O(N)$ vector sigma models can be understood in a similar manner as describing statistical ensembles of branched polymers [4]-[7]. These models can also be submitted to a double scaling limit. This is done with N-renormalization group techniques based on exact recursion relations in N [4]-[10].

Instead we propose to start from saddle point integrals. Partition functions are then to leading order represented in the form of generalized Airy functions ($D = 0$, see (66)) or as a partition function with a new field theoretic action ($D > 0$, see (93)). We shall refer to the function (respectively: functional) in the exponential as "Airy action" (respectively: "Airy field action").

The saddle point integrals arise from singularities in the original action when the limit $N \to \infty$ is performed. Such singularities can be classified [11, 12] and form s-dimensional families. Each familiy possesses s moduli as continuous parameters. If $s = 0$, the families are discrete and are grouped by their symmetry into A, D and E series. The A-series can be realized in single-vector sigma models and is the object of our interest in this article. It has been shown [13, 14] that D and E series can be realized by two-vector sigma models. In the field theoretic literature only A-series singularities have been identified before us (the triple scaling cases (i) and (ii) in [15] are separable as $A_1 \times A_2$, respectively $A_1 \times A_3$).

By application of diffeomorphisms a singularity can be brought to canonical form. This canonical form contains the full information defining a "universality class of multicritical behaviour" (A_k: k-critical). Our aim in this article is to extract universal quantities such as critical indices and the universal part of the beta functions for the whole A-series in dimensions $0 \leq D < 4$. For $D > 0$ two kinds of boundary conditions are considered: finite cube periodic boundary conditions and infinite volume. The spacetime dimension D is interpolated whenever possible. We obtain closed algebraic expressions in each case (no infinite sums or integrals).

Our treatment of $D \geq 2$ field theories is restricted to "naive double scaling", i.e. renormalizations are neglected. These imply logarithmic modifications, namely N^σ is multiplied with a polynomial in $\log N$ [8, 9]. We expect that these modifications can also be calculated explicitly (i.e. in terms of algebraic expressions) for all cases A_k. Some preliminary discussions are presented in [9] (e.g. introduction of counter terms). These logarithmic modifications are also of interest for mathematics, they go beyond the concepts of Arnold's school.

The authors of [7, 9] treated the cases $A_k,\ k \geq 3$, incorrectly. They eliminated Gaussian degrees of freedom connected with nonvanishing eigenvalues of the Hessian by orthogonal projection along the zero mode eigenvectors. The orthogonal structure is produced by the Hessian itself, which looses its meaning at higher orders of the Taylor expansion. In fact, it is not difficult to see that additional "curvature terms" arise first at fourth order ($k = 3$). The correct method is explained in the text. It is based on the "splitting lemma" ([12], Theorem 4.5) which is proved by the implicit function theorem. Gaussian degrees of freedom, which each belong to an A_1 singularity, have to be integrated out before the relevant singularity is isolated.

The Airy functions are central and unambiguously derived in our approach. In the N-renormalization group approach a constrained Airy function depending on only one variable is obtained by solving a differential equation. Namely, let the sum in (66) run over $1 \leq n \leq k-1$ and set

$$\zeta = -\zeta_1, \ \zeta_2 = \zeta_3 = \cdots = \zeta_{k-1} = 0, \ \epsilon = +1$$

then the resulting function $Y(\zeta)$ satisfies

$$\left(\zeta - \left(\frac{d}{d\zeta} \right)^k \right) Y(\zeta) = 0.$$

In the same way one can derive a system of differential equations for the general case [13, 14].

Asymptotic expansions for large ζ play an important role in the N-renormalization group approach [7, 10, 15]. For the singularity A_2 one can use the known expansions of the standard Airy functions Ai and Bi [16]. For the generalized Airy functions of the singularities $A_k, \ k \geq 3$, different orderings of the large arguments $\{\zeta_l\}$ are possible, each one by repeated application of saddle point integrations along a chain of reductions

$$A_k \to A_{k-1} \to \cdots \to A_2 \to A_1$$

implying a different asymptotic expansion. Degenerate reduction chains with intermediate singularities skipped can also occur.

In Section 2 the single-O(N)-vector sigma model is formulated and transformed into an effective field theory of two scalar fields. The Hessian of this effective field theory is the basis for the discussion of singularities. Its corank must be one in order to admit singularities of type $\{A_k\}_2^\infty$. It was shown in [14] that sigma models containing r O(N)-vectors can be formulated such that the Hessian is of corank r.

The elimination of the Gaussian degrees of freedom which are all degrees except one for $\{A_k\}_2^\infty$, is a major algebraic issue. It is formulated and solved in Section 3 for finite volume. Moreover we calculate critical coupling constants and the position of the saddle point (r_0 or b(0)). Both these results can be carried over to the infinite volume case (Section 4).

The deviations of the coupling constants from their critical value map to lowest order linearly on the deformation space. This linear map is denoted as "susceptibility matrix". Its calculation is the major topic of Section 4. It can also be inverted explicitly. The double scaling limit is defined as the combined limit when N tends to infinity and the coupling constants approach their critical values. In detail it involves the susceptibility matrix and the critical indices. Both enter also the linear terms of the beta functions which can thus be given explicitly for all $\{A_k\}_2^\infty$.

In Section 5 we study the case when the sigma model is carried by the whole of $D < 2$ dimensional space time. The momenta form a continuous spectrum. A momentum scale Λ which is tied to the renormalized mass of the theory separates small momenta $\{|p| < \Lambda\}$ from large momenta $\{|p| > \Lambda\}$. The latter belong to Gaussian degrees of freedom that are integrated out, whereas the former are additional deformation parameters which under double scaling induce the kinetic energy term in the Airy field action. The critical indices are modified but the susceptibility matrix remains unchanged as compared with Section 4.

If the sigma model is carried by infinite spacetime of dimensions $2 \leq D < 4$, the double scaling limit can be performed provided the field theory resulting from the saddle point integral is renormalizable. Counter terms have to be introduced [17, 9] in the course of the limit and the quantity $N\Lambda$ is the ultraviolet cutoff. This is studied in Section 6. If we use dimensional regularization for $2 < D < 4$, all critical objects can be shown to be analytic continuations of the corresponding quantities for $0 < D < 2$. However, for $D = 2$ we use a subtraction scheme depending on a mass parameter μ^2 and all critical quantities are recalculated.

The type A_k of the singularity is restricted by renormalizability to

$$k \leq \frac{D+2}{D-2}.$$

Surprisingly we observe that for "exceptional dimensions"

$$D_n = 2\frac{n}{n-1}, \quad n \in \{3, 4, 5, ...\}$$

k is further restricted by

$$k \leq \begin{cases} n-1, \text{n odd}, \\ n-2, \text{n even}. \end{cases}$$

This result is a consequence of the analytic form found for the critical quantities.

In Section 7 we return to the unstable field theories resulting from saddle point integrals. Though we make suggestions of how to ascribe a meaning to them, their properties are unclear. Nevertheless, realistic systems of statistical mechanics may be described by them and it would be wrong to neglect them.

2 The model

We study conventional sigma models with the action

$$S = \int d^D x \left[\frac{1}{2}(\partial_\nu \vec{S})(\partial_\nu \vec{S}) + \frac{1}{2}\beta^2 \vec{S}^2 + U(\vec{S}^2)\right] \tag{1}$$

($\vec{S} \in \mathbb{R}_N$)
with the potential

$$U(\sigma) = \sum_{r=2}^{\infty} \frac{f_r}{r} \sigma^r. \tag{2}$$

We shall interpolate the dimension D. For the purpose of our investigation it is not relevant whether the series (2) is finite, analytic or formal.

By a standard functional Fourier transformation and performing some of the functional integrations we can transform the action (1) into an effective action

$$\begin{aligned} S_{\text{eff}} &= \int d^D x \, [U(\sigma(x)) - i\rho(x)\sigma(x)] \\ &\quad + \frac{1}{2} \mathrm{Tr} \log[-\Delta + \beta^2 + 2i\rho] \end{aligned} \tag{3}$$

with partition function

$$Z = \int D\sigma D\rho \exp[-N S_{\text{eff}}(\sigma, \rho)]. \tag{4}$$

This partition function is to be evaluated in the limit $N \to \infty$. Application of singularity theory amounts to evaluation of (4) by saddle point integrals.

The system may either be considered on unbounded spacetime or on a cube with volume $V = L^D$ and periodic boundary conditions. Fourier transforms are defined by

$$\hat{\alpha}(p) = \int d^D x \, e^{-ipx} \alpha(x) \tag{5}$$

in either case, but the inverse transformations involve either integrations

$$\int \frac{d^D p}{(2\pi)^D}$$

or summations

$$\frac{1}{V} \sum_{\substack{p \text{ from one} \\ \text{Brillouine zone}}} .$$

In explicit formulas we will always write integrals.

Let us assume that the saddle point of (4)

$$(\sigma_0, \rho_0)$$

is constant over spacetime. The saddle point is then determined by

$$U'(\sigma_0) = i\rho_0 \tag{6}$$

$$\sigma_0 = \int \frac{d^D p}{(2\pi)^D}(p^2 + m^2)^{-1} \tag{7}$$

where

$$m^2 = \beta^2 + 2i\rho_0 \tag{8}$$

is assumed positive. In order to render the integral (7) convergent, we limit D to $0 \leq D < 2$. Only in the sixth section we will abandon this constraint.

The fields σ and ρ fluctuate

$$\sigma(x) = \sigma_0(1 + \alpha(x)) \tag{9}$$

$$\rho(x) = \rho_0(1 + \beta(x)) \tag{10}$$

and the n'th order term of S_{eff} in the fluctuations is denoted $S_{\text{eff}}^{(n)}$. Then

$$\begin{aligned} S_{\text{eff}}^{(2)} = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}(\hat{\alpha}(-p), \hat{\beta}(-p)) \\ \begin{pmatrix} \sigma_0^2 U''(\sigma_0) & \sigma_0 r_0 \\ \sigma_0 r_0 & -2r_0^2\Sigma(p) \end{pmatrix} \begin{pmatrix} \hat{\alpha}(p) \\ \hat{\beta}(p) \end{pmatrix} \end{aligned} \tag{11}$$

where the real quantity

$$r_0 = -i\rho_0 \tag{12}$$

has been introduced and

$$\Sigma(p) = \int \frac{d^D q}{(2\pi)^D}[(q^2 + m^2)((q-p)^2 + m^2)]^{-1}. \tag{13}$$

The n'th order term $S_{\text{eff}}^{(n)}$ is in coordinate space integrals

$$\begin{aligned} S_{\text{eff}}^{(n)} = \frac{\sigma_0^n U^{(n)}(\sigma_0)}{n!}\int d^D x \alpha(x)^n \\ -\frac{1}{2n}(2r_0)^n \text{Tr}\left[(-\Delta + m^2)^{-1}\beta(x)\right]^n. \end{aligned} \tag{14}$$

The Hessian $S_{\text{eff}}^{(2)}$ is diagonalized by

$$\begin{pmatrix} \hat{\alpha}(p) \\ \hat{\beta}(p) \end{pmatrix} = \begin{pmatrix} a(p) \\ 1 \end{pmatrix}\hat{\xi}(p) + \begin{pmatrix} b(p) \\ 1 \end{pmatrix}\hat{\eta}(p) \tag{15}$$

where the two eigenvectors are orthogonal implying

$$a(p)b(p) = -1 \tag{16}$$

and have norms squared

$$\begin{aligned} N_+(p) &= a(p)^2 + 1 \\ N_-(p) &= b(p)^2 + 1. \end{aligned} \tag{17}$$

Then the Hessian assumes the form

$$\begin{aligned} S_{\mathrm{eff}}^{(2)} = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} &[\lambda_+(p) N_+(p) \hat{\xi}(-p) \hat{\xi}(p) \\ &+ \lambda_-(p) N_-(p) \hat{\eta}(-p) \hat{\eta}(p)] \end{aligned} \tag{18}$$

with eigenvalues

$$\begin{aligned} \lambda_\pm(p) = \frac{1}{2} \Big\{ &\sigma_0^2 U''(\sigma_0) - 2r_0^2 \Sigma(p) \\ \mp \Big[(\sigma_0^2 U''(\sigma_0) + 2r_0^2 \Sigma(p))^2 &+ 4\sigma_0^2 r_0^2 \Big]^{\frac{1}{2}} \Big\}. \end{aligned} \tag{19}$$

The p-dependence of $N_\pm(p), a(p), b(p), \lambda_\pm(p)$ originates from $\Sigma(p)$. Since

$$\Sigma(p) = \Sigma(-p) \tag{20}$$

all these quantities have the same symmetry. In the infinite volume case $\Sigma(p)$ depends only on p^2 and decreases monotonously from $p^2 = 0$ to $p^2 = \infty$ where it vanishes.

We will discuss only the case that the Hessian becomes singular at $p = 0$. In this case we must have

$$2\Sigma(0) U''(\sigma_0) + 1 = 0 \tag{21}$$

implying

$$\lambda_-(0) = 0 \tag{22}$$

$$\lambda_+(0) < 0. \tag{23}$$

From (19) follows moreover

$$\lambda_-(p) > 0, \quad p \neq 0. \tag{24}$$

3 The critical behaviour for fixed finite volume

The critical behaviour of a singularity of type A_k is produced by the limit $N \to \infty$ and is essentially independent of whether the volume is finite or infinite. Of course the finite size leads to finite size corrections which we shall neglect here. If the volume is finite, the momentum spectrum is discrete and the corank of the Hessian is one. The possible singularities are of type $A_k, k \in \mathbb{N}$. The fluctuations

$$\begin{aligned} \hat{\xi}(p), &\ \text{all } p \\ \hat{\eta}(p), &\ \text{all } p \neq 0 \end{aligned}$$

are Gaussian and must be integrated out first. The saddle point around which the Gaussian integration is performed is fixed by

$$\frac{\partial S_{\text{eff}}}{\partial \hat{\xi}(p)} = 0, \text{ all } p \tag{25}$$

$$\frac{\partial S_{\text{eff}}}{\partial \hat{\eta}(p)} = 0, \text{ all } p \neq 0 \tag{26}$$

so that its location depends on $\hat{\eta}(0)$. Because of translational invariance these conditions imply

$$\hat{\xi}(p) = \hat{\eta}(p) = 0, \quad p \neq 0. \tag{27}$$

Inserting this into S_{eff} gives a new action

$$\tilde{S}_{\text{red}}(\xi_0, \eta_0) = S_{\text{eff}}(\hat{\xi}, \hat{\eta})\Big|_{\substack{\hat{\xi}(p)=\hat{\eta}(p)=0 \\ (p\neq 0)}} \tag{28}$$

where we introduced the new variables

$$\xi_0 = \frac{\hat{\xi}(0)}{V}, \eta_0 = \frac{\hat{\eta}(0)}{V}. \tag{29}$$

From (25) remains

$$\frac{\partial \tilde{S}_{\text{red}}}{\partial \xi_0}(\xi_0, \eta_0) = 0 \tag{30}$$

or explicitly

$$-\lambda_+(0) N_+(0) \xi_0 = \sum_{n=3}^{\infty} \frac{\partial}{\partial \xi_0} \tilde{S}_{\text{red}}^{(n)}(\xi_0, \eta_0). \tag{31}$$

A splitting lemma [12] asserts that this elimination equation can be solved iteratively

$$\xi_0 = H(\eta_0) = \sum_{n=2}^{\infty} a_n \eta_0^n \tag{32}$$

and that H exists in a neighborhood of zero as a function. An explicit determination of the coefficients $\{a_n\}$ at the critical point (the $\{a_n\}$ are functions of $\{f_r\}$ from (2) which assume critical values $\{f_r^c\}$) is crucial for any further explicit determination of critical quantities.

Inserting (32) into $\tilde{S}_{\text{red}}$ (28) gives the reduced action (neglecting a constant)

$$\begin{aligned} S_{\text{red}}(\eta_0) &= \sum_{n=2}^{\infty} \frac{g_n}{n} \eta_0^n \\ &= \tilde{S}_{\text{red}}(H(\eta_0), \eta_0). \end{aligned} \tag{33}$$

A singularity A_k arises if

$$\begin{aligned} g_n = 0, \quad 2 \leq n \leq k \\ g_{k+1} \neq 0. \end{aligned} \tag{34}$$

Now we start the explicit calculation of the critical quantities. Some technicalities are unavoidable in this context. The function $\Sigma(p)$ (13) can be expanded in a power series of p^2 (convergence radius $4m^2$, only the first terms are needed)

$$\Sigma(p) = \sum_{n=0}^{\infty} B_n (p^2)^n. \tag{35}$$

Moreover, we use the standard integrals

$$\begin{aligned} \Pi_n &= \int \frac{d^D p}{(2\pi)^D} (p^2 + m^2)^{-n} \\ &= (4\pi)^{-\mu} \frac{\Gamma(n-\mu)}{\Gamma(n)} (m^2)^{\mu - n} \end{aligned} \tag{36}$$

$$(\mu = \tfrac{1}{2} D).$$

Then B_n can be expressed in terms of Π_{n+2} by

$$B_n = \frac{(-1)^n n! (n+1)!}{(2n+1)!} \Pi_{n+2}. \tag{37}$$

We can use Π_1 and Π_2 to express all fractional powers of momentum dimensions, e.g.

$$\Pi_{n+2} = \frac{\Pi_2^{n+1}}{\Pi_1^n} \delta_n \tag{38}$$

so that δ_n is a function of D only (or $\mu = \frac{1}{2} D$)

$$\delta_n = \frac{(2-\mu)_n}{(n+1)!(1-\mu)^n}. \tag{39}$$

Noting that $\sigma_0 = \Pi_1$ by (7) and (36), we normalize the derivation of the potential at the critical point by

$$\Pi_1^n U^{(n)}(\Pi_1)\Big|_{\text{crit}} = \frac{\Pi_1^2}{2\Pi_2} v_n. \tag{40}$$

Then the following result can be derived

$$v_{n+2} = \sum_{\text{partitions of } n} (-1)^{\ell-1}(n+\ell)! \prod_{j=1}^{\infty} \frac{\delta^{n_j}}{n_j!} \tag{41}$$

$(n \geq 0, v_2 = -1$ from (21))
where ℓ is the length of the partition and n_j is the repetition number of j

$$n = \sum_{j=1}^{\infty} j n_j \tag{42}$$

$$\ell = \sum_{j=1}^{\infty} n_j. \tag{43}$$

This formula has been checked by computer up to $n = 8$. Inserting δ_n (39) into (41) gives the simple expression

$$v_{n+2} = (-1)^{n+1} \left(\frac{2-\mu}{1-\mu}\right)_n . \tag{44}$$

In the case in which the volume is finite, the definitions (38),(40) and the purely algebraic result (41) remain valid. But δ_n (39) obtains a finite size correction which is neglected in (44). We shall neglect such corrections also in the sequel.

The next issue is to calculate the critical coupling constants $\{f_r^c\}$ from all v_n. For a singularity A_k we normalize the coupling constants to

$$A_k : f_r = 0 \text{ for } r > k. \tag{45}$$

In analogy with (40) we define

$$\Pi_1^r f_r^c = \frac{\Pi_1^2}{2\Pi_2} p_r^{(k)}(\mu). \tag{46}$$

Inverting the system of equations ($n \in \{1, 2, ..., k-1\}$)

$$v_{n+1} = (-1)^n \sum_{r=2}^{k} (1-r)_n p_r^{(k)}(\mu) \tag{47}$$

gives ($r \in \{2, 3, ..., k\}$)

$$p_r^{(k)}(\mu) = \frac{(-1)^{r-1}}{(r-1)!(k-r)!} \cdot \frac{1-\mu}{(1-\mu)(r-1)+1} \cdot \left(\frac{2-\mu}{1-\mu}\right)_{k-1} . \tag{48}$$

In the same context we can calculate the critical value for

$$\begin{aligned} b(0) &= \frac{2\Pi_2 r_0}{\Pi_1} = -a(0)^{-1} \\ &= -\left(\frac{\Pi_1^2}{2\Pi_2}\right)^{-1} \Pi_1 U'(\Pi_1) \end{aligned} \tag{49}$$

which gives

$$\begin{aligned} b(0) &= p_1^{(k)}(\mu) \\ &= (1-\mu)\left[\frac{1}{(k-1)!}\left(\frac{2-\mu}{1-\mu}\right)_{k-1} - 1\right] \end{aligned} \tag{50}$$

(if $r = 1$ is inserted into (48) we obtain $p_1^{(k)}(\mu) + (1-\mu)$).

At the critical point also g_{k+1} (34) is fixed

$$\frac{g_{k+1}^c}{k+1} = (-1)^{k+1} \frac{(p_1^{(k)}(\mu))^{k+1}}{(k+1)!} \left(\frac{2-\mu}{1-\mu}\right)_{k-1} \frac{\Pi_1^2}{2\Pi_2}. \tag{51}$$

It is important to remark that the second condition (34) is satisfied indeed by (51), since for $0 \leq \mu < 1$ $(0 \leq D < 2)$

$$\left(\frac{2-\mu}{1-\mu}\right)_{k-1} > (k-1)! \tag{52}$$

$$p_1^{(k)}(\mu) > 0 \tag{53}$$

$$\mathrm{sign}\ g_{k+1}^c = (-1)^{k+1}. \tag{54}$$

4 Deformation of the singularity and the double scaling limit for fixed finite volume

Singularities can be deformed [11, 12]. We assume that this is achieved for A_k by

$$f_r = f_r^c(1+\Theta_r) \tag{55}$$

$$2 \leq r \leq k$$

whereas the parameter m^2 is invariant. This is a nonstandard way of deforming: the standard way would be to keep $f_k = f_k^c$ fixed and let m^2 (or f_1) vary. The quantity m^2 is kept constant to simplify the following discussion and this is achieved by compensation of the variation of r_0 and β^2 in (8)

$$m^2 = \beta^2 - 2r_0. \tag{56}$$

Invariance of m^2 implies that

$$\sigma_0 = \Pi_1(m^2) \tag{57}$$

is not varied either, so that in

$$r_0 = -U'(\sigma_0) \tag{58}$$

the variation comes only from (55). We conclude that from the k quantities $\{f_1 = \frac{\beta^2}{2}, f_2, \ldots, f_k\}$ only $k-1$ are varied.

In any case the Hessian is diagonalized exactly and the fluctuations $\hat{\xi}, \hat{\eta}$ are considered independent of the deformation.

Correspondingly the coupling constants $\{g_n\}$ in (33) deviate from the critical values,

$$\frac{g_n}{n} = \sum_{r=2}^{k} \alpha_{nr}^{(k)} \Theta_r + O_2(\Theta) \tag{59}$$
$$(2 \leq n \leq k).$$

The double scaling limit is obtained by coupling two processes

(1) $N \to \infty$

(2) $\Theta_r \to 0$, $\forall r$

in a particular way. In the momentum spectrum any neighboring eigenvalue p_1 of $p = 0$ has a fixed distance $O(L^{-1})$ of $p = 0$ and therefore the deformation parameters can be restricted to a neighborhood of zero so small that $\lambda_-(p_1)$ and $\lambda_-(0)$ have values in non-intersecting intervals on this neighborhood.

We study the singular partition function

$$Z_{\text{sing}} = \int d\eta_0 e^{-NS_{\text{red}}(\eta_0)}. \tag{60}$$

Since $g_{k+1}^c \neq 0$ we can approximate g_{k+1} by g_{k+1}^c over the whole deformation neigborhood. We introduce a new variable y by requiring

$$y^{k+1} = N|g_{k+1}^c|\eta_0^{k+1} \tag{61}$$

$$\eta_0 = y \left(N|g_{k+1}^c|\right)^{-\frac{1}{k+1}}. \tag{62}$$

Inserting (62) into $S_{\text{red}}(\eta_0)$, we see that we have to perform the limit

$$\zeta_n = \lim N \left(N|g^c_{k+1}|\right)^{-\frac{n}{k+1}} \sum_r \alpha^{(k)}_{nr} \Theta_r \tag{63}$$

$$(2 \leq n \leq k).$$

Thus the double scaling limit is defined via the "susceptibility matrix" $\alpha^{(k)}$, and each linear combination $(\alpha^{(k)}\Theta)_n$ scales as $N^{-\sigma^{(k)}_n}$

$$\sigma^{(k)}_n = 1 - \frac{n}{k+1}. \tag{64}$$

These are the "critical indices". The result of the double scaling limit is to leading order

$$Z_{\text{sing}} = \left(N|g^c_{k+1}|\right)^{-\frac{1}{k+1}} Y_\epsilon(\zeta_2, \zeta_3, ...\zeta_k) \tag{65}$$

where Y_ϵ is the generalized Airy function,

$$Y_\epsilon(\zeta_2, \zeta_3, ..., \zeta_k) = \int_{C^{(k)}} dy \exp\left\{-\sum_{n=2}^{k} \zeta_n y^n - \epsilon \frac{y^{k+1}}{k+1}\right\} \tag{66}$$

$$\epsilon = \text{sign } g^c_{k+1}. \tag{67}$$

The contour $C^{(k)}$ is the real axis if

$$\epsilon = +1, \ k \text{ odd} \tag{68}$$

and a combination of complex contours, running from infinity to infinity along which the integral converges exponentially, in all other cases. By a translation $y \to y + a$ we can eliminate the term y^k and produce a term y^1 obtaining the standard form of a generalized Airy function.

The function Y_ϵ or any function of Y_ϵ such as $F = \log Y_\epsilon$ satisfy a renormalization group equation

$$\left(N\frac{\partial}{\partial N} - \sum_{n=2}^{k} \beta_n(\Theta)\frac{\partial}{\partial \Theta_n}\right) F(\zeta_2, \zeta_3, ...\zeta_k) = 0 \tag{69}$$

where each ζ_n is considered as a function of N and $\{\Theta_r\}$ which in the neighborhood of the singularity is determined by (63). The beta functions $\{\beta_n(\Theta)\}$ are determined from $(2 \leq n \leq k)$

$$N\frac{\partial \zeta_n}{\partial N} - \sum_{r=2}^{k} \beta_r(\Theta)\frac{\partial \zeta_n}{\partial \Theta_r} = 0. \tag{70}$$

For small $\{\Theta_r\}$ this is satisfied if

$$\beta_r(\Theta) = \sum_{n=2}^{k} \mathcal{N}^{(k)}_{rn} \Theta_n + O_2(\Theta) \tag{71}$$

$$\mathcal{N}^{(k)} = \alpha^{(k),-1}\mathrm{diag}\ \sigma^{(k)}\alpha^{(k)}. \tag{72}$$

The susceptibility matrix must therefore be invertible. We will verify this by an explicit calculation. This calculation starts from the following observation. We can use

$$u_n = U^{(n)}(\Pi_1) - U^{(n)}(\Pi_1)\Big|_{crit} \tag{73}$$

as parameters of deformation instead of the $\{\Theta_n\}$. Then $\tilde{S}_{\text{red}}$ depends on

$$\tilde{S}_{\text{red}}(\xi_0, \eta_0; u_2, u_3, ...u_k). \tag{74}$$

From the constraint (30) we obtain the elimination function

$$\xi_0 = H(\eta_0; u_2, u_3, ..., u_k) \tag{75}$$

and

$$S_{\text{red}}(\eta_0; u_2, u_3, ..., u_k) = \tilde{S}_{\text{red}}(H(\eta_0; u_2, u_3, ..., u_k), \eta_0; u_2...u_k). \tag{76}$$

It follows from (30) that in

$$\frac{\partial S_{\text{red}}}{\partial u_n} = \frac{\partial \tilde{S}_{\text{red}}}{\partial \xi_0}\frac{\partial H}{\partial u_n} + \frac{\partial \tilde{S}_{\text{red}}}{\partial u_n} \tag{77}$$

the first term vanishes. The variation of u_n enters the second term either directly or via r_0. However, it can be shown, that at the critical point and for constant $\{a_n\}$ the derivative with respect to r_0 vanishes. Thus only the direct dependence is left and we obtain

$$\frac{1}{n}\frac{\partial g_n}{\partial u_\ell} = \frac{1}{\ell!}\left(-\frac{\Pi_1}{b(0)}\right)^{\ell} \sum_{\substack{\text{partitions of n} \\ \text{of length } \ell}} \begin{pmatrix} \ell \\ n_1 n_2 n_3 ... \end{pmatrix} (-b(0)^2)^n \prod_{j=2}^{\infty} a_j^{n_j} \tag{78}$$

where ℓ is the length and n_j the repetition number of j (as in (42),(43)). The coefficients $\{a_j\}$ of the elimination function H (32) at the critical point can be expressed as functions of D (or μ) by

$$a_{n+1} = -b(0)^{n+2} \sum_{\text{partitions of } n} \frac{(n+\ell)!}{(n+1)!}(1+b(0)^2)^{-\ell} \cdot \prod_{j=1}^{\infty} \frac{1}{n_j!}\left(\frac{v_{j+2}}{(j+1)!}\right)^{n_j} \tag{79}$$

with ℓ and n_j as in (42), (43). This formula has been verified for $n \leq 8$.

Next we reduce the susceptibility matrix

$$\alpha_{nr}^{(k)} = \frac{\Pi_1^2}{2\Pi_2}(p_1^{(k)}(\mu))^n p_r^{(k)}(\mu)\tilde{\alpha}_{nr}^{(k)} \tag{80}$$

and find from inserting (79) into $\tilde{S}_{\text{red}}$

$$\alpha_{nr}^{(k)} = \sum_{s=2}^{r} S_{ns}^{(k)} \binom{r-1}{s-1}. \tag{81}$$

Here $S^{(k)}$ is a lower left triangular matrix

$$S_{nr}^{(k)} = 0, \quad r > n \tag{82}$$

and

$$B_{sr} = \binom{r-1}{s-1} \tag{83}$$

is an upper right triangular matrix. Moreover we have

$$S_{nn}^{(k)} = \frac{1}{n} \quad (n \geq 2) \tag{84}$$

and for all other elements $n > r \geq 2$

$$S_{nr}^{(k)} = \sum_{\text{partitions of } n-r} (1 - \frac{n}{n+\ell-1}\delta_{r2}) \frac{(n+\ell-1)!}{n!}(1+b(0)^2)^{-\ell} \cdot \prod_{j=1}^{\infty} \frac{1}{n_j!}\left(\frac{v_{j+2}}{(j+1)!}\right)^{n_j} \tag{85}$$

with ℓ the length and n_j the repetition number of j of the partition of $n-r$.

The representation (81) gives for the inverse

$$\tilde{\alpha}^{(k),-1} = B^{-1}S^{(k),-1} \tag{86}$$

with

$$B_{sr}^{-1} = (-1)^{s-r}\binom{r-1}{s-1} \tag{87}$$

and

$$\begin{aligned} S_{rn}^{(k),-1} &= -rnS_{rn}^{(k)} + \sum_{\substack{s \\ (r>s>n)}} rsnS_{rs}^{(k)}S_{sn}^{(k)} \\ &\quad - \sum_{\substack{s_1,s_2 \\ (r>s_1>s_2>n)}} rs_1s_2nS_{rs_1}^{(k)}S_{s_1s_2}^{(k)}S_{s_2n}^{(k)} \\ &\quad \mp \ldots \quad (r > n) \end{aligned} \tag{88}$$

$$S_{nn}^{(k),-1} = n. \tag{89}$$

Finally we obtain by inserting (80),(81), and (86) into (72)

$$\mathcal{N}_{rn}^{(k)} = \frac{p_n^{(k)}(\mu)}{p_r^{(k)}(\mu)}\left(B^{-1}S^{(k),-1}\text{diag }\sigma^{(k)}S^{(k)}B\right)_{rn}. \tag{90}$$

5 The case of unbounded volume

In the case of unbounded volume the momentum spectrum is continuous. The domain of small momenta

$$|p| < \Lambda \tag{91}$$

is considered as an additional deformation. This leads to an additional term in the generalized Airy function integral (the kinetic energy term) and modified critical indices

$$\sigma_n^{(k)} \to \chi_n^{(k)}(\mu). \tag{92}$$

The generalized Airy function (66) is replaced by a field theoretic partition function

$$\begin{aligned} Y_\phi = \int D\phi \exp\Big\{ -\frac{1}{2} \int d^D x \phi(x)(-\Delta + M^2)\phi(x) \\ - \sum_{n=3}^{k} \zeta_k \int d^D x \phi(x)^k - \frac{F_{k+1}}{k+1} \int d^D x \phi(x)^{k+1} \Big\}. \end{aligned} \tag{93}$$

The kinetic energy term has been normalized in (93) instead of the $(k+1)$-st order term. The dimension D is still in the interval $0 \leq D < 2$.

The reduced action S_{red} depends only on $\{\hat{\eta(p)} \,|\, |p| < \Lambda\}$ so that all other degrees of freedom must be integrated out by performing a Gaussian saddle point integration. As the first step we obtain the half-reduced action

$$\tilde{S}_{\text{red}}(\hat{\xi}(p), \hat{\eta}(p)) = S_{\text{eff}}(\hat{\xi}, \hat{\eta})\Big|_{\substack{\hat{\xi}(p)=\hat{\eta}(p)=0 \\ |p|>\Lambda}}. \tag{94}$$

We are left with the issue to solve

$$\frac{\delta}{\delta \hat{\xi}(p)} \tilde{S}_{\text{red}} = 0 \text{ for all } |p| < \Lambda \tag{95}$$

for $\hat{\xi}(p)$. If $\Lambda \ll m$ the trace terms

$$\text{Tr}[(-\Delta + m^2)^{-1} \beta(x)]^n = \int \prod_{i=1}^{n} \frac{d^D q_i}{(2\pi)^D} \frac{\hat{\beta}(q_i - q_{i+1})}{q_i^2 + m^2} \tag{96}$$

$$(q_{n+1} = q_1)$$

reduce to the n-fold convolution product of $\hat{\beta}$ at argument zero times a constant (see (36)), namely

$$= \Pi_n \hat{\beta}_*^n(0). \tag{97}$$

Thus the elimination (95) differs from (30), (31) only by the replacement of $\xi_0^{n_1} \eta_0^{n_2}$ by

$$\left(\hat{\xi}_*^{n_1} * \hat{\eta}_*^{n_2} \right)(0). \tag{98}$$

Its solution is

$$\hat{\xi}(p) = \sum_{n=2}^{\infty} a_n \hat{\eta}_*^n(p) \tag{99}$$

with $\{a_n\}$ as in (32). Correspondingly the reduced action is

$$S_{\text{red}}(\hat{\eta}) = \sum_{n=2}^{\infty} \frac{g_n}{n} \hat{\eta}_*^n(0) \tag{100}$$

with $\{g_n\}$ as in (33). The condition (34) for the appearance of a singularity A_k remains unchanged.

Momenta and coordinates are scaled by [8, 9]

$$p = N^{-\lambda} p' \tag{101}$$

$$x = N^{+\lambda} x' \tag{102}$$

where $\lambda > 0$ is necessary in order that in the limit $N \to \infty$ the domain Λ is mapped onto $\mathbb{R}_D$. The fields are renormalized by

$$\phi(x') = C^{(k)} N^{\frac{1+D\lambda}{k+1}} \eta(x) \tag{103}$$

so that the power of order $k+1$ in (93) obtains a finite coefficient in the limit $N \to \infty$. Both $C^{(k)}$ and λ are determined from the kinetic energy term.

We return to

$$\frac{1}{2} \int\limits_{|p|<\Lambda} \frac{d^D p}{(2\pi)^D} \lambda_-(p) N_-(p) \hat{\eta}(-p) \hat{\eta}(p) \tag{104}$$

and expand into deformation parameters $\{\Theta_n\}$ and p

$$\begin{aligned} \lambda_-(p) N_-(0) &= \frac{\Pi_1^2}{2\Pi_2} b(0)^2 \Big\{ \frac{1}{6}(2-\mu) \frac{p^2}{m^2} \\ &\quad + \sum_{r=2}^{k} (r-1) p_r^{(k)}(\mu) \Theta_r \Big\} + \text{higher order terms} \end{aligned} \tag{105}$$

where (19), (52), (46) have been used. This implies (see (50))

$$C^{(k)} = p_1^{(k)}(\mu) \left[\frac{\Pi_1^2}{2\Pi_2} \frac{1}{6} (2-\mu) \frac{1}{m^2} \right]^{\frac{1}{2}} \tag{106}$$

and [9]

$$\lambda = \frac{k-1}{2(k+1) - D(k-1)}. \tag{107}$$

Positivity of λ is satisfied as long as

$$D < D_\infty = 2 \frac{k+1}{k-1}. \tag{108}$$

This is trivially fulfilled for $D < 2$. From the second term in (102) we obtain the double scaling limit

$$\lim_{\substack{N\to\infty \\ \Theta_r\to 0,\ \text{all r}}} N^{2\lambda}\left(\sum_{r=2}^{k}(r-1)p_r^{(k)}(\mu)\Theta_r\right) = \frac{1}{6}(2-\mu)\frac{M^2}{m^2}. \tag{109}$$

Since this limiting procedure is independent of the other double scaling limits described below (due to the invertibility of the susceptibility matrix) we can ascribe to M^2 any value, in particular any positive value.

The susceptibility matrix $\alpha^{(k)}$ (53) enters all other double scaling limits as usual. For all $3 \leq n \leq k$ we have

$$\zeta_n = \lim_{\substack{N\to\infty \\ \Theta_r\to 0,\ \text{all r}}} (C^{(k)})^{-n} N^{\chi_n^{(k)}} \sum_{r=2}^{k} \alpha_{nr}^{(k)}\Theta_r \tag{110}$$

where the critical indices are now

$$\chi_n^{(k)} = \frac{k+1-n}{(k+1)-\mu(k-1)} \tag{111}$$

so that

$$\chi_n^{(k)}(\mu=0) = \sigma_n^{(k)} \tag{112}$$

(see (64)). Moreover we find from (111)

$$\chi_2^{(k)} = 2\lambda. \tag{113}$$

If we identify

$$\zeta_2 = \frac{1}{2}M^2 \tag{114}$$

we can incorporate the limit (109) into the set of limits (110) as in the case $n = 2$.

Finally we note that (see (51), (103))

$$F_{k+1} = (C^{(k)})^{-k-1} g_{k+1}^c. \tag{115}$$

The partition function is written as function

$$Y_\phi(\zeta_2, \zeta_3, ..., \zeta_k)$$

of the double scale invariant quantities $\{\zeta_n\}$. Any function of Y_ϕ satisfies a renormalization group equation (69) with beta functions obeying (71), where, however,

$$\mathcal{N}^{(k)} = \alpha^{(k),-1}\ \mathrm{diag}\chi^{(k)}\alpha^{(k)}. \tag{116}$$

6 Dimensions $D \geq 2$

At $D = 2$ the integral Π_1 (36) exhibits a pole of first order whereas $\Pi_n, n \geq 2$, are holomorphic

$$\Pi_1 = \frac{1}{4\pi}\frac{1}{1-\mu} + \frac{1}{4\pi}\log\frac{4\pi e^{\Gamma'(1)}}{m^2} + O(1-\mu). \tag{117}$$

To obtain a regular expression in $2 < D < 4$ we can simply analytically continue Π_1 in D:

$$\Pi_1^{\mathrm{an}} = \int \frac{d^D p}{(2\pi)^D}\left[(p^2+m^2)^{-1} - (p^2)^{-1}\right] \tag{118}$$

is the convergent integral representation for this analytic continuation in $2 < D < 4$. We can thus renormalize the position of the saddle point $\sigma_0 \to \sigma_0^{\mathrm{ren}}$

$$\sigma_0^{\mathrm{ren}} = \Pi_1^{\mathrm{an}}. \tag{119}$$

The purely formal subtraction formula

$$\sigma_0^{\mathrm{ren}} = \sigma_0 - \sigma_\infty \tag{120}$$

$$\sigma_\infty = \int \frac{d^D p}{(2\pi)^D}\cdot\frac{1}{p^2} \qquad \text{(divergent)} \tag{121}$$

suggest how to renormalize coupling constants and mass [17]. We set

$$\begin{aligned} U'(\sigma_0) &= \sum_{r=2}^{\infty} f_r \sigma_0^{r-1} \\ &= f_1^{\mathrm{ren}} + \sum_{r=2}^{\infty} f_r^{\mathrm{ren}} (\sigma_0^{\mathrm{ren}})^{r-1} \end{aligned} \tag{122}$$

where

$$f_n^{\mathrm{ren}} = \sum_{r=n}^{\infty} \binom{r-1}{n-1} \sigma_\infty^{r-n} f_r \tag{123}$$

and $f_n^{\mathrm{ren}}, 2 \leq n \leq k$, are assumed to be finite. This can be achieved when (45) is valid by adjusting $\{f_r \,|\, 2 \leq r \leq k\}$ correspondingly. We renormalize ρ_0 by

$$i\rho^{\mathrm{ren}} = i\rho_0 - f_1^{\mathrm{ren}} \tag{124}$$

and

$$(m^{\mathrm{ren}})^2 = m^2 - 2f_1^{\mathrm{ren}}. \tag{125}$$

Finally we skip the label "ren" (and "an") and end up with the rule: replace σ_0 by Π_1^{an} and obtain all critical quantities in the interval $2 < D < 4$ by analytic continuation. The consistency of this rule has still to be investigated.

First we inspect from (105) that the sign of the kinetic energy term remains unchanged and $C^{(k)}$ (106) stays real. There remains, however, a problem with the sign of

$$\lambda = \frac{1}{2}\chi_2^{(k)} \tag{126}$$

((107), (111), (114)). We mentioned already that λ is positive if $D < D_\infty(k)$. On the other hand the field theory (93) is superrenormalizable for $D < D_\infty(k)$ and renormalizable for $D = D_\infty(k)$. Thus $D_\infty(k)$ is an absolute limit which we cannot overcome.

In the case $D < D_\infty(k)$ we have to subtract some low order Green functions, for $D = D_\infty(k)$ we need a finite number of subtractions (counter terms) in the action. The subtractions are by momentum cutoff

$$|p'| \leq N^\lambda \Lambda, \quad D < D_\infty(k). \tag{127}$$

For $D = D_\infty(k)$ the limit $N \to \infty$ cannot be performed at all but N has to be renormalized

$$N' = N^{\frac{1}{2(k+1)-D(k-1)}}, \quad D < D_\infty(k) \tag{128}$$

$$|p'| \leq (N')^{k-1}\Lambda \tag{129}$$

and then the limit $N' \to \infty, D \to D_\infty$ is executed. In any way we conclude that:

1. the parameter Λ introduced in the scaling (101), (102) transforms the $N \to \infty$ limit to the UV-cutoff removal limit;

2. the double scaling limit makes sense only if the necessary subtractions dictated by standard renormalization theory are performed in the limiting procedure.

New insights on the renormalization procedure in general cannot be expected.

For $D = 2$ we replace analytic regularization by subtraction of the pole term, i.e. from (117) we derive

$$\Pi_1^{\text{ren}} = \frac{1}{4\pi}\log\frac{\mu^2}{m^2}. \tag{130}$$

Identifying σ_∞ in (120) with the pole term (117) we can proceed exactly as for $D > 2$ and renormalize coupling constants and mass. The calculation of the critical quantities has to be redone in view of (130) and we will outline the results.

For $n \geq 2$ we obtain

$$\Pi_n = \frac{(m^2)^{1-n}}{4\pi(n-1)}. \tag{131}$$

Inserting this into an equivalent version of (41) we obtain

$$U^{(n)}(\Pi_1) = \frac{1}{2}m^2(-4\pi)^{n-1} \tag{132}$$

and correspondingly from (40), (131)

$$v_n = -\left(-\log\frac{\mu^2}{m^2}\right)^{n-2}. \qquad (133)$$

The critical coupling constants follow from (132)

$$f_n^c = \frac{1}{2}m^2\frac{(-4\pi)^{n-1}}{(n-1)!}j_{k-n}(\exp;\log\frac{\mu^2}{m^2}) \qquad (134)$$

where

$$j_n(f;z) \qquad (135)$$

is the Taylor polynomial ("jet") of degree n for the function f with variable z. From (134) we deduce

$$b(0) = \frac{j_{k-1}(\exp;\log\frac{\mu^2}{m^2})-1}{\log\frac{\mu^2}{m^2}} \qquad (136)$$

and

$$\frac{g_{k+1}^c}{k+1} = \frac{m^2}{8\pi}\frac{(1-j_{k-1}(\exp;\log\frac{\mu^2}{m^2}))^{k+1}}{(k+1)!}. \qquad (137)$$

The coefficients $\{a_n\}$ of the elimination function H (32) at the critical point are obtained from (79) and (133)

$$\begin{aligned} a_{n+1} &= (-1)^{n+1}\frac{b(0)^{n+2}}{(n+1)!}(\log\frac{\mu^2}{m^2})^n \\ &\cdot\sum_{\ell=1}^{n}(-1)^\ell\frac{K_{n\ell}}{(1+b(0)^2)^\ell} \end{aligned} \qquad (138)$$

where $K_{n\ell}$ are integers

$$K_{n\ell} = (n+\ell)!\sum_{\substack{\text{partitions of } n\\ \text{of length } \ell}}\prod_{j=1}^{\infty}\frac{1}{n_j!}\left(\frac{1}{(j+1)!}\right)^{n_j} \qquad (139)$$

(n_j is the repetition number of j in the partition). Instead of (80) we reduce the susceptibility matrix by

$$\alpha_{nr}^{(k)} = (\Pi_1 b(0))^n(4\pi)^r f_r^c\tilde{\alpha}_{nr}^{(k)} \qquad (140)$$

and for the reduced matrix (81)-(85) remain valid. Instead of (85) we find an analogous formula with v_{j+2} replaced by $(-1)^{j+1}$ leading to

$$S_{nr}^{(k)} = \sum_{\ell=1}^{n-r}(-1)^{n+\ell-r}(1-\frac{n}{n+\ell-1}\delta_{r2})\frac{(n+\ell-1)!}{n!}\frac{K_{n-r,\ell}}{(n+\ell-r)!}(1+b(0)^2)^{-\ell}. \qquad (141)$$

Since the double scaling limit for $D \geq 2$ necessitates regularization in the UV momentum domain, as dictated by the known renormalization theory, we refrain from dealing with the case $D = 4$ ($k = 3$) here. It has been argued that a double scaling limit does not exist at $D = 4$ [18, 19]. We emphasize that $D = 4$ is an "exceptional dimension" in the sense described below ($n = 2$ in (144), $k_{\text{max}} = 0$ in (146)). This hints also to the nonexistence of the standard double scaling limit.

Now we return to the second condition (34) in the case $2 \leq D < \infty$. In fact from (51) we can see that g^c_{k+1} vanishes if and only if

1. $\left(\frac{2-\mu}{1-\mu}\right)_{k-1} = 0$, which is fulfilled for

$$\mu = \frac{n}{n-1}, \quad 2 \leq n \leq k \tag{142}$$

2. $p_1^{(k)}(\mu) = 0$, which occurs only if k is odd and

$$\mu = \frac{k+1}{k} \tag{143}$$

(that there are no other zeros of $p_1^{(k)}(\mu)$ in the interval $1 < \mu < 2$ has been verified by computer up to $k = 20$).

Thus there exist exceptional dimensions

$$\mu_n = \frac{n}{n-1}, \quad n \geq 3 \ (\in \mathbb{Z}) \tag{144}$$

for which the type of the singularity A_k has k not constrained by the renormalizability limit (see (109))

$$k \leq k_{\text{ren}} = \frac{\mu+1}{\mu-1} \tag{145}$$

but by the stronger bound

$$k \leq k_{\text{max}} = \begin{cases} n-1, & n \text{ odd}, \\ n-2, & n \text{ odd}. \end{cases} \tag{146}$$

In particular the physically very interesting case $D = 3$ is in this set of exceptional dimensions with $n = 3$ and

$$k_{\text{max}} = 2. \tag{147}$$

7 Remark: The unstable cases

All actions (93) for which

$$\text{sign } F^c_{k+1} = +1, \quad k \text{ odd} \tag{148}$$

is <u>not</u> satisfied are unstable field theories if interpreted conventionally. However, our derivation of these conditions from saddle point integrals implies that the fields ϕ range over complex contours. For example, for $k = 2$, these are the standard Airy function contours

$$C = C_0 - \frac{1}{2}(C_{\frac{1}{3}} + C_{\frac{2}{3}}) \qquad (\epsilon = +1) \tag{149}$$

$$C = -C_{\frac{1}{2}} + \frac{1}{2}(C_{\frac{1}{6}} + C_{\frac{5}{6}}) \qquad (\epsilon = -1) \tag{150}$$

where $C_q,\ q \in \mathbb{Q}$, denotes the oriented ray along the argument

$$\arg C_q = 2\pi q \tag{151}$$

from zero to infinity. Again from $k = 2,\ D = 0$ we know that there exists a domain of parameters

$$(\zeta_2, \zeta_3, \ldots, \zeta_k, F_{k+1}) \tag{152}$$

where the partition function is positive and another one where it oscillates so that both domains are separated by a hypersurface on which the partition function vanishes. Whether in the domain of positivity the action defines a reasonable renormalizable field theory is unknown.

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