Exponentially exact hyperbolic systems

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Abstract

Starting with general hyperbolic systems of conservation laws, a special sub–class is extracted in which classical solutions can be expressed in terms of a linear transport equation. A characterizing property of this sub–class which contains, for example, all linear systems and non–linear scalar equations, is the existence of so called exponentially exact entropies.

Keywords. hyperbolic conservation laws, classical solutions, solution formula, kinetic approach, special entropies

AMS subject classifications. 35L45, 82C40

1 Introduction

The considerations in this article are motivated by a result of Lions, Perthame and Tadmor [12] which allows a reformulation of scalar conservation laws in terms of kinetic equations. The surprising aspect of this reformulation is that non–linear differential equations turn out to be equivalent to equations with a linear transport operator and a source term. This observation can be used to derive numerical schemes for the nonlinear equations by discretizing the linear operator of the reformulation [3, 6, 15, 10, 16], and it is useful to prove analytical properties of solutions of conservation laws (see [12]).

To illustrate the basic ideas we consider a specific example. According to [12], the problem to find the entropy solution of the Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0, \qquad u(0,x) = u^0(x), \qquad x \in \mathbb{R} \tag{1}$$

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can be restated as finding a solution $f(t,x,v)$ of the transport equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \frac{\partial m}{\partial v} \qquad \text{in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+) \tag{2}$$

where m is a non–negative bounded measure which is chosen to ensure a particular v–dependence of f

$$f(t,x,v) = \mu(u(t,x);v) \qquad \text{for some function } u(t,x). \tag{3}$$

Here, μ is the difference of two Heaviside functions $\mu(u;v) = H(v) - H(v-u)$. The relation between (1) and (2), (3) is as follows (for details see [12]): if u is the entropy solution of (1) then $f(t,x,v) = \mu(u(t,x);v)$ solves (2) for some non–negative bounded measure m. Conversely, if f, m solve (2), (3) then the v–average of f

$$\langle f(t,x,v), 1\rangle_v := \int_{\mathbb{R}} f(t,x,v)\, dv = u(t,x)$$

is the entropy solution of the Burgers equation. To illustrate this structural property, we test (2) with the function $\phi(v)\equiv 1$. In view of the constraint (3) and the fact that $\langle \partial_v m, 1\rangle_v = -\langle m, \partial_v 1\rangle_v = 0$, we get

$$\frac{\partial}{\partial t}\langle \mu, 1\rangle_v + \frac{\partial}{\partial x}\langle \mu, v\rangle_v = 0. \tag{4}$$

An easy calculation yields

$$\langle \mu(u;v), 1\rangle_v = u, \qquad \langle \mu(u;v), v\rangle_v = \frac{1}{2}u^2 \qquad \forall u\in\mathbb{R}$$

so that (4) really turns into (1) showing that $u = \langle f, 1\rangle_v$ solves the Burgers equation.

The measure m which serves as a Lagrange multiplier to ensure the constraint $f=\mu$ has the interesting property that its (t,x) support is concentrated on the points of discontinuity of u. In other words, for smooth solutions of the conservation law, f automatically keeps the form μ. This leads to the following statement which is the basis for the investigations in this article: the smooth solutions u of equation (1) can be obtained as average $u = \langle f, 1\rangle_v$ where f follows the evolution of free transport

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0, \qquad f(0,x,v) = \mu(u^0(x);v). \tag{5}$$

Since (5) is easily solvable, we obtain a formula for smooth solutions of the conservation law in terms of the initial value

$$u(t,x) = \left\langle \mu(u^0(x-vt);v), 1\right\rangle_v. \tag{6}$$

Note that (5) describes the free streaming of a particle ensemble ($f(t,x,v)$ is the number of particles which have the velocity v at time t and position x). Hence, the relation between (1) and (5) can also be regarded as a particle model for the conservation law. In this context it is remarkable that the nonlinear behavior of the solution to (1) can be described by an extremely simple, linear particle dynamics. On the other hand, it is also clear that the simple free streaming leads to wrong results as soon as shocks appear in the solution. In fact, shocks are naturally connected to a deceleration of the flow (e.g. in the Burgers equation the shock speed is the average of the speeds to the left and to the right of the discontinuity) but this effect can not be captured with a model where the particles are not subject to any force. Hence, the "collision" term $\partial_v m$ is required to replace, for example, high particle velocities by the shock velocity.

In the following, we want to classify those hyperbolic systems of conservation laws which allow a similar kinetic approach as the Burgers equation. More precisely, we study general, autonomous hyperbolic problems of the form

$$\partial_t \boldsymbol{U}(t,\boldsymbol{x}) + \partial_{x_j} \boldsymbol{F}^j(\boldsymbol{U}(t,\boldsymbol{x})) = \boldsymbol{0}, \qquad \boldsymbol{U}(0,\boldsymbol{x}) = \boldsymbol{U}^0(\boldsymbol{x}) \tag{7}$$

with $\boldsymbol{x} \in \mathbb{R}^d$ and $\boldsymbol{U} = (U_1, \dots, U_m)^T$ (Einstein's summation convention for repeated indices will be used throughout). The aim is to characterize systems which are equivalent to a transport equation for the vector density $\boldsymbol{f} = (f_1, \dots, f_m)^T$

$$\frac{\partial \boldsymbol{f}}{\partial t} + v_j \frac{\partial \boldsymbol{f}}{\partial x_j} = \boldsymbol{Q}$$

where $\boldsymbol{Q}$ is a Lagrange multiplier taking the role of $\partial m/\partial v$ in (2). In particular, we assume that $\boldsymbol{Q}$ assures a special functional form $\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{v}) = \boldsymbol{\mu}(\boldsymbol{U}(t,\boldsymbol{x});\boldsymbol{v})$, satisfies $\langle \boldsymbol{Q}, 1\rangle_{\boldsymbol{v}} = \boldsymbol{0}$, and is supported on the points of discontinuity of $\boldsymbol{U}(t,\boldsymbol{x})$.

Here, we will focus only on the necessary condition for such a representation, that smooth solutions can be written as velocity averages $\boldsymbol{U}(t,\boldsymbol{x}) = \langle \boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{v}), 1\rangle_{\boldsymbol{v}}$ where $\boldsymbol{f}$ satisfies the simple evolution

$$\frac{\partial \boldsymbol{f}}{\partial t} + v_j \frac{\partial \boldsymbol{f}}{\partial x_j} = \boldsymbol{0}, \qquad \boldsymbol{f}(0,\boldsymbol{x},\boldsymbol{v}) = \boldsymbol{\mu}(\boldsymbol{U}^0(\boldsymbol{x});\boldsymbol{v}) \tag{8}$$

giving rise to the solution formula

$$\boldsymbol{U}(t,\boldsymbol{x}) = \left\langle \boldsymbol{\mu}(\boldsymbol{U}^0(\boldsymbol{x} - t\boldsymbol{v});\boldsymbol{v}), 1\right\rangle_{\boldsymbol{v}}. \tag{9}$$

Such a representation is obtained for a class of systems which we call *exponentially exact.* The property which characterizes these systems is related to the exponential matrix $\hat{E}(\boldsymbol{U};\boldsymbol{\xi}) = \exp(-i\xi_j A^j(\boldsymbol{U}))$ where $A^j = \nabla \boldsymbol{F}^j$ is the Jacobian matrix of the flux $\boldsymbol{F}^j$ (note that this exponential matrix occurs, for example, if linear hyperbolic systems are solved with the Fourier method). More precisely, we call a hyperbolic system *exponentially exact*, if for any $\boldsymbol{\xi} \in \mathbb{R}^d$ and $k = 1, \dots, d$ the matrix valued function $\boldsymbol{U} \mapsto \hat{E}(\boldsymbol{U};\boldsymbol{\xi}) A^k(\boldsymbol{U})$ possesses a primitive $\hat{\boldsymbol{\mu}}(\boldsymbol{U};\boldsymbol{\xi})$. The equivalence between (7) and (8) is then obtained if $\boldsymbol{\mu}$ is chosen as inverse Fourier transform of $\hat{\boldsymbol{\mu}}$ with respect to $\boldsymbol{\xi}$.

In the case of Burgers equation, we have for example $\hat{E}(u;\xi) = \exp(-i\xi u)$ which has the primitive

$$\hat{\mu}(u;\xi) = \int_0^u \exp(-i\xi s)\, ds \tag{10}$$

and the inverse Fourier transform

$$\mu(u;v) = \int_0^u \delta(v-s)\, ds = -(H(v-u) - H(v)).$$

Note that this is exactly the constraint used in our introductory example from [12].

Similarly, we find that all non–linear scalar conservation laws are exponentially exact because primitives of $\hat{E}A^k$ can always be found by a simple integration as in (10). Also, the class of exponentially exact systems naturally includes all linear equations since $\hat{E}\,A^k$ is $\boldsymbol{U}$–independent and thus is the Jacobian of the linear function $\hat{\boldsymbol{\mu}}(\boldsymbol{U};\boldsymbol{\xi}) = \hat{E}(\boldsymbol{\xi})\,A^k\boldsymbol{U}$. Apart from these major examples, there are also some non–linear hyperbolic systems which are exponentially exact. As examples, we mention the systems proposed by Brenier and Corrias [1] as well as the isentropic Euler equations with constant pressure.

We conclude the introduction with an outline of the article. In Section 2, we present the main results together with the required definitions and assumptions. Proofs are contained in Sections 3 to 5. Finally, applications of the results are presented in Section 6.

2 Statement of the result

2.1 Assumptions on the hyperbolic system

In the following, we will consider general hyperbolic problems

$$\partial_t \boldsymbol{U}(t,\boldsymbol{x}) + \partial_{x_j}\boldsymbol{F}^j(\boldsymbol{U}(t,\boldsymbol{x})) = \boldsymbol{0}, \qquad \boldsymbol{U}(0,\boldsymbol{x}) = \boldsymbol{U}^0(\boldsymbol{x}) \tag{11}$$

with $\boldsymbol{x} \in \mathbb{R}^d$. We assume that the unknowns $\boldsymbol{U} = (U_1,\ldots,U_m)^T$ are contained in a connected open set $\mathcal{S} \subset \mathbb{R}^m$ (the *state space*) and that $\boldsymbol{F}^j : \mathcal{S} \mapsto \mathbb{R}^m$ are C^1–functions. In the generic case $d > 1$ and $m > 1$, we also assume that $\mathcal{S}$ is simply connected. Note that (11) is *hyperbolic* if all linear combinations $\xi_j A^j(\boldsymbol{U})$ of the Jacobian matrices $A^j(\boldsymbol{U}) = \nabla \boldsymbol{F}^j(\boldsymbol{U})$ of the fluxes have only real eigenvalues for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and all $\boldsymbol{U} \in \mathcal{S}$.

Concerning classical solutions of (11), we consider the spaces $\mathscr{J}_T^k$ of $\mathcal{S}$–valued functions $\boldsymbol{U} \in C^k([-T,T]\times\mathbb{R}^d,\mathcal{S})$ which have uniformly bounded derivatives and for which $\boldsymbol{U}([-T,T]\times\mathbb{R}^d)$ is a compact subset of $\mathcal{S}$. Using this notation, our assumption can be stated in the following way: for any $\boldsymbol{U}^0 \in \mathscr{J}_0^\infty$ there exists $T > 0$ such that (11) admits a classical solution $\boldsymbol{U} \in \mathscr{J}_T^1$. (For symmetric hyperbolic systems, existence of smooth solutions is shown, for example, in [13].)

2.2 Formulation of the problem

The original idea to rewrite smooth solutions of (11) in terms of a kinetic formulation will be extended to general entropies. Here, a scalar function $\eta : \mathcal{S} \mapsto \mathbb{R}$ is an entropy function with entropy fluxes $\theta^j : \mathcal{S} \mapsto \mathbb{R}$ provided

$$\nabla^T \eta \, \nabla \boldsymbol{F}^j = \nabla^T \theta^j, \qquad j = 1, \ldots, d \tag{12}$$

where $\nabla^T \eta = (\nabla \eta)^T$. Of course, differentiability of η and θ^j is required. If $\boldsymbol{U}$ is a smooth solution of (11), relation (12) implies that the entropy $\eta(\boldsymbol{U})$ satisfies an additional conservation law

$$\partial_t \eta(\boldsymbol{U}) + \partial_{x_j} \theta^j(\boldsymbol{U}) = 0. \tag{13}$$

Our aim is to find a representation of $\eta(\boldsymbol{U})$ as velocity average of a function $f(t, \boldsymbol{x}, \boldsymbol{v})$ which follows the simple evolution of free transport

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} = 0, \qquad f(0, \boldsymbol{x}, \boldsymbol{v}) = \mu(\boldsymbol{U}^0(\boldsymbol{x}); \boldsymbol{v}),$$

or in other words,

$$\eta(\boldsymbol{U}(t, \boldsymbol{x})) = \left\langle \mu(\boldsymbol{U}^0(\boldsymbol{x} - t\boldsymbol{v}); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}. \tag{14}$$

The question is, whether such a representation exists at all and how the kernel μ has to be chosen.

Note that the approach includes the problem to find a representation of the solution $\boldsymbol{U}$ itself. In fact, with the linear entropies $\eta_i(\boldsymbol{U}) = U_i$ and corresponding fluxes $\theta_i^j(\boldsymbol{U}) = F_i^j(\boldsymbol{U})$ for $j = 1, \ldots, d$, the conservation law (13) is just the i^{th} member of the system (11). If suitable kernels μ_i exist, we will collect them in a vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)^T$ in accordance with our notation in Section 1.

In order to give (14) a precise mathematical meaning, we have to require some properties of μ. For fixed $\boldsymbol{U} \in \mathcal{S}$, we assume that $\mu(\boldsymbol{U})$ is a compactly supported distribution (we also write $\mu(\boldsymbol{U}; \boldsymbol{v})$ to indicate that $\mu(\boldsymbol{U})$ acts on the variable $\boldsymbol{v}$). Introducing as usual $\mathscr{E}(\mathbb{R}^d)$ as the space of C^∞ functions with a topology generated by the seminorms

$$q_n(\varphi) = \max_{|\boldsymbol{\alpha}| \le n} \sup_{|\boldsymbol{v}| \le n} |\nabla^{\boldsymbol{\alpha}} \varphi(\boldsymbol{v})|, \qquad \varphi \in \mathscr{E}(\mathbb{R}^d),\ n \in \mathbb{N},$$

(we use standard multi–index notation) the compactly supported distributions $\mathscr{E}'(\mathbb{R}^d)$ are the continuous linear functionals on $\mathscr{E}$. Using this notation, we require that $\boldsymbol{U} \mapsto \mu(\boldsymbol{U})$ is a continuous mapping with values in $\mathscr{E}'$ which has some locally uniform properties.

Definition 1 *By $\mathscr{K}$ we denote the set of all continuous functions $\mu : \mathcal{S} \mapsto \mathscr{E}'(\mathbb{R}^d)$ which satisfy for any compact $K \subset \mathcal{S}$ and any $\varphi \in \mathscr{E}(\mathbb{R}^d)$*

$$|\langle \mu(\boldsymbol{U}), \varphi \rangle| \leq C_K q_{N_K}(\varphi), \qquad \forall \boldsymbol{U} \in K$$

where N_K and C_K depend on μ and K. The subset $\mathscr{K}^1 \subset \mathscr{K}$ contains all μ for which there exists a continuous mapping $\nabla\mu : \mathcal{S} \mapsto [\mathscr{E}'(\mathbb{R}^d)]^m$ such that

$$\nabla \langle \mu(\boldsymbol{U}), \varphi \rangle = \langle \nabla\mu(\boldsymbol{U}), \varphi \rangle \qquad \forall \varphi \in \mathscr{E}(\mathbb{R}^d).$$

In Proposition 8, we show that the right hand side in (14) is a tempered distribution in $\boldsymbol{x}$ for every $t \in \mathbb{R}$. We use $\mathscr{S}'(\mathbb{R}^d)$ to denote the set of tempered distributions where $\mathscr{S}(\mathbb{R}^d)$ are the rapidly decaying test functions with a topology generated by the seminorms

$$p_n(\psi) = \max_{|\boldsymbol{\alpha}| \leq 2n} \sup_{\boldsymbol{x} \in \mathbb{R}^d} |(1 + |\boldsymbol{x}|^2)^n \nabla^{\boldsymbol{\alpha}} \psi(\boldsymbol{x})|, \qquad \psi \in \mathscr{S}(\mathbb{R}^d),\ n \in \mathbb{N}.$$

On the other hand, if $\boldsymbol{U} \in \mathscr{J}_T^1$, the function $\boldsymbol{x} \mapsto \eta(\boldsymbol{U}(t, \boldsymbol{x}))$ is uniformly bounded for every $t \in [-T, T]$ so that it can also be viewed as a tempered distribution. A more precise formulation of (14) is based on the following

Definition 2 *Let $\eta \in C^1(\mathcal{S}, \mathbb{R})$ and $\mu \in \mathscr{K}$. We call μ a kinetic representation of η if for all $\boldsymbol{U}^0 \in \mathscr{J}_0^\infty$ with corresponding solution $\boldsymbol{U} \in \mathscr{J}_T^1$ of (11), the equality*

$$\eta(\boldsymbol{U}(t, \boldsymbol{x})) = \left\langle \mu(\boldsymbol{U}^0(\boldsymbol{x} - t\boldsymbol{v}); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}$$

holds in $\mathscr{S}'(R^d)$ for all $t \in [-T, T]$.

Our basic questions can now be stated as follows: When does an entropy η possess a kinetic representation μ? How does μ look like, if it exists?

2.3 The result

To answer the basic questions from the previous section, we completely characterize those entropies having a kinetic representation. The characterization is related to an integrability property of a function involving the entropy and the hyperbolic system.

Definition 3 *A function $\eta \in C^1(\mathcal{S}, \mathbb{R})$ is called exponentially exact entropy for the system (11) if*

$$\boldsymbol{U} \mapsto \nabla^T \eta(\boldsymbol{U}) \exp(-i\xi_j A^j(\boldsymbol{U})) A^k(\boldsymbol{U}) \tag{15}$$

has a primitive for every $\boldsymbol{\xi} \in \mathbb{R}^d$ and $k = 1, \ldots, d$. If all linear entropies satisfy this condition, the system (11) is called exponentially exact.

We remark that exponentially exact entropies are also entropies in the usual sense. This is easily seen by setting $\boldsymbol{\xi} = \mathbf{0}$ in (15), in which case the primitives are just the entropy fluxes. According to Lemma 12, exponential exactness requires in addition that also the functions $\nabla^T\eta A^j A^k$, $\nabla^T\eta(A^j A^i + A^i A^j)A^k$, etc. have primitives. We will show in Section 5 that exponential exactness of η is equivalent to the existence of a kinetic representation.

Theorem 4 *Assume the system* (11) *satisfies the conditions in Section 2.1 and* $\eta \in C^1(\mathcal{S}, \mathbb{R})$. *Then,* η *has a kinetic representation if and only if* η *is exponentially exact.*

The sufficiency part of Theorem 4, which is proven in Section 5.1, also yields a complete characterization of the kinetic representation.

Theorem 5 *Let* η *be an exponentially exact entropy. Then there exists a kinetic representation* $\mu \in \mathscr{K}^1$ *with the property*

$$\nabla^T\mu(\boldsymbol{U}) = \nabla^T\eta(\boldsymbol{U})E(\boldsymbol{U}), \qquad E(\boldsymbol{U}) = \mathcal{F}_{\boldsymbol{\xi}}^{-1}\exp(-i\xi_j A^j(\boldsymbol{U})).$$

and $\langle\mu(\boldsymbol{U}), 1\rangle = \eta(\boldsymbol{U})$ *for all* $\boldsymbol{U} \in \mathcal{S}$. *Any other kinetic representation differs from* μ *only by a compactly supported distribution* $C \in \mathscr{E}'(\mathbb{R}^d)$ *which is independent of* $\boldsymbol{U}$ *and satisfies* $\langle C, 1\rangle = 0$.

Since the integrability condition (15) is satisfied for any smooth η if (11) is a single conservation law, we see that kinetic representations are particularly well suited for the scalar case. For systems, however, exponential exactness is a real restriction and kinetic representations are less natural.

3 The kinetic formulation

3.1 Basic observations

In this section we study properties of $\tilde{\eta}(t, \boldsymbol{x}) = \langle\mu(\boldsymbol{U}(\boldsymbol{x} - \boldsymbol{v}t); \boldsymbol{v}), 1\rangle_{\boldsymbol{v}}$, where $\boldsymbol{U} \in \mathscr{I}_0^0$ and $t \in \mathbb{R}$. For any fixed t, $\tilde{\eta}$ is a distribution in $\boldsymbol{x}$ which is defined by

$$\langle\tilde{\eta}(t, \boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{x}} := \int_{\mathbb{R}^d} \langle\mu(\boldsymbol{U}(\boldsymbol{x}); \boldsymbol{v}), \psi(\boldsymbol{x} + \boldsymbol{v}t)\rangle_{\boldsymbol{v}}\, d\boldsymbol{x}. \tag{16}$$

We will see that ψ can be chosen from the space $\mathscr{S}$ of rapidly decaying test functions. Since t–derivatives of $\tilde{\eta}$ lead to additional $\boldsymbol{v}$–factors, we immediately consider the more general functional

$$G_\varphi(t, \boldsymbol{x}) = \langle\mu(\boldsymbol{U}(\boldsymbol{x} - \boldsymbol{v}t); \boldsymbol{v}), \varphi(\boldsymbol{v})\rangle_{\boldsymbol{v}}$$

based on $\varphi \in \mathscr{E}(\mathbb{R}^d)$ which is defined according to

$$\langle G_\varphi(t,\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{x}} := \int_{\mathbb{R}^d} \langle \mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v}), \varphi(\boldsymbol{v})\psi(\boldsymbol{x}+\boldsymbol{v}t)\rangle_{\boldsymbol{v}} \, d\boldsymbol{x}. \tag{17}$$

The proof that (17) defines a tempered distribution splits into two steps. First, we need an estimate of the integrand in (17) which will be formulated in terms of the $\mathscr{E}$–seminorms q_n and the $\mathscr{S}$–seminorms p_n introduced in Section 2.2. We also use $(\tau_{\boldsymbol{h}}\psi)(\boldsymbol{x}) := \psi(\boldsymbol{x}+\boldsymbol{h})$ to denote the shift operator.

Lemma 6 *Let $\bar{\boldsymbol{x}}, \boldsymbol{x} \in \mathbb{R}^d$, $|t| \le T$ for some $T > 0$, $\boldsymbol{U} \in \mathscr{J}_0^0$, $\mu \in \mathscr{K}$, $\psi \in \mathscr{S}(\mathbb{R}^d)$ and $\varphi \in \mathscr{E}(\mathbb{R}^d)$. Then, the estimate*

$$|\langle \mu(\boldsymbol{U}(\bar{\boldsymbol{x}});\boldsymbol{v}), \varphi(\boldsymbol{v})\psi(\boldsymbol{x}+\boldsymbol{v}t)\rangle_{\boldsymbol{v}}| \le \frac{C}{(1+|\boldsymbol{x}|^2)^M} q_M(\varphi) \sup_{|\boldsymbol{v}|\le M} p_M(\tau_{\boldsymbol{v}t}\psi)$$

holds for some $M \in \mathbb{N}$ with $M \ge d$ and $C > 0$ depending on $\mu, \boldsymbol{U}, T$ and d.

Proof: The composition of $\boldsymbol{U} \in \mathscr{J}_0^0$ with $\mu \in \mathscr{K}$, which is a continuous $\mathscr{E}'$–valued mapping (see Definition 1), leads to $\boldsymbol{x} \mapsto \mu(\boldsymbol{U}(\boldsymbol{x})) \in C^0(\mathbb{R}^d, \mathscr{E}'(\mathbb{R}^d))$. Since $\boldsymbol{U} \in \mathscr{J}_0^0$ has a range which is contained in a compact subset K of $\mathcal{S}$, we find the uniform estimate

$$|\langle \mu(\boldsymbol{U}(\bar{\boldsymbol{x}});\boldsymbol{v}), \phi(\boldsymbol{v})\rangle_{\boldsymbol{v}}| \le C q_M(\phi), \qquad \forall \phi \in \mathscr{E}(\mathbb{R}^d),\ \bar{\boldsymbol{x}} \in \mathbb{R}^d. \tag{18}$$

Here, $M \in \mathbb{N}$ is chosen larger than d and N_K from Definition 1. The constant C_K has been included in C which, from now on, is taken as a generic constant which depends on $\mu, \boldsymbol{U}, T$ and the dimension d.

In view of (17), we apply (18) to the special test function $\phi(\boldsymbol{v}) = \varphi(\boldsymbol{v})\psi(\boldsymbol{x}+\boldsymbol{v}t)$. Using chain and product rule, we immediately get for $|\boldsymbol{\alpha}| \le M$

$$|\nabla_{\boldsymbol{v}}^{\boldsymbol{\alpha}}\phi(\boldsymbol{v})| \le C \max_{|\boldsymbol{\alpha}|\le M} |\nabla_{\boldsymbol{v}}^{\boldsymbol{\alpha}}\varphi(\boldsymbol{v})| \max_{|\boldsymbol{\alpha}|\le M} |\nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\psi(\boldsymbol{x}+\boldsymbol{v}t)|. \tag{19}$$

Since ψ is rapidly decreasing, it is bounded in each of the $\mathscr{S}$–seminorms. Hence,

$$|\nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\psi(\boldsymbol{x}+\boldsymbol{v}t)| \le \frac{p_M(\tau_{\boldsymbol{v}t}\psi)}{(1+|\boldsymbol{x}|^2)^M}$$

so that the result follows with (18) and (19). ∎

In the next step, we combine the estimate with continuity properties of the shift operator

$$p_M(\tau_{\boldsymbol{h}}\psi) \le (2+4|\boldsymbol{h}|^2)^M p_M(\psi) \tag{20}$$

$$p_M(\tau_{\boldsymbol{h}}\psi - \psi) \le C|\boldsymbol{h}| p_{M+1}(\psi), \qquad |\boldsymbol{h}| \le 1 \tag{21}$$

to show that (17) defines a tempered distribution.

Lemma 7 *Let $\mu \in \mathscr{K}$, $\boldsymbol{U} \in \mathscr{J}_0^0$, $\varphi \in \mathscr{E}(\mathbb{R}^d)$ and $|t| \leq T$ for some $T > 0$. Then, the functional $\psi \mapsto \langle G_\varphi(t,\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{x}}$ given by* (17) *defines a tempered distribution which satisfies the estimate*

$$|\langle G_\varphi(t,\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{x}}| \leq C q_M(\varphi) p_M(\psi)$$

for some $M \in \mathbb{N}$ and $C \geq 0$ depending on $\mu, \boldsymbol{U}, T$ and d.

Proof: We first show that $F(t,\boldsymbol{x}) := \langle \mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v}), \varphi(\boldsymbol{v})\psi(\boldsymbol{x}+\boldsymbol{v}t)\rangle_{\boldsymbol{v}}$ is a continuous function in $\boldsymbol{x}$ which gives measurability. Integrability of F over $\mathbb{R}^d$ then follows from Lemma 6 so that (17) is well defined. To show continuity, we split

$$\begin{aligned}|F(t,\boldsymbol{x}+\boldsymbol{h}) - F(t,\boldsymbol{x})| \leq &\, |\langle \mu(\boldsymbol{U}(\boldsymbol{x}+\boldsymbol{h});\boldsymbol{v}) - \mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v}), \varphi(\boldsymbol{v})\tau_{\boldsymbol{v}t}\psi(\boldsymbol{x})\rangle_{\boldsymbol{v}}| \\ &+ |\langle \mu(\boldsymbol{U}(\boldsymbol{x}+\boldsymbol{h});\boldsymbol{v}), \varphi(\boldsymbol{v})\tau_{\boldsymbol{v}t}(\tau_{\boldsymbol{h}}\psi - \psi)(\boldsymbol{x})\rangle_{\boldsymbol{v}}|.\end{aligned}$$

The first term vanishes for $\boldsymbol{h} \to 0$ because of the assumed continuity of μ. For the other term we need the estimates (20) and (21) which yield

$$\sup_{|\boldsymbol{v}|\leq M} p_M(\tau_{\boldsymbol{v}t}(\tau_{\boldsymbol{h}}\psi - \psi)) \leq C(2 + 4|TM|^2)^M p_{M+1}(\psi)|\boldsymbol{h}|.$$

In connection with Lemma 6, convergence of the second term follows.

For the integral $\int F(t,\boldsymbol{x})\,d\boldsymbol{x} = \langle G_\varphi(t,\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{x}}$, we similarly obtain with (20) and Lemma 6

$$|\langle G_\varphi(t,\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{x}}| \leq C(2 + 4|TM|^2)^M q_M(\varphi) p_M(\psi)$$

which finally proves that $G_\varphi(t,\cdot)$ is a tempered distribution for every $\varphi \in \mathscr{E}(\mathbb{R}^d)$. ■

Our next result concerns differentiability properties of $\tilde{\eta}(t,\boldsymbol{x})$ which is, by definition, equal to $G_1(t,\boldsymbol{x})$. It requires a refined version of (21) which is easily obtained by a Taylor expansion argument

$$p_M(\tau_{\boldsymbol{h}}\psi - \psi - h_i\partial_{x_i}\psi) \leq C|\boldsymbol{h}|^2 p_{M+2}(\psi), \qquad |\boldsymbol{h}| \leq 1. \tag{22}$$

Proposition 8 *The mapping $\tilde{\eta}$ defined in* (16) *is contained in $C^\infty(\mathbb{R}, \mathscr{S}'(\mathbb{R}^d))$ with derivatives*

$$\left(\frac{\partial}{\partial t}\right)^n \tilde{\eta}(t,\boldsymbol{x}) = (-1)^n \sum_{|\boldsymbol{\alpha}|=n} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}} \langle \mu(\boldsymbol{U}(\boldsymbol{x}-\boldsymbol{v}t);\boldsymbol{v}), \boldsymbol{v}^{\boldsymbol{\alpha}}\rangle_{\boldsymbol{v}}.$$

Proof: Using again $F(t,\boldsymbol{x}) = \langle \mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v}), \varphi(\boldsymbol{v})\psi(\boldsymbol{x}+\boldsymbol{v}t)\rangle_{\boldsymbol{v}}$ we define

$$\begin{aligned}\Delta_s :=& |F(t+s,\boldsymbol{x}) - F(t,\boldsymbol{x}) - s\langle \mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v}), \varphi(\boldsymbol{v})v_i\partial_{x_i}\tau_{\boldsymbol{v}t}\psi(\boldsymbol{x})\rangle_{\boldsymbol{v}}| \\ =& |\langle \mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v}), \varphi(\boldsymbol{v})\tau_{\boldsymbol{v}t}(\tau_{\boldsymbol{v}s}\psi - \psi - sv_i\partial_{x_i}\psi)(\boldsymbol{x})\rangle_{\boldsymbol{v}}|.\end{aligned}$$

With Lemma 6 and relations (20) and (22) we get for $|sM| \leq 1$

$$\Delta_s \leq \frac{C}{(1+|\boldsymbol{x}|^2)^M} q_M(\varphi)(2+4|tM|^2)^M |sM|^2 p_{M+2}(\psi).$$

This yields both $\Delta_s/s \to 0$ and an s–independent integrable majorant so that $t \mapsto \int F(t,\boldsymbol{x})\,d\boldsymbol{x} = \langle G_\varphi(t,\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{v}}$ is differentiable with derivative

$$\frac{d}{dt}\langle G_\varphi(t,\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{x}} = \langle G_{v_i\varphi}(t,\boldsymbol{x}), \partial_{x_i}\psi(\boldsymbol{x})\rangle_{\boldsymbol{x}} = \langle -\partial_{x_i} G_{v_i\varphi}(t,\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{x}},$$

or in other words $\partial_t G_\varphi(t,\boldsymbol{x}) = -\partial_{x_i} G_{v_i\varphi}(t,\boldsymbol{x})$. Since $\boldsymbol{v} \mapsto v_i\varphi(\boldsymbol{v})$ is still contained in $\mathscr{E}(\mathbb{R}^d)$ we immediately get by induction that $G_\varphi \in C^\infty(\mathbb{R}, \mathscr{S}'(\mathbb{R}^d))$ and

$$\left(\frac{\partial}{\partial t}\right)^n G_\varphi(t,\boldsymbol{x}) = (-1)^n \sum_{|\boldsymbol{\alpha}|=n} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}} G_{\boldsymbol{v}^{\boldsymbol{\alpha}}\varphi}(t,\boldsymbol{x}).$$

where we have used (59) in the appendix. The result follows by setting $\varphi \equiv 1$. ■

Using Lemma 7, one can easily show in connection with the Schwartz kernel theorem [7] that

$$\langle \mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v}), \varphi(\boldsymbol{v})\psi(\boldsymbol{x})\rangle_{(\boldsymbol{x},\boldsymbol{v})} := \int_{\mathbb{R}^d} \langle \mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v}), \varphi(\boldsymbol{v})\rangle_{\boldsymbol{v}}\, \psi(\boldsymbol{x})\,d\boldsymbol{x}$$

defines a unique distribution in $\mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ (which is the dual of $\mathscr{D} = C_0^\infty$ with the usual convergence of compactly supported test functions). Using results form the proof of Proposition 8, we finally see that $\mu(\boldsymbol{U}(\boldsymbol{x}-\boldsymbol{v}t);\boldsymbol{v})$ (the pullback of $\mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v})$ by the mapping $(\boldsymbol{x},\boldsymbol{v}) \mapsto (\boldsymbol{x}-\boldsymbol{v}t, \boldsymbol{v})$) is infinitely often differentiable with respect to t and satisfies the free transport problem

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} = 0, \qquad f(0,\boldsymbol{x},\boldsymbol{v}) = \mu(\boldsymbol{U}(\boldsymbol{x});\boldsymbol{v}).$$

The $\boldsymbol{v}$–average of this solution is, of course, $\tilde{\eta}(t,\boldsymbol{x}) = \langle \mu(\boldsymbol{U}(\boldsymbol{x}-\boldsymbol{v}t);\boldsymbol{v}), 1\rangle_{\boldsymbol{v}}$.

3.2 Regularization

We study a regularization of $\tilde{\eta}$

$$\tilde{\eta}_\epsilon(t,\boldsymbol{x}) := \langle \tilde{\eta}(t,\boldsymbol{y}), h_\epsilon(\boldsymbol{x}-\boldsymbol{y})\rangle_{\boldsymbol{y}} \tag{23}$$

which is based on a particular family of functions h_ϵ. This family is obtained in the following way: choosing $H \in C_0^\infty(\mathbb{R}^d)$ with $H(\mathbf{0}) = 1$ and $H(-\boldsymbol{\xi}) = H(\boldsymbol{\xi})$, we set

$$H_\epsilon(\boldsymbol{\xi}) := H(\epsilon\boldsymbol{\xi}) \qquad \text{and} \qquad h_\epsilon := \mathcal{F}^{-1}H_\epsilon, \qquad \epsilon > 0. \tag{24}$$

Since H_ϵ approximates the one–function for $\epsilon \to 0$, the inverse Fourier transform h_ϵ is a δ–approximating sequence. The compact support of H_ϵ assures that h_ϵ is analytic. For our subsequent investigations, we need the following properties.

Lemma 9 *The functions h_ϵ are contained in $\mathscr{S}(\mathbb{R}^d)$ and for any $\Lambda \in \mathscr{S}'(\mathbb{R}^d)$, the convolution $\Lambda * h_\epsilon$ converges to Λ in the $\mathscr{S}'$–topology as $\epsilon \to 0$. If $|\boldsymbol{x}| \leq R$ and $M \in \mathbb{N}$ then the derivative $g(\boldsymbol{y}) := \nabla_{\boldsymbol{y}}^{\boldsymbol{\alpha}} h_\epsilon(\boldsymbol{x}-\boldsymbol{y})$ satisfies*

$$p_M(g) \leq C\gamma^{|\boldsymbol{\alpha}|}$$

where C, γ depend on H, M, R and ϵ.

Proof: In order to show that the convolution of $\psi \in \mathscr{S}(\mathbb{R}^d)$ with h_ϵ converges to ψ in $\mathscr{S}(\mathbb{R}^d)$, one can use $\mathscr{S}$–continuity of the Fourier transform

$$\psi * h_\epsilon - \psi = \mathcal{F}^{-1}\mathcal{F}\psi * (h_\epsilon - 1) = \mathcal{F}^{-1}(\hat{\psi}(H_\epsilon - 1)).$$

It remains to show $\mathscr{S}$–convergence of $\hat{\psi}(H_\epsilon - 1)$. Using product rule, the following estimate can be derived

$$p_M(\hat{\psi}(H_\epsilon - 1)) \leq C p_{M+1}(\hat{\psi}) \max_{|\boldsymbol{\alpha}|\leq 2M} \sup_{\boldsymbol{x}\in\mathbb{R}^d} \frac{|\nabla^{\boldsymbol{\alpha}}(H_\epsilon - 1)(\boldsymbol{x})|}{1+|\boldsymbol{x}|^2}. \tag{25}$$

In the case $\boldsymbol{\alpha} = \mathbf{0}$, we have $H_\epsilon - 1 \to 0$ locally uniformly so that convergence of (25) is obtained in connection with the decay of $1/(1+|\boldsymbol{x}|^2)$. For $\boldsymbol{\alpha} \neq \mathbf{0}$, the derivative leads to a factor $\epsilon^{|\boldsymbol{\alpha}|}$ and thus gives convergence. Using the relation

$$\langle \Lambda * h_\epsilon(\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{\boldsymbol{x}} = \langle \Lambda(\boldsymbol{y}), \psi * h_\epsilon(\boldsymbol{y})\rangle_{\boldsymbol{y}}$$

the $\mathscr{S}'$–convergence follows immediately.

To estimate the derivative $\nabla_{\boldsymbol{y}}^{\boldsymbol{\alpha}} h_\epsilon(\boldsymbol{x}-\boldsymbol{y})$, we use

$$g(\boldsymbol{y}) = \nabla_{\boldsymbol{y}}^{\boldsymbol{\alpha}} h_\epsilon(\boldsymbol{x}-\boldsymbol{y}) = \mathcal{F}_{\boldsymbol{\xi}}^{-1}\left(e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}(-i\boldsymbol{\xi})^{\boldsymbol{\alpha}} H_\epsilon(-\boldsymbol{\xi})\right)$$

and the $\mathscr{S}$–continuity of $\mathcal{F}^{-1}$ in the form $p_M(\mathcal{F}^{-1}\psi) \leq C p_{M+d}(\psi)$. Since p_{M+d} involves derivatives of order $\bar{M} = 2(M+d)$, This leads to a factor $(1+|\boldsymbol{x}|)^{\bar{M}}$ from the $\boldsymbol{\xi}$–derivatives of $\exp(-i\boldsymbol{x}\cdot\boldsymbol{\xi})$ and another factor $|\boldsymbol{\alpha}|^{\bar{M}}(1+S/\epsilon)^{|\boldsymbol{\alpha}|}$ from the maximum of the derivatives of $(-i\boldsymbol{\xi})^{\boldsymbol{\alpha}}$ over the support of H_ϵ which is contained in a ball of radius $S > 0$. Finally,

$$p_{M+d}(H_\epsilon) \leq C(1+(S/\epsilon)^2)^{M+d} q_{\bar{M}}(H_1)$$

so that

$$p_M(g) \leq C(1+|\boldsymbol{x}|)^{\bar{M}}(1+S/\epsilon)^{\bar{M}}|\boldsymbol{\alpha}|^{\bar{M}}(1+S/\epsilon)^{|\boldsymbol{\alpha}|}.$$

With the remark that $|\boldsymbol{\alpha}|^{\bar{M}} \leq \exp(\bar{M})^{|\boldsymbol{\alpha}|}$, the result follows. ■

Using these properties of the mollifying kernels h_ϵ, we can derive the following result for the regularization of $\tilde{\eta}$.

Proposition 10 *For any* $\boldsymbol{x} \in \mathbb{R}^d$*, the function* $\tilde{\eta}_\epsilon(\cdot, \boldsymbol{x})$ *defined in* (23) *can be written as*

$$\tilde{\eta}_\epsilon(t, \boldsymbol{x}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left. \frac{\partial^n \tilde{\eta}_\epsilon}{\partial t^n} \right|_{(0,\boldsymbol{x})}$$

For $\epsilon \to 0$*,* $\tilde{\eta}_\epsilon(t, \cdot)$ *converges to* $\tilde{\eta}(t, \cdot)$ *in* $\mathscr{S}'(\mathbb{R}^d)$ *for every* $t \in \mathbb{R}$*.*

Proof: We assume that $|t| \le T$ and $|\boldsymbol{x}| \le R$ for arbitrary $T, R > 0$. According to Proposition 8, the n–th time derivative of $\tilde{\eta}$ is given by

$$\left(\frac{\partial}{\partial t}\right)^n \tilde{\eta}_\epsilon(t, \boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=n} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \left\langle G_{\boldsymbol{v}^{\boldsymbol{\alpha}}}(t, \boldsymbol{y}), \nabla_{\boldsymbol{y}}^{\boldsymbol{\alpha}} h_\epsilon(\boldsymbol{x} - \boldsymbol{y}) \right\rangle_{\boldsymbol{y}}$$

where we have set $G_{\boldsymbol{v}^{\boldsymbol{\alpha}}}(t, \boldsymbol{y}) = \langle \mu(\boldsymbol{U}(\boldsymbol{y} - \boldsymbol{v}t); \boldsymbol{v}), \boldsymbol{v}^{\boldsymbol{\alpha}} \rangle_{\boldsymbol{v}}$. Considering one of the terms in the sum and setting $g(\boldsymbol{y}) = \nabla_{\boldsymbol{y}}^{\boldsymbol{\alpha}} h_\epsilon(\boldsymbol{x} - \boldsymbol{y})$, we get with Lemma 7

$$|\langle G_{\boldsymbol{v}^{\boldsymbol{\alpha}}}(t, \boldsymbol{y}), g(\boldsymbol{y}) \rangle_{\boldsymbol{y}}| \le C q_M(\boldsymbol{v}^{\boldsymbol{\alpha}}) p_M(g).$$

With the estimate of $p_M(g)$ in Lemma 9 and $q_M(\boldsymbol{v}^{\boldsymbol{\alpha}}) \le C |\boldsymbol{\alpha}|^M M^{|\boldsymbol{\alpha}|}$ which can be estimated further by $q_M(\boldsymbol{v}^{\boldsymbol{\alpha}}) \le C(M \exp(M))^{|\boldsymbol{\alpha}|}$, we obtain

$$|\langle G_{\boldsymbol{v}^{\boldsymbol{\alpha}}}(t, \boldsymbol{y}), g(\boldsymbol{y}) \rangle_{\boldsymbol{y}}| \le C(\gamma M \exp(M))^{|\boldsymbol{\alpha}|}.$$

Since $\sum_{|\boldsymbol{\alpha}|=n} |\boldsymbol{\alpha}|!/\boldsymbol{\alpha}! = d^n$ (see (57) in the appendix) we conclude

$$\left| \left(\frac{\partial}{\partial t}\right)^n \tilde{\eta}_\epsilon(t, \boldsymbol{x}) \right| \le C(d\gamma M \exp(M))^n.$$

This relation allows to estimate the remainder term in the Taylor expansion of $\tilde{\eta}_\epsilon$ on $[-T, T]$. The convergence of $\tilde{\eta}_\epsilon$ to $\tilde{\eta}$ follows from Lemma 9. ■

4 The classical solution

As we have seen in Proposition 8, $\tilde{\eta} \in C^\infty(\mathbb{R}, \mathscr{S}'(\mathbb{R}^d))$. Thus, the entropy $\eta(\boldsymbol{U}(t, \boldsymbol{x}))$ based on the classical solution $\boldsymbol{U}$ of the system of conservation laws (11) can only coincide with $\tilde{\eta}$ if also $\eta(\boldsymbol{U}) \in C^\infty([-T, T], \mathscr{S}'(\mathbb{R}^d))$. A sufficient condition for this smoothness property is exponential exactness. Before we can state a corresponding result, we need a notion of symmetric products of the Jacobian matrices $A^1, \dots A^d$. We define all n–fold symmetric products $\boldsymbol{A}^{\boldsymbol{\alpha}}$ through the relation

$$\frac{1}{n!}(\xi_j A^j)^n = \sum_{|\boldsymbol{\alpha}|=n} \frac{1}{\boldsymbol{\alpha}!} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \boldsymbol{A}^{\boldsymbol{\alpha}}. \tag{26}$$

For example, if $\boldsymbol{e}_i$ are the standard unit vectors, then

$$\boldsymbol{A}^{\boldsymbol{0}} = I, \qquad \boldsymbol{A}^{\boldsymbol{e}_i} = A^i, \qquad \boldsymbol{A}^{\boldsymbol{e}_i+\boldsymbol{e}_j} = \frac{1}{2}(A^iA^j + A^jA^i).$$

Note that the exponential matrix $\exp(-i\xi_j A^j)$ can be written as

$$\exp(-i\xi_j A^j) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}(\xi_j A^j)^n = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^d} \frac{1}{\boldsymbol{\alpha}!}(-i\boldsymbol{\xi})^{\boldsymbol{\alpha}}\boldsymbol{A}^{\boldsymbol{\alpha}}. \tag{27}$$

We thus have an alternative definition

$$\boldsymbol{A}^{\boldsymbol{\alpha}} = (i\nabla_{\boldsymbol{\xi}})^{\boldsymbol{\alpha}} \exp(-i\xi_j A^j)\big|_{\boldsymbol{\xi}=\boldsymbol{0}}. \tag{28}$$

Definition 11 *A function $\eta \in C^1(\mathcal{S},\mathbb{R})$ is called entropy of order $n_0 \in \mathbb{N}_0$ for the system* (11) *if the continuous mappings*

$$\boldsymbol{U} \mapsto \nabla^T\eta(\boldsymbol{U})\boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U})A^k(\boldsymbol{U}), \qquad 0 \le |\boldsymbol{\alpha}| < n_0,\ k = 1,\dots,d$$

have primitives.

We remark that entropies of order zero are just smooth functions with values in $\mathcal{S}$ (due to an empty assumption) and that usual entropies for (11) are recovered as first order entropies. Also, the entropies of infinite order are exponentially exact.

Lemma 12 *An entropy η of* (11) *is exponentially exact if and only if it is of infinite order.*

Proof: Assume η is exponentially exact. Then, for any piecewise smooth, closed curve Γ in $\mathcal{S}$ and every $\boldsymbol{\xi} \in \mathbb{R}^d$, $k \in \{1,\dots,d\}$ we have

$$\int_\Gamma \nabla^T\eta(\boldsymbol{U}) \exp(-i\xi_j A^j(\boldsymbol{U}))A^k(\boldsymbol{U})\, d\boldsymbol{U} = 0$$

Taking the derivative $\nabla_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}}$ of this integral at $\boldsymbol{\xi} = \boldsymbol{0}$ yields according to (28)

$$\int_\Gamma \nabla^T\eta(\boldsymbol{U})\boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U})A^k(\boldsymbol{U})\, d\boldsymbol{U} = 0 \tag{29}$$

which shows that $\boldsymbol{U} \mapsto \nabla^T\eta(\boldsymbol{U})\boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U})A^k(\boldsymbol{U})$ has a primitive.

For the converse direction we note that the exponential series in (27) is absolutely convergent locally uniformly in $\boldsymbol{U} \in \mathcal{S}$. Starting with (29) for any $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ and $k \in \{1,\dots,d\}$, we find

$$\int_\Gamma \nabla^T\eta \exp(-i\xi_j A^j)A^k\, d\boldsymbol{U} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^d} \frac{(-i\boldsymbol{\xi})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \int_\Gamma \nabla^T\eta\boldsymbol{A}^{\boldsymbol{\alpha}}A^k\, d\boldsymbol{U} = 0$$

for any piecewise smooth, closed curve Γ in $\mathcal{S}$ which completes the proof. ■

Now, we connect the order of η with smoothness of $t \mapsto \eta(\boldsymbol{U}(t,\boldsymbol{x}))$.

Proposition 13 *Let η be an entropy of order $n_0 \in \mathbb{N}_0$ for (11) and $\boldsymbol{U} \in \mathscr{J}_T^1$ a classical solution for some $T > 0$. Then $\eta(\boldsymbol{U})$ can be viewed as a mapping from $[-T,T]$ to $\mathscr{S}'(\mathbb{R}^d)$ which is (n_0+1)–times continuously differentiable. The derivatives are given by*

$$\left(\frac{\partial}{\partial t}\right)^n \eta(\boldsymbol{U}) = (-1)^n \sum_{|\boldsymbol{\alpha}|=n-1} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}} \left[\nabla^T \eta(\boldsymbol{U}) \boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U}) A^k(\boldsymbol{U}) \partial_{x_k} \boldsymbol{U}\right] \tag{30}$$

and satisfy the estimate

$$\left| \left\langle \left(\frac{\partial}{\partial t}\right)^n \eta(\boldsymbol{U}(t,\boldsymbol{x})), \psi(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}} \right| \leq C\gamma^n \max_{|\boldsymbol{\alpha}|<n} p_d(\nabla^{\boldsymbol{\alpha}} \psi)$$

for some constants C and γ depending on η and $\boldsymbol{U}$.

Proof: Since $\boldsymbol{U} \in \mathscr{J}_T^1$ has a compact range in $\mathcal{S}$, we find that $\eta(\boldsymbol{U})$ is uniformly bounded in $[-T,T] \times \mathbb{R}^d$. Thus, $\eta(\boldsymbol{U})$ defines a mapping with values in $\mathscr{S}'(\mathbb{R}^d)$. The time derivative

$$\frac{\partial}{\partial t} \eta(\boldsymbol{U}(t,\boldsymbol{x})) = \nabla^T \eta(\boldsymbol{U}(t,\boldsymbol{x})) A^k(\boldsymbol{U}(t,\boldsymbol{x})) \frac{\partial}{\partial x_k} \boldsymbol{U}(t,\boldsymbol{x})$$

is also uniformly bounded. Indeed, compactness of the range of $\boldsymbol{U}$ implies $|\nabla^T \eta(\boldsymbol{U})| \leq \bar{C}$ as well as $|A^k(\boldsymbol{U})| \leq \bar{C}$ and the gradient of $\boldsymbol{U}$ is uniformly bounded by assumption, $|\partial_{x_k} \boldsymbol{U}| \leq \bar{C}$. Hence, $\eta(\boldsymbol{U}) \in C^1([-T,T], \mathscr{S}'(\mathbb{R}^d))$ and

$$\left| \left\langle \frac{\partial}{\partial t} \eta(\boldsymbol{U}(t,\boldsymbol{x})), \psi(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}} \right| \leq C d \bar{C}^3 p_d(\psi).$$

We continue with an induction argument, assuming that $\eta(\boldsymbol{U})$ is contained in $C^{n_0}([-T,T], \mathscr{S}'(\mathbb{R}^d))$ for some $n_0 \geq 1$ with derivatives (30) for $1 \leq n \leq n_0$. Since η is of order n_0, we can find primitives $\Pi^{(\boldsymbol{\alpha},k)}$ of $\nabla^T \eta \boldsymbol{A}^{\boldsymbol{\alpha}} A^k$ so that

$$\nabla^T \eta(\boldsymbol{U}) \boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U}) A^k(\boldsymbol{U}) \partial_{x_k} \boldsymbol{U} = \partial_{x_k} \Pi^{(\boldsymbol{\alpha},k)}(\boldsymbol{U}), \qquad |\boldsymbol{\alpha}| = n_0 - 1,\ k \in \{1, \ldots, d\}.$$

Hence, for any test function $\psi \in \mathscr{S}(\mathbb{R}^d)$, (30) implies

$$\left(\frac{\partial}{\partial t}\right)^{n_0} \langle \eta(\boldsymbol{U}), \psi \rangle = \sum_{|\boldsymbol{\alpha}|=n_0-1} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \left\langle \Pi^{(\boldsymbol{\alpha},k)}(\boldsymbol{U}), \partial_{x_k} \nabla^{\boldsymbol{\alpha}} \psi \right\rangle. \tag{31}$$

Since $\boldsymbol{U}$ has compact range, the composition $\Pi^{(\boldsymbol{\alpha},k)}(\boldsymbol{U})$ is uniformly bounded. To calculate the time derivative, we use that $\boldsymbol{U}$ satisfies (11)

$$\frac{\partial}{\partial t} \Pi^{(\boldsymbol{\alpha},k)}(\boldsymbol{U}) = -\nabla^T \Pi^{(\boldsymbol{\alpha},k)}(\boldsymbol{U}) A^l(\boldsymbol{U}) \frac{\partial}{\partial x_l} \boldsymbol{U} = \nabla^T \eta(\boldsymbol{U}) \boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U}) A^k(\boldsymbol{U}) A^l(\boldsymbol{U}) \frac{\partial}{\partial x_l} \boldsymbol{U}.$$

Taking into account that the $|\boldsymbol{\alpha}|$–fold product $\boldsymbol{A}^{\boldsymbol{\alpha}}$ is bounded by $\bar{C}^{|\boldsymbol{\alpha}|}$ (see (56) in the appendix), we find that $\Pi^{(\boldsymbol{\alpha},k)}(\boldsymbol{U}) \in C^1([-T,T],\mathscr{S}'(\mathbb{R}^d))$ with

$$\left|\left\langle \frac{\partial}{\partial t}\Pi^{(\boldsymbol{\alpha},k)}(\boldsymbol{U}(t,\boldsymbol{x})),\psi(\boldsymbol{x})\right\rangle_{\boldsymbol{x}}\right| \leq Cd\bar{C}^{n_0+3}p_d(\psi).$$

In view of (31), this implies $\eta(\boldsymbol{U}) \in C^{n_0+1}([-T,T],\mathscr{S}'(\mathbb{R}^d))$ with

$$\left(\frac{\partial}{\partial t}\right)^{n_0+1}\langle\eta(\boldsymbol{U}),\psi\rangle = (-1)^{n_0+1}\sum_{|\boldsymbol{\alpha}|=n_0-1}\frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!}\left\langle \partial_{x_k}\nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\left[\nabla^T\eta\boldsymbol{A}^{\boldsymbol{\alpha}}A^kA^l\partial_{x_l}\boldsymbol{U}\right],\psi\right\rangle.$$

Using the relation

$$\sum_{|\boldsymbol{\alpha}|=n_0-1}\frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!}\partial_{x_k}\nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\nabla^T\eta\boldsymbol{A}^{\boldsymbol{\alpha}}A^k = \sum_{|\boldsymbol{\beta}|=n_0}\frac{|\boldsymbol{\beta}|!}{\boldsymbol{\beta}!}\nabla_{\boldsymbol{x}}^{\boldsymbol{\beta}}\nabla^T\eta\boldsymbol{A}^{\boldsymbol{\beta}}$$

which is obtained from (62) in the appendix, we get (30) with $n = n_0+1$. Finally, $\sum_{|\boldsymbol{\alpha}|=n_0-1}|\boldsymbol{\alpha}|!/\boldsymbol{\alpha}! = d^{n_0-1}$ shown in (57) leads to the estimate

$$\left|\left\langle\left(\frac{\partial}{\partial t}\right)^{n_0+1}\eta(\boldsymbol{U}(t,\boldsymbol{x})),\psi(\boldsymbol{x})\right\rangle_{\boldsymbol{x}}\right| \leq C(d\bar{C})^{n_0+1}\max_{|\boldsymbol{\alpha}|<n_0+1}p_d(\nabla^{\boldsymbol{\alpha}}\psi).$$

■

As in the previous section, we consider special regularizations of $\eta(\boldsymbol{U})$.

Proposition 14 *Assume η is an exponentially exact entropy and let $\boldsymbol{U} \in \mathscr{I}_T^1$ solve (11) for some $T > 0$. Then, $\eta(\boldsymbol{U}) \in C^\infty([-T,T],\mathscr{S}'(\mathbb{R}^d))$ and the regularization $\eta_\epsilon(t,\boldsymbol{x}) = \langle\eta(\boldsymbol{U}(t,\boldsymbol{y})),h_\epsilon(\boldsymbol{x}-\boldsymbol{y})\rangle_{\boldsymbol{y}}$ with h_ϵ defined in (24) can be written as*

$$\eta_\epsilon(t,\boldsymbol{x}) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left.\frac{\partial^n\eta_\epsilon}{\partial t^n}\right|_{(0,\boldsymbol{x})}$$

For $\epsilon \to 0$, we get $\eta_\epsilon(t,\cdot) \to \eta(\boldsymbol{U}(t,\cdot))$ in $\mathscr{S}'(\mathbb{R}^d)$ for every $t \in [-T,T]$.

Proof: Since η is exponentially exact, Lemma 12 shows that η is an entropy of infinite order which gives smoothness according to Proposition 13. We also get

$$\left|\left(\frac{\partial}{\partial t}\right)^n\eta_\epsilon(t,\boldsymbol{x})\right| \leq C\gamma^n\max_{|\boldsymbol{\alpha}|<n}p_d(g_{\boldsymbol{\alpha}})$$

where $g_\alpha(\boldsymbol{y}) = \nabla_{\boldsymbol{y}}^{\boldsymbol{\alpha}}h_\epsilon(\boldsymbol{x}-\boldsymbol{y})$. In connection with Lemma 9 this proves that for $t \in [-T,T]$ and $|\boldsymbol{x}| \leq R$

$$\left|\left(\frac{\partial}{\partial t}\right)^n\eta_\epsilon(t,\boldsymbol{x})\right| \leq \bar{C}\bar{\gamma}^n$$

so that η_ϵ coincides with its Taylor series. The convergence $\eta_\epsilon(t,\cdot) \to \eta(\boldsymbol{U}(t,\cdot))$ follows from Lemma 9. ■

5 Proof of the main result

Our aim is to show that the exponential exactness of an entropy η is in one–to–one correspondence with the existence of a kinetic representation (Theorem 4). Moreover, we completely characterize the kinetic representations (Theorem 5).

5.1 Sufficiency part of Theorem 4

Assuming that η is exponentially exact, we construct a particular kinetic representation μ which also yields the existence part of Theorem 5. In the construction of μ, the hyperbolicity of the system (11) plays a decisive role. It is used to show that the exponential matrix $\exp(-i\xi_j A^j(\boldsymbol{U}))$ is a tempered distribution in $\boldsymbol{\xi}$ and that its inverse Fourier transform $E(\boldsymbol{U};\boldsymbol{v})$ is a compactly supported distribution in $\boldsymbol{v}$. The proof of this fact (Lemma 16) differs from the existence proof for linear systems [18] only in the treatment of the $\boldsymbol{U}$–dependence.

Then, in Lemma 17, we show that a primitive $\mu(\boldsymbol{U})$ of $\nabla^T\eta(\boldsymbol{U})E(\boldsymbol{U})$ is a kinetic representation of the exponentially exact entropy η. The idea in the proof is to show that all time derivatives of $\tilde{\eta}(t,\boldsymbol{x}) = \left\langle \mu(\boldsymbol{U}^0(\boldsymbol{x}-\boldsymbol{v}t);\boldsymbol{v},1\right\rangle_{\boldsymbol{v}}$ at $t=0$ coincide with the time derivatives of $\eta(\boldsymbol{U})$. The result then follows with the help of the Taylor series representations in Propositions 10 and 14.

We start by quoting a result from [18].

Lemma 15 *Let M be any $m \times m$ matrix. There is a constant C_m depending only on m such that*

$$|\exp(iM)| \le C_m\,(1+|M|)^m\,e^{I(M)}$$

where $I(M)$ is the largest absolute value of the imaginary parts of the eigenvalues of M.

Lemma 15 enables us to apply the Paley–Wiener theorem which characterizes the Fourier transform of compactly supported distributions.

Lemma 16 *Let* $\hat{E}(\boldsymbol{U};\boldsymbol{\xi}) := \exp(-i\xi_j A^j(\boldsymbol{U}))$ *and* $K \subset \mathcal{S}$ *be compact. There is a constant* $\tilde{C}_K$ *such that*

$$|\hat{E}(\boldsymbol{U};\boldsymbol{\xi})| \le \tilde{C}_K(1+|\boldsymbol{\xi}|)^m, \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d, \boldsymbol{U} \in K$$

and $\boldsymbol{U} \mapsto \hat{E}(\boldsymbol{U})$ *can be viewed as a matrix valued mapping where each entry is a regular tempered distribution which depends continuously on* $\boldsymbol{U} \in \mathcal{S}$*. The inverse Fourier transform* $E(\boldsymbol{U};\boldsymbol{v})$ *is a matrix of compactly supported distributions in* $\boldsymbol{v}$ *which also depends continuously on* $\boldsymbol{U}$*. The support is contained in a ball with radius* $r_K < \infty$ *for* $\boldsymbol{U} \in K$ *and for any* $\phi \in \mathscr{E}(\mathbb{R}^d)$ *we have*

$$|\left\langle E(\boldsymbol{U}), \phi \right\rangle| \le C_K \sum_{|\alpha| \le N} \sup_{|\boldsymbol{v}| \le r_K+1} |\nabla^\alpha \phi(\boldsymbol{v})|, \qquad \forall \boldsymbol{U} \in K,$$

where N *depends only on the dimensions of* $\boldsymbol{x}$*–space and state space.*

Proof: For a vector $\boldsymbol{\eta} = \boldsymbol{\xi} + i\boldsymbol{\zeta} \in \mathbb{C}^d$ we have $|\hat{E}(\boldsymbol{U};\boldsymbol{\eta})| \le |\hat{E}(\boldsymbol{U};\boldsymbol{\xi})|\,|\hat{E}(\boldsymbol{U};i\boldsymbol{\zeta})|$ so that with Lemma 15 and the estimate

$$|\hat{E}(\boldsymbol{U};i\boldsymbol{\zeta})| \le \exp(|\zeta_j A^j(\boldsymbol{U})|) \le \exp(r_K|\boldsymbol{\zeta}|), \qquad r_K = \sup_{\boldsymbol{U}\in K} \sum_{j=1}^{d} |A^j(\boldsymbol{U})|$$

we get

$$|\hat{E}(\boldsymbol{U};\boldsymbol{\eta})| \le \tilde{C}_K(1+|\boldsymbol{\xi}|)^m \exp(r_K|\boldsymbol{\zeta}|), \qquad \tilde{C}_K = (r_K+1)^m.$$

Observe that we have used the hyperbolicity which implies that $\xi_j A^j(\boldsymbol{U})$ has only real eigenvalues so that $I(\xi_j A^j) = 0$ in Lemma 15. If $\boldsymbol{\zeta} = \boldsymbol{0}$, we recover the bound for $|\hat{E}(\boldsymbol{U};\boldsymbol{\xi})|$ which also shows that each component of $\hat{E}(\boldsymbol{U})$ is a tempered distribution. An application of Paley–Wiener theorem [17] yields $\mathcal{F}_{\boldsymbol{\xi}}^{-1}(\hat{E}(\boldsymbol{U};\boldsymbol{\xi})) \in [\mathscr{E}']^{m\times m}$ as well as the result on the support. The continuity of E in any bounded open set B with $\bar{B} \subset S$ follows from the continuous $\boldsymbol{U}$–dependence of

$$\left\langle \hat{E}(\boldsymbol{U}), \psi \right\rangle = \int_{\mathbb{R}^d} \psi(\boldsymbol{\xi}) \hat{E}(\boldsymbol{U};\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

where $\psi \in \mathscr{S}$ is a test function. The $\boldsymbol{U}$–independent majorant is given by $\tilde{C}_{\bar{B}}(1 + |\boldsymbol{\xi}|)^m \psi(\boldsymbol{\xi})$. To show continuity of the inverse Fourier transform $E(\boldsymbol{U})$ we pick a smooth cut–off function χ with the properties $0 \le \chi \le 1$, $\chi(\boldsymbol{v}) = 1$ for $|\boldsymbol{v}| \le r_{\bar{B}} + 1/2$ and $\chi(\boldsymbol{v}) = 0$ for $|\boldsymbol{v}| > r_{\bar{B}} + 1$. Then

$$\left\langle E(\boldsymbol{U}), \varphi \right\rangle = \left\langle E(\boldsymbol{U}), \varphi\chi \right\rangle = \left\langle \hat{E}(\boldsymbol{U}), \mathcal{F}^{-1}(\varphi\chi) \right\rangle$$

for all $\boldsymbol{U} \in B$ which gives continuity with the previous result. Using the relation again together with the polynomial growth of $\hat{E}$ in $\boldsymbol{\xi}$, we derive the estimate

$$|\left\langle E(\boldsymbol{U}), \varphi \right\rangle| \le C p_{m+d}(\mathcal{F}^{-1}(\varphi\chi))$$

where C depends on the set B and the dimensions d, m. Finally, with the continuity of the Fourier transform, we have $|\langle E(\boldsymbol{U}), \varphi\rangle| \leq C p_{m+2d}(\varphi\chi)$ so that the result follows by using that the cut–off function is supported on $r_{\bar{B}} + 1$. ■

Lemma 17 *Let η be an exponentially exact entropy. Then there exists a kinetic representation $\mu \in \mathscr{K}^1$ with the property $\nabla^T\mu(\boldsymbol{U}) = \nabla^T\eta(\boldsymbol{U})E(\boldsymbol{U})$ and $\langle\mu(\boldsymbol{U}), 1\rangle = \eta(\boldsymbol{U})$ for all $\boldsymbol{U} \in \mathcal{S}$.*

Proof: According to Lemma 12, exponential exactness of η implies that all products $\nabla^T\eta \boldsymbol{A}^{\boldsymbol{\alpha}} A^k$ have primitives. Multiplying with $\boldsymbol{\xi}^{\boldsymbol{\alpha}}\xi_k$ and summing over all $|\boldsymbol{\alpha}| = n-1$ and $k = 1, \ldots, d$ we get with (26) that also

$$\nabla^T\eta(\xi_j A^j)^n = (n-1)! \sum_{|\boldsymbol{\alpha}|=n-1} \frac{1}{\boldsymbol{\alpha}!} \nabla^T\eta \boldsymbol{\xi}^{\boldsymbol{\alpha}} \boldsymbol{A}^{\boldsymbol{\alpha}} (\xi_k A^k)$$

has a primitive for every $n \in \mathbb{N}$. With an argument as in the proof of Lemma 12 it follows that $\hat{\omega}(\boldsymbol{U}; \boldsymbol{\xi}) = \nabla^T\eta(\boldsymbol{U})\hat{E}(\boldsymbol{U}; \boldsymbol{\xi})$ is an exact one–form in $\boldsymbol{U}$ and hence

$$\int_\Gamma \langle\hat{\omega}(\boldsymbol{U}; \boldsymbol{\xi}), \psi(\boldsymbol{\xi})\rangle_{\boldsymbol{\xi}} \, d\boldsymbol{U} = 0, \qquad \forall\psi \in \mathscr{S}(\mathbb{R}^d) \tag{32}$$

for any closed and piecewise smooth curve Γ in $\mathcal{S}$. Based on the inverse Fourier transform $\omega(\boldsymbol{U}) = \nabla^T\eta(\boldsymbol{U})E(\boldsymbol{U})$ of $\hat{\omega}$, we define

$$\langle\mu(\bar{\boldsymbol{U}}), \varphi\rangle = \int_{\boldsymbol{U}^*}^{\bar{\boldsymbol{U}}} \langle\omega(\boldsymbol{U}; \boldsymbol{v}), \varphi(\boldsymbol{v})\rangle_{\boldsymbol{v}} \, d\boldsymbol{U} + \eta(\boldsymbol{U}^*)\varphi(\boldsymbol{0}) \tag{33}$$

where the integral is along an arbitrary, piecewise smooth curve which connects some fixed point $\boldsymbol{U}^*$ in $\mathcal{S}$ with $\bar{\boldsymbol{U}}$ (independence of the value from the chosen curve follows from (32)). If $\bar{\boldsymbol{U}}$ varies in a closed ball $B \subset \mathcal{S}$, one can choose, for example, a combination of a curve Γ_B to the center of B and line segments. Together with Lemma 16 we find for $K = B \cup \Gamma_B$

$$|\langle\mu(\bar{\boldsymbol{U}}), \varphi\rangle| \leq \bar{C}_K q_{N_K}(\varphi), \qquad \forall\bar{\boldsymbol{U}} \in K. \tag{34}$$

The constant N_K is chosen as maximum of N and $r_K + 1$ in Lemma 16 and $\bar{C}_K$ contains the length of Γ_B as well as the maximum of $|\nabla^T\eta|$ on K and the constant C_K from Lemma 16. If $\bar{\boldsymbol{U}}$ varies in an arbitrary compact set $K \subset \mathcal{S}$ we get a similar estimate by covering K with a finite number of balls to reduce the argument to the previous special case. Together with the obvious linearity, relation (34) shows that $\mu(\boldsymbol{U})$ defined in (33) defines a compactly supported distribution. Since $\langle\mu(\boldsymbol{U}), \varphi\rangle$ is even continuously differentiable in $\boldsymbol{U}$ we conclude that $\mu \in \mathscr{K}^1$ with derivative

$\nabla^T\mu(\boldsymbol{U}) = \nabla^T\eta(\boldsymbol{U})E(\boldsymbol{U})$. Since $\langle\omega(\boldsymbol{U};\boldsymbol{v}),1\rangle_{\boldsymbol{v}} = \hat{\omega}(\boldsymbol{U};\boldsymbol{0}) = \nabla^T\eta(\boldsymbol{U})$, we also have by definition of μ

$$\left\langle\mu(\bar{\boldsymbol{U}}),1\right\rangle = \int_{\boldsymbol{U}^*}^{\bar{\boldsymbol{U}}} \nabla^T\eta(\boldsymbol{U})\,d\boldsymbol{U} + \eta(\boldsymbol{U}^*) = \eta(\bar{\boldsymbol{U}}) \tag{35}$$

for all $\bar{\boldsymbol{U}} \in \mathcal{S}$. To complete the proof, it remains to show that μ is a kinetic representation of η. In view of the convergence information in Propositions 10 and 14 this can be accomplished by showing that $\eta_\epsilon = \tilde{\eta}_\epsilon$ for all $\epsilon > 0$ or, due to the Taylor expansions, that

$$\left(\frac{\partial}{\partial t}\right)^n \eta_\epsilon\bigg|_{t=0} = \left(\frac{\partial}{\partial t}\right)^n \tilde{\eta}_\epsilon\bigg|_{t=0} \qquad \forall\epsilon > 0 \tag{36}$$

While the case $n = 0$ follows immediately from (35)

$$\tilde{\eta}_\epsilon(0,\boldsymbol{x}) = \left\langle\left\langle\mu(\boldsymbol{U}^0(\boldsymbol{y})),1\right\rangle, h_\epsilon(\boldsymbol{x}-\boldsymbol{y})\right\rangle_{\boldsymbol{y}} = \left\langle\eta(\boldsymbol{U}^0(\boldsymbol{y})), h_\epsilon(\boldsymbol{x}-\boldsymbol{y})\right\rangle_{\boldsymbol{y}} = \eta_\epsilon(0,\boldsymbol{x})$$

the cases $n \geq 1$ require the explicit formulas for the time derivatives of $\eta(\boldsymbol{U})$ and $\tilde{\eta}$ in Propositions 13 and 8. We can thus reduce (36) to the problem of showing the $\mathscr{S}'$ relation $\Delta_n(\boldsymbol{U}^0) = 0$ for all $n \geq 1$ and $\boldsymbol{U}^0 \in \mathscr{I}_0^\infty$, where

$$\Delta_n(\boldsymbol{U}^0) = \sum_{|\boldsymbol{\alpha}|=n-1} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!}\nabla^{\boldsymbol{\alpha}}\Big(\partial_{x_k}\left\langle\mu(\boldsymbol{U}^0), v_k\boldsymbol{v}^{\boldsymbol{\alpha}}\right\rangle - \nabla^T\eta(\boldsymbol{U}^0)\boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U}^0)A^k(\boldsymbol{U}^0)\partial_{x_k}\boldsymbol{U}^0\Big).$$

To calculate the derivative $\partial_{x_k}\left\langle\mu(\boldsymbol{U}^0), v_k\boldsymbol{v}^{\boldsymbol{\alpha}}\right\rangle$, we note that according to (33)

$$\nabla^T\Big\langle\mu(\boldsymbol{U};\boldsymbol{v}),\boldsymbol{v}^{\boldsymbol{\beta}}\Big\rangle_{\boldsymbol{v}} = \Big\langle\omega(\boldsymbol{U};\boldsymbol{v}),\boldsymbol{v}^{\boldsymbol{\beta}}\Big\rangle_{\boldsymbol{v}} = (i\nabla_{\boldsymbol{\xi}})^{\boldsymbol{\beta}}\hat{\omega}(\boldsymbol{U};\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}=\boldsymbol{0}}$$

and due to (28), this yields

$$\nabla^T\Big\langle\mu(\boldsymbol{U};\boldsymbol{v}),\boldsymbol{v}^{\boldsymbol{\beta}}\Big\rangle_{\boldsymbol{v}} = \nabla^T\eta(\boldsymbol{U})\boldsymbol{A}^{\boldsymbol{\beta}}(\boldsymbol{U}).$$

Since η is exponentially exact, we can find primitives $\Pi^{(\boldsymbol{\alpha},k)}$ of $\nabla^T\eta\boldsymbol{A}^{\boldsymbol{\alpha}}A^k$ and with the definition $\Pi^{\boldsymbol{\beta}} := \sum_{k=1}^d \beta_k\Pi^{(\boldsymbol{\beta}-\boldsymbol{e}_k,k)}/|\boldsymbol{\beta}|$ we find with (60)

$$\nabla^T\Pi^{\boldsymbol{\beta}} = \frac{1}{|\boldsymbol{\beta}|}\sum_{k=1}^d \beta_k\nabla^T\eta\boldsymbol{A}^{\boldsymbol{\alpha}}A^k = \nabla^T\eta\boldsymbol{A}^{\boldsymbol{\beta}} = \nabla^T\Big\langle\mu(\boldsymbol{U}),\boldsymbol{v}^{\boldsymbol{\beta}}\Big\rangle_{\boldsymbol{v}}.$$

Hence, $\partial_{x_k}\left\langle\mu(\boldsymbol{U}^0),\boldsymbol{v}^{\boldsymbol{\alpha}}v_k\right\rangle_{\boldsymbol{v}} = \partial_{x_k}\Pi^{\boldsymbol{\alpha}+\boldsymbol{e}_k}(\boldsymbol{U}^0)$ and we can write

$$\Delta_n(\boldsymbol{U}^0) = \sum_{|\boldsymbol{\alpha}|=n-1}\frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!}\nabla^{\boldsymbol{\alpha}}\partial_{x_k}\Big(\Pi^{\boldsymbol{\alpha}+\boldsymbol{e}_k}(\boldsymbol{U}^0) - \Pi^{(\boldsymbol{\alpha},k)}(\boldsymbol{U}^0)\Big).$$

Using relation (62) in the appendix and the definition of $\Pi^{\boldsymbol{\beta}}$, we obtain

$$\Delta_n(\boldsymbol{U}^0) = \sum_{|\boldsymbol{\beta}|=n} \frac{|\boldsymbol{\beta}|!}{\boldsymbol{\beta}!} \nabla^{\boldsymbol{\beta}} \left(\Pi^{\boldsymbol{\beta}}(\boldsymbol{U}^0) - \frac{1}{|\boldsymbol{\beta}|} \sum_{k=1}^{d} \beta_k \Pi^{(\boldsymbol{\beta}-\boldsymbol{e}_k,k)}(\boldsymbol{U}^0) \right) = 0.$$

■

5.2 Necessity part of Theorem 4

We now assume that a kinetic representation μ exists for $\eta \in C^1(\mathcal{S}, \mathbb{R})$ or, in other words, for η being an entropy of order zero (see Definition 11). The proof is then carried out by induction over the order n_0 of η.

According to Proposition 13, we know that for an entropy η of order n_0, the composition $\eta(\boldsymbol{U})$ with $\boldsymbol{U} \in \mathscr{J}_T^1$ is (n_0+1) times continuously differentiable. The relation $\eta(\boldsymbol{U}(t,\boldsymbol{x})) = \langle \mu(\boldsymbol{U}^0(\boldsymbol{x}-\boldsymbol{v}t); \boldsymbol{v}), 1 \rangle_{\boldsymbol{v}}$ then implies equality of the derivatives at $t=0$, which we use to derive information about the $\boldsymbol{v}$–moments of μ in Lemma 18. The trick is to use special initial values $\boldsymbol{U}^0(\boldsymbol{x}) = \boldsymbol{\phi}(\boldsymbol{z}\cdot\boldsymbol{x})$ with $\boldsymbol{z} \in \mathbb{R}^d$ being a fixed vector.

The induction step is then completed in Lemma 19 where we show that η is actually of order n_0+1 if a kinetic representation exists (again by choosing special initial values which are constructed based on homotopies). This shows that η is of infinite order and, in view of Lemma 12, that η is exponentially exact.

The section ends with a uniqueness result (Lemma 20) which completes the proof of Theorem 5 in connection with Lemma 17.

Lemma 18 *Assume η is an entropy of order $n_0 \in \mathbb{N}_0$ with kinetic representation $\mu \in \mathscr{K}$. Then $\langle \mu(\bar{\boldsymbol{U}}), 1 \rangle = \eta(\bar{\boldsymbol{U}})$ and $\nabla^T \langle \mu(\bar{\boldsymbol{U}}; \boldsymbol{v}), \boldsymbol{v}^{\boldsymbol{\beta}} \rangle_{\boldsymbol{v}} = \nabla^T \eta(\bar{\boldsymbol{U}}) \boldsymbol{A}^{\boldsymbol{\beta}}(\bar{\boldsymbol{U}})$ for all $\bar{\boldsymbol{U}} \in \mathcal{S}$, $|\boldsymbol{\beta}| \le n_0+1$ and, if $n_0 \ge 1$, we get for any $\boldsymbol{U}^0 \in \mathscr{J}_0^\infty$*

$$\sum_{|\boldsymbol{\alpha}|=n_0} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}} \left[\nabla^T \eta(\boldsymbol{U}^0)(\boldsymbol{A}^{\boldsymbol{\alpha}+\boldsymbol{e}_k}(\boldsymbol{U}^0) - \boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U}^0) A^k(\boldsymbol{U}^0)) \partial_{x_k} \boldsymbol{U}^0 \right] = 0. \tag{37}$$

Proof: If μ is a kinetic representation of η, we know in particular that the equalities

$$\left(\frac{\partial}{\partial t} \right)^n \eta(\boldsymbol{U}) \bigg|_{t=0} = \left(\frac{\partial}{\partial t} \right)^n \tilde{\eta} \bigg|_{t=0} \qquad \forall n \in \mathbb{N}_0 \tag{38}$$

hold in $\mathscr{S}'(\mathbb{R}^d)$. First, we consider the case $n = 0$ which implies for all $\boldsymbol{U}^0 \in \mathscr{J}_0^\infty$

$$\eta(\boldsymbol{U}^0(\boldsymbol{x})) = \left\langle \mu(\boldsymbol{U}^0(\boldsymbol{x}); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}.$$

Choosing the constant function $\boldsymbol{U}^0(\boldsymbol{x}) = \bar{\boldsymbol{U}}$ for any $\bar{\boldsymbol{U}} \in \mathcal{S}$, we obtain one of the claimed relations. Next, we exploit (38) for $n \geq 1$. If η is an entropy of order $n_0 \in \mathbb{N}_0$, time derivatives up to order $n = n_0 + 1$ are available with Proposition 13. Together with Proposition 8 for the derivatives of $\tilde{\eta}$, we get from (38)

$$\sum_{|\boldsymbol{\alpha}|=n_0} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}} \left(\partial_{x_k} \left\langle \mu(\boldsymbol{U}^0), v_k \boldsymbol{v}^{\boldsymbol{\alpha}} \right\rangle - \nabla^T \eta(\boldsymbol{U}^0) \boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U}^0) A^k(\boldsymbol{U}^0) \partial_{x_k} \boldsymbol{U}^0 \right) = 0. \qquad (39)$$

We start by choosing a special initial value $\boldsymbol{U}^0$. For given $\bar{\boldsymbol{U}} \in \mathcal{S}$ and $\boldsymbol{e} \in \mathbb{R}^m$ with $|\boldsymbol{e}| = 1$, we pick $\boldsymbol{\phi} \in C^\infty(\mathbb{R}, \mathcal{S})$ with range in a small ball around $\bar{\boldsymbol{U}}$ such that for ϵ sufficiently small

$$\boldsymbol{\phi}(s) = \bar{\boldsymbol{U}} + s\boldsymbol{e} \qquad 0 \leq |s| < \epsilon \qquad (40)$$

and $\boldsymbol{\phi}(s) = \bar{\boldsymbol{U}}$ for $|s| > 1$. Then, for any $\boldsymbol{z} \in \mathbb{R}^d$, $\boldsymbol{U}^0(\boldsymbol{x}) = \boldsymbol{\phi}(\boldsymbol{z} \cdot \boldsymbol{x})$ is contained in $\mathscr{J}_0^\infty$. Note that $\boldsymbol{U}^0$ is the pullback of $\boldsymbol{\phi}$ under the mapping $H_{\boldsymbol{z}} : \mathbb{R}^d \mapsto \mathbb{R}$ defined by $H_{\boldsymbol{z}}(\boldsymbol{x}) := \boldsymbol{z} \cdot \boldsymbol{x}$ which we also denote by $\boldsymbol{U}^0 = H_{\boldsymbol{z}}^* \boldsymbol{\phi}$. Since $\boldsymbol{z} \neq \boldsymbol{0}$, $\nabla H_{\boldsymbol{z}}$ is a surjective mapping from $\mathbb{R}^d$ to $\mathbb{R}$ and the usual chain rule carries over to the calculus of generalized functions [7]. Thus

$$\partial_{x_k} \left\langle \mu(\boldsymbol{U}^0), \boldsymbol{v}^{\boldsymbol{\alpha}} v_k \right\rangle = \partial_{x_k} H_{\boldsymbol{z}}^* \left\langle \mu(\boldsymbol{\phi}), \boldsymbol{v}^{\boldsymbol{\alpha}} v_k \right\rangle = H_{\boldsymbol{z}}^* \frac{d}{ds} \left\langle \mu(\boldsymbol{\phi}), \boldsymbol{v}^{\boldsymbol{\alpha}} v_k \right\rangle z_k.$$

Repeating the argument for the $\nabla^{\boldsymbol{\alpha}}$ derivative and using the linearity of the pullback operation, we obtain from (39)

$$H_{\boldsymbol{z}}^* \sum_{|\boldsymbol{\alpha}|=n_0} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \boldsymbol{z}^{\boldsymbol{\alpha}+\boldsymbol{e}_k} \left(\frac{d}{ds} \right)^{n_0} \left(\frac{d}{ds} \left\langle \mu(\boldsymbol{\phi}), v_k \boldsymbol{v}^{\boldsymbol{\alpha}} \right\rangle - \nabla^T \eta(\boldsymbol{\phi}) \boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{\phi}) A^k(\boldsymbol{\phi}) \boldsymbol{\phi}' \right) = 0.$$

Assuming that $z_j \neq 0$, setting $\bar{\boldsymbol{x}} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)^T$, and suppressing Einstein's summation convention over the index j, we obtain the following representation of the pullback of any $\Lambda \in \mathscr{D}'(\mathbb{R})$ (see [7])

$$\langle H_{\boldsymbol{z}}^* \Lambda, \psi \rangle_{\boldsymbol{x}} = \frac{1}{z_j} \int_{\mathbb{R}^{d-1}} \left\langle \Lambda(x_j), \psi\left(\frac{x_j}{z_j} - \frac{\bar{z}_k}{z_j} \bar{x}_k \right) \right\rangle_{x_j} d\bar{\boldsymbol{x}}.$$

With test functions of tensorial structure $\psi(\boldsymbol{x}) = \psi_j(x_j)\bar{\psi}(\bar{\boldsymbol{x}})$ it easily follows that $H_{\boldsymbol{z}}^*$ is injective so that $H_{\boldsymbol{z}}^* \Lambda = 0$ implies $\Lambda = 0$. Hence, we obtain for all $\boldsymbol{0} \neq \boldsymbol{z} \in \mathbb{R}^d$

$$\sum_{|\boldsymbol{\alpha}|=n_0} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \boldsymbol{z}^{\boldsymbol{\alpha}+\boldsymbol{e}_k} \left(\frac{d}{ds} \right)^{n_0} \left(\frac{d}{ds} \left\langle \mu(\boldsymbol{\phi}), v_k \boldsymbol{v}^{\boldsymbol{\alpha}} \right\rangle - \nabla^T \eta(\boldsymbol{\phi}) \boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{\phi}) A^k(\boldsymbol{\phi}) \boldsymbol{\phi}' \right) = 0.$$

Using (61), we rewrite the sum in terms of multi–indices $\boldsymbol{\beta}$ of length n_0+1

$$\sum_{|\boldsymbol{\beta}|=n_0+1} \frac{|\boldsymbol{\beta}|!}{\boldsymbol{\beta}!} \boldsymbol{z}^{\boldsymbol{\beta}} \left(\frac{d}{ds}\right)^{n_0} \left(\frac{d}{ds}\left\langle \mu(\boldsymbol{\phi}), \boldsymbol{v}^{\boldsymbol{\beta}} \right\rangle - \nabla^T \eta(\boldsymbol{\phi}) \boldsymbol{A}^{\boldsymbol{\beta}}(\boldsymbol{\phi}) \boldsymbol{\phi}'\right) = 0.$$

Now, taking the derivative $\nabla_{\boldsymbol{z}}^{\boldsymbol{\beta}_0}$ with $|\boldsymbol{\beta}_0| = n_0+1$, it follows that

$$\left(\frac{d}{ds}\right)^{n_0} \left(\frac{d}{ds}\left\langle \mu(\boldsymbol{\phi}), \boldsymbol{v}^{\boldsymbol{\beta}_0} \right\rangle - \nabla^T \eta(\boldsymbol{\phi}) \boldsymbol{A}^{\boldsymbol{\beta}_0}(\boldsymbol{\phi}) \boldsymbol{\phi}'\right) = 0, \qquad \forall |\boldsymbol{\beta}| = n_0.$$

This, on the other hand, implies that $\frac{d}{ds}\left\langle \mu(\boldsymbol{\phi}), \boldsymbol{v}^{\boldsymbol{\beta}} \right\rangle - \nabla^T \eta(\boldsymbol{\phi}) \boldsymbol{A}^{\boldsymbol{\beta}}(\boldsymbol{\phi}) \boldsymbol{\phi}'$ is a polynomial in s of degree n_0-2 (see [7]) for any $\boldsymbol{\beta}$ of length n_0+1. Since $\boldsymbol{\phi}(s)$ is constant for large $|s|$, it follows that the polynomial vanishes on an open set and thus must be identically zero. Hence,

$$\frac{d}{ds}\left\langle \mu(\boldsymbol{\phi}), \boldsymbol{v}^{\boldsymbol{\beta}} \right\rangle = \nabla^T \eta(\boldsymbol{\phi}) \boldsymbol{A}^{\boldsymbol{\beta}}(\boldsymbol{\phi}) \boldsymbol{\phi}' \tag{41}$$

and since the right hand side is contained in $C(\mathbb{R})$, we have $\left\langle \mu(\boldsymbol{\phi}), \boldsymbol{v}^{\boldsymbol{\beta}} \right\rangle \in C^1(\mathbb{R})$ and (41) is satisfied in the classical sense [7]. Due to (40), we conclude that all directional derivatives of $\boldsymbol{U} \mapsto \left\langle \mu(\boldsymbol{U}), \boldsymbol{v}^{\boldsymbol{\beta}} \right\rangle$ are continuous, so that for any $\boldsymbol{\beta}$ of length n_0+1 the claimed relation

$$\nabla^T \left\langle \mu(\boldsymbol{U}), \boldsymbol{v}^{\boldsymbol{\beta}} \right\rangle_{\boldsymbol{v}} = \nabla^T \eta(\boldsymbol{U}) \boldsymbol{A}^{\boldsymbol{\beta}}(\boldsymbol{U})$$

holds. Inserting this partial result back into (39), we now get with the abbreviation $\Lambda_{\boldsymbol{\alpha},k} = \nabla^T \eta(\boldsymbol{A}^{\boldsymbol{\alpha}+\boldsymbol{e}_k} - \boldsymbol{A}^{\boldsymbol{\alpha}} A^k)$

$$\sum_{|\boldsymbol{\alpha}|=n_0} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}} \left[\Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}^0) \partial_{x_k} \boldsymbol{U}^0\right] = 0.$$

which completes the proof. ■

Lemma 19 *Assume η is an entropy of order $n_0 \in \mathbb{N}_0$ with a kinetic representation. Then η is of order n_0+1.*

Proof: We have to show that $\nabla^T \eta \boldsymbol{A}^{\boldsymbol{\alpha}} A^k$ has a primitive for every $\boldsymbol{\alpha}$ of length n_0 and $k \in \{1, \ldots, d\}$. According to Lemma 18, we know that $\nabla^T \eta \boldsymbol{A}^{\boldsymbol{\alpha}+\boldsymbol{e}_k}$ has the primitive $\langle \mu(\boldsymbol{U}; \boldsymbol{v}); \boldsymbol{v}^{\boldsymbol{\alpha}} v_k \rangle_{\boldsymbol{v}}$ if μ is a kinetic representation of η so that it suffices to show that

$$\Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}) = \nabla^T \eta(\boldsymbol{U}) (\boldsymbol{A}^{\boldsymbol{\alpha}+\boldsymbol{e}_k}(\boldsymbol{U}) - \boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U}) A^k(\boldsymbol{U}))$$

has a primitive. Note that in the one dimensional case ($d=1$) we have $\Lambda_{\boldsymbol{\alpha},k} = 0$ since $\boldsymbol{A}^{\boldsymbol{\alpha}+\boldsymbol{e}_1} = (A^1)^{|\boldsymbol{\alpha}|+1} = \boldsymbol{A}^{\boldsymbol{\alpha}} A^1$. In this case, the proof is complete since the zero function has a primitive. In the general scalar case ($m = 1, d \geq 1$) the function $\Lambda_{\boldsymbol{\alpha},k}$ is a continuous scalar function depending on a scalar variable U and thus has a primitive by the fundamental theorem of calculus. Hence, we can focus on the case $m > 1, d > 1$ for which we have assumed that $\mathcal{S}$ is simply connected. From Lemma 18 we use the relation

$$\sum_{|\boldsymbol{\alpha}|=n_0} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}} \left[\Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}^0)\partial_{x_k}\boldsymbol{U}^0\right] = 0 \tag{42}$$

and start with the observation that $\Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}^0)\partial_{x_k}\boldsymbol{U}^0$ is compactly supported if $\nabla\boldsymbol{U}^0 = 0$ outside a compact set. Assuming this property of $\boldsymbol{U}^0$, we can test (42) with the function $\psi(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{\beta}}/\boldsymbol{\beta}!$ where $|\boldsymbol{\beta}| = n_0$. Then, $\nabla^{\boldsymbol{\alpha}}\psi$ is zero unless $\boldsymbol{\alpha} = \boldsymbol{\beta}$ in which case it identically one. Thus (42) implies for any $\boldsymbol{U}^0 \in \mathscr{J}_0^\infty$ with vanishing Jacobian for large $|\boldsymbol{x}|$

$$\left\langle\Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}^0)\partial_{x_k}\boldsymbol{U}^0, 1\right\rangle_{\boldsymbol{x}} = \mathbf{0} \qquad \forall|\boldsymbol{\alpha}| = n_0. \tag{43}$$

Using the continuity of $\boldsymbol{U} \mapsto \Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U})$ and a density argument, we can even show that (43) holds for any $\boldsymbol{U}^0 \in C^1(\mathbb{R}^d)$ with compact range in $\mathcal{S}$ if $\nabla\boldsymbol{U}^0$ is compactly supported. The remainder of the argument relies on the observation that (43) can be converted into an integral over an arbitrary closed C^2 curve $\boldsymbol{\gamma} : [0,1] \mapsto \mathcal{S}$

$$\int_0^1 \Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{\gamma}(s))\dot{\boldsymbol{\gamma}}(s)\, ds = \mathbf{0} \tag{44}$$

by choosing a special sequence of initial values. Relation (44) then implies that $\Lambda_{\boldsymbol{\alpha},k}$ is exact (see, for example, [4] for a proof of this result in the case $d = 3$ which is based exactly on our smoothness assumptions and which can easily be extended to general dimensions).

In the construction of a sequence of initial values we make use of the assumption that $\mathcal{S}$ is simply connected which implies the existence of a mapping $\boldsymbol{H} \in C^2([0,1]^2; \mathcal{S})$ and a point $\bar{\boldsymbol{U}}$ such that $s \mapsto \boldsymbol{H}(s,1)$ is the given curve $s \mapsto \boldsymbol{\gamma}(s)$ and that $\boldsymbol{H}(s,0) = \bar{\boldsymbol{U}}$ for all $s \in [0,1]$. We then take a sequence $\varphi_n \in \mathscr{D}(\mathbb{R})$ with supports in $(-2,2)$ such that φ_n' tends to the indicator function of $[0,1]$ which we denote by $\mathcal{X}_{[0,1]}$. Next, we pick a sequence of smooth functions $\tilde{\psi}_n \in C^\infty(\mathbb{R})$ such that $0 \leq \tilde{\psi}_n \leq 1$, $\tilde{\psi}_n(-x) = \tilde{\psi}_n(x)$, $\tilde{\psi}_n(x) = 1$ on $|x| < n$, $\tilde{\psi}_n(x) = 0$ on $|x| > n+2$ and satisfies the uniform bound $|\tilde{\psi}_n'| \leq 1$. We then set $\psi_n(\boldsymbol{x}) = \tilde{\psi}_n(x_1)\cdots\tilde{\psi}_n(x_d)$ and

$$\boldsymbol{U}_n^0(\boldsymbol{x}) := \boldsymbol{H}(\varphi_n(x_1), \psi_n(\boldsymbol{x}))$$

which is contained in C^2 and satisfies $\boldsymbol{U}_n^0(\boldsymbol{x}) = \bar{\boldsymbol{U}}$ (and hence $\nabla\boldsymbol{U}_n^0(\boldsymbol{x}) = 0$) for large enough $|\boldsymbol{x}|$. Moreover, we have the point-wise convergence $\boldsymbol{U}_n^0(\boldsymbol{x}) \to \boldsymbol{\gamma}(x_1)\mathcal{X}_{[0,1]}(x_1)$ and, with $|\boldsymbol{x}|_\infty = \max_{i=1}^d |x_i|$,

$$\nabla\boldsymbol{U}_n^0(\boldsymbol{x}) = \Big(\dot{\boldsymbol{\gamma}}(\varphi_n(x_1))\varphi_n'(x_1), \mathbf{0}, \ldots, \mathbf{0}\Big), \qquad |\boldsymbol{x}|_\infty < n$$

which converges to $(\dot{\boldsymbol{\gamma}}(x_1)\mathcal{X}_{[0,1]}(x_1), \mathbf{0}, \dots, \mathbf{0})$. We now split the integration in (43) into contributions from the sets $|x_1| < 2$ and $|x_1| \geq 2$. To estimate the latter, we use the fact that $\psi_n(-\boldsymbol{x}) = \psi_n(\boldsymbol{x})$ and that $\boldsymbol{U}_n^0(\boldsymbol{x}) = H(0, \psi_n(\boldsymbol{x}))$ for $|x_1| \geq 2$. Hence also $\boldsymbol{U}_n^0(-\boldsymbol{x}) = \boldsymbol{U}_n^0(\boldsymbol{x})$ and $(\partial_{x_k}\boldsymbol{U}_n^0)(-\boldsymbol{x}) = -\partial_{x_k}\boldsymbol{U}_n^0(\boldsymbol{x})$ so that

$$\int_{|x_1|\geq 2} \Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}_n^0(\boldsymbol{x}))\partial_{x_k}\boldsymbol{U}_n^0(\boldsymbol{x})\, d\boldsymbol{x} = \mathbf{0}.$$

The remaining set $|x_1| < 2$ is split further into the part where $|\boldsymbol{x}|_\infty < n$ so that

$$\Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}_n^0(\boldsymbol{x}))\partial_{x_k}\boldsymbol{U}_n^0(\boldsymbol{x}) = \Lambda_{\boldsymbol{\alpha},1}(\boldsymbol{\gamma}(\varphi_n(x_1)))\dot{\boldsymbol{\gamma}}(\varphi_n(x_1))\varphi_n'(x_1)$$

and in the part $|\boldsymbol{x}|_\infty > n$. Note that $\nabla \boldsymbol{U}_n^0$ is nonzero only in $|\boldsymbol{x}|_\infty < n+2$ and that the volume of $S_n = \{\boldsymbol{x} : |x_1| < 2, n \leq |\boldsymbol{x}|_\infty < n+2\}$ is of order n^{d-2}. Since $\Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}_n^0)\partial_{x_k}\boldsymbol{U}_n^0$ is bounded uniformly in n, we see that

$$\lim_{n\to\infty} \frac{1}{n^{d-1}} \int_{S_n} \Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}_n^0)\partial_{x_k}\boldsymbol{U}_n^0\, d\boldsymbol{x} = \mathbf{0}.$$

Hence, we find with (43) and the considerations above

$$\mathbf{0} = \lim_{n\to\infty} \frac{1}{n^{d-1}} \int_{|\boldsymbol{x}|<n} \Lambda_{\boldsymbol{\alpha},k}(\boldsymbol{U}_n^0)\partial_{x_k}\boldsymbol{U}_n^0\, d\boldsymbol{x} = \int_0^1 \Lambda_{\boldsymbol{\alpha},1}(\boldsymbol{\gamma}(s))\dot{\boldsymbol{\gamma}}(s)\, ds$$

which is (44) with $k = 1$. For other choices of k, the same idea can be applied. ■

Lemma 20 *Let μ_1, μ_2 be two kinetic representations of an entropy η. Then $C = \mu_1 - \mu_2 \in \mathscr{E}'(\mathbb{R}^d)$ is independent of $\boldsymbol{U}$ and satisfies $\langle C, 1\rangle_{\boldsymbol{v}} = 0$.*

According to the Paley–Wiener theorem, $\mu(\boldsymbol{U}) \in \mathscr{E}'$ implies that the Fourier transform $\mathcal{F}\mu(\boldsymbol{U})$ of $\mu(\boldsymbol{U})$ is an analytic function, i.e.

$$\mathcal{F}\mu(\boldsymbol{U};\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^d} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \nabla^{\boldsymbol{\alpha}}\mathcal{F}\mu(\boldsymbol{U};\boldsymbol{\xi})|_{\boldsymbol{\xi}=\mathbf{0}}\,.$$

Since the derivatives of $\mathcal{F}\mu(\boldsymbol{U})$ at $\boldsymbol{\xi} = \mathbf{0}$ are related to velocity moments

$$\nabla^{\boldsymbol{\alpha}}\mathcal{F}\mu(\boldsymbol{U};\boldsymbol{\xi})|_{\boldsymbol{\xi}=\mathbf{0}} = (-i)^{|\boldsymbol{\alpha}|}\langle \mu(\boldsymbol{U}), \boldsymbol{v}^{\boldsymbol{\alpha}}\rangle$$

we find with Lemma 18 that the series of $\boldsymbol{U}$ derivatives is

$$\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^d} \frac{(-i\boldsymbol{\xi})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\nabla^T\eta(\boldsymbol{U})\boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U}) = \nabla^T\eta(\boldsymbol{U})\hat{E}(\boldsymbol{U};\boldsymbol{\xi}). \tag{45}$$

With the estimate

$$\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^d} \left| \frac{(-i\boldsymbol{\xi})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \nabla^T\eta(\boldsymbol{U})\boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U}) \right| \le |\nabla^T\eta(\boldsymbol{U})| \exp(|\xi_j|\ |A^j(\boldsymbol{U})|)$$

we see that the absolute convergence is locally uniform in $\boldsymbol{U}$ so that $\mathcal{F}\mu(\boldsymbol{U};\boldsymbol{\xi})$ is $\boldsymbol{U}$–differentiable with derivative $\nabla^T\mathcal{F}\mu = \nabla^T\eta\hat{E}$. If we have two kinetic representations $\mu_1, \mu_2 \in \mathscr{K}$, we thus find that $\nabla^T\mathcal{F}\mu_1 = \nabla^T\mathcal{F}\mu_2$ or, in other words, that $\mathcal{F}(\mu_1 - \mu_2)$ is independent of $\boldsymbol{U}$. Denoting $C = \mu_1 - \mu_2$ we conclude that $\langle C, \psi\rangle = \left\langle \mathcal{F}(\mu_1 - \mu_2), \mathcal{F}^{-1}\psi \right\rangle$ is also independent of $\boldsymbol{U}$ for every $\psi \in \mathscr{S}$. Using the result from Lemma 18 that $\left\langle \mu_i(\bar{\boldsymbol{U}}), 1 \right\rangle = \eta(\bar{\boldsymbol{U}})$ for all $\bar{\boldsymbol{U}} \in \mathcal{S}$ we see that $\langle C, 1\rangle = 0$.

■

6 Examples

6.1 Scalar equations

In the case of scalar conservation laws, the state space $\mathcal{S}$ is an interval (open and connected subset of $\mathbb{R}$) and we assume that $0 \in \mathcal{S}$ which can always be achieved by a simple transformation.

A crucial property which distinguishes scalar equations from the case of systems is the abundance of entropies and – in our case – the abundance of exponentially exact entropies. In fact, any function $\eta \in C^1(\mathcal{S}, \mathbb{R})$ is exponentially exact since

$$\eta'(U)\exp(-i\xi_j A^j(U))A^k(U) = \frac{d}{dU}\int_0^U \eta'(s)\exp(-i\xi_j A^j(s))A^k(s)\, ds$$

(note that A^j are 1×1 matrices, i.e. scalars). Setting $\boldsymbol{F} = (F^1, \ldots, F^d)^T$, so that $\boldsymbol{F}' = (A^1, \ldots, A^d)^T$, a kinetic representation of η is given by

$$\mu(U;\boldsymbol{v}) = \int_0^U \eta'(s)\delta(\boldsymbol{v} - \boldsymbol{F}'(s))\, ds + \eta(0)\delta(\boldsymbol{v}) \tag{46}$$

Since $\delta(\boldsymbol{v} - \boldsymbol{F}'(s))$ is the inverse Fourier transform of $\exp(-i\xi_j A^j(s))$. We remark that the relation $\eta(U(t,\boldsymbol{x})) = \left\langle \mu(U^0(\boldsymbol{x} - \boldsymbol{v}t); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}$ can also be related to the s–average of the solution

$$f(t,\boldsymbol{x},s) = \eta'(s)\mathcal{X}(s; U^0(\boldsymbol{x} - \boldsymbol{F}'(s)t))$$

of the kinetic equation

$$\frac{\partial f}{\partial t} + A^j(s)\frac{\partial f}{\partial x_j} = 0, \qquad f(0,\boldsymbol{x},s) = \eta'(s)\mathcal{X}(s; U^0(\boldsymbol{x})) \tag{47}$$

where $\mathcal{X}(s;a) = H(s) - H(s-a)$ is the difference of two Heaviside functions. Indeed, for any test function $\psi \in \mathscr{D}(\mathbb{R})$, we find with (16) and (46)

$$\left\langle \left\langle \mu(U^0(\boldsymbol{x} - \boldsymbol{v}t); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}, \psi(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}} = \int_{\mathbb{R}} \int_0^{U^0(\boldsymbol{x})} \eta'(s) \psi(\boldsymbol{x} + \boldsymbol{F}'(s)t)\, ds\, d\boldsymbol{x}.$$

Taking into account that $\int_0^a g(s)\, ds = \int_{\mathbb{R}} \mathcal{X}(s;a) g(s)\, ds$ for any integrable function g, we obtain with the coordinate transformation $(\boldsymbol{x}, s) \mapsto (\boldsymbol{x} - \boldsymbol{F}'(s)t, s)$

$$\left\langle \left\langle \mu(U^0(\boldsymbol{x} - \boldsymbol{v}t); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}, \psi(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}} = \langle \langle f(t, \boldsymbol{x}, s), 1 \rangle_s, \psi(\boldsymbol{x}) \rangle_{\boldsymbol{x}}$$

which shows that also $\eta(U(t, \boldsymbol{x})) = \langle f(t, \boldsymbol{x}, s), 1 \rangle_s$. The reformulation (47) has been used in [5, 14, 12] and turns out to be very appropriate in the scalar case.

6.2 Linear systems

For any linear function $\eta : \mathbb{R}^m \mapsto \mathbb{R}$, the vector $\boldsymbol{C}^T = \nabla^T \eta \exp(-i\xi_j A^j) A^k$ is independent of $\boldsymbol{U}$ and thus has the linear function $\boldsymbol{U} \mapsto \boldsymbol{C} \cdot \boldsymbol{U}$ as primitive. This implies, in particular, that all linear systems are exponentially exact and that the solution can be represented by $\boldsymbol{U}(t, \boldsymbol{x}) = \left\langle \boldsymbol{\mu}(\boldsymbol{U}^0(\boldsymbol{x} - \boldsymbol{v}t); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}$ where component i of $\boldsymbol{\mu}$ is a kinetic representation of $\eta_i(\boldsymbol{U}) = U_i$.

More general exponentially exact entropies can be found in the one–dimensional case if $\mathcal{S}$ is simply connected. In fact, any C^2 entropy η with C^2 entropy flux ϕ is exponentially exact. To see this, we note that

$$B_{ij} = \frac{\partial^2 \eta}{\partial U_i \partial U_j}$$

is a symmetric matrix and $\nabla^T \eta A = \nabla^T \phi$ implies, by taking another $\boldsymbol{U}$ derivative,

$$B_{ij} A_{jk} = \frac{\partial^2 \phi}{\partial U_i \partial U_j}$$

(We write A instead of A^1 in the 1D case). Hence, BA is symmetric giving rise to $BA = (BA)^T = A^T B^T = A^T B$ and by induction

$$BA^n = (A^n)^T B = (BA^n)^T. \tag{48}$$

Finally, we conclude that

$$B(\boldsymbol{U}) \exp(-i\xi A) A = \sum_{n=0}^{\infty} \frac{(-i\xi)^n}{n!} B(\boldsymbol{U}) A^{n+1}$$

is also symmetric and, with the Lemma of Poincaré, that $\boldsymbol{U} \mapsto \nabla^T \eta(\boldsymbol{U}) \exp(-i\xi A) A$ has a primitive. A similar argument in the general case of linear systems is only possible if the matrices A^j commute. This is due to the relation

$$B(\xi_j A^j)^n A^k = (B A^k (\xi_j A^j)^n)^T$$

which generalizes (48) and which yields symmetry only if $A^k A^j = A^j A^k$.

We conclude that linear systems are always exponentially exact and that kinetic representations can also be found for non–linear entropies in the one–dimensional case and in the general case if the Jacobian matrices A^j commute.

6.3 Non–linear systems in one dimension

For general non–linear systems in one space dimension, the assumption of exponential exactness is very restrictive. From Lemma 12 we know that it is equivalent to the requirement that all powers of the Jacobian matrix $A = A^1$ have primitives. Therefore, it is not surprising that most systems are not exponentially exact [8, 9]. In this case, one can still define

$$\boldsymbol{\mu}(\boldsymbol{U}; v) := \int_{\boldsymbol{0}}^{\boldsymbol{U}} E(\boldsymbol{W})\, d\boldsymbol{W} \qquad E = \mathcal{F}_\xi^{-1} \exp(-i\xi A)$$

but the the solution formula is satisfied only approximately

$$\boldsymbol{U}(t,x) = \left\langle \boldsymbol{\mu}(\boldsymbol{U}^0(x - v\Delta t); v), 1 \right\rangle_v + \mathcal{O}(\Delta t^2). \tag{49}$$

However, if the main motivation is to find a numerical approximation method for the hyperbolic system, then relation (49) can still be valuable [8, 9].

Although it is difficult to find exponentially exact non–linear systems, it is not impossible. Simple examples of exponential exactness are given by completely decoupled systems and linear transformations thereof. The flux vector is of the form $\boldsymbol{F}(\boldsymbol{U}) = (F_1(U_1), \ldots, F_m(U_m))$ so that $A(\boldsymbol{U})$ and all its powers are diagonal matrices for which primitives are found by integration.

While decoupled systems are essentially a repetition of the scalar case, there are also more interesting examples like special isentropic Euler systems. The variables in these 2×2 systems are mass density $\rho > 0$ and momentum density $m \in \mathbb{R}$. For abbreviation, we also introduce the velocity $u = m/\rho$. Then, the systems have the form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial m}{\partial x} &= 0 \\ \frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(\rho u^2 + p(\rho)\right) &= 0 \end{aligned} \tag{50}$$

where the pressure p is a given function of ρ which satisfies $p' \geq 0$. Setting up the exponential matrix based on $A = \nabla \boldsymbol{F}$ with $\boldsymbol{F} = (m, m^2/\rho + p(\rho))^T$, a straight forward calculation shows that the isentropic Euler system is exponentially exact under the condition that $p'(\rho)/\rho = p''(\rho)/2$ which singles out the pressure laws of the form

$$p(\rho) = C + D\rho^3 \qquad C, D \geq 0.$$

In the case $p(\rho) = D\rho^3$ it is known that (50) can be decoupled into two independent Burgers' equations which shows again that exponential exactness is related to a certain simplicity of the system. A kinetic representation is in this case

$$\boldsymbol{\mu}(\rho, u; v) = \begin{pmatrix} 1 \\ v \end{pmatrix} \alpha \mathcal{X}(|v-u|; c(\rho)), \qquad c(\rho) = \sqrt{p'(\rho)}$$

with α depending only on D and $\mathcal{X}(s; a) = H(s) - H(s-a)$ defined as in the scalar case.

The case of constant pressure $p(\rho) = C$ leads to a representation of the form $\boldsymbol{\mu}(\rho, u; v) = \left(\begin{smallmatrix} 1 \\ v \end{smallmatrix}\right) \rho\delta(v-u)$ which is considered more detailed in the next section.

Apart from the examples above, a whole class of exponentially exact systems has been reported by Brenier and Corrias [1]. They derive an infinite hierarchy of conservation systems for which the exact solution can be written in the form of a kinetic representation as long as the solution is smooth. Each system in this hierarchy (which contains Burgers equation ($m = 1$) and the isentropic Euler equation ($m = 2$) with $p(\rho) = D\rho^3$ as special members) can therefore be taken as nontrivial example.

We conclude with a remark on exponentially exact entropies. Similar to the linear case, one can show [8] that any strictly convex entropy of the one–dimensional hyperbolic system is exponentially exact. As requirement one needs that the state space is simply connected and that the system itself is exponentially exact.

6.4 Non–linear systems in higher dimensions

Motivated by the one–dimensional case, we check whether special isentropic Euler equations also lead to exponentially exact systems in the case of higher dimensions. Already for $d = 2$, we find that exponential exactness can only be achieved if p is constant [8]. Let us therefore consider the system in $d \geq 1$ dimensions

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial m_j}{\partial x_j} &= 0 \\ \frac{\partial m_i}{\partial t} + \frac{\partial}{\partial x_j} \rho u_i u_j &= 0 \end{aligned}$$

After some calculations we find that the linear combination $\xi_j A^j$ can be written as block matrix

$$\xi_j A^j = \begin{pmatrix} 0 & \boldsymbol{\xi}^T \\ -(\boldsymbol{\xi}\cdot\boldsymbol{u})\boldsymbol{u} & \boldsymbol{u}\otimes\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\boldsymbol{u})I \end{pmatrix}$$

and the exponential matrix as

$$\exp(-i\xi_j A^j) = e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}} \begin{pmatrix} 1 + i\boldsymbol{u}\cdot\boldsymbol{\xi} & -i\boldsymbol{\xi}^T \\ i(\boldsymbol{\xi}\cdot\boldsymbol{u})\boldsymbol{u} & I - i\boldsymbol{u}\otimes\boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} 1 \\ i\nabla_{\boldsymbol{\xi}} \end{pmatrix} e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}} \begin{pmatrix} 1 + i\boldsymbol{u}\cdot\boldsymbol{\xi} & -i\boldsymbol{\xi}^T \end{pmatrix}.$$

The products $\exp(-i\xi_j A^j)A^k$ are obtained from the expression

$$\exp(-i\xi_j A^j)(\zeta_k A^k) = \begin{pmatrix} 1 \\ i\nabla_{\boldsymbol{\xi}} \end{pmatrix} e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}} \begin{pmatrix} i(\boldsymbol{\xi}\cdot\boldsymbol{u})(\boldsymbol{\zeta}\cdot\boldsymbol{u}) & \boldsymbol{\zeta}^T - i(\boldsymbol{\zeta}\cdot\boldsymbol{u})\boldsymbol{\xi}^T \end{pmatrix} \tag{51}$$

by setting $\boldsymbol{\zeta} = \boldsymbol{e}_k$ for $k = 1, \ldots, d$. To check that (51) has a primitive, it suffices to show

$$\frac{\partial}{\partial\rho} e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}}(\zeta_k - i(\boldsymbol{\zeta}\cdot\boldsymbol{u})\xi_k) = \frac{\partial}{\partial m_k} e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}} i(\boldsymbol{\xi}\cdot\boldsymbol{u})(\boldsymbol{\zeta}\cdot\boldsymbol{u})$$

and

$$\frac{\partial}{\partial m_k} e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}}(\zeta_l - i(\boldsymbol{\zeta}\cdot\boldsymbol{u})\xi_l) = \frac{\partial}{\partial m_l} e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}}(\zeta_k - i(\boldsymbol{\zeta}\cdot\boldsymbol{u})\xi_k)$$

which is an easy calculation. Thus, the system of isentropic Euler equations with constant pressure is exponentially exact in any space dimension and therefore is intimately related to a kinetic formulation (see also Brenier [2]). According to Theorem 5, the kernel $\boldsymbol{\mu}$ is obtained as inverse Fourier transform of a primitive of the exponential matrix. It is easy to check that $\left(\begin{smallmatrix} 1 \\ i\nabla_{\boldsymbol{\xi}} \end{smallmatrix}\right) e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}}$ is a primitive having the inverse Fourier transform

$$\boldsymbol{\mu}(\rho, \boldsymbol{u}; \boldsymbol{v}) = \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \rho\delta(\boldsymbol{v} - \boldsymbol{u}).$$

7 Conclusion

In this article, we have investigated a remarkable relation between hyperbolic systems and the simple transport equation of free flow. The considerations are based on the observation that solutions of linear systems can be written as velocity averages of solutions of the free flow equation. Such a kinetic representation even carries over to smooth solutions of certain non–linear equations or, more generally to corresponding entropy conservation laws. The property which is necessary and sufficient for an entropy η to have a kinetic representation is a generalization of the usual condition that the functions $\nabla^T \eta A^k$ have primitives which are called entropy fluxes. The additional requirement is that all functions $\nabla^T \eta \boldsymbol{A}^{\boldsymbol{\alpha}} A^k$ have primitives where $\boldsymbol{A}^{\boldsymbol{\alpha}}$ are $|\boldsymbol{\alpha}|$–fold symmetric products of the matrices A^k. Any entropy η satisfying these additional integrability conditions is called exponentially exact and the attribute is given to the underlying system if all linear entropies have this property.

From the definition it is obvious that all entropies for scalar conservation laws are exponentially exact because primitives can be found by simple integration (for the same reason, usual entropies are easily obtained in the scalar case). This explains from a new point of view why kinetic representations are particularly well suited for scalar equations. It also explains the difficulties in finding kinetic representations for systems of conservation laws because the integrability conditions are no longer trivial (for the same reason, usual entropies are difficult to obtain for systems).

Although the class of non–linear exponentially exact systems is not empty, as we have shown in the example section, it seems to be quite small. This suggests that the representation of general hyperbolic systems by a free transport equation with a source term which is only supported on the points of discontinuity of the solution is too restrictive. Kinetic representations for general systems therefore require either a more complicated source term or modifications in the transport operator (see [11] for an approach in this direction).

A Symmetric products

Let $A^1, \dots, A^d$ be $m \times m$ matrices and $\boldsymbol{\alpha}$ any multi–index of length $n \geq 0$. We define an n–fold symmetric product $\boldsymbol{A}^{\boldsymbol{\alpha}}$ by the relation

$$\frac{1}{n!}(\xi_j A^j)^n = \sum_{|\boldsymbol{\alpha}|=n} \frac{1}{\boldsymbol{\alpha}!} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \boldsymbol{A}^{\boldsymbol{\alpha}}. \tag{52}$$

Using Einstein's summation convention over the repeated indices $\lambda_1, \dots, \lambda_n$, we have explicitly

$$(\xi_j A^j)^n = \xi_{\lambda_1} \cdots \xi_{\lambda_n} A^{\lambda_1} \cdots A^{\lambda_n}.$$

If $I(\boldsymbol{\lambda})$ is the multi–index which counts in its entry j the number of components $\lambda_k = j$, we can write

$$(\xi_j A^j)^n = \sum_{|\boldsymbol{\alpha}|=n} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \sum_{I(\boldsymbol{\lambda})=\boldsymbol{\alpha}} A^{\lambda_1} \cdots A^{\lambda_n}.$$

Inserting this relation into (52), we get

$$\boldsymbol{A}^{\boldsymbol{\alpha}} = \frac{\boldsymbol{\alpha}!}{|\boldsymbol{\alpha}|!} \sum_{I(\boldsymbol{\lambda})=\boldsymbol{\alpha}} A^{\lambda_1} \cdots A^{\lambda_n}. \tag{53}$$

By taking all A^j equal to the identity matrix I, we find together with (52)

$$\frac{1}{n!}(\xi_1 + \cdots + \xi_d)^n = \sum_{|\boldsymbol{\alpha}|=n} \frac{1}{\boldsymbol{\alpha}!}\boldsymbol{\xi}^{\boldsymbol{\alpha}} \left(\frac{\boldsymbol{\alpha}!}{|\boldsymbol{\alpha}|!} \sum_{I(\boldsymbol{\lambda})=\boldsymbol{\alpha}} 1 \right). \tag{54}$$

One checks easily that $\nabla_{\boldsymbol{\xi}}^{\boldsymbol{\beta}}(\xi_1 + \cdots + \xi_d)^n/n! = 1$ if $|\boldsymbol{\beta}| = n$. Applied to (54) this yields

$$\frac{\boldsymbol{\beta}!}{|\boldsymbol{\beta}|!} \sum_{I(\boldsymbol{\lambda})=\boldsymbol{\beta}} 1 = 1. \tag{55}$$

Now, if $|\cdot|$ is a sub-multiplicative matrix product, we conclude from (53) and (55)

$$|\boldsymbol{A}^{\boldsymbol{\alpha}}| \leq \left(\max_{j=1}^{d} |A^j| \right)^{|\boldsymbol{\alpha}|}. \tag{56}$$

Using the special vector $\boldsymbol{\xi} = (1, \ldots, 1)^T$ in (54), we also find the important relation

$$\sum_{|\boldsymbol{\alpha}|=n} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} = d^n. \tag{57}$$

Similar to relation (53), we can define the symmetrization of any quantity F depending on n indices $\lambda_1, \ldots, \lambda_n$

$$F^{\boldsymbol{\alpha}} := \frac{\boldsymbol{\alpha}!}{|\boldsymbol{\alpha}|!} \sum_{I(\boldsymbol{\lambda})=\boldsymbol{\alpha}} F(\lambda_1, \ldots, \lambda_n).$$

We then have

$$\xi_{\lambda_1} \cdots \xi_{\lambda_n} F(\lambda_1, \ldots, \lambda_n) = \sum_{|\boldsymbol{\alpha}|=n} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} \boldsymbol{\xi}^{\boldsymbol{\alpha}} F^{\boldsymbol{\alpha}}. \tag{58}$$

If F depends on some additional variable $\boldsymbol{x}$, a similar relation holds for n–fold derivatives

$$\frac{\partial}{\partial x_{\lambda_1}} \cdots \frac{\partial}{\partial x_{\lambda_n}} F(\lambda_1, \ldots, \lambda_n) = \sum_{|\boldsymbol{\alpha}|=n} \frac{1}{\boldsymbol{\alpha}!} \nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}} F^{\boldsymbol{\alpha}}. \tag{59}$$

(In fact, any multiplication with an object which is completely symmetric gives rise to such a representation.) If G depends on $n+1$ indices, the symmetrization with fixed index $\lambda_{n+1} = k$ is denoted

$$G^{\boldsymbol{\alpha}}(k) := \frac{\boldsymbol{\alpha}!}{|\boldsymbol{\alpha}|!} \sum_{I(\boldsymbol{\lambda})=\boldsymbol{\alpha}} G(\lambda_1, \ldots, \lambda_n, k).$$

A relation between the completely symmetrized expression $G^{\boldsymbol{\alpha}+\boldsymbol{e}_k}$ and $G^{\boldsymbol{\alpha}}(k)$ is given by

$$G^{\boldsymbol{\beta}} = \frac{1}{|\boldsymbol{\beta}|}\sum_{k=1}^{d}\beta_k G^{\boldsymbol{\alpha}}(k), \qquad \boldsymbol{\beta} = \boldsymbol{\alpha} + \boldsymbol{e}_k. \tag{60}$$

Indeed, with (58), we find

$$\sum_{|\boldsymbol{\beta}|=n+1}\frac{|\boldsymbol{\beta}|!}{\boldsymbol{\beta}!}\boldsymbol{\xi}^{\boldsymbol{\beta}}G^{\boldsymbol{\beta}} = \sum_{k=1}^{d}\sum_{|\boldsymbol{\alpha}|=n}\frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!}\boldsymbol{\xi}^{\boldsymbol{\alpha}}\xi_k G^{\boldsymbol{\alpha}}(k) \tag{61}$$

and since $\nabla^{\boldsymbol{\beta}}(\boldsymbol{\xi}^{\boldsymbol{\alpha}}\xi_k) = \boldsymbol{\alpha}!\beta_k$, (60) follows from (61) by taking $\boldsymbol{\xi}$–derivatives. Replacing formally $\boldsymbol{\xi}$ by $\nabla_{\boldsymbol{x}}$, we get similar to (61)

$$\sum_{|\boldsymbol{\beta}|=n+1}\frac{|\boldsymbol{\beta}|!}{\boldsymbol{\beta}!}\nabla_{\boldsymbol{x}}^{\boldsymbol{\beta}}G^{\boldsymbol{\beta}} = \sum_{k=1}^{d}\sum_{|\boldsymbol{\alpha}|=n}\frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!}\nabla_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\frac{\partial}{\partial x_k}G^{\boldsymbol{\alpha}}(k). \tag{62}$$

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