Consistency analysis of mesh-free methods for conservation laws

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Summary Based on general partitions of unity and standard numerical flux functions, a class of mesh-free methods for conservation laws is derived. A Lax-Wendroff type consistency analysis is carried out for the general case of moving partition functions. The analysis leads to a set of conditions which are checked for the finite volume particle method FVPM. As a by-product, classical finite volume schemes are recovered in the approach for special choices of the partition of unity.

1 Introduction

The need for mesh-free methods typically arises in connection with problems posed in time depending or very complicated geometries where the handling of mesh discretizations becomes technically complicated or very time consuming. If interesting features in solutions should be captured with maximal computational speed and minimal memory requirements, dynamic adaption of the resolution is necessary. In mesh-based methods, refinement or coarsening techniques require programming of complicated data structures which reflect the hierarchical connectivity relations in the refined mesh. If the mesh points are allowed to move, as in Lagrangian methods, large deviations lead to degenerate mesh cells and stability problems can occur because the neighborhood structure may no longer reflect the actual relative positions of the nodes. Other examples where usual mesh structures are not applicable are high dimensional problems because of memory limitations. A typical example for this situation arises in connection with the Boltzmann equation where particle methods are classically used to construct approximate solutions [10]. In gas and fluid dynamics, the SPH method [9] has been successfully applied to problems with free boundaries and large deviations. For variants of the SPH method, we refer to [6,14]. A detailed analysis can be found in [2] and [5,11]. Another classical application of particle methods is the simulation of vortex dynamics in incompressible Euler or Navier-Stokes flows [12,4,3]. Recent developments in the area of mesh-free methods for hyperbolic problems include the finite mass method (FMM) [15, 16] and the partition of unity method (PUM) [7] (see also the references therein for mesh-free finite element methods and [1] for a general overview on mesh-free methods).

In this article, we analyze the finite volume particle method (FVPM) [8]. In fact, we are going to embed this method into a more general framework which also includes classical finite volume schemes. Since we will use a modification of the original approach in [8], let us briefly outline the construction for the case of scalar conservation laws in one space dimension

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad u(0,x) = u^0(x) \tag{1}$$

In standard finite difference discretizations of the Cauchy problem, approximate values u_i are calculated at regularly spaced points $x_i = ih$, $i \in \mathbb{Z}$ with distance $h > 0$. The value u_i typically represents the integral average of u over a volume $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ attached to x_i. In terms of the indicator function $\psi_i(x)$ of that interval, we can write the cell average as

$$u_i = \frac{1}{h}\int_{\mathbb{R}} \psi_i u \, dx = \frac{1}{V_i}\langle u, \psi_i\rangle, \qquad V_i = \langle 1, \psi_i\rangle$$

where $\langle\cdot,\cdot\rangle$ abbreviates x-integration. Note that $\{\psi_i : i \in \mathbb{Z}\}$ is a partition of unity, i.e. $\sum_{i\in\mathbb{Z}} \psi_i(x) = 1$ for all $x \in \mathbb{R}$.

As extension of this concept, we are going to introduce a particle method with *particle positions* x_i which may be irregularly spaced and moving. To each x_i we associate a function ψ_i, the *particle*. As in the finite difference approach, $\{\psi_i : i \in \mathbb{Z}\}$ will be a partition of unity but the supports of the functions ψ_i may overlap. More precisely, we assume that the particles ψ_i are smooth functions which are localized around the particle positions $x_i(t)$ and satisfy $\sum_{i\in\mathbb{Z}} \psi_i(t,x) = 1$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}^+ = [0,\infty)$ (for details of the construction, we refer to Section 4). The positions are supposed to move according to a differential equation $\dot{x} = a(t,x)$ with a given field a. As we will see, this movement implies that ψ_i satisfies the relations

$$\frac{\partial \psi_i}{\partial x} = \sum_{j\in\mathbb{Z}} \left(\Gamma_{ji} - \Gamma_{ij}\right), \qquad \frac{\partial \psi_i}{\partial t} = -\sum_{j\in\mathbb{Z}} (\dot{x}_i \Gamma_{ji} - \dot{x}_j \Gamma_{ij}) \tag{2}$$

where the function Γ_{ij} is localized on the intersection of the supports of particle i and particle j. Using (2), we find that ψ_i satisfies the transport equation

$$\frac{\partial \psi_i}{\partial t} + \dot{x}_i \frac{\partial \psi_i}{\partial x} = \sum_{j\in\mathbb{Z}} (\dot{x}_j - \dot{x}_i)\Gamma_{ij}. \tag{3}$$

Note that the left hand side in (3) describes the movement of the particle while the right hand side is related to a deformation of ψ_i. Deformations arise if particles move relative to each other so that the function values have to change in order to keep the property that the sum of all ψ_i is equal to one. For the local averages $u_i = \langle u, \psi_i\rangle / V_i$ of the solution u of equation (1) we find

$$\frac{d}{dt}(u_i V_i) = \left\langle \frac{\partial u}{\partial t}, \psi_i \right\rangle + \left\langle u, \frac{\partial \psi_i}{\partial t} \right\rangle = \left\langle f(u), \frac{\partial \psi_i}{\partial x} \right\rangle + \left\langle u, \frac{\partial \psi_i}{\partial t} \right\rangle$$

and with (2), we get

$$\frac{d}{dt}(u_i V_i) = \sum_{j\in\mathbb{Z}} \left(\langle f(u) - \dot{x}_i u, \Gamma_{ji}\rangle - \langle f(u) - \dot{x}_j u, \Gamma_{ij}\rangle\right).$$

For abbreviation, we introduce the Lagrangian flux

$$G(t,x,u) = f(u) - ua(t,x)$$

which consists of the flux in (1) as well as a contribution ua due to the particle movement with velocity a. Setting $G_i = G(t,x_i,u_i)$ and $\gamma_{ij} = \langle \Gamma_{ij}, 1\rangle$, we have approximately

$$\frac{d}{dt}(u_i V_i) \approx \sum_{j\in\mathbb{Z}} (G_i\gamma_{ji} - G_j\gamma_{ij})$$

since Γ_{ij} are localized close to x_i and x_j. Now, we use the splitting $ac - bd = (a-b)(c+d)/2 + (a+b)(c-d)/2$ which yields

$$\begin{aligned}\sum_{j\in\mathbb{Z}} (G_i\gamma_{ji} - G_j\gamma_{ij}) &= \sum_{j\in\mathbb{Z}} \frac{1}{2}(G_i - G_j)(\gamma_{ij} + \gamma_{ji})\\ &\quad - \sum_{j\in\mathbb{Z}} \frac{1}{2}(G_i + G_j)(\gamma_{ij} - \gamma_{ji})\end{aligned}$$

Assuming $G_i \approx G_j$ for $\gamma_{ij} + \gamma_{ji} \neq 0$ (i.e. for nearby particles), we conclude further

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}(u_i V_i) \approx -\sum_j |\beta_{ij}| \frac{G_i + G_j}{2} n_{ij}$$

where $\beta_{ij} = \gamma_{ij} - \gamma_{ji}$ and $n_{ij} = \operatorname{sign}(\beta_{ij})$. Note that $\frac{1}{2}(G_i + G_j)n_{ij}$ is the numerical flux function of *central differencing*. A more general approach is obtained if we replace this particular expression by a general numerical flux function $g_{ij} = g(t, x_i, u_i, x_j, u_j, n_{ij})$ for $G(t,x,u)$.

We end up with a system of ordinary differential equations

$$\frac{d}{dt}(u_i V_i) = -\sum_j |\beta_{ij}| g_{ij}, \qquad u_i(0) = \left\langle u^0, \psi_i(0)\right\rangle / V_i(0). \tag{4}$$

Based on the solution $u_i(t)$ of (4) we construct an approximate solution $\tilde{u}$ of the original problem (1) by setting

$$\tilde{u}(t,x) = \sum_{i\in\mathbb{Z}} u_i(t)\psi_i(t,x). \tag{5}$$

Conservativity of the scheme follows from the property $|\beta_{ij}|g_{ij} = -|\beta_{ji}|g_{ji}$ which implies

$$\begin{aligned}\frac{d}{dt}\langle \tilde{u}, 1\rangle &= \frac{d}{dt}\sum_{i\in\mathbb{Z}} u_i V_i = -\sum_{i,j\in\mathbb{Z}} |\beta_{ij}| g_{ij}\\ &= -\frac{1}{2}\sum_{i,j\in\mathbb{Z}} (|\beta_{ij}| g_{ij} + |\beta_{ji}| g_{ji}) = 0.\end{aligned}$$

Remark 1 Choosing $a \equiv 0$, $x_i = ih$, ψ_i as indicator functions of $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $\beta_{i\,i+1} = 1$, $\beta_{i\,i-1} = -1$, $\beta_{ij} = 0$ otherwise, and $n_{ij} = \mathrm{sign}(\beta_{ij})$, then (4) turns into a usual finite difference scheme for (1) provided that the time derivative is discretized by Euler's method.

In [14,2] schemes of a structure similar to (4) are considered but the coefficients β_{ij} in this approach are of a very special form and do not exactly satisfy the requirements that will be introduced here. Using overlapping particles ψ_i and $\beta_{ij} = \gamma_{ij} - \gamma_{ji}$ as introduced above, the method turns into the finite volume particle method which has been tested for scalar conservation laws like (1) and for the system of Euler equations in [8].

Here, our aim is to show the consistency of (4) with a Lax–Wendroff type result: assuming that (5) is close in a suitable sense to some function $u : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$, it already follows that u is a weak solution of the problem (1).

Definition 1 *A function $u \in \mathbb{L}^1_{loc}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$ is called weak solution of the Cauchy problem* (1) *with $u^0 \in \mathbb{L}^1_{loc}(\mathbb{R})$ if*

$$\int_0^\infty \left\langle u(t), \frac{\partial \phi}{\partial t}(t) \right\rangle + \left\langle f(u(t)), \frac{\partial \phi}{\partial x}(t) \right\rangle \, dt + \langle u^0, \phi(0) \rangle = 0$$

for all $\phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R})$. Here, $\phi(t)$ and $u(t)$ denote the functions $x \mapsto \phi(t,x)$ and $x \mapsto u(t,x)$ respectively.

While the detailed consistency proof will be given in Section 3, we can already outline the main steps. We start with the relation

$$\begin{aligned} \left\langle \frac{\partial \tilde{u}}{\partial t}, \phi \right\rangle &= \sum_{i \in \mathbb{Z}} \left\langle \psi_i \frac{du_i}{dt}, \phi \right\rangle + \sum_{i \in \mathbb{Z}} \left\langle u_i \frac{\partial \psi_i}{\partial t}, \phi \right\rangle \\ &= - \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left\langle |\beta_{ij}| g_{ij} \frac{\psi_i}{V_i}, \phi \right\rangle \\ &\quad + \sum_{i \in \mathbb{Z}} \left(\left\langle u_i \frac{\partial \psi_i}{\partial t}, \phi \right\rangle - \left\langle u_i \frac{\dot{V}_i}{V_i} \psi_i, \phi \right\rangle \right). \end{aligned} \tag{6}$$

Using again the conservation property $|\beta_{ij}| g_{ij} = -|\beta_{ji}| g_{ji}$, we can rewrite the flux term as

$$- \sum_{i,j \in \mathbb{Z}} \left\langle |\beta_{ij}| g_{ij} \frac{\psi_i}{V_i}, \phi \right\rangle = - \sum_{i,j \in \mathbb{Z}} \left\langle \frac{1}{2} |\beta_{ij}| g_{ij} \left(\frac{\psi_i}{V_i} - \frac{\psi_j}{V_j} \right), \phi \right\rangle.$$

The consistency of the numerical flux and the fact that $\beta_{ij} \neq 0$ only for particles i, j which are close to each other (i.e. $x_i \approx x_j$ and $u_i \approx u_j$), implies that we can approximate g_{ij} by $G_i n_{ij}$ for such pairs

$$- \sum_{i,j \in \mathbb{Z}} \left\langle \frac{1}{2} |\beta_{ij}| g_{ij} \left(\frac{\psi_i}{V_i} - \frac{\psi_j}{V_j} \right), \phi \right\rangle \approx - \sum_{i,j \in \mathbb{Z}} \left\langle G_i \frac{1}{2} \beta_{ij} \left(\frac{\psi_i}{V_i} - \frac{\psi_j}{V_j} \right), \phi \right\rangle.$$

A crucial observation is that the right hand side is a weak derivative

$$\sum_{i,j\in\mathbb{Z}} \left\langle G_i \frac{1}{2}\beta_{ij}\left(\frac{\psi_i}{V_i}-\frac{\psi_j}{V_j}\right), \phi\right\rangle = -\left\langle \sum_{i\in\mathbb{Z}} G_i H_i, \frac{\partial \phi}{\partial x}\right\rangle$$

where the functions $\{H_i : i \in \mathbb{Z}\}$ are constructed from β_{ij} and $\{\psi_i : i \in \mathbb{Z}\}$ and form again a partition of unity. In the special case of finite difference schemes (see Remark 1), the partitions $\{\psi_i\}$ and $\{H_i\}$ are depicted in Fig. 1.

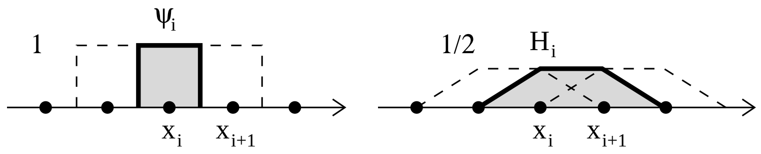

Fig. 1. The partitions of unity in the case of finite difference schemes

Since the sum $\sum_{i\in\mathbb{Z}} G_i H_i$ can be viewed as an approximation of the Lagrangian flux G, we obtain

$$-\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \left\langle |\beta_{ij}| g_{ij} \frac{\psi_i}{V_i}, \phi\right\rangle \approx \left\langle f(\tilde{u}) - a\tilde{u}, \frac{\partial \phi}{\partial x}\right\rangle. \tag{7}$$

For the second sum in (6) we get with (3)

$$\sum_{i\in\mathbb{Z}} \left\langle u_i \frac{\partial \psi_i}{\partial t}, \phi\right\rangle = \left\langle \sum_{i\in\mathbb{Z}} u_i \dot{x}_i \psi_i, \frac{\partial \phi}{\partial x}\right\rangle + \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} u_i(\dot{x}_j - \dot{x}_i)\left\langle \mathit{\Gamma}_{ij}, \phi\right\rangle.$$

Here, the first term approximates $\langle \tilde{u}a, \partial_x \phi\rangle$ and the second one is related to the change of shape of the functions ψ_i. It turns out that this term is approximately compensated by the contribution due to the volume change $\sum \left\langle u_i \dot{V}_i / V_i \psi_i, \phi\right\rangle$ in (6). Hence

$$\sum_{i\in\mathbb{Z}} \left(\left\langle u_i \frac{\partial \psi_i}{\partial t}, \phi\right\rangle - \left\langle u_i \frac{\dot{V}_i}{V_i}\psi_i, \phi\right\rangle\right) \approx \left\langle a\tilde{u}, \frac{\partial \phi}{\partial x}\right\rangle.$$

Combining this result with (7) and (6), the term $\langle a\tilde{u}, \partial_x \phi\rangle$ vanishes so that

$$\left\langle \frac{\partial \tilde{u}}{\partial t}, \phi\right\rangle \approx \left\langle f(\tilde{u}), \frac{\partial \phi}{\partial x}\right\rangle. \tag{8}$$

If now $\tilde{u}$ converges in a suitable sense to a function u, the relation (8) is the essential part in showing that u is a weak solution of the problem (1).

We conclude the introductory remarks with an outline of the article. In Section 2, the general consistency result is presented together with some definitions and the assumptions on the partition $\{\psi_i\}$, the geometric coefficients β_{ij}, and the numerical flux function g_{ij}. The proof of the main result is contained in Section 3. Finally, we check that the finite volume particle method (FVPM) satisfies all requirements and thus is consistent.

2 A Lax-Wendroff type result

Our aim is to derive a consistency result for the finite volume particle method which has been introduced in the previous section. It turns out that the result is largely independent of the form of the chosen partition of unity and the exact structure of the geometric coefficients β_{ij} and therefore, we base the proof on general assumptions which are listed below. In setting up these conditions, we have taken care that standard finite volume (resp. finite difference) methods on fixed regular or irregular grids are also contained in the considerations. For example, the choice of parameters mentioned in Remark 1 obviously satisfies all the requirements.

Before listing the assumptions, we need the notion of *locally finite* families.

Definition 2 *Let $\mathcal{M}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$ be the set of strongly measurable functions on $\mathbb{R}^+$ with values in $\mathbb{L}^1_{loc}(\mathbb{R})$ and let*

$$\mathcal{F} = \{ F_i \in \mathcal{M}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R})) : i \in \mathbb{Z} \}.$$

For $f \in \mathbb{L}^1_{loc}(\mathbb{R})$ let $\operatorname{supp} f$ be the complement of the largest open set on which f vanishes in the sense of distributions. We introduce

$$I_{\mathcal{F}}(t,x) := \{ i \in \mathbb{Z} : x \in \operatorname{supp} F_i(t) \}$$

which is the set of indices of those F_i which are non-zero at (t,x). If we replace t or x in $I_{\mathcal{F}}(t,x)$ by sets, this abbreviates the union

$$I_{\mathcal{F}}(A,B) := \bigcup_{t \in A} \bigcup_{x \in B} I_{\mathcal{F}}(t,x) \qquad A, B \subset \mathbb{R}.$$

The indices of the functions F_i whose support is completely contained in an interval $B_\epsilon(x)$ of radius $\epsilon > 0$ around x at time t are collected in

$$I_{\mathcal{F}}(t,x,\epsilon) := \{ i \in \mathbb{Z} : \operatorname{supp} F_i(t) \subset B_\epsilon(x) \}.$$

The set $\mathcal{F}$ is called locally finite if $I_{\mathcal{F}}([0,T],K)$ is finite for any compact set $K \subset \mathbb{R}$ and any $T > 0$.

2.1 The particle clouds

A set of functions $\Psi = \{\psi_i : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R} : i \in \mathbb{Z}\}$ will be called a moving cloud of particles if the following conditions are satisfied:

Regularity properties

- ψ_i is measurable on $\mathbb{R}^+ \times \mathbb{R}$,
- $\psi_i \in C^1(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$,
- $\operatorname{diam} \operatorname{supp} \psi_i(t) \leq S$ for some $S > 0$,
- $\psi_i(t,x) = 0$ for all $x \notin \operatorname{supp} \psi_i(t)$,

Partition of unity properties

- Ψ is locally finite,
- $0 \leq \psi_i(t,x) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{R}$,

– $\sum_{i\in\mathbb{Z}} \psi_i(t,x) = 1$ for all $t \geq 0$, $x \in \mathbb{R}$,

Position and volume properties

– for some $\alpha > 0$, the volume $V_i(t) = \langle 1, \psi_i(t)\rangle$ satisfies $V_i(t) \geq \alpha$ for all $i \in \mathbb{Z}$ and $t \geq 0$,
– there exists a continuous function $x_i : \mathbb{R}^+ \mapsto \mathbb{R}$ such that $x_i(t) \in \operatorname{supp} \psi_i(t)$ which is called the position of ψ_i,

Movement properties

– there exists a field $a \in C^0(\mathbb{R}^+, C^1(\mathbb{R}))$ such that with $a_i(t) = a(t, x_i(t))$, the relation

$$\frac{\partial \psi_i}{\partial t} + a_i \frac{\partial \psi_i}{\partial x} = \nu_i \psi_i$$

holds for some $\nu_i \in \mathbb{L}^\infty_{loc}(\mathbb{R}^+, \mathbb{L}^1(\mathbb{R}))$ satisfying

$$\sup_{0\leq t\leq T} \|\nu_i^{(h)}(t)\|_{\mathbb{L}^1(\mathbb{R})} \leq C_T$$

in the sense of distributions on $\mathbb{R}$ (since $\psi_i \in C^1(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$ is compactly supported, we can view ψ_i as a differentiable mapping with values in the space of compactly supported distributions $\mathscr{E}'(\mathbb{R})$).

A sequence $\Psi_* = \{\Psi_h : 0 < h \leq 1\}$ of moving particle clouds is called uniformly regular (or short "urp–sequence") if the above assumptions hold for $\Psi = \Psi_h$ with S, α and C_T replaced by $S_{\Psi_*}h$, $\alpha_{\Psi_*}h$ and $C_{\Psi_*,T}h$. Here, $S_{\Psi_*}, \alpha_{\Psi_*}$ and $C_{\Psi_*,T}$ are assumed to be uniform constants for the sequence Ψ_*. In addition, we require that

$$\sup_{h>0} \sup_{t\geq 0} \sup_{x\in\mathbb{R}} |I_{\Psi_h}(t,x,rh)| < \infty \qquad \forall r > 0.$$

2.2 The geometric coefficients

Let $\Psi = \{\psi_i : i \in \mathbb{Z}\}$ be a moving cloud of particles. A family of measurable functions $\Theta = \{\beta_{ij} : \mathbb{R}^+ \mapsto \mathbb{R} : i,j \in \mathbb{Z}\}$ is called Ψ–admissible if

– $|\beta_{ij}(t)| \leq C$ for all $i,j \in \mathbb{Z}$ and $t \geq 0$,
– $\beta_{ij} = -\beta_{ji}$,
– there exists $B > 0$ such that $\beta_{ij}(t) = 0$ if $|x_i(t) - x_j(t)| > B$,
– $\sum_{j\in\mathbb{Z}} \beta_{ij}(t) = 0$ for all $i \in \mathbb{Z}$ and $t \geq 0$,
– for each $t \geq 0$ there exists $\bar{x} \in \mathbb{R}$ such that

$$\sum_{x_i(t)\geq\bar{x}} \sum_{x_j(t)\geq\bar{x}} \beta_{ij}(t) = 1$$

Let $\Psi_* = \{\Psi_h : 0 < h \leq 1\}$ be a urp–sequence and $\Theta_* = \{\Theta_h : 0 < h \leq 1\}$ a sequence of families $\Theta_h = \{\beta_{ij}^{(h)} : i,j \in \mathbb{Z}\}$. Then Θ_* is called Ψ_*–admissible sequence of geometric coefficients if each Θ_h is Ψ_h–admissible with B replaced by $B_{\Theta_*}h$ and $\beta_{ij}^{(h)}$ being uniformly bounded also in h.

2.3 The numerical flux function

If $a \in C^0(\mathbb{R}^+, C^1(\mathbb{R}))$ is a given velocity field and f the Lipschitz continuous flux of the conservation law, we first introduce the modified flux

$$G(t,x,u) = f(u) - ua(t,x).$$

We then assume that $g(t,x_1,u_1,x_2,u_2,n)$ with $t \geq 0$, $x_1,x_2,u_1,u_2 \in \mathbb{R}$ and $n \in \{-1,1\}$ is a numerical flux function for G which satisfies

Consistency

- $g(t,x,u,x,u,n) = G(t,x,u)n$

Conservativity

- $g(t,x,u,y,v,-n) = -g(t,y,v,x,u,n)$

Continuity

- $|g(t,x,u,y,v,n) - g(t,\bar{x},\bar{u},\bar{y},\bar{v},n)| \leq L(|x-\bar{x}|+|y-\bar{y}|+|u-\bar{u}|+|v-\bar{v}|)$, where L depends monotonically on t and $\max\{|u|,|\bar{u}|,|v|,|\bar{v}|\}$. Also, g is assumed to be continuous in $t \in \mathbb{R}^+$.

2.4 The particle method

Let $\Psi = \{\psi_i : i \in \mathbb{Z}\}$ be a moving particle cloud, g a numerical flux function satisfying the assumptions of Section 2.3, and $\Theta = \{\beta_{ij} : i,j \in \mathbb{Z}\}$ a Ψ–admissible family of geometric coefficients. Further let $u^0 \in \mathbb{L}^1_{loc}(\mathbb{R})$. A set of functions $\{u_i \in C^1(\mathbb{R}^+) : i \in \mathbb{Z}\}$ is called solution of the (Ψ,Θ,g)–particle method (or simply (Ψ,Θ,g)–solution) if for all $i \in \mathbb{Z}$

$$\frac{d}{dt}(u_i V_i) = -\sum_{j\in\mathbb{Z}} |\beta_{ij}| g_{ij}, \qquad u_i(0) = \left\langle u^0, \psi_i(0)\right\rangle / V_i(0).$$

where

$$g_{ij}(t) = g(t, x_i(t), u_i(t), x_j(t), u_j(t), n_{ij}(t)), \qquad n_{ij} = \operatorname{sign}\beta_{ij}$$

and x_i is the position of particle ψ_i.

If Ψ_* is a urp–sequence, Θ_* a Ψ_*–admissible sequence of geometric coefficients, then a sequence $\{u_i^{(h)} \in C^1(\mathbb{R}^+) : i \in \mathbb{Z}\}$, $h > 0$ is called solution of the (Ψ_*,Θ_*,g)–particle method if for fixed $h > 0$ the set $\{u_i^{(h)} : i \in \mathbb{Z}\}$ is a (Ψ_h,Θ_h,g)–solution.

2.5 Ψ_*-convergence

The particle method presented in the previous section includes an approximation of the initial value u^0. We now study in which sense, for example, $u^0 \in \mathbb{L}^\infty(\mathbb{R})$ is approximated by $\tilde{u}(0,x) = \sum u_i(0)\psi_i(0,x)$ where $u_i(0)$ are the local averages $\left\langle u^0, \psi_i(0)\right\rangle / V_i(0)$. The resulting notion of Ψ_*-convergence will then be assumed also for $t > 0$ to get the consistency result. We start with a preparatory remark.

Lemma 1 *Let $\Psi = \{\psi_i : i \in \mathbb{Z}\}$ be a moving particle cloud and $\{u_i : \mathbb{R}^+ \mapsto \mathbb{R} : i \in \mathbb{Z}\}$ a family of measurable functions. Then,*

$$u(t,x) = \sum_{i \in I_\Psi(t,x)} u_i(t)\psi_i(t,x)$$

is measurable on $\mathbb{R}^+ \times \mathbb{R}$ and can be identified with $\sum_{i\in\mathbb{Z}} u_i\psi_i \in \mathcal{M}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$. If $|u_i(t)| \le C(t)$ for some increasing function $C : \mathbb{R}^+ \mapsto \mathbb{R}^+$, then u is contained in $\mathbb{L}^\infty_{loc}(\mathbb{R}^+, \mathbb{L}^\infty(\mathbb{R}))$. If $u_i \in C^1(\mathbb{R}^+)$ then $u \in C^1(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$ with derivative

$$\frac{\partial u}{\partial t} = \sum_{i\in\mathbb{Z}} \psi_i \frac{du_i}{dt} + \sum_{i\in\mathbb{Z}} u_i \frac{\partial \psi_i}{\partial t}.$$

Proof The truncated series

$$S_n(t,x) = \sum_{i=-n}^{n} u_i(t)\psi_i(t,x)$$

is clearly measurable on $\mathbb{R}^+ \times \mathbb{R}$ and converges point-wise to

$$u(t,x) = \sum_{i \in I_\Psi(t,x)} u_i(t)\psi_i(t,x)$$

which is therefore measurable. Since Ψ is locally finite, we have for any compact $K \subset \mathbb{R}$ that $|I_\Psi(t,K)| < \infty$. For any $x \in K$ it follows $I_\Psi(t,x) \subset I_\Psi(t,K)$ which leads to the estimate

$$|u(t,x)| \le \max_{i\in I_\Psi(t,K)} |u_i(t)| \sum_{i\in I_\Psi(t,K)} \psi_i(t,x) \le \max_{i\in I_\Psi(t,K)} |u_i(t)|$$

for all $x \in K$ so that $u(t) \in \mathbb{L}^\infty_{loc}(\mathbb{R}) \subset \mathbb{L}^1_{loc}(\mathbb{R})$ Because of point-wise convergence $S_n \to u$ and the uniform bound for $x \in K \subset \mathbb{R}$, it follows that $S_n(t) \to u(t)$ in $\mathbb{L}^1_{loc}(\mathbb{R})$ for $n \to \infty$. Hence, we can identify $\sum_{i\in\mathbb{Z}} u_i\psi_i$ with the function u.

Under the condition $|u_i(t)| \le C(t)$, we find immediately $|u(t,x)| \le C(t)$. Finally, if $u_i \in C^1(\mathbb{R}^+)$, then $u_i\psi_i \in C^1(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$ still makes up a locally finite family. Hence, if x is restricted to a given compact $K \subset \mathbb{R}$ and $t \in [0,T]$, we can replace u by a finite sum so that the result follows.

Proposition 1 *Assume $\Psi_* = \{\Psi_h : 0 < h \le 1\}$ is a urp–sequence and let $u \in \mathbb{L}^\infty_{loc}(\mathbb{R}^+, \mathbb{L}^\infty(\mathbb{R}))$ be given. Then, the coefficients*

$$u_i^{(h)}(t) := \left\langle \psi_i^{(h)}(t), u(t) \right\rangle / V_i^{(h)}(t), \qquad \psi_i^{(h)} \in \Psi_h$$

are measurable on $\mathbb{R}^+$ and, for a.e. $t \in \mathbb{R}^+$,
satisfy $|u_i^{(h)}(t)| \le \|u(t)\|_{\mathbb{L}^\infty(\mathbb{R})}$ and

$$\max_{i\in I_{\Psi_h}(t,x,rh)} |u_i(t) - u(t,x)| \to 0 \qquad \text{as } h \to 0$$

for a.e. $x \in \mathbb{R}$ and all $r \ge S_{\Psi_*}$. If $u \in C^1_0(\mathbb{R}^+ \times \mathbb{R})$, we even find

$$\max_{i\in I_{\Psi_h}(t,x,rh)} |u_i(t) - u(t,x)| \le Ch \qquad \forall t \ge 0.$$

Proof Since u is strongly measurable, $u(t)$ is the $\mathbb{L}^1_{loc}(\mathbb{R})$ limit of simple functions $s_n(t)$. The product $t \mapsto \langle \psi_i(t), s_n(t) \rangle$ is obviously measurable so that the same holds for the limit, giving rise to measurability of u_i (we suppress the index h for ease of notation). The bound on u_i simply follows from

$$|u_i(t)| \leq \|u(t)\|_{\mathbb{L}^\infty(\mathbb{R})} \langle \psi_i(t), 1 \rangle / V_i(t) = \|u(t)\|_{\mathbb{L}^\infty(\mathbb{R})}.$$

To show the convergence, we pick $x \in \mathbb{R}, t \geq 0$ and $r \geq S_{\Psi_*}$ For any $i \in I_{\Psi_h}(t, x, rh)$ the conditions $\operatorname{diam} \operatorname{supp} \psi_i \leq S_{\Psi_*} h$ and $0 \leq \psi_i \leq 1$ then imply that $\psi_i(t)$ is bounded from above by the indicator function $\mathcal{X}_{B_{rh}(x)}$ of a ball with radius rh around x. Hence,

$$\begin{aligned} |u_i(t) - u(t,x)| &= \frac{1}{V_i(t)} | \langle u(t) - u(t,x), \psi_i(t) \rangle | \\ &\leq \frac{2rh}{V_i(t)} \frac{1}{2rh} \left\langle |u(t) - u(t,x)|, \mathcal{X}_{B_{rh}(x)} \right\rangle \\ &\leq \frac{2rh}{\alpha_{\Psi_*} h} \operatorname{av}(|u(t) - u(t,x)|, B_{rh}(x)) \end{aligned}$$

where $\operatorname{av}(f, A) = \frac{1}{|A|} \int_A f(y)\, dy$ is the averaging operator. It is known [13] that for all Lebesgue–points of $u(t)$ (and thus a.e. in x) the average of $|u(t,y) - u(t,x)|$ over the ball $y \in B_{rh}(x)$ tends to zero for $h \to 0$ which leads to the claimed convergence. If $u \in C^1(\mathbb{R}^+ \times \mathbb{R})$, uniform Lipschitz continuity yields at once

$$\begin{aligned} |u_i(t) - u(t,x)| &= | \langle \psi_i(t), u(t) - u(t,x) \rangle | / V_i(t) \\ &\leq L \operatorname{diam} \operatorname{supp} \psi_i \leq L S_{\Psi_*} h. \end{aligned}$$

The convergence result of Proposition 1 motivates the following definition of Ψ_*–convergence.

Definition 3 *Let $\Psi_* = \{\Psi_h : 0 < h \leq 1\}$ be a urp–sequence. A sequence of families of measurable functions $\{u_i^{(h)} : \mathbb{R}^+ \mapsto \mathbb{R} : i \in \mathbb{Z}\}$, $0 < h \leq 1$, Ψ_*–converges to $u : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ if for a.e. $t \geq 0$*

$$\max_{i \in I_{\Psi_h}(t,x,rh)} |u_i^{(h)}(t) - u(t,x)| \xrightarrow{h \to 0} 0$$

for a.e. $x \in \mathbb{R}$ and every $r \geq S_{\Psi_*}$.

2.6 The consistency result

Using the Definitions from above, we can now state

Theorem 1 *Let $\Psi_* = \{\Psi_h : 0 < h \leq 1\}$ be a urp–sequence, g a numerical flux function, and Θ_* a Ψ_*–admissible sequence of geometric coefficients. If $\{u_i^{(h)} \in C^1(\mathbb{R}^+) : i \in \mathbb{Z}\}$, $0 < h \leq 1$ is a solution of the (Ψ_*, Θ_*, g)–particle method which satisfies the uniform bound $|u_i^{(h)}(t)| \leq C(t)$ for some function $C : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and Ψ_*–converges to some $u : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$, then u is a weak solution of the Cauchy problem, satisfies $\|u(t)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq C(t)$ and $\tilde{u}^{(h)} := \sum_{i \in \mathbb{Z}} u_i^{(h)} \psi_i^{(h)}$ converges to u in $\mathbb{L}^1_{loc}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$.*

3 Proof of Theorem 1

We split the proof into several sub-steps. Since there is no danger of ambiguity, the superscript h is dropped in all proofs for ease of notation.

In the first step, we show the bound on u and convergence in the space $\mathbb{L}^1_{loc}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$.

Lemma 2 *Under the conditions of Theorem 1, we have*

$$\tilde{u}^{(h)} = \sum_{i\in\mathbb{Z}} u_i^{(h)}\psi_i^{(h)} \xrightarrow[h\to 0]{} u \qquad \text{in } \mathbb{L}^1_{loc}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$$

and $\|u(t)\|_{\mathbb{L}^\infty(\mathbb{R})} \le C(t)$*. For* $t = 0$*, we find* $\tilde{u}^{(h)}(0) \to u^0$ *in* $\mathbb{L}^1_{loc}(\mathbb{R})$*. More generally, if* $A : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ *is continuous, we obtain*

$$\sum_{i\in\mathbb{Z}} A(t, x_i^{(h)}(t), u_i^{(h)}(t))\psi_i^{(h)}(t,x) \xrightarrow{h\to 0} A(t,x,u(t,x))$$

in $\mathbb{L}^1_{loc}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$*.*

Proof Let u be the Ψ_* limit and note that, according to Lemma 1, the index set in the definition of $\tilde{u}$ can be replaced by $I_{\Psi_h}(t,x)$. Using the relation $I_{\Psi_h}(t,x) \subset I_{\Psi_h}(t,x,rh)$ for $r \ge S_{\Psi_*}$ as well as the partition of unity property, we find

$$\begin{aligned} |\tilde{u}(t,x) - u(t,x)| &= \left| \sum_{i\in I_{\Psi_h}(t,x,rh)} (u_i(t) - u(t,x))\psi_i(t,x) \right| \\ &\le \max_{i\in I_{\Psi_h}(t,x,rh)} |u_i(t) - u(t,x)| \end{aligned} \tag{9}$$

which tends to zero as $h \to 0$ for a.e. $t \ge 0$ almost everywhere in $x \in \mathbb{R}$. Assuming a bound on u_i, it is easy to see from the above estimate that u is also bounded. In this case, we obtain from the Lebesgue theorem $\tilde{u}(t) \to u(t)$ in $\mathbb{L}^1_{loc}(\mathbb{R})$ for a.e. $t \ge 0$ so that u is strongly measurable, i.e. $u \in \mathcal{M}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$. Using again the bound on $u(t)$ and $\tilde{u}(t)$, we conclude $\tilde{u} \to u$ in $\mathbb{L}^1_{loc}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$. Proposition 1 applied to the function $u(t,x) = u^0(x)$ shows that $\{u_i(0) : i \in \mathbb{Z}\}$ Ψ_*-converges and is uniformly bounded. Hence, the above argument shows that $\tilde{u}(0,x) \to u^0(x)$ in $\mathbb{L}^1_{loc}(\mathbb{R})$. Repeating estimate (9) again for $\tilde{A}(t,x) = \sum_{i\in\mathbb{Z}} A(t, x_i(t), u_i(t))\psi_i(t,x)$, we find

$$|\tilde{A}(t,x) - A(t,x,u(t,x))| \le \max_{i\in I_{\Psi_h}(t,x,rh)} |A(t, x_i(t), u_i(t)) - A(t,x,u(t,x))|$$

where x_i is the position of the particle ψ_i. Note that, due to uniform continuity of A in a neighborhood of $(t,x,u(t,x))$, we get convergence for a.e. $t \ge 0$ a.e. in $x \in \mathbb{R}$. If x is restricted to a compact set and $t \in [0,T]$, we conclude that $A(t,x,u(t,x))$ and $A(t,x_i,u_i)$ are bounded. Hence, with the same argument as above, convergence in $\mathbb{L}^1_{loc}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$ follows.

Since the approximation $\tilde{u}^{(h)}$ of u is t-differentiable we just have to show convergence of the flux terms to get consistency, as the following Lemma indicates.

Lemma 3 *With the assumptions of Theorem 1, we find that u is a weak solution of the Cauchy problem if the approximation $\tilde{u}^{(h)}$ satisfies*

$$\int_0^\infty \left\langle \frac{\partial \tilde{u}^{(h)}}{\partial t}, \phi \right\rangle dt \xrightarrow{h\to 0} \int_0^\infty \left\langle f(u), \frac{\partial \phi}{\partial x} \right\rangle dt$$

for every $\phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R})$.

Proof The convergence $\tilde{u} \to u$ obtained in Lemma 2 implies at once

$$\begin{aligned}\int_0^\infty \left\langle u, \frac{\partial \phi}{\partial t} \right\rangle dt &= \lim_{h\to 0} \int_0^\infty \frac{\partial}{\partial t} \langle \tilde{u}, \phi \rangle - \left\langle \frac{\partial \tilde{u}}{\partial t}, \phi \right\rangle dt \\ &= -\lim_{h\to 0} \langle \tilde{u}(0), \phi(0) \rangle - \lim_{h\to 0} \int_0^\infty \left\langle \frac{\partial \tilde{u}}{\partial t}, \phi \right\rangle dt.\end{aligned}$$

Using Lemma 2 again, we get convergence of the initial value and with the assumption for the second limit, it follows that u is a weak solution.

The result of Lemma 1 implies

$$\frac{\partial \tilde{u}}{\partial t} = \sum_{i\in\mathbb{Z}} \psi_i \frac{du_i}{dt} + \sum_{i\in\mathbb{Z}} u_i \frac{\partial \psi_i}{\partial t}$$

and with u_i being a (Ψ_h, Θ_h, g)–solution, we obtain

$$\frac{\partial \tilde{u}}{\partial t} = -\sum_{i,j\in\mathbb{Z}} |\beta_{ij}| g_{ij} \frac{\psi_i}{V_i} + \sum_{i\in\mathbb{Z}} u_i \left(\frac{\partial \psi_i}{\partial t} - \frac{\dot{V}_i}{V_i} \psi_i \right). \tag{10}$$

In the next lemma, we consider convergence of the second term on the right hand side of (10).

Lemma 4 *Under the conditions of Theorem 1, we find for* $\phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R})$

$$\int_0^\infty \left\langle \sum_{i\in\mathbb{Z}} u_i^{(h)} \left(\frac{\partial \psi_i^{(h)}}{\partial t} - \frac{\dot{V}_i^{(h)}}{V_i^{(h)}} \psi_i^{(h)} \right), \phi \right\rangle dt \xrightarrow{h\to 0} \int_0^\infty \left\langle au, \frac{\partial \phi}{\partial x} \right\rangle dt.$$

Proof Using the assumption on the time derivatives of ψ_i, we get

$$\sum_{i\in\mathbb{Z}} \left\langle u_i \frac{\partial \psi_i}{\partial t}, \phi \right\rangle = \sum_{i\in\mathbb{Z}} \left\langle u_i a(t, x_i) \psi_i, \frac{\partial \phi}{\partial x} \right\rangle + \sum_{i\in\mathbb{Z}} \langle u_i \nu_i \psi_i, \phi \rangle. \tag{11}$$

In view of Lemma 2, the first term on the right hand side gives the desired limit

$$\sum_{i\in\mathbb{Z}} u_i(t) a(t, x_i(t)) \psi_i(t, x) \xrightarrow{h\to 0} u(t, x) a(t, x)$$

in $\mathbb{L}^1_{loc}(\mathbb{R}^+, \mathbb{L}^1_{loc}(\mathbb{R}))$. Hence, it suffices to show that the second term in (11) vanishes in connection with the contribution due to $\dot{V}_i/V_i$. We first observe that

$$\dot{V}_i = \left\langle \frac{\partial \psi_i}{\partial t}, 1 \right\rangle = -a(t, x_i) \left\langle \frac{\partial \psi_i}{\partial x}, 1 \right\rangle + \langle \nu_i \psi_i, 1 \rangle$$

where the first term on the right equals zero. It remains to show that

$$\int_0^\infty \sum_{i \in \mathbb{Z}} u_i \left(\langle \nu_i \psi_i, \phi \rangle - \langle \nu_i \psi_i, 1 \rangle \frac{\langle \psi_i, \phi \rangle}{V_i} \right) dt \xrightarrow{h \to 0} 0. \tag{12}$$

Since ϕ is compactly supported, we first note that for a given $t \geq 0$, the summation can be restricted to the indices $I_{\Psi_h}(t, 0, R)$ for R sufficiently large. The number of indices in this set can be estimated by $|I_{\Psi_h}(t, 0, R)| \leq C/h$. Indeed, this bound is obtained by covering $(-R, R)$ with intervals of length $(S_{\Psi_*} + 1)h$ which requires a number of $\mathcal{O}(1/h)$ since the particle number in each of the small intervals is bounded

$$|I_{\Psi_h}(t, \bar{x}, (S_{\Psi_*} + 1)h)| \leq \sup_{h>0} \sup_{t \geq 0} \sup_{x \in \mathbb{R}} |I_{\Psi_h}(t, x, (S_{\Psi_*} + 1)h)| < \infty, \qquad \bar{x} \in \mathbb{R}.$$

Hence, convergence of (12) follows if we can bound each term in the sum by an expression of order h^2. Rearranging the bracket in (12), we get with Proposition 1 and the assumptions on ν_i

$$\left| \left\langle \nu_i \psi_i, \phi - \frac{\langle \psi_i, \phi \rangle}{V_i} \right\rangle \right| \leq \|\nu_i\|_{\mathbb{L}^1(\mathbb{R})} C h \leq \tilde{C} h^2.$$

Since u_i are uniformly bounded in h, the result follows.

Before we focus on the convergence of the flux terms in (10), we need some auxiliary result which covers a central argument in the consistency proof.

Lemma 5 *Let $\Psi = \{\psi_i : i \in \mathbb{Z}\}$ be a moving particle cloud and $\{\beta_{ij} : i, j \in \mathbb{Z}\}$ a Ψ–admissible family of geometric coefficients. The functions*

$$\Pi_{ij}(t, x) = \frac{1}{2} \beta_{ij}(t) \int_{-\infty}^{x} \frac{\psi_i(t, s)}{V_i(t)} - \frac{\psi_j(t, s)}{V_j(t)} \, ds, \qquad i, j \in \mathbb{Z}$$

form a locally finite family of x–differentiable functions which satisfy $|\Pi_{ij}(t, x)| \leq \sup_{i,j \in \mathbb{Z}} |\beta_{ij}(t)|$. Moreover $\Pi_{ij}(t, x) \neq 0$ implies $i, j \in I_\Psi(t, x, D)$ where $D = 3S + B$ is related to the maximal diameter S of the supports of ψ_i as well as the constant B characterizing the indices i, j for which $\beta_{ij} = 0$. Based on Π_{ij}, another locally finite family of functions

$$H_i(t, x) := \sum_{j \in \mathbb{Z}} \Pi_{ij}(t, x)$$

can be defined which is a partition of unity

$$\sum_{i \in \mathbb{Z}} H_i(t, x) = 1 \qquad \forall t \geq 0, x \in \mathbb{R}.$$

Each H_i satisfies a bound

$$|H_i(t,x)| \leq \sup_{i,j\in\mathbb{Z}} |\beta_{ij}(t)|\,|I_\Psi(t,x,D)|.$$

Proof According to the definition, Π_{ij} is a function with compact support in the convex hull of the supports of ψ_i and ψ_j. Indeed, if we denote this convex hull by $[a,b]$, we see that for $x \leq a$ the integrand is identically zero and for $x \geq b$, we have

$$\Pi_{ij}(t,x) = \frac{1}{2}\beta_{ij}\left(\int_{-\infty}^{\infty} \frac{\psi_i(t,s)}{V_i(t)}\,ds - \int_{-\infty}^{\infty} \frac{\psi_j(t,s)}{V_j(t)}\,ds\right) = 0.$$

Moreover, we have the bound $|\Pi_{ij}| \leq |\beta_{ij}|/2$ since

$$-1 \leq -\int_{-\infty}^{x} \frac{\psi_j}{V_j}\,ds \int_{-\infty}^{x} \frac{\psi_i}{V_i} - \frac{\psi_j}{V_j}\,ds \leq \int_{-\infty}^{x} \frac{\psi_i}{V_i}\,ds \leq 1.$$

Since x_i is contained in $\operatorname{supp}\psi_i$, we have $x_i, x_j \in [a,b]$ and since the support of ψ_i has a diameter less than S, we find (with $x_i \leq x_j$) that $[a,b] \subset [x_i - S, x_j + S]$. Since β_{ij} is different from zero only for $|x_i - x_j| \leq B$, the condition $\Pi_{ij}(t,x) \neq 0$ implies that $|b-a| < 2S+B$ as well as $x \in [a,b] \subset x + [-(2S+B), 2S+B]$. In other words, $i,j \in I_\Psi(t,x,D)$ with $D = 3S+B$. To show that $\mathcal{L} := \{\Pi_{ij} : i,j \in \mathbb{Z}\}$ is locally finite, we take $T > 0$ and $K \subset \mathbb{R}$ compact. For a ball $\hat{K}$ which contains K in such a way that the boundaries have at least a distance D, we know that $I_\Psi([0,T],\hat{K})$ is finite and if $i \notin I_\Psi([0,T],\hat{K})$ then the support of $\Pi_{ij}(t)$ does not intersect K for any $j \in \mathbb{Z}$ and $t \in [0,T]$. Hence, $|I_\mathcal{L}([0,T],K)|$ can be estimated by $|I_\Psi([0,T],\hat{K})| \sup_{t\geq 0} \sup_{x\in\mathbb{R}} |I_\Psi(t,x,D)|$. Based on the locally finite family $\mathcal{L}$, we now introduce $\mathcal{H} := \{H_i : i \in \mathbb{Z}\}$ according to

$$H_i(t,x) := \sum_{j\in\mathbb{Z}} \Pi_{ij}(t,x), \qquad i \in \mathbb{Z}.$$

With the same argument as above, one can show that

$$|I_\mathcal{H}([0,T],K)| \leq |I_\Psi([0,T],\hat{K})|$$

so that $\mathcal{H}$ is also locally finite. Moreover, each H_i satisfies the bound $|H_i(t,x)| \leq \sup_{ij} |\beta_{ij}|\ |I_\Psi(t,x,D)|$ since the sum involves at every point at most $|I_\Psi(t,x,D)|$ many terms. For the space derivative of the sum of all H_i, we find

$$\frac{\partial}{\partial x}\sum_{i\in\mathbb{Z}} H_i = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \frac{1}{2}\beta_{ij}\left(\frac{\psi_i}{V_i} - \frac{\psi_j}{V_j}\right) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \beta_{ij}\frac{\psi_i}{V_i}.$$

Since, by assumption, $\sum_{j\in\mathbb{Z}} \beta_{ij} = 0$, we conclude that $\sum_{i\in\mathbb{Z}} H_i$ is a constant $c \in \mathbb{R}$. To determine c, we need the assumption on the geometric coefficients that $\sum_{x_i\geq\bar{x}}\sum_{x_j\geq\bar{x}} \beta_{ij} = 1$ for some $\bar{x} \in \mathbb{R}$. The idea is to pick a test function $\phi \in C_0^\infty(\mathbb{R})$ which satisfies $0 \leq \phi \leq 1$, $\langle 1, \phi\rangle = 1$ and which is supported sufficiently far to the right of $\bar{x}$, say $a := \inf\operatorname{supp}\phi > \bar{x} + 2D$. We then have

$$c = \left\langle \sum_{i\in\mathbb{Z}} H_i, \phi \right\rangle = \left\langle \sum_{x_i\geq\bar{x}} H_i, \phi \right\rangle$$

because for $x_i < \bar{x}$, the support of H_i is disjoint with the one of ϕ. Introducing the function

$$\Phi(x) = \int_x^\infty \phi(s)\,ds, \qquad x \in \mathbb{R}$$

we conclude that $\Phi' = -\phi$ and $\langle 1, \phi\rangle = 1$ implies that $\Phi(x) = 1$ for $x \in (-\infty, a]$. Integration by parts yields

$$c = \left\langle \sum_{x_i \geq \bar{x}} H_i', \Phi \right\rangle = \left\langle \sum_{x_i \geq \bar{x}} \sum_{j \in \mathbb{Z}} \frac{\partial \Pi_{ij}}{\partial x}, \Phi \right\rangle.$$

Note that for index pairs (i, j) with $x_i \geq \bar{x}$, $x_j < \bar{x}$ and $\beta_{ij} \neq 0$, the function Π_{ij} must be supported close to $\bar{x}$ so that, by construction, it is supported in $x \in (-\infty, a]$ where $\Phi = 1$. Hence, the corresponding integrals $\langle \partial_x \Pi_{ij}, \Phi\rangle$ vanish and

$$c = \left\langle \sum_{x_i \geq \bar{x}} \sum_{x_j \geq \bar{x}} \frac{1}{2}\beta_{ij} \left(\frac{\psi_i}{V_i} - \frac{\psi_j}{V_j}\right), \Phi \right\rangle = \left\langle \sum_{x_i \geq \bar{x}} \sum_{x_j \geq \bar{x}} \beta_{ij} \frac{\psi_i}{V_i}, \Phi \right\rangle.$$

Finally, our construction assures that for indices i with $\langle \psi_i, \Phi\rangle < V_i$, x_i is close to the support of ϕ and thus sufficiently far from $\bar{x}$ to assure that all $j \in \mathbb{Z}$ with $\beta_{ij} \neq 0$ satisfy $x_j \geq \bar{x}$. Using $\sum_{j \in \mathbb{Z}} \beta_{ij} = 0$, we get

$$\sum_{x_j \geq \bar{x}} \beta_{ij} = 0 = \sum_{x_j \geq \bar{x}} \beta_{ij} \frac{\langle \psi_i, \Phi\rangle}{V_i}$$

For indices i with $\langle \psi_i, \Phi\rangle = V_i$, we also have

$$\sum_{x_j \geq \bar{x}} \beta_{ij} = \sum_{x_j \geq \bar{x}} \beta_{ij} \frac{\langle \psi_i, \Phi\rangle}{V_i}$$

and hence

$$c = \sum_{x_i \geq \bar{x}} \sum_{x_j \geq \bar{x}} \beta_{ij} = 1.$$

Now, we can conclude the proof of Theorem 1 by showing the convergence of the flux terms in (10).

Lemma 6 *With the assumptions of Theorem 1, we find for any $\phi \in C_0^1(\mathbb{R}^+ \times \mathbb{R})$*

$$\int_0^\infty \left\langle - \sum_{i,j \in \mathbb{Z}} |\beta_{ij}^{(h)}| g_{ij}^{(h)} \frac{\psi_i^{(h)}}{V_i^{(h)}}, \phi \right\rangle dt \xrightarrow{h \to 0} \int_0^\infty \left\langle f(u) - au, \frac{\partial \phi}{\partial x} \right\rangle dt.$$

Proof We first note that the double sum is actually finite since both indices can be restricted to $I_{\Psi_h}([0,T], K)$ for $T > 0$ and $K \subset \mathbb{R}$ sufficiently large so that $\operatorname{supp}\phi$ is well contained in the compact set $[0, T] \times K$.

We then exploit the relation $n_{ji} = \operatorname{sign}(\beta_{ji}) = -\operatorname{sign}(\beta_{ij}) = -n_{ij}$ together with the conservativity of g which leads to $|\beta_{ij}|g_{ij} = -|\beta_{ji}|g_{ji}$. This allows us to write

$$\sum_{i,j\in\mathbb{Z}} |\beta_{ij}|g_{ij}\frac{\psi_i}{V_i} = \sum_{i,j\in\mathbb{Z}} \frac{1}{2}|\beta_{ij}|g_{ij}\left(\frac{\psi_i}{V_i} - \frac{\psi_j}{V_j}\right).$$

According to Lemma 5, we have

$$\frac{1}{2}\beta_{ij}\left(\frac{\psi_i}{V_i} - \frac{\psi_j}{V_j}\right) = \frac{\partial \Pi_{ij}}{\partial x}$$

so that with integration by parts

$$\int_0^\infty \left\langle -\sum_{i,j\in\mathbb{Z}} |\beta_{ij}|g_{ij}\frac{\psi_i}{V_i}, \phi\right\rangle dt = \int_0^\infty \left\langle \sum_{i,j\in\mathbb{Z}} g_{ij}n_{ij}\Pi_{ij}, \frac{\partial\phi}{\partial x}\right\rangle dt$$

Using consistency of g, we write

$$g_{ij}n_{ij} = G_i + R_{ij}, \qquad G_i(t) = G(t, x_i(t), u_i(t)). \tag{13}$$

The remainder $R_{ij} = g(t, x_i, u_i, x_i, u_i, n_{ij})n_{ij} - g_{ij}n_{ij}$ can be estimated with the help of Lipschitz continuity of g

$$|R_{ij}| \leq L(|u_i - u_j| + |x_i - x_j|).$$

Defining

$$R(t,x) = \sum_{i,j\in\mathbb{Z}} R_{ij}(t)\Pi_{ij}(t,x)\frac{\partial\phi}{\partial x}(t,x)$$

and using the fact that $\Pi_{ij}(t,x) \neq 0$ only if $i,j \in I_{\Psi_h}(t,x,D_{\Psi_*}h)$ with $D_{\Psi_*} = 3S_{\Psi_*} + B_{\Psi_*}$, we have the estimate

$$\begin{aligned}|R(t,x)| \leq \sum_{i,j\in I_{\Psi_h}(t,x,D_{\Psi_*}h)} \sup_{i,j\in\mathbb{Z}} |\beta_{ij}(t)|L(|u_i(t) - u_j(t)| \\ + |x_i(t) - x_j(t)|)|\partial_x\phi(t,x)|\end{aligned}$$

Due to the uniform bound on $|\beta_{ij}|$, u_i and u as well as the estimate $|x_i - x_j| \leq 2D_{\Psi_*}h$ and the assumed Ψ_*–convergence, we conclude

$$\int_0^\infty \langle R, 1\rangle\, dt \xrightarrow{h\to 0} 0.$$

In view of (13), it remains to show that

$$\int_0^\infty \left\langle \sum_{i\in\mathbb{Z}} G_iH_i, \frac{\partial\phi}{\partial x}\right\rangle dt \xrightarrow{h\to 0} \int_0^\infty \left\langle G, \frac{\partial\phi}{\partial x}\right\rangle dt \tag{14}$$

where $G = G(t, x, u(t,x))$ and H_i is defined as in Lemma 5. Using the fact that H_i is a partition of unity, (14) reduces to the condition

$$\int_0^\infty \left\langle \sum_{i\in\mathbb{Z}} (G_i - G)H_i, \frac{\partial\phi}{\partial x}\right\rangle dt \xrightarrow{h\to 0} 0. \tag{15}$$

The Lipschitz continuity of G leads to the estimate

$$|G_i - G| \leq L(|u_i - u| + |x_i - x|).$$

Since $H_i(t,x) \neq 0$ implies $i \in I_{\Psi_h}(t,x,D_{\Psi_*}h)$

$$\sum_{i\in\mathbb{Z}} L(|u_i - u| + |x_i - x|)|H_i \partial_x \phi(t,x)|$$
$$\leq L|I_{\Psi_h}(t,x,D_{\Psi_*}h)| \max_{i\in I_{\Psi_h}(t,x,D_{\Psi_*}h)} (|u_i - u| + |x_i - x|)|H_i \partial_x \phi(t,x)|$$

so that (15) follows with the uniform bounds on u_i, $|H_i|$, $|I_{\Psi_h}(t,x,D_{\Psi_*}h)|$ and the assumed convergence of u_i.

In view of (10), Lemma 4 and Lemma 6 show that the assumption of Lemma 3 is satisfied which completes the proof of Theorem 1.

4 The finite volume particle method

Based on a given set of particle positions x_i which move according to $\dot{x}_i = a(t,x_i)$, we construct a partition of unity $\{\psi_i\}$ and geometric coefficients β_{ij}. Together with (4) this defines the finite volume particle method (FVPM). For suitable sequences of particle positions and the associated partitions and coefficients we then check the conditions of Section 2. Applying Theorem 1, we conclude that FVPM is consistent to (1).

4.1 Point clouds

In order to obtain reasonable approximation properties with a cloud of points $C = \{\, x_i \in \mathbb{R} : i \in \mathbb{Z} \,\}$, it is clear that some regularity of C has to be assumed. To quantify gaps in the cloud C, we introduce the functional

$$\gamma(C) := \sup_{x\in\mathbb{R}} \inf_{p\in C} |x - p|$$

and to control the clustering of points, we use

$$N(r,C) := \sup_{x\in\mathbb{R}} |\{p \in C : |x - p| < r\,\}| \qquad r > 0$$

where $|\cdot|$ is the counting measure. Obviously, $N(r,C)$ is the largest number of points $p \in C$ in an interval of radius r around any $x \in \mathbb{R}$.

Definition 4 *The set $C = \{\, x_i \in \mathbb{R} : i \in \mathbb{Z} \,\}$ is called a regular point cloud, if $\gamma(C) < \infty$ and $N(r,C) < \infty$ for all $r > 0$.*

If we study families of point clouds we will assume a certain uniformity.

Definition 5 *A family $\{\, C_h : h > 0 \,\}$ is called uniformly regular if*

$$\sup_{h>0} \gamma(C_h/h) < \infty \qquad \text{and} \qquad \sup_{h>0} N(r, C_h/h) < \infty \quad \forall r > 0.$$

Note that $\gamma(C_h/h) < \infty$ assures that the maximal distance between neighboring points in C_h is of order h. Indeed, if we assume the points x_i of a cloud C to be ordered according to $x_i \leq x_{i+1}$, we can write

$$\gamma(C) = \frac{1}{2} \sup_{i \in \mathbb{Z}} |x_{i+1} - x_i| \tag{16}$$

so that $\gamma(C_h/h) \leq \alpha$ implies $|x_{i+1} - x_i| \leq 2\alpha h$ for all $i \in \mathbb{Z}$.

It is a simple matter to check that

$$N(rh, C_h) = N(r, C_h/h)$$

so that the second condition in Definition 4 assures that in an interval of radius rh the points from C_h cannot cluster in such a way that their number becomes infinite as $h \to 0$.

4.2 Moving point clouds

In the next step, we consider point clouds which move along a prescribed velocity field $a \in C^0(\mathbb{R}^+, C^1(\mathbb{R}))$. If $C = C(0)$ is the initial point configuration, we define $C(t) = X(t; C, 0)$, where $X(t; \bar{x}, \tau)$ is the solution at time t of the problem $\dot{x} = a(t, x)$, $x(\tau) = \bar{x}$.

Lemma 7 *Let $a \in C^0(\mathbb{R}^+, C^1(\mathbb{R}))$ and $C(0)$ be a regular cloud of points. Then, for any $T > 0$ there exists $K > 0$ such that*

$$|X(t; p, \tau) - X(t; q, \tau)| \leq K|p - q| \qquad \forall t, \tau \in [0, T],\ p, q \in \mathbb{R}.$$

For $0 \leq t \leq T$, the set $C(t)X(t; C(0), 0)$ is a regular cloud with

$$\gamma(C(t)) \leq K\gamma(C(0)) \quad \text{and} \qquad N(r, C(t)) \leq N(rK, C(0)) \quad \forall r > 0.$$

Proof Due to our smoothness assumptions on a, the flow map X is well defined and the constant K is obtained with a standard Gronwall estimate. Assuming that $x_i \leq x_{i+1}$ for all $x_i \in C(0)$, we note that the ordering is not changed in the evolution. According to (16), we have

$$\gamma(C(t)) = \frac{1}{2} \sup_{i \in \mathbb{Z}} |X(t; x_{i+1}, 0) - X(t; x_i, 0)| \leq K\gamma(C(0))$$

With a similar argument for the backward movement, we conclude with the relation $p = X(0; X(t; p, 0), t)$ that

$$|X(0; x, t) - p| = |X(0; x, t) - X(0; X(t; p, 0), t)| \leq K|x - X(t; p, 0)|.$$

Hence $|x - X(t; p, 0)| < r$ implies $|X(0; x, t) - p| < Kr$ so that $N(r, C(t)) \leq N(rK, C(0))$.

Mainly to avoid working with finite time intervals, we introduce the notion of fields with finite strain.

Definition 6
A function a is called field of finite strain if $a \in C^0(\mathbb{R}^+, C^1(\mathbb{R}))$ gives rise to a flow map X which satisfies for some $K > 0$

$$|X(t;p,\tau) - X(t;q,\tau)| \le K|p-q| \qquad \forall t,\tau \in \mathbb{R}^+,\, p,q \in \mathbb{R}$$

Corollary 1 *Let a be a field of finite strain and $\{\, C_h(0) : h > 0 \,\}$ a uniformly regular family of point clouds. Then $\{\, C_h(t) : h > 0 \,\}$ is uniformly regular and*

$$\sup_{t\in\mathbb{R}} \sup_{h>0} \gamma(C_h(t)/h) < \infty$$
$$\sup_{t\in\mathbb{R}} \sup_{h>0} N(r, C_h(t)/h) < \infty \quad \forall r > 0.$$

Proof We just note that the assumptions on the field a guarantee existence of solutions to $\dot{x} = a(t,x)$, $x(0) = \bar{x}$ for all times. The uniform regularity follows immediately from Lemma 7.

4.3 Construction of particles

To explain the construction of particles, we first restrict to the case of a single, non–moving point cloud C. Taking a Lipschitz continuous function $W : \mathbb{R} \mapsto \mathbb{R}^+$ which is strictly positive for $|x| \le \lambda = \gamma(C)$, say $W(x) \ge \sigma_{\min} > 0$, and which vanishes for $|x| \ge \kappa\lambda$ with some $\kappa > 1$, we define

$$\psi_i(x) = \frac{W_i(x)}{\sigma(x)}, \quad \sigma(x) = \sum_{i\in\mathbb{Z}} W_i(x), \quad W_i(x) = W(x - x_i), \quad i \in \mathbb{Z}.$$

In Fig. 2, this construction is visualized. The symbols on the x-axis indicate the particle positions. Around each position x_i, the function W_i is plotted.

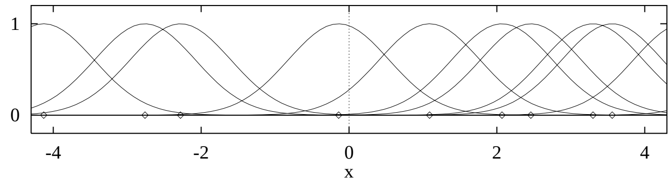

Fig. 2. Irregular particle positions x_i and functions W_i

The sum σ of all the functions W_i is shown in Fig. 3.

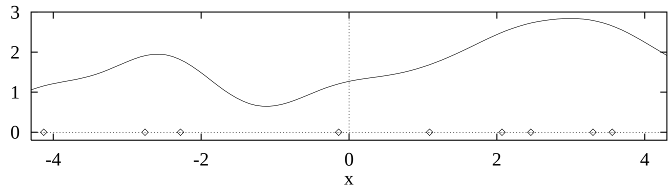

Fig. 3. The function σ corresponding to Fig. 2

After dividing W_i by σ, we get the partition functions ψ_i which, in contrast to W_i, may be non-symmetric and of different height (see Fig. 4).

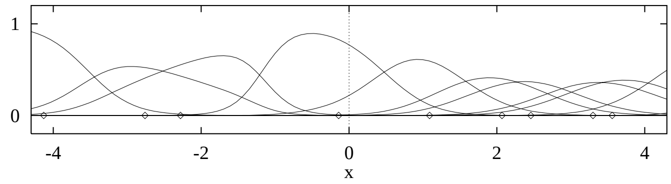

Fig. 4. The partition of unity corresponding to Fig. 2

We remark that the sum defining σ is finite at each $x \in \mathbb{R}$ because it involves only those points $x_i \in C$ with $|x_i - x| \leq \kappa\lambda$ which are at most $N(\kappa\lambda, C)$ many. We also know that $\sigma \geq \sigma_{\min}$ because the biggest gap in the particle cloud is of length $\lambda = \gamma(C)$ but the functions W_i are bigger than $\sigma_{\min}$ over intervals of at least that length. Hence the maximal possible gap is still covered by at least one of the W_i.

If the regular cloud is moving along a field of finite strain, the same construction is carried out with $\lambda = \sup_{t \geq 0} \gamma(C(t))$ and

$$\psi_i(t,x) = \frac{W_i(t,x)}{\sigma(t,x)}, \ \sigma(t,x) = \sum_{i \in \mathbb{Z}} W_i(t,x), \ W_i(t,x) = W(x - x_i(t)) \tag{17}$$

Finally, for a moving, uniformly regular family $\{ C_h : h > 0 \}$ we introduce particles using $\lambda = \sup_{h>0} \sup_{t \geq 0} \gamma(C_h(t)/h)$ and

$$W_i^{(h)}(t,x) = W\left(\frac{x - x_i(t)}{h}\right)$$

giving rise to $\psi_i^{(h)}$ and $\sigma^{(h)}$ as in (17).

Proposition 2 *Let $\Psi_* = \{ \Psi_h : 0 < h \leq 1 \}$ be a family of particle clouds $\Psi_h = \{ \psi_i^{(h)} : i \in \mathbb{Z} \}$ which are constructed based on a uniformly regular family of point clouds moving along a field of finite strain. Then, each Ψ_h is locally finite with*

$$\sup_{h>0} \sup_{t \geq 0} \sup_{x \in \mathbb{R}} |I_{\Psi_h}(t,x,rh)| < \infty \qquad \forall r > 0$$

and

$$\psi_i^{(h)} \in C^0(\mathbb{R}^+, W^{1,1}(\mathbb{R})) \cap C^1(\mathbb{R}^+, \mathbb{L}^1(\mathbb{R})).$$

Further, there exists $S_{\Psi_*} > 0$ such that

$$\operatorname{diam} \operatorname{supp} \psi_i^{(h)}(t) \leq S_{\Psi_*} h \qquad \forall i \in \mathbb{Z}, \, h > 0$$

and $\alpha_{\Psi_*} > 0$ such that

$$V_i^{(h)}(t) := \left\langle \psi_i^{(h)}(t), 1 \right\rangle \geq \alpha_{\Psi_*} h$$

for all $t \geq 0$ *and* $h > 0$*. The derivatives of* $\psi_i^{(h)}$ *are given by*

$$\frac{\partial \psi_i^{(h)}}{\partial x} = \sum_{j \in \mathbb{Z}} \left(\Gamma_{ji}^{(h)} - \Gamma_{ij}^{(h)} \right),$$

$$\frac{\partial \psi_i^{(h)}}{\partial t} = -\sum_{j \in \mathbb{Z}} (\dot{x}_i^{(h)} \Gamma_{ji}^{(h)} - \dot{x}_j^{(h)} \Gamma_{ij}^{(h)})$$

where

$$\Gamma_{ij}^{(h)} = \frac{\psi_i^{(h)}}{\sigma^{(h)}} \frac{\partial W_j^{(h)}}{\partial x}.$$

We also have

$$\frac{\partial \psi_i^{(h)}}{\partial t} + \dot{x}_i^{(h)} \frac{\partial \psi_i^{(h)}}{\partial x} = \nu_i^{(h)} \psi_i^{(h)}$$

where $\nu_i^{(h)} \in \mathbb{L}^\infty_{loc}(\mathbb{R}^+, \mathbb{L}^1(\mathbb{R}))$ *allows the estimate*

$$\sup_{0 \leq t \leq T} \|\nu_i^{(h)}(t)\|_{\mathbb{L}^1(\mathbb{R})} \leq C_{\Psi_*, T} h \qquad \forall h > 0.$$

Proof We assume that Ψ_h is based on a moving point cloud $C_h(t)$. Due to our construction of ψ_i (we suppress the superscript h), the support is contained in a ball of radius $\kappa\lambda h$, giving rise to $S_{\Psi_*} = 2\kappa\lambda$. Since the number of points in a ball of radius rh certainly dominates the number of particles which are completely contained in that ball, we also conclude that $|I_{\Psi_h}(t, x, rh)| \leq N(rh, C_h(t))$ which is uniformly bounded by assumption. Since for any compact $K \subset \mathbb{R}$, we can find $R > 0$ such that K is contained in a ball of radius Rh around the origin, we get at once

$$|I_{\Psi_h}(t, K)| \leq |I_{\Psi_h}(t, 0, Rh)| \leq N(Rh, C_h(t)) \leq N(\bar{K}Rh, C_h(0))$$

where the last inequality follows from Lemma 7.
Altogether, $|I_{\Psi_h}([0, T], K)|$ is uniformly bounded for any $T > 0$ which shows that Ψ_h (and thus also $\mathcal{W}_h := \{ W_i : i \in \mathbb{Z} \}$) is locally finite.

The estimate on the diameter of the supports also implies $I_{\Psi_h}(t, x) \subset I_{\Psi_h}(t, x, S_{\Psi_*} h)$, giving rise to a uniform bound

$$\sigma(t, x) \leq \sup_{t \geq 0} \sup_{x \in \mathbb{R}} |I_{\Psi_h}(t, x, S_{\Psi_*} h)| \max_{x \in \mathbb{R}} W(x) =: \sigma_{\max}.$$

Since $W(x/h) \geq \sigma_{\min}$ on $|x| \leq \lambda h$, we obtain

$$\psi_i(t, x) \geq \frac{\sigma_{\min}}{\sigma_{\max}}, \qquad |x| \leq \lambda h.$$

Consequently, the volume can be estimated from below

$$V_i \geq \int_{|x| \leq \lambda h} \psi_i \, dx \geq \frac{\sigma_{\min}}{\sigma_{\max}} 2\lambda h$$

and we set $\alpha_{\Psi_*} = 2\lambda \sigma_{\min}/\sigma_{\max}$.

To show smoothness properties of ψ_i, we continue with the observation that the support of $W_i(t)$ stays in a compact set $K \subset \mathbb{R}$ if t varies in a compact interval $[0,T]$. Hence, we can replace σ by

$$\tilde{\sigma}(t,x) = \sum_{i\in\tilde{I}} W_i(t,x), \qquad \tilde{I} = I_{\mathcal{W}_h}([0,T],K)$$

which is only a finite sum. Since Lipschitz continuity implies differentiability almost everywhere, we immediately get

$$\frac{\partial \psi_i}{\partial x} = \frac{1}{\tilde{\sigma}}\frac{\partial W_i}{\partial x} - \frac{W_i}{\tilde{\sigma}}\frac{1}{\tilde{\sigma}}\sum_{k\in\tilde{I}}\frac{\partial W_k}{\partial x}. \tag{18}$$

At this point, we remark that $\psi_i \in C^0(\mathbb{R}^+, W^{1,1}(\mathbb{R}))$. Indeed, a small change in time leads to a little translation of the participating functions W_k which is a continuous operation in $\mathbb{L}^1(\mathbb{R})$.

Multiplying equation (18) by ψ_j, summing over all j and replacing $\tilde{\sigma}$ again by σ, we arrive at

$$\frac{\partial \psi_i}{\partial x} = \frac{1}{\sigma}\sum_{j\in\mathbb{Z}}\left(\psi_j\frac{\partial W_i}{\partial x} - \psi_i\frac{\partial W_j}{\partial x}\right). \tag{19}$$

Using the fact that $\partial_t W_i = -\dot{x}_i \partial_x W_i$, we obtain in an analogous way

$$\frac{\partial \psi_i}{\partial t} = -\frac{1}{\sigma}\sum_{j\in\mathbb{Z}}\left(\dot{x}_i\psi_j\frac{\partial W_i}{\partial x} - \dot{x}_j\psi_i\frac{\partial W_j}{\partial x}\right). \tag{20}$$

Considering W_i as an $\mathbb{L}^1(\mathbb{R})$ valued function on $\mathbb{R}^+$, we have $W_i \in C^1(\mathbb{R}^+, \mathbb{L}^1(\mathbb{R}))$ where continuity of the first derivative is again due to the continuity of the translation operator in $\mathbb{L}^1(\mathbb{R})$. A straight forward estimate of difference quotients shows that also $\psi_i \in C^1(\mathbb{R}^+, \mathbb{L}^1(\mathbb{R}))$. Combining (19) and (20), we end up with

$$\frac{\partial \psi_i}{\partial t} + \dot{x}_i\frac{\partial \psi_i}{\partial x} = \left(\frac{1}{\sigma}\sum_{j\in\mathbb{Z}}(\dot{x}_j - \dot{x}_i)\frac{\partial W_j}{\partial x}\right)\psi_i. \tag{21}$$

Introducing

$$\nu_i = \left(\frac{1}{\sigma}\sum_{j\in\mathbb{Z}}(\dot{x}_j - \dot{x}_i)\frac{\partial W_j}{\partial x}\right)\zeta_i$$

with ζ_i being the indicator function of a ball of radius $\kappa\lambda h$ around $x_i(t)$, the right hand side in (21) can also be written as $\nu_i\psi_i$. To estimate the $\mathbb{L}^1$–norm of $\nu_i(t)$, we note that $\sigma \geq \sigma_{\min}$ and, since the velocity field $a(t)$ is uniformly bounded in $C^1(\mathbb{R})$ if t ranges in a compact interval, we get $|\dot{x}_j - \dot{x}_i| \leq L(t)|x_j - x_i|$. Note that this estimate is only needed if $\zeta_i\partial_x W_j \neq 0$ which may happen if $|x_j - x_i| \leq 2\kappa\lambda h$ (otherwise the supports are disjoint). The number of involved points x_j is estimated by $N(2\kappa\lambda h, C_h(t)) \leq N(2K\kappa\lambda h, C_h(0))$ according to Lemma 7 so that

$$\|\nu_i(t)\|_{\mathbb{L}^1(\mathbb{R}))} \leq L(t)N(2K\kappa\lambda h, C_h(0)) \left\| \frac{\partial W(\cdot/h)}{\partial x} \right\|_{\mathbb{L}^1(\mathbb{R})} 2\kappa\lambda h.$$

With the change of variables $y = x/h$, we find that $\|\partial_x W(\cdot/h)\|_{\mathbb{L}^1(\mathbb{R})} = \|\partial_x W\|_{\mathbb{L}^1(\mathbb{R})}$ for all $h > 0$ so that the result follows with

$$C_{\Psi_*}(T) := \sup_{0\leq t\leq T} \sup_{h>0} 2\kappa\lambda L(t)N(2K\kappa\lambda h, C_h(0)).$$

4.4 Geometric coefficients

Motivated by the derivation of the method in Section 1, we define the coefficients

$$\beta_{ij}^{(h)}(t) = \left\langle \Gamma_{ij}^{(h)}(t) - \Gamma_{ji}^{(h)}(t), 1 \right\rangle \tag{22}$$

where $\Gamma_{ij}^{(h)}$ are taken from Proposition 2

$$\Gamma_{ij}^{(h)} = \frac{\psi_i^{(h)}}{\sigma^{(h)}} \frac{\partial W_j^{(h)}}{\partial x}$$

Proposition 3 *The coefficients $\beta_{ij}^{(h)}$ are uniformly bounded and satisfy $\beta_{ij}^{(h)} = -\beta_{ji}^{(h)}$ as well as $\sum_{j\in\mathbb{Z}} \beta_{ij}^{(h)} = 0$ for all $i \in \mathbb{Z}$. There exists a constant $B > 0$ such that $|x_i^{(h)}(t) - x_j^{(h)}(t)| \geq Bh$ implies $\beta_{ij}^{(h)}(t) = 0$. Finally, for every $\bar{x} \in \mathbb{R}$, we have*

$$\sum_{x_i^{(h)}(t)\geq\bar{x}} \sum_{x_j^{(h)}(t)\geq\bar{x}} \beta_{ij}^{(h)}(t) = 1 \qquad \forall t \geq 0,\, h > 0.$$

Proof We again suppress the superscript h in the proof. ¿From the definition (22) of β_{ij}, the skew symmetry follows at once.
Since $|\psi_i/\sigma| \leq 1/\sigma_{\min}$, we find

$$\int_{\mathbb{R}} |\Gamma_{ij}(t,x)|\, dx \leq \frac{1}{\sigma_{\min}} \int_{\mathbb{R}} \left| \frac{\partial W(x/h)}{\partial x} \right| dx = \frac{1}{\sigma_{\min}} \int_{\mathbb{R}} \left| \frac{\partial W}{\partial x}(y) \right| dy$$

which is a uniform bound giving rise to $|\beta_{ij}^{(h)}(t)| \leq 2\|\partial_x W\|_{\mathbb{L}^1(\mathbb{R})}/\sigma_{\min}$. Taking into account that $\operatorname{diam}\operatorname{supp}\psi_i \leq S_{\Psi_*}h$, we conclude that for $|x_i(t) - x_j(t)| \geq 2S_{\Psi_*}h$, the supports of W_j and ψ_i are disjoint and hence $\beta_{ij} = 0$. The remaining two properties are shown based on a useful reformulation of the formula for β_{ij}

$$\beta_{ij} = 2\left\langle \psi_i, \frac{\partial \psi_j}{\partial x} \right\rangle. \tag{23}$$

Equation (23) follows immediately from

$$\frac{\partial \psi_j}{\partial x} = \frac{1}{\sigma}\frac{\partial W_j}{\partial x} - \psi_j \frac{1}{\sigma}\frac{\partial \sigma}{\partial x}$$

so that

$$\langle \Gamma_{ij}, 1\rangle = \left\langle \psi_i, \frac{\partial \psi_j}{\partial x}\right\rangle + \left\langle \psi_i\psi_j, \left(\frac{\partial \sigma}{\partial x}\right)/\sigma\right\rangle$$

and

$$\beta_{ij} = \langle \Gamma_{ij} - \Gamma_{ji}, 1\rangle = \left\langle \psi_i \frac{\partial \psi_j}{\partial x} - \psi_j \frac{\partial \psi_i}{\partial x}, 1\right\rangle = 2\left\langle \psi_i, \frac{\partial \psi_j}{\partial x}\right\rangle.$$

It implies that

$$\sum_{j\in\mathbb{Z}} \beta_{ij} = 2\left\langle \psi_i, \frac{\partial}{\partial x}\sum_{j\in\mathbb{Z}} \psi_j\right\rangle = 0$$

and with $\Psi_{\bar{x}}(t,x) := \sum_{x_i \geq \bar{x}} \psi_i(t,x)$,

$$\sum_{x_i\geq\bar{x}}\sum_{x_j\geq\bar{x}} \beta_{ij} = 2\left\langle \Psi_{\bar{x}} \frac{\partial}{\partial x}\Psi_{\bar{x}}, 1\right\rangle = \left\langle \frac{\partial}{\partial x}\Psi_{\bar{x}}^2, 1\right\rangle = \Psi_{\bar{x}}^2\Big|_{x=-\infty}^{x=\infty}.$$

Note that $\Psi_{\bar{x}}(t,x) = 0$ for $x \to -\infty$ since all $\psi_i(t,x)$ with $x_i \geq \bar{x}$ vanish for $x < \bar{x} - Bh$. On the other hand, for $x > \bar{x} + Bh$, the function $\Psi_{\bar{x}}(t,x)$ coincides with $\sum_{i\in\mathbb{Z}} \psi_i(t,x) = 1$, so that $\Psi_{\bar{x}}^2\big|_{x=-\infty}^{x=\infty} = 1$.

5 Conclusion

We have presented a consistency result for a general class of conservative, meshfree methods based on partitions of unity. Apart from the partition and a standard numerical flux function, the schemes are characterized by the parameters β_{ij} which contain geometrical information about relative position of particles and the amount of overlap. For example, in classical finite volume methods, which are recovered in the approach for a special choice of the partition of unity, the coefficients β_{ij} are related to the surface area of the cell faces (in the multi-dimensional case) and the corresponding normal directions. In the finite volume particle method (FVPM), which can be viewed as a generalization of classical finite volume methods to the case of overlapping and moving grid cells, the coefficients are calculated based on the partition functions according to $\beta_{ij} = 2\langle \psi_i, \partial_x \psi_j\rangle$. Since the proof of the consistency result requires only little regularity of the partition of unity functions and is mainly based on some general assumptions on the coefficients β_{ij}, it applies at the same time to FVPM and standard finite volume methods.

The advantage of FVPM to work for general distributions of particle positions and overlapping partition functions has to be paid with the calculation of V_i and β_{ij} which involve integration over ψ_i and $\partial_x \psi_j$. The goal is to discretize the integrals in such a way that the evaluation becomes fast without violating the consistency conditions presented here (note that for consistency, the specific form $\beta_{ij} = 2\langle \psi_i, \partial_x \psi_j\rangle$ is not necessary). Since additional restrictions on β_{ij} may arise from stability considerations, a convergence analysis is naturally the next step in the investigation of the method. Apart from that, the treatment of bounded domains is most important, because the main applications of particle methods will be in complicated and time depending geometries.

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