On the minimal energy state of a mixture of charged classical and quantum fluids

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Abstract

The paper concerns the equilibrium state of ultra small semiconductor devices. Due to the quantum drift diffusion model, electrons and holes behave as a mixture of charged quantum fluids. Typically the involved scaled Planck's constants of holes, ξ, is significantly smaller than the scaled Planck's constant of electrons. By setting formally $\xi = 0$ a well-posed differential-algebraic system arises. Existence and uniqueness of an equilibrium solution is proved. A rigorous asymptotic analysis shows that this equilibrium solution is the limit (in a rather strong sense) of quantum systems as $\xi \to 0$. In particular the ground state energies of the quantum systems converge to the ground state energy of the differential-algebraic system as $\xi \to 0$.

Key words. thermal equilibrium state; charged fluids; mixture of quantum fluids and classical fluids; semi-classical limits.

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1 Introduction

Quantum Drift Diffusion models (QDDs) [3, 6] give a rather accurate account of the macroscopic behavior of ultra small semiconductor devices in terms of particle densities, current densities and electric fields. In particular, QDDs allow for the modelization of several quantum effects on macroscopic scales [7]. These features of QDDs encourage investigations of several scaling limits which arise in practical situations [8, 9].

The scaling limits investigated so far are settled on the assumption that both charge carriers - electrons and holes - exhibit quantum properties. However, the effective masses of electrons and holes may differ significantly [2]. As a consequence, the scaled Planck's constants of electrons and holes are in several situations of different orders of magnitude.

In particular the case "scaled Planck's constant for holes" $\ll$ "scaled Planck's constant for electrons" is of distinguished importance [2]. In this situation one has to expect that holes behave as a classical charged fluid while electrons exhibit quantum phenomena.

It is the aim of this paper to discuss the thermal equilibrium state of a bipolar charged quantum fluid when the scaled Planck's constant of one of the two components is significantly smaller than the other.

To fix ideas let us give the precise definition of the problem.

We consider the QDD of a bipolar charged quantum fluid confined to Ω where we assume

(A1) $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or $d = 3$ is a non void, bounded domain.
$H^1(\Omega)$ is continuously embedded in $L^6(\Omega)$ and $H^1(\Omega)$ is compactly embedded in $L^4(\Omega)$.

Remark 1. *Since $d \leq 3$, the embeddings of $H^1(\Omega)$ hold, e.g., for domains Ω with Lipschitz boundary [1].*

The term "bipolar" refers to two particle types, namely negatively charged "electrons" and positively charged "holes". The particle density of electrons is $n_\xi = n_\xi(x) \geq 0$ and the particle density of holes is $p_\xi = p_\xi(x) \geq 0$, where x ranges in Ω.

The thermal equilibrium state of the system is governed by the semi linear elliptic system [2, 8]

$$\varepsilon^2 \Delta \sqrt{n_\xi} = \sqrt{n_\xi}\,(\log(n_\xi) + V_\xi + \alpha_\xi) \tag{1}$$

$$\varepsilon^2 \xi \Delta \sqrt{p_\xi} = \sqrt{p_\xi}\,(\log(p_\xi) - V_\xi + \beta_\xi) \tag{2}$$

$$-\lambda \Delta V_\xi = n_\xi - p_\xi - C, \tag{3}$$

where $V_\xi = V_\xi(x)$ is the electrostatic potential, the real numbers $\varepsilon > 0$ and $\varepsilon\xi > 0$ are the scaled Planck's constants of electrons and holes, respectively, $\lambda > 0$ is the scaled minimal Debye length and the function

(A2) $C \in L^\infty(\Omega)$

represents a fixed background ion distribution.

System (1)-(3) is supplied with the constraints

$$\int_\Omega n_\xi(x)\,dx = N > 0, \tag{4}$$

$$\int_\Omega p_\xi(x)\,dx = P > 0, \tag{5}$$

$$\int_\Omega V_\xi(x)\,dx = 0, \tag{6}$$

and

$$V_\xi \text{ satisfies homogeneous Neumann boundary conditions,} \tag{7}$$

i.e. we have to impose global charge neutrality

(A3) $N - P = \int_\Omega C(x)\,dx$.

In (1) - (7) the functions n_ξ, p_ξ, V_ξ and the real numbers α_ξ, β_ξ are unknown.

Since our main conclusions do not depend on the particular choice of ε, λ we set

$$\varepsilon = \lambda = 1$$

henceforth.

Following [8] the analysis of (1)-(7) proceeds by introducing the corresponding energy functional J_ξ which is constructed in several steps.

J_ξ is the sum of a ξ-dependent quantum energy term and ξ-independent enthalpy and electrostatic energy terms.

In the general context considered here we have to take a closer look at the electrostatic energy E_∞. We introduce the sets

$$\mathcal{C}_0 := \left\{ \phi \in H^1(\Omega) : \int_\Omega \phi\,dx = 0 \right\},$$

and

$$\mathcal{C}_{el} := \left\{ f \in L^1(\Omega) : \right.$$
$$\left. \exists V \in \mathcal{C}_0 : \left(\forall \phi \in \mathcal{C}_0 \cap L^\infty(\Omega) : \int_\Omega \nabla V \cdot \nabla \phi\,dx = \int_\Omega f \phi\,dx \right) \right\}.$$

It is easy to see: for each $f \in \mathcal{C}_{el}$ there is exactly one $V \in \mathcal{C}_0$ such that $\int_\Omega \nabla V \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx$ for all $\phi \in \mathcal{C}_0 \cap L^\infty(\Omega)$.

We denote this $V \in \mathcal{C}_0$ by $V[f]$ henceforth.

Remark 2. *a) If $f \in \mathcal{C}_{el}$, then $V = V[f]$ is the unique weak solution of*

$$-\Delta V = f - [f], \quad V \in \mathcal{C}_0,$$

where

$$[f] := \frac{1}{\text{meas}(\Omega)} \int_\Omega f \, dx,$$

subject to homogeneous Neumann boundary conditions, i.e.

$$\int_\Omega \nabla V \cdot \nabla \phi \, dx = \int_\Omega (f - [f]) \, \phi \, dx,$$

for all $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$.
b) In particular we have $L^2(\Omega) \subset \mathcal{C}_{el}$.
c) By a density argument one easily verifies

$$\forall h \in L^2(\Omega), \ \forall f \in \mathcal{C}_{el}: \quad \int_\Omega \nabla V[h] \cdot \nabla V[f] \, dx = \int_\Omega h \, V[f] \, dx.$$

d) The set $\mathcal{C}_{el}$ is not closed with respect to weak convergence in $L^1(\Omega)$.

We assume

(A4) There exists a $K_\infty > 0$ only depending on Ω such that

$$\|V[f]\|_{L^\infty(\Omega)} \le K_\infty \, \|f\|_{L^2(\Omega)}, \qquad \text{for all } f \in L^2(\Omega) \text{ with } \int_\Omega f \, dx = 0.$$

Remark 3. *Assumption* (A4) *is a requirement on the smoothness of $\partial\Omega$. For instance it is well known, see e.g. [4], that for $\partial\Omega \in C^\infty$ the estimate*

$$\|V[f]\|_{H^2(\Omega)} \le K \|f\|_{L^2(\Omega)}$$

holds. This estimate implies for domains Ω with Lipschitz boundary in dimensions $d \le 3$ assumption (A4)*, because in this situation the embedding $H^2(\Omega) \to C_B(\Omega)$ is continuous [1].*

Now we introduce the electrostatic energy functional

$$E_\infty : L^1(\Omega) \to \mathbb{R}_0^+ \cup \{\infty\}, \quad E_\infty(f) = \begin{cases} \frac{1}{2} \int |\nabla V[f]|^2 \, dx & , \ f \in \mathcal{C}_{el} \\ \infty & , \ f \notin \mathcal{C}_{el} \end{cases}$$

and the energy functional corresponding to (1)-(7),

$$J_\xi : \Gamma \to \mathbb{R}, \quad J_\xi(n,p) = \int_\Omega |\nabla \sqrt{n}|^2 \, dx + \xi \int_\Omega |\nabla \sqrt{p}|^2 \, dx$$
$$+ \int_\Omega n(\log(n) - 1) \, dx + \int_\Omega p(\log(p) - 1) \, dx + E_\infty(n - p - C),$$

where

$$\Gamma := \Big\{ (n,p) \in L^1(\Omega) \times L^1(\Omega) : \; n, p \geq 0, \; \sqrt{n}, \sqrt{p} \in H^1(\Omega),$$
$$\int_\Omega n \, dx = N, \; \int_\Omega p \, dx = P \Big\}.$$

Remark 4. *If* $(n,p) \in \Gamma$*, then* $\sqrt{n}, \sqrt{p} \in H^1(\Omega)$*. Hence* $n, p \in L^2(\Omega)$ *and therefore* $\Gamma \subset \mathcal{C}_{el} \times \mathcal{C}_{el}$*, i.e.* $E_\infty(n - p - C) < \infty$*. Furthermore, due to sub quadratic growth of* $t \log(t)$ *as* $t \to \infty$*, we have* $n \log(n), p \log(p) \in L^1(\Omega)$*. Hence* $J_\xi(n,p) < \infty$ *for all* $\xi \in (0, \infty)$*.*

Following [8] we have

Theorem 1. *[8] Assume* (A1)-(A4)*. Then* J_ξ *has for each* $\xi \in (0,\infty)$ *exactly one minimizer* $(n_\xi, p_\xi) \in \Gamma$*, i.e.*

$$J_\xi(n_\xi, p_\xi) = \min_\Gamma J_\xi.$$

Furthermore, for all $\xi \in (0, \infty)$*,*

1. *there are real numbers* α_ξ, β_ξ *such that the quintuple* $(n_\xi, p_\xi, V_\xi, \alpha_\xi, \beta_\xi)$ *- where* $V_\xi = V[n_\xi - p_\xi - C]$ *- is a solution of* (1)-(7)*,*
2. *there are real numbers* $m_\xi, M_\xi \in (0, \infty)$ *such that*

$$m_\xi \leq n_\xi, p_\xi \leq M_\xi, \quad |V_\xi| \leq M_\xi,$$

3. $\sqrt{n_\xi}, \sqrt{p_\xi}$ *satisfy homogeneous Neumann boundary conditions, i.e.*

$$\sqrt{n_\xi} = -V[\sqrt{n_\xi}\,(\log(n_\xi) + V_\xi + \alpha_\xi)], \; \sqrt{p_\xi} = -V[\sqrt{p_\xi}\,(\log(p_\xi) - V_\xi + \beta_\xi)].$$

Remark 5. *a) The real numbers* α_ξ, β_ξ *are the Lagrange multipliers of the minimization problem* $J_\xi \to \mathrm{Min}$ *on* Γ*.*
b) As a consequence of 1. and of 3. we have for each $\xi \in (0, \infty)$*,*

$$\int_\Omega \sqrt{n_\xi}\,(\log(n_\xi) + V_\xi + \alpha_\xi) \, dx = \int_\Omega \sqrt{p_\xi}\,(\log(p_\xi) - V_\xi + \beta_\xi) \, dx = 0.$$

In the sequel we shall be concerned with the limit $\xi \to 0$, in particular with the following questions.

a) Setting $\xi = 0$ in (1)-(7) we obtain

$$\Delta\sqrt{n_0} = \sqrt{n_0}\,(\log(n_0) + V_0 + \alpha_0) \quad (8)$$

$$0 = \log(p_0) - V_0 + \beta_0 \quad (9)$$

$$-\Delta V_0 = n_0 - p_0 - C, \quad (10)$$

$$\int_\Omega n_0(x)\, dx = N > 0, \quad (11)$$

$$\int_\Omega p_0(x)\, dx = P > 0, \quad (12)$$

$$\int_\Omega V_0(x)\, dx = 0, \quad (13)$$

and

$$V_0 \text{ satisfies homogeneous Neumann boundary conditions.} \quad (14)$$

The question is: Does this differential-algebraic system has an equilibrium solution $(n_0, p_0, V_0, \alpha_0, \beta_0)$, i.e. is there a minimizer (n_0, p_0) of the energy

$$J_0 : \Gamma_0 \to \mathbb{R}_0^+ \cup \{\infty\},$$
$$J_0(n,p) = \int_\Omega |\nabla\sqrt{n}|^2\, dx + \int_\Omega n(\log(n) - 1)\, dx + \int_\Omega p(\log(p) - 1)\, dx + E_\infty(n - p - C),$$

where

$$\Gamma_0 := \left\{ (n,p) \in L^1(\Omega) \times \mathcal{C}_{el} :\ n, p \geq 0,\ \sqrt{n} \in H^1(\Omega),\ \int_\Omega n\, dx = N,\ \int_\Omega p\, dx = P \right\},$$

such that $(n_0, p_0, V[n_0 - p_0 - C], \alpha_0, \beta_0)$ - where the real numbers α_0, β_0 are the Lagrange multipliers of minimizing J_0 in Γ_0 - is a solution of (8)-(14) ?

b) In which sense - if at all - do we have

$$n_\xi \to n_0,\ p_\xi \to p_0,\ V[n_\xi - p_\xi - C] \to V[n_0 - p_0 - C], \alpha_\xi \to \alpha_0,\ \beta_\xi \to \beta_0,$$

as $\xi \to 0$?

c) Do we have $\lim_{\xi\to 0}\min_{\Gamma} J_\xi = \min_{\Gamma_0} J_0$?

The paper is organized as follows.

In section 2 we investigate system (8)-(14). The main result is theorem 2 which is an analogon to theorem 1. In particular we prove existence and uniqueness of an equilibrium solution minimizing J_0 in Γ_0. In contrast to the proof of theorem 1 the derivation of a priori estimates for p_0 is a bit delicate. Since p_0 satisfies an algebraic equation no maximum principle can be applied. Instead, one has to use monotonicity operators for a semi linear elliptic operator acting on the potential V_0. As a consequence of theorem 2, question a) has an affirmative answer.

Section 3 addresses to the convergence of the minimizers of J_ξ to the minimizer of J_0 as $\xi \to 0$. In theorem 3 the corresponding results are collected. For the particle densities uniform pointwise bounds away from zero and away from ∞ are proved. This result extends theorem 1 where the corresponding constants are ξ-dependent. Questions b) and c) are given affirmative answers. Furthermore the convergence of and of $\min_\Gamma J_\xi$ to $\min_{\Gamma_0} J_0$ as $\xi \to 0$ is proved. This result is rather important for physical reasons: The ground state energy of the approximating quantum/classical system is the limiting value of the ground state energies of quantum/quantum systems as the scaled Planck's constant of the respective particle type tends to zero.

The proof of theorem 2 is deferred to the Appendix.

2 Variational Analysis of J_0

In this section we are concerned with the minimization of J_0 in Γ_0 and with the derivation of the corresponding Euler Lagrange equations.

The main result is

Theorem 2. *Assume* (A1)-(A4). *Then J_0 has exactly one minimizer (n_0, p_0) in Γ_0, i.e.*

$$J_0(n_0, p_0) = \min_{\Gamma_0} J_0.$$

Furthermore,

1. *there are real numbers α_0, β_0 such that the quintuple $(n_0, p_0, V_0, \alpha_0, \beta_0)$ - where $V_0 = V[n_0 - p_0 - C]$ - is a solution of* (8)-(14),
2. *there are real numbers $m_0, M_0 \in (0, \infty)$ such that*
$$m_0 \le n_0, p_0 \le M_0, \quad |V_0| \le M_0,$$
3. *$\sqrt{n_0}, \sqrt{p_0}$ satisfy homogeneous Neumann boundary conditions, i.e.*
$$\sqrt{n_0} = -V[\sqrt{n_0}\,(\log(n_0) + V_0 + \alpha_0)], \ \sqrt{p_0} = -V[\sqrt{p_0}\,(\log(p_0) - V_0 + \beta_0)].$$

4. $p_0 \in H^1(\Omega)$ *and* $\sqrt{p_0} \in H^1(\Omega)$.

The proof is deferred to the appendix.

Remark 6. *a) Theorem 2 settles question a) of the introduction.*
b) For the subsequent investigations of $\xi \to 0$ *the result* $\sqrt{p_0} \in H^1(\Omega)$ *will be most important.*

3 Approximation by Quantum Models

In this section we are concerned with the limiting behavior $\xi \to 0$ of the quintuple $(n_\xi, p_\xi, V_\xi, \alpha_\xi, \beta_\xi)$. Physically speaking the investigation addresses to the question whether the mixed quantum/classical state can be approximated by quantum/quantum states.

The main result is

Theorem 3. *Assume* (A1)-(A4). *Then*

1. *The value of the ground energy of* J_ξ *converges as* $\xi \to 0$ *to the value of the ground energy of* J_0, *i.e.*

$$\lim_{\xi\to 0}\left(\min_{\Gamma} J_\xi\right) = \min_{\Gamma_0} J_0,$$

and therefore

$$\lim_{\xi\to 0} J_\xi(n_\xi, p_\xi) = J_0(n_0, p_0).$$

2. *There are positive* m, M *such that*

$$\forall \xi \in [0,\infty): \quad m \le n_\xi, p_\xi \le M.$$

3. $V_\xi \to V_0 = V[n_0 - p_0 - C]$ *strongly in* $H^1(\Omega)$ *and strongly in* $L^\infty(\Omega)$ *as* $\xi \to 0$.

4. $n_\xi \to n_0$ *strongly in* $H^1(\Omega)$ *and strongly in* $L^\infty(\Omega)$ *as* $\xi \to 0$.

5. $p_\xi \to p_0$ *strongly in* $H^1(\Omega)$ *and strongly in* $L^p(\Omega)$, $p \in [1,\infty)$, *as* $\xi \to 0$. *Furthermore,* $p_\xi \rightharpoonup p_0$ *weak* in* $L^\infty(\Omega)$ *as* $\xi \to 0$.

6. $\lim_{\xi\to 0}\alpha_\xi = \alpha_0$ *and* $\lim_{\xi\to 0}\beta_\xi = \beta_0$.

Remark 7. *a) Theorem 3 settles questions b) and c) of the introduction.*
b) The estimates of theorem 3 2. are independent of ξ. *Thus the corresponding result of theorem 1 of [8] is improved.*

Proof. We observe $\Gamma \subseteq \Gamma_0$.
1. We have for all $\xi \in (0,\infty)$ and for all $(n,p) \in \Gamma$,

$$J_0(n,p) \le J_\xi(n,p), \quad \text{thus} \quad \min_{\Gamma_0} J_0 \le \inf_{\Gamma} J_0 \le \min_{\Gamma} J_\xi,$$

and therefore

$$\min_{\Gamma_0} J_0 \le \liminf_{\xi\to 0} \left(\min_{\Gamma} J_\xi\right). \tag{15}$$

On the other hand we have $(n_0,p_0) \in \Gamma$ due to theorem 2 4. Hence for all $\xi \in (0,\infty)$,

$$\min_{\Gamma} J_\xi \le J_\xi(n_0,p_0). \tag{16}$$

Since $\lim_{\xi\to 0} J_\xi(n_0,p_0) = J_0(n_0,p_0)$, we have

$$\limsup_{\xi\to 0} \left(\min_{\Gamma} J_\xi\right) \le J_0(n_0,p_0) = \min_{\Gamma_0} J_0, \tag{17}$$

and therefore, due to (15) and (17), $\lim_{\xi\to 0}(\min_\Gamma J_\xi) = \min_{\Gamma_0} J_0$.
2. We shall derive some a priori estimates. Positive constants which are independent of ξ are denoted by $K_1, K_2, K_3, \ldots$. For the sake of brevity we put for $t \ge 0$, $H(t) := t(\log(t)-1)$ (with $H(0)=0$). Then $H(t) \ge -1$ for all $t \ge 0$. We put $N_0 := \frac{N}{\mathrm{meas}(\Omega)}$ and $P_0 := \frac{P}{\mathrm{meas}(\Omega)}$. Then $(N_0,P_0) \in \Gamma \subseteq \Gamma_0$ and therefore, if $\xi \in (0,\infty)$ or $\xi = 0$,

$$\begin{aligned} 0 \le J_\xi(n_\xi,p_\xi) + 2\mathrm{meas}(\Omega) &= \int_\Omega |\nabla\sqrt{n_\xi}|^2\,dx + \xi\int_\Omega |\nabla\sqrt{p_\xi}|^2\,dx \\ &+ \int_\Omega (H(n_\xi)+1)\,dx + \int_\Omega (H(p_\xi)+1)\,dx + E_\infty(n_\xi - p_\xi - C) \\ &\le J_\xi(N_0,P_0) + 2\mathrm{meas}(\Omega) \\ &=: K_1 = \mathrm{meas}(\Omega)(H(N_0)+H(P_0)+2) + E_\infty(N_0 - P_0 - C). \end{aligned} \tag{18}$$

By non-negativity of the terms of the left-hand side we deduce

$$\forall \xi \in [0,\infty): \quad \|\sqrt{n_\xi}\|_{H^1(\Omega)} \le K_2, \quad \|V_\xi\|_{H^1(\Omega)} \le K_2, \tag{19}$$

where $V_\xi = V[n_\xi - p_\xi - C]$. Furthermore, we have due to $(n_0,p_0) \in \Gamma$ for all $\xi \in (0,\infty)$,

$$\begin{aligned} J_0(n_\xi,p_\xi) + \xi\int_\Omega |\nabla\sqrt{p_\xi}|^2\,dx &= J_\xi(n_\xi,p_\xi) \\ &\le J_\xi(n_0,p_0) = J_0(n_0,p_0) + \xi\int_\Omega |\nabla\sqrt{p_0}|^2\,dx, \end{aligned} \tag{20}$$

while due to $\Gamma \subseteq \Gamma_0$,

$$J_0(n_0, p_0) \leq J_0(n_\xi, p_\xi). \tag{21}$$

We deduce from (20) and from (21) the estimate

$$\forall \xi \in [0, \infty): \quad \|\sqrt{p_\xi}\|_{H^1(\Omega)} \leq K_3 := \|\sqrt{p_0}\|_{H^1(\Omega)}. \tag{22}$$

From (19) and (22) we deduce via (A1)

$$\forall \xi \in [0, \infty): \quad \|n_\xi\|_{L^2(\Omega)} \leq K_4, \quad \|p_\xi\|_{L^2(\Omega)} \leq K_4. \tag{23}$$

Since $V_\xi = V[n_\xi - p_\xi - C]$ we deduce from (A4) via (23)

$$\forall \xi \in [0, \infty): \quad \|V_\xi\|_{L^\infty(\Omega)} \leq K_5. \tag{24}$$

Since $\sqrt{n_\xi} \in H^1(\Omega) \cap L^\infty(\Omega)$, $\xi \in (0, \infty)$ or $\xi = 0$, we can use $\sqrt{n_\xi}$ as test function in (2) to obtain

$$\alpha_\xi N = -\int_\Omega |\nabla \sqrt{n_\xi}|^2 \, dx - \int_\Omega n_\xi \log(n_\xi) \, dx - \int_\Omega V_\xi \, n_\xi \, dx, \tag{25}$$

hence by using $|t \log(t)| \leq 1 + t + t^2$ for all $t \geq 0$,

$$|\alpha_\xi| \, N \leq \|\sqrt{n_\xi}\|^2_{H^1(\Omega)} + 1 + N + \|n_\xi\|^2_{L^2(\Omega)} + \|V_\xi\|_{L^2(\Omega)} \|n_\xi\|_{L^2(\Omega)},$$

and therefore via (19), (23),

$$\forall \xi \in [0, \infty): \quad |\alpha_\xi| \leq K_6. \tag{26}$$

Now we use for $\xi \in (0, \infty)$ (the case $\xi = 0$ is excluded for the moment) the function $\sqrt{p_\xi} \in H^1(\Omega) \cap L^\infty(\Omega)$ as test function in (3) to obtain

$$\beta_\xi P = -\xi \int_\Omega |\nabla \sqrt{p_\xi}|^2 \, dx - \int_\Omega p_\xi \log(p_\xi) \, dx - \int_\Omega V_\xi \, p_\xi \, dx, \tag{27}$$

and we obtain by summing up (25) and (27),

$$\begin{aligned} &\alpha_\xi N + \beta_\xi P \\ &\quad = -J_\xi(n_\xi, p_\xi) + \int_\Omega (n_\xi + p_\xi) \, dx + \frac{1}{2} \| |\nabla V_\xi| \|^2_{L^2(\Omega)} - \int_\Omega V_\xi \, (n_\xi + p_\xi) \, dx, \end{aligned}$$

hence

$$\begin{aligned} &|\alpha_\xi N + \beta_\xi P| \\ &\quad \leq |J_\xi(n_\xi, p_\xi)| + N + P + \|V_\xi\|^2_{H^1(\Omega)} + \|V_\xi\|_{L^2(\Omega)} (\|n_\xi\|_{L^2(\Omega)} + \|p_\xi\|_{L^2(\Omega)}), \end{aligned}$$

such that we obtain via (19), (23), (26) the estimate

$$\forall \xi \in [0,\infty): \quad |\beta_\xi| \le K_7. \tag{28}$$

Now we can apply standard methods to deduce via (2), (3) for all $\xi \in (0,\infty)$,

$$-\|V_\xi\|_{L^\infty(\Omega)} - |\alpha_\xi| \le \log(n_\xi) \le \|V_\xi\|_{L^\infty(\Omega)} + |\alpha_\xi|,$$

$$-\|V_\xi\|_{L^\infty(\Omega)} - |\beta_\xi| \le \log(p_\xi) \le \|V_\xi\|_{L^\infty(\Omega)} + |\beta_\xi|,$$

and therefore via (24), (26), (28),

$$\forall \xi \in [0,\infty): \quad 0 < K_8 \le n_\xi, p_\xi \le K_9, \tag{29}$$

where the estimate for $\xi = 0$ follows from theorem 2.

It remains to prove the convergence statements. Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence in $(0,\infty)$ with $\lim_{k\to\infty} \xi_k = 0$. For the sake of brevity we set

$$n_k := n_{\xi_k},\ p_k := p_{\xi_k},\ V_k := V_{\xi_k},\ \alpha_k := \alpha_{\xi_k},\ \beta_k := \beta_{\xi_k}.$$

Due to (A1), (19), (22), (24), (26), (28), (29) we have - possibly after extracting a subsequence but without changing notations - $n_*, p_* \in H^1(\Omega) \cap L^\infty(\Omega)$ with $K_8 \le n_*, p_* \le K_9$, $\alpha_*, \beta_* \in \mathbb{R}$ and $V_* \in H^1(\Omega) \cap L^\infty(\Omega)$ such that

$$V_k \rightharpoonup V_*,\ \sqrt{n_k} \rightharpoonup \sqrt{n_*},\ \sqrt{p_k} \rightharpoonup \sqrt{p_*} \quad \text{weakly in } H^1(\Omega) \text{ as } k \to \infty, \tag{30}$$

$$V_k \to V_*,\ n_k \to n_*,\ p_k \to p_* \quad \text{strongly in } L^2(\Omega) \text{ as } k \to \infty, \tag{31}$$

$$V_k \to V_*,\ n_k \to n_*,\ p_k \to p_* \quad \text{almost everywhere as } k \to \infty, \tag{32}$$

$$V_k \rightharpoonup V_*,\ n_k \rightharpoonup n_*,\ p_k \rightharpoonup p_* \quad \text{weak* in } L^\infty(\Omega) \text{ as } k \to \infty, \tag{33}$$

$$\alpha_k \to \alpha_*,\ \beta_k \to \beta_* \quad \text{as } k \to \infty. \tag{34}$$

We readily deduce from (32) and from (29):

$$\forall p \in [1,\infty): \quad V_k \to V_*,\ n_k \to n_*,\ p_k \to p_* \quad \text{strongly in } L^p(\Omega) \text{ as } k \to \infty, \tag{35}$$

and therefore due to (A4),

$$V_k \to V_* \quad \text{strongly in } L^\infty(\Omega) \text{ as } k \to \infty. \tag{36}$$

Passing to the limit $k \to \infty$ in the weak formulation of the Poisson equation (3) we obtain

$$V_* = V[n_* - p_* - C],$$

as well as

$$V_k \to V_* \quad \text{strongly in } H^1(\Omega) \text{ as } k \to \infty, \tag{37}$$

while by weak lower semi continuity,

$$J_0(n_*, p_*) \leq \liminf_{k \to \infty} J_{\xi_k}(n_k, p_k) = \min_{\Gamma_0} J_0, \tag{38}$$

hence by $(n_*, p_*) \in \Gamma_0$ (which follows from (30) and (35))

$$J_0(n_*, p_*) = \min_{\Gamma_0} J_0,$$

and therefore $(n_*, p_*) = (n_0, p_0)$ due to uniqueness of the minimizer in Γ_0. As a consequence, $V_* = V_0 = V[n_0 - p_0 - C]$ and via (38),

$$\int_\Omega |\nabla \sqrt{n_k}|^2 \, dx \to \int_\Omega |\nabla \sqrt{n_0}|^2 \, dx, \; \int_\Omega |\nabla \sqrt{p_k}|^2 \, dx \to \int_\Omega |\nabla \sqrt{p_0}|^2 \, dx \quad \text{as } k \to \infty,$$

hence

$$\sqrt{n_k} \to \sqrt{n_0}, \; \sqrt{p_k} \to \sqrt{p_0} \quad \text{strongly in } H^1(\Omega) \text{ as } k \to \infty, \tag{39}$$

and therefore by (29),

$$n_k \to n_0, \; p_k \to p_0 \quad \text{strongly in } H^1(\Omega) \text{ as } k \to \infty, \tag{40}$$

as well. $\alpha_* = \alpha_0$ and $\beta_* = \beta_0$ follows from the previous estimates, from (25) and from (27). Finally, we deduce

$$\sqrt{n_k}\,(\log(n_k) + V_k + \alpha_k) \to \sqrt{n_0}\,(\log(n_0) + V_0 + \alpha_0) \qquad \text{strongly in } L^2(\Omega) \text{ as } k \to \infty,$$

hence, due to $\sqrt{n_\xi} = V[\sqrt{n_k}\,(\log(n_k) + V_k + \alpha_k)]$, and due to $\sqrt{n_0} = V[\sqrt{n_0}\,(\log(n_0) + V_0 + \alpha_0)]$, we deduce from (A4), $n_k \to n_0$ strongly in $L^\infty(\Omega)$ as $k \to \infty$. □

4 Appendix: Minimization of J_0 in Γ_0

Proof. (Theorem 2) The verification that J_0 has a minimizer in Γ_0 may be settled on more or less standard arguments involving coercitivity and weak lower semi-continuity. However the derivation of the associated Euler Lagrange is due to the lack of differentiability of the function $t \mapsto t\log(t)$ at $t = 0$ *not* straight-forward.

We proceed by an approximation argument.

For $\delta \in (0,1)$ and $t \in \mathbb{R}$ let

$$\log_\delta(t) = \begin{cases} \log(t) & \text{if} \quad t > \delta \\ 2t - 2\delta + \log(\delta) & \text{if} \quad t \le \delta \end{cases},$$

and

$$H_\delta(t) = \begin{cases} t(\log(t) - 1) & \text{if} \quad t > \delta \\ t^2 + (\log(\delta) - 2\delta)\, t + \delta^2 - \delta \log(\delta) & \text{if} \quad t \le \delta \end{cases},$$

such that H_δ is continuously differentiable on $\mathbb{R}$ with derivative $\log_\delta$. We note

$$-1 \le H_\delta(t) \le H(t) := t\,(\log(t) - 1), \quad \forall t \ge 0, \forall \delta \in (0,1). \tag{41}$$

We introduce [6]

$$\Gamma^* := \Big\{ (\rho, p) \in H^1(\Omega) \times \mathcal{C}_{el} : \\ \Big(\int_\Omega \rho\rho^+ \, dx = N \Big) \wedge (p \ge 0) \wedge \Big(\int_\Omega p \, dx = P \Big) \Big\},$$

where

$$\rho^+ = \max\{\rho, 0\}, \quad \rho^- = \max\{-\rho, 0\}.$$

For $\delta \in (0,1)$ we define the functional

$$E_\delta : \Gamma^* \to \mathbb{R} \cup \{\infty\},$$
$$E_\delta(\rho, p) = \int_\Omega |\nabla \rho|^2 \, dx + \int_\Omega \rho\rho^+ (\log(\rho\rho^+) - 1) \, dx + \int_\Omega H_\delta(p) \, dx \\ + E_\infty(\rho\rho^+ - p - C).$$

Furthermore, we introduce

$$E_0 : \Gamma^* \to \mathbb{R} \cup \{\infty\},$$
$$E_0(\rho, p) = \int_\Omega |\nabla \rho|^2 \, dx + \int_\Omega \rho\rho^+ (\log(\rho\rho^+) - 1) \, dx + \int_\Omega p(\log(p) - 1) \, dx \\ + E_\infty(\rho\rho^+ - p - C).$$

Our strategy is to prove existence and uniqueness of a minimizer (ρ_δ, p_δ) of E_δ in Γ^*, to justify the validity of the corresponding Euler Lagrange equations, to derive δ-independent estimates and to pass to the limit $\delta \to 0$ then.

Step 1: Uniqueness of a minimizer. Let $\delta \in (0,1)$ or $\delta = 0$ be fixed for the moment. Since E_δ is *not* convex, uniqueness of a minimizer of E_δ in Γ^* is a priori not clear. Assume (ρ_δ, p_δ) and $(\rho_\circ, p_\circ)$ are minimizers of E_δ in Γ^*, i.e.

$$E_\delta(\rho_\delta, p_\delta) = E_\delta(\rho_\circ, p_\circ) = \inf_{\Gamma^*} E_\delta,$$

where we note $\inf_{\Gamma^*} E_\delta \geq -2$.

We set

$$\rho_\theta := \sqrt{(1-\theta)\rho_\delta\rho_\delta^+ + \theta\rho_\circ\rho_\circ^+}, \quad p_\theta := (1-\theta)p_\delta + \theta p_\circ, \quad \theta \in (0,1).$$

Then $(\rho_\theta, p_\theta) \in \Gamma^*$ for all $\theta \in (0,1)$. Since $\rho_\theta\rho_\theta^+ = (1-\theta)\rho_\delta\rho_\delta^+ + \theta\rho_\circ\rho_\circ^+$ by strict convexity,

$$\begin{aligned}
&\int_\Omega \rho_\theta\rho_\theta^+(\log(\rho_\theta\rho_\theta^+) - 1)\,dx + \int_\Omega H_\delta(p_\theta)\,dx + E_\infty(\rho_\theta\rho_\theta^+ - p_\theta - C) \\
&< (1-\theta)\left(\int_\Omega \rho_\delta\rho_\delta^+(\log(\rho_\delta\rho_\delta^+) - 1)\,dx + \int_\Omega H_\delta(p_\delta)\,dx + E_\infty(\rho_\delta\rho_\delta^+ - p_\delta - C)\right) + \\
&\qquad \theta\left(\int_\Omega \rho_\circ\rho_\circ^+(\log(\rho_\circ\rho_\circ^+) - 1)\,dx + \int_\Omega H_\delta(p_\circ)\,dx + E_\infty(\rho_\circ\rho_\circ^+ - p_\circ - C)\right),
\end{aligned}$$

for all $\theta \in (0,1)$ unless $(\rho_\delta, p_\delta) = (\rho_\circ, p_\circ)$. Furthermore, we have for all $\theta \in (0,1)$ the inequality

$$\begin{aligned}
&(1-\theta)\int_\Omega |\nabla\rho_\delta|^2\,dx + \theta\int_\Omega |\nabla\rho_\circ|^2\,dx - \int_\Omega |\nabla\rho_\theta|^2\,dx \\
&\qquad = \theta(1-\theta)\int_{\{\rho_\circ^+>0\}\cup\{\rho_\delta^+>0\}} \frac{\left(\rho_\circ^+\nabla\rho_\delta^+ - \rho_\delta^+\nabla\rho_\circ^+\right)^2}{(1-\theta)\rho_\delta\rho_\delta^+ + \theta\rho_\circ\rho_\circ^+}\,dx \geq 0,
\end{aligned}$$

hence

$$E_\delta(\rho_\theta, p_\theta) < (1-\theta)E_\delta(\rho_\delta, p_\delta) + \theta E_\delta(\rho_\circ, p_\circ), \quad \theta \in (0,1),$$

unless $(\rho_\delta, p_\delta) = (\rho_\circ, p_\circ)$.

Step 2: Existence of a minimizer. Let $\delta \in (0,1)$ or $\delta = 0$ be fixed for the moment. We consider a minimizing sequence $(\rho_k, p_k)_{k\in\mathbb{N}}$. Since $E(\rho_k^+, p_k) \leq E(\rho_k, p_k)$ and since $(\rho_k^+, p_k) \in \Gamma^*$ whenever $(\rho_k, p_k) \in \Gamma^*$, we can assume $\rho_k = \rho_k^+ \geq 0$ for all $k \in \mathbb{N}$.

We observe $(\sqrt{N_0}, P_0) \in \Gamma^*$, where $N_0 := N/\text{meas}(\Omega)$ and $P_0 := P/\text{meas}(\Omega)$. Hence

$$\begin{aligned}
&\inf_{\Gamma^*} E_\delta \leq E_\delta(\sqrt{N_0}, P_0) < E_\delta(\sqrt{N_0}, P_0) + 1 = K_1 \\
&:= 1 + (N_0(\log(N_0) - 1) + P_0(\log(P_0) - 1))\text{meas}(\Omega) + E_\infty(N_0 - P_0 - C), \quad (42)
\end{aligned}$$

where we have made use of (41). Due to assumption (A4) we have $K_1 \in \mathbb{R}$. We note: K_1 does not depend on δ.

We assume $E_\delta(\rho_k, p_k) \le K_1$ for all $k \in \mathbb{N}$. Then we have due to (42) and due to (41) the a priori estimate

$$\int_\Omega |\nabla \rho_k|^2\, dx + \int_\Omega \left(\rho_k^2(\log(\rho_k^2) - 1) + 1\right)\, dx + \int_\Omega (H_\delta(p_k) + 1)\, dx + E_\infty(\rho_k^2 - p_k - C) \le K_2 := K_1 + 2\,\mathrm{meas}(\Omega), \quad (43)$$

where K_2 is independent of δ.

Since all terms on the left hand side or this inequality ro non negatve we have by passing if necessary to a subsequence but without changing notations

$$\rho_k = \rho_k^+ \rightharpoonup \rho_\delta \quad \text{weakly in } H^1(\Omega) \text{ as } k \to \infty,$$

$$p_k \rightharpoonup p_\delta \quad \text{weakly in } L^1(\Omega) \text{ as } k \to \infty,$$

and therefore by assumption (A1),

$$\rho_k \to \rho_\delta \quad \text{strongly in } L^4(\Omega) \text{ as } k \to \infty,$$

$$\rho_k^2 \to \rho_\delta^2 \quad \text{strongly in } L^2(\Omega) \text{ as } k \to \infty.$$

As a consequence, $\rho_\delta \ge 0$ and $(\rho_\delta, p_\delta) \in \Gamma^*$. We certainly have

$$\int_\Omega |\nabla \rho_\delta|^2\, dx \le \liminf_{k\to\infty} \int_\Omega |\nabla \rho_k|^2\, dx, \quad (44)$$

and due to $\rho_k \rho_k^+ = \rho_k^2$, $k \in \mathbb{N}$, due to $\rho_\delta \rho_\delta^+ = \rho_\delta^2$, by convexity

$$\int_\Omega \rho_\delta^2(\log(\rho_\delta^2) - 1)\, dx + \int_\Omega H_\delta(p_\delta)\, dx \le \liminf_{k\to\infty} \left(\int_\Omega \rho_k^2(\log(\rho_k^2) - 1)\, dx + \int_\Omega H_\delta(p_k)\, dx \right). \quad (45)$$

Now let us consider $E_\infty(\rho_k^2 - p_k - C)$ as $k \to \infty$. Since a priori only $\rho_k^2 - p_k - C \rightharpoonup \rho_\delta^2 - p_\delta - C$ weakly in $L^1(\Omega)$ as $k \to \infty$, the statement $\rho_\delta^2 - p_\delta - C \in \mathcal{C}_{el}$ (compare remark 2 c) it is not immediate. We set $V_k := V[\rho_k^2 - p_k - C]$ for $k \in \mathbb{N}$. Since the sequence $(\|\nabla V_k\|_{L^2(\Omega):\mathbb{R}^d})_{k\in\mathbb{N}}$ is bounded (see (43)) we have (possibly after passing to a subsequence, but without changing notations) due to $\int_\Omega V_k\, dx = 0$ for all $k \in \mathbb{N}$,

$$V_k \rightharpoonup V_\delta, \quad \text{weakly in } H^1(\Omega) \text{ as } k \to \infty,$$

which implies

$$\int_\Omega |\nabla V_\delta|^2 \, dx \le \liminf_{k\to\infty} \int_\Omega |\nabla V_k|^2 \, dx, \tag{46}$$

while on the other hand for all $\phi \in \mathcal{C}_0 \cap L^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega \nabla V_\delta \cdot \nabla \phi \, dx &= \lim_{k\to\infty} \int_\Omega \nabla V_k \cdot \nabla \phi \, dx \\ &= \lim_{k\to\infty} \int_\Omega (\rho_k^2 - p_k - C)\phi \, dx = \int_\Omega (\rho_\delta^2 - p_\delta - C)\phi \, dx,\end{aligned}$$

such that we deduce $\rho_\delta^2 - p_\delta - C \in \mathcal{C}_{el}$, $V_\delta = V[\rho_\delta^2 - p_\delta - C]$, via (46), and via

$$E_\infty(\rho_\delta^2 - p_\delta - C) \le \liminf_{k\to\infty} E_\infty(\rho_k^2 - p_k - C),$$

the estimate

$$E_\delta(\rho_\delta, p_\delta) \le \liminf_{k\to\infty} E_\delta(\rho_k, p_k),$$

i.e. (ρ_δ, p_δ) is a minimizer of E_δ in Γ^*.

Step 3: Derivation of the Euler Lagrange equations. Here we make use of the differentiability of H_δ. Hence the case $\delta = 0$ is excluded.

Let $\delta \in (0,1)$ be fixed for the moment. We shall derive the Euler Lagrange equations for (ρ_δ, p_δ).

Let $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$. For $t \in (0,1)$ let

$$N(t) := \sqrt{\int_\Omega (\rho_\delta + t\phi)(\rho_\delta + t\phi)^+ \, dx}.$$

We certainly have $N(0+) = N$. In particular, there is $t_0 \in (0,1)$ such that $N(t) \ge N/2$ for all $t \in (0, t_0]$. We set

$$\phi_t := \frac{N}{N(t)} \, (\rho_\delta + t\phi), \quad t \in (0, t_0].$$

Obviously, $\phi_t \in H^1(\Omega)$ with $\int_\Omega \phi_t \phi_t^+ \, dx = N$, $t \in (0, t_0]$. As a consequence, we have $(\phi_t, p_\delta) \in \Gamma^*$ for all $t \in (0, t_0]$. Furthermore it is easy to see

$$\infty > E_\delta(\phi_t, p_\delta) - E_\delta(\rho_\delta, p_\delta) \ge 0, \quad t \in (0, t_0],$$

thus

$$\liminf_{t\to 0+} \frac{E_\delta(\phi_t, p_\delta) - E_\delta(\rho_\delta, p_\delta)}{t} \ge 0. \tag{47}$$

Since

$$N^2(t) = N^2 + 2t \int_\Omega \rho_\delta \phi \, dx + t^2 \int_\Omega A(\rho_\delta(x), \phi(x)) \phi^2(x) \, dx, \quad t \in (0, t_0]$$

for a measurable function $A : \mathbb{R}^2 \to \mathbb{R}$ with $|A(\sigma_1, \sigma_2)| \leq 2$ for all $(\sigma_1, \sigma_2) \in \mathbb{R}^2$, we obtain

$$\lim_{t \to 0+} \frac{\int_\Omega \left(|\nabla \phi_t|^2 - |\nabla \rho_\delta|^2 \right) \, dx}{t} = -\frac{1}{N^2} \int_\Omega \rho_\delta \, \phi \, dx \int_\Omega |\nabla \rho_\delta|^2 \, dx + 2 \int_\Omega \nabla \rho_\delta \cdot \nabla \phi \, dx. \quad (48)$$

We deduce from the estimate

$$\forall \sigma \geq 0 : \; 4|\sigma \log(\sigma)| \leq 4(1 + \sigma + \sigma^2),$$

due to $\rho_\delta, \phi \in L^6(\Omega)$, see (A1), for all $t \in (0, t_0]$ the estimate

$$\left| \frac{H(\phi_t) - H(\rho_\delta^2)}{t} \right| \leq \left(1 + \frac{N}{N(t)} \right)^3 (a_0 + a_1(\rho_\delta^6 + |\phi|^6)),$$

where $H(\sigma) = \sigma^2(\log(\sigma^2) - 1)$, $\sigma \geq 0$, and a_0, a_1 are some positive constants which are independent of ρ_δ, Φ.

Now it is easy to deduce

$$\lim_{t \to 0+} \int_\Omega \frac{H(\phi_t) - H(\rho_\delta)}{t} \, dx = \int_\Omega (2\rho_\delta \log(\rho_\delta^2)) \, \phi \, dx - \frac{1}{N^2} \int_\Omega \rho_\delta \phi \, dx \int_\Omega 2\rho_\delta^3 \log(\rho_\delta^2) \, dx, \quad (49)$$

via Lebesgue's dominated convergence theorem.

Furthermore, we obtain with the aid of remark 2 c) that

$$\lim_{t \to 0+} \frac{E_\infty(\phi_t - p_\delta - C) - E_\infty(\rho_\delta^2 - p_\delta)}{t} = 2 \int_\Omega V_\delta \, \rho_\delta \, \phi \, dx - \frac{1}{N^2} \int_\Omega \rho_\delta \phi \, dx \int_\Omega V_\delta \, \rho_\delta^2 \, dx. \quad (50)$$

Equations (48), (49), (50) hold for all $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$. Hence we have actually equality in (47) and therefore

$$\int_\Omega \nabla \rho_\delta \cdot \nabla \phi \, dx + \int_\Omega \rho_\delta (\log(\rho_\delta^2) + V_\delta + \alpha_\delta) \phi \, dx = 0, \quad (51)$$

where $\alpha_\delta \in \mathbb{R}$ is given by

$$\alpha_\delta = -\frac{1}{2\,N^2}\left(\int_\Omega |\nabla \rho_\delta|^2\,dx + 2\int_\Omega \rho_\delta^3 \log(\rho_\delta^2)\,dx + \int_\Omega V_\delta\,\rho_\delta^2\,dx\right). \tag{52}$$

By a similiar, but standard procedure [5] (here the differentiability of H_δ enters) we deduce the variational inequalities

$$\begin{aligned} \log_\delta(p_\delta) - V_\delta + \beta_\delta = 0 \quad &\text{on} \quad \{p_\delta > 0\} \\ \log_\delta(0) - V_\delta + \beta_\delta \geq 0 \quad &\text{on} \quad \{p_\delta = 0\} \end{aligned} \tag{53}$$

with $\beta_\delta \in \mathbb{R}$ follow.

Introducing the generalized inverse of $\log_\delta^+$,

$$g_\delta : \mathbb{R} \to [0,\infty), \quad g_\delta(\sigma) = \begin{cases} \log_\delta^{-1}(\sigma) & \text{if} \quad \sigma > -2\delta + \log(\delta) \\ 0 & \text{if} \quad \sigma \leq -2\delta + \log(\delta) \end{cases}$$

we rewrite (53) as

$$p_\delta = g_\delta(V_\delta - \beta_\delta). \tag{54}$$

Step 4: Estimates independent of $\delta \in (0,1)$. Several constants which are independent of δ are denoted by $K_3, K_4, \ldots$.

We deduce from (43), (44), (45) and (46) the estimate

$$\int_\Omega |\nabla \rho_\delta|^2\,dx + \int_\Omega \left(\rho_\delta^2(\log(\rho_\delta^2) - 1) + 1\right)\,dx + \int_\Omega (H_\delta(p_\delta) + 1)\,dx + E_\infty(\rho_\delta^2 - p_\delta - C) \\ \leq K_2, \tag{55}$$

where K_2 is independent of δ. In particular we deduce

$$\|\rho_\delta\|_{H^1(\Omega)}^2 \leq K_3 := N + K_2, \quad \|V_\delta\|_{H^1(\Omega)} \leq K_4,$$

from which we deduce due to (52), and due to (A1) for some (products of) imbedding constants $a_0, a_1, a_2, a_3 \in (0,\infty)$ (which are intrinisically independent of δ),

$$\begin{aligned} |\alpha_\delta| &\leq \frac{1}{2\,N^2}\left(\|\rho_\delta\|_{H^1(\Omega)}^2 + \int_\Omega (1 + \rho_\delta^3 + \rho_\delta^6)\,dx + \|V_\delta\|_{L^2(\Omega)}\|\rho_\delta\|_{L^4(\Omega)}\right) \\ &\leq \frac{1}{2\,N^2}\left(\|\rho_\delta\|_{H^1(\Omega)}^2 + a_0 + a_1\|\rho_\delta\|_{H^1(\Omega)}^3 + a_2\|\rho_\delta\|_{H^1(\Omega)}^6 + a_3\|V_\delta\|_{H^1(\Omega)}\|\rho_\delta\|_{H^1(\Omega)}\right) \\ &\leq K_5 := \frac{1}{2\,N^2}\left(K_3 + a_0 + a_1 K_3^{3/2} + a_2 K_3^3 + a_3\sqrt{K_3}K_4\right). \end{aligned} \tag{56}$$

In a similiar we proceed to prove

$$\forall \delta \in (0,1): \quad \rho_\delta(\log(\rho_\delta^2) + V_\delta + \alpha_\delta) \in L^2(\Omega),$$

with

$$\|\rho_\delta(\log(\rho_\delta^2) + V_\delta + \alpha_\delta)\|_{L^2(\Omega)} \leq K_6. \tag{57}$$

Furthermore, (51) holds for all $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$. Hence

$$\rho_\delta = V[\rho_\delta(\log(\rho_\delta^2) + V_\delta + \alpha_\delta)],$$

and therefore due to (A4),

$$\|\rho_\delta\|_{L^\infty(\Omega)} \leq K_7 := K_\infty K_6. \tag{58}$$

Due to (54) we have

$$-\Delta V_\delta = \rho_\delta^2 - g_\delta(V_\delta - \beta_\delta) - C. \tag{59}$$

Since $\lim_{\sigma\to\infty} g_\delta(\sigma) = \infty$, there is $m \in (0,\infty)$ with $g_\delta(m) = 1 + K_7^2 + \|C\|_{L^\infty(\Omega)}$. The function $[V_\delta - \beta_\delta - m]^+$ belongs to $H^1(\Omega)$. We take for $k \in \mathbb{N}$,

$$\phi_k := \min\{k, [V_\delta - \beta_\delta - m]^+\} \in H^1(\Omega) \cap L^\infty(\Omega).$$

Certainly, $\phi_k \to [V_\delta - \beta_\delta - m]^+$ as $k \to \infty$, strongly in $H^1(\Omega)$ with $0 \leq \phi_k \leq \phi_{k+1}$ for all $k \in \mathbb{N}$. Hence for all $k \in \mathbb{N}$,

$$\begin{aligned} -\int_\Omega \nabla V_\delta \cdot \nabla \phi_k \, dx &= -\int_\Omega (\rho_\delta^2 - g_\delta(V_\delta - \beta_\delta) - C)\phi_k \, dx \\ &= \int_{\{V_\delta - \beta_\delta \geq m\}} (-\rho_\delta^2 + g_\delta(V_\delta - \beta_\delta) + C)\phi_k \, dx. \end{aligned} \tag{60}$$

By strong convergence in $H^1(\Omega)$ we have

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega \nabla V_\delta \cdot \nabla \phi_k \, dx &= \int_\Omega \nabla V_\delta \cdot \nabla [V_\delta - \beta_\delta - m]^+ \, dx \\ &= \int_\Omega |\nabla [V_\delta - \beta_\delta - m]^+|^2 \, dx, \end{aligned}$$

hence by Beppo Levi's theorem (we note $(-\rho_\delta^2 + g_\delta(V_\delta - \beta_\delta) + C)\phi_k \geq 0$ on $\{V_\delta - \beta_\delta \geq m\}$),

$$\begin{aligned} &\lim_{k\to\infty} \int_{\{V_\delta - \beta_\delta \geq m\}} (-\rho_\delta^2 + g_\delta(V_\delta - \beta_\delta) + C)\phi_k \, dx \\ &\qquad = \int_{\{V_\delta - \beta_\delta \geq m\}} (-\rho_\delta^2 + g_\delta(V_\delta - \beta_\delta) + C) \, [V_\delta - \beta_\delta - m]^+ \, dx. \end{aligned}$$

Now we deduce from (60),

$$0 \geq \int_{\{V_\delta - \beta_\delta \geq m\}} (-\rho_\delta^2 + g_\delta(V_\delta - \beta_\delta) + C)\,[V_\delta - \beta_\delta - m]^+ \, dx \geq 0,$$

and therefore $[V_\delta - \beta_\delta - m]^+ = 0$ almost everywhere on Ω. Thus, $V_\delta \leq m + \beta_\delta$ almost everywhere on Ω. As a consequence, since g_δ is strictly increasing,

$$p_\delta = g_\delta(V_\delta - \beta_\delta) \leq g_\delta(m) = K_8 := 1 + K_7^2 + \|C\|_{L^\infty(\Omega)}. \tag{61}$$

On the other hand, $V_\delta = V[\rho_\delta^2 - p_\delta - C]$, such that we deduce from (A4)

$$\begin{aligned} \|V_\delta\|_{L^\infty(\Omega)} &\leq K_\infty \|\rho_\delta^2 - p_\delta - C\|_{L^2(\Omega)} \\ &\leq K_9 := K_\infty \sqrt{\text{meas}(\Omega)}(K_7^2 + K_8 + \|C\|_{L^\infty(\Omega)}). \end{aligned} \tag{62}$$

Since $\{p_\delta > 0\}$ has nonzero measure we deduce from (53),

$$-\beta_\delta \leq K_{10} := \|V_\delta\|_{L^\infty(\Omega)} + \log_\delta(K_8) = \|V_\delta\|_{L^\infty(\Omega)} + \log(K_8), \tag{63}$$

because due to (61) we have $K_8 \geq 1$ and by construction we have $\log_\delta(\sigma) = \log(\sigma)$ for all $\sigma \geq 1$.

On the other hand it is easy to see that

$$g_\delta(\sigma) \leq \exp(\sigma), \quad \sigma \in \mathbb{R}.$$

Hence

$$P = \int_\Omega g_\delta(V_\delta - \beta_\delta)\, dx \leq \int_\Omega \exp(V_\delta - \beta_\delta)\, dx \leq \int_\Omega \exp(K_9 - \beta_\delta)\, dx,$$

from which we deduce

$$\beta_\delta \leq K_{11} := |\log(P)| + K_9 + |\log(\text{meas}(\Omega))|. \tag{64}$$

We can summarize (63) and (64) as

$$|\beta_\delta| \leq K_{12} := K_{10} + K_{11}. \tag{65}$$

Step 5: Passing from E_δ to E_0. According to (53) there is $\delta^* \in (0,1)$ such that $p_\delta > 0$ almost everywhere on Ω for all $\delta \in (0, \delta^*)$, because otherwise $-2\delta + \log(\delta) \geq -K_{12} - K_9$ for all $\delta \in (0,1)$. We therefore have $\log_\delta(p_\delta) - V_\delta + \beta_\delta = 0$ for all $\delta \in (0, \delta^*)$. Since $\log_\delta(\sigma) \leq \log(\delta)$ for all $\sigma \leq \delta$ and since $\|V_\delta\|_{L^\infty(\Omega)} + |\beta_\delta| \leq K_9 + K_{12}$, we deduce: there is $\delta_1 \in (0, \delta^*)$ such that $p_\delta \geq \delta_1$ for all $\delta \in (0, \delta_1)$. Hence $\log_\delta(p_\delta) = \log(p_\delta)$ for all $\delta \in (0, \delta_1)$. As a consequence, we have

$$p_\delta(\log(p_\delta) - 1) = H_0(p_\delta) = H_\delta(p_\delta), \quad \delta \in (0, \delta_1).$$

Hence

$$E_\delta(\rho_\delta, p_\delta) = E_0(\rho_\delta, p_\delta), \quad \delta \in (0, \delta_1).$$

On the other hand, for all $(\rho, p) \in \Gamma^*$, we have due to (41)

$$E_\delta(\rho, p) \leq E_0(\rho, p).$$

Hence for all $\delta \in (0, \delta_1)$,

$$\inf_{\Gamma^*} E_0 \leq E_0(\rho_\delta, p_\delta) = E_\delta(\rho_\delta, p_\delta) = \inf_{\Gamma^*} E_\delta \leq \inf_{\Gamma^*} E_0,$$

and as a conclusion: There is $\delta_\circ \in (0, 1)$ such that $(\rho_{\delta_\circ}, p_{\delta_\circ})$ is the unique minimizer of E_0 in Γ^*. We set $(\rho_2, p_2) := (\rho_{\delta_\circ}, p_{\delta_\circ})$.

Step 6: Finishing the proof. We have to prove that (ρ_2^2, p_2) is the unique minimizer of J_0 in Γ_0. Since $(\sqrt{n}, p) \in \Gamma^*$ and since $E_0(\sqrt{n}, p) = J_0(n, p)$ for all $(n, p) \in \Gamma^*$ we have

$$\inf_{\Gamma^*} E_0 \leq \inf_{\Gamma_0} J_0.$$

On the other hand we have $(\rho_2^2, p_2) \in \Gamma_0$. Thus

$$\inf_{\Gamma_0} J_0 \leq J_0(\rho_2^2, p_2) = \inf_{\Gamma^*} E_0,$$

such that $\inf_{\Gamma^*} E_0 = \inf_{\Gamma_0} J_0$ and as a consequence,

$$(n_0, p_0) := (\rho_2^2, p_2)$$

is a minimizer of J_0 in Γ_0. Furthermore, if $(n_3, p_3) \in \Gamma_0$ is a minimizer of J_0 in Γ_0 then $(\sqrt{n_3}, p_3)$ is a minimizer of E_0 in Γ^*. Hence - since E_0 has exactly one minimizer in Γ^* - $\sqrt{n_3} = \rho_2$ and $p_3 = p_2$ which gives $(n_3, p_3) = (n_0, p_0)$, i.e. (n_0, p_0) is the unique minimizer of J_0 in Γ_0.

1. Setting $V_0 = V[n_0 - p_0 - C]$ and $\alpha_0 := \alpha_{\delta_\circ}, \beta_0 := \beta_{\delta_\circ}$ (see the end of Step 5), then - due to $\log_\delta(p_0) = \log(p_0)$ - the quintuple $(n_0, p_0, V_0, \alpha_0, \beta_0)$ is a solution of (8)-(14).

2. The inequalities $0 < \delta_1 \leq p_0 \leq K_8$, $n_0 \leq K_7^2$ and $|V_0| \leq K_9$ have already been established in Step 4. It remains to prove that n_0 is uniformly bounded away from 0. This follows from a standard truncation argument from (8) employing $\|n_0\|_{L^\infty(\Omega)} \leq K_7^2$ and $|V_0| \leq K_9$.

3. has already been proved in Step 4.

4. We have $p_0 = \exp(V_0 - \beta_0)$. Since $V_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ we have $p_0 \in H^1(\Omega)$. In analogy, $\sqrt{p_0} = \exp((V_0 - \beta_0)/2) \in H^1(\Omega)$. □

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