On the Multiscale Solution of Satellite Problems by Use of Locally Supported Kernel Functions Corresponding to Equidistributed Data on Spherical Orbits

by

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Abstract

Being interested in (rotation-)invariant pseudodifferential equations of satellite problems corresponding to spherical orbits, we are reasonably led to generating kernels that depend only on the spherical distance, i. e. in the language of modern constructive approximation form spherical radial basis functions. In this paper approximate identities generated by such (rotation-invariant) kernels which are additionally locally supported are investigated in detail from theoretical as well as numerical point of view. So-called spherical difference wavelets are introduced. The wavelet transforms are evaluated by the use of a numerical integration rule, that is based on Weyl's law of equidistribution. This approximate formula is constructed such that it can cope with millions of (satellite) data. The approximation error is estimated on the orbital sphere. Finally, we apply the developed theory to the problems of satellite-to-satellite tracking (SST) and satellite gravity gradiometry (SGG).

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Introduction

In approximation theory, locally supported trial functions are nothing new, having been discussed already in 1910 by Haar (cf. [Ha]), long before anyone began speaking of wavelets. The primary importance of locally supported basis functions in one-dimensional Euclidean space is the 'birth' to an entire family of wavelets by means of two operations, viz. (dyadic) dilations and (integer) translations. In other words, an entire set of approximants is available from a single locally supported 'mother wavelet' function, and this set provides useful 'building block' functions that enable fast decorrelation of data. In consequence, a 'multiscale analysis' in terms of so-called Haar wavelets consists of studying the 'detail signals', i.e. the difference in approximations made at adjacent resolution levels. Once a full understanding of multiscale approximation using the Haar basis is ensured, the extension to other (smooth) locally supported wavelet bases is just a matter of changing the mother wavelet. The fundamental principles remain the same. (For more details the reader is referred, for example, to [Chu] [Da], [Ho], [LoMaRi], [Ni], and the references therein.)

In modern satellite problems the orbits are quite attractive for mathematical modelling: A circular orbit implies that the data are lying on a sphere; the measurements offer a global data coverage and an extremely dense and uniform distribution; the measurements (achieved by employing the significant principles of satellite-to-satellite tracking (SST) and/or satellite gravity gradiometry (SGG)) provide global information about the first and/or second radial derivatives of the gravitational potential at a moderate altitude. The radial derivatives on spherical orbits are representable by rotation-invariant pseudodifferential equations (for more details on pseudodifferential operators on the sphere see [Sv]; their role in modern satellite problems is described in [Fr]).

This is the reason why it is of basic interest to construct locally supported wavelets on the sphere reflecting the rotational invariance of the operators of the SST/SGG-observables. In consequence, multiscale modelling of the data has to be formulated by a rotation-invariant multiscale approach, and the features of any signal on the sphere have to be examined by some process of 'spherical cap windowing'. In this respect it should be noted that the basic framework of rotation-invariant wavelet approximation has been developed by the Geomathematics Group of the University of Kaiserslautern during the last years. (Note that other approaches to spherical wavelets are due to e.g. [Sw1], [Sw2], [Va]). Localizing and even locally supported scaling and wavelet functions have been shown to act as adequate approximants in rotation-invariant approximation on the sphere (confer [FrGeSchr] and the references therein). However, if the approximants happen to constitute locally supported (rotation-invariant) bases and the data points are of huge number and uniform distribution, then the sequence of consecutive differences within the multiscale process admits a more efficient and economical study. This situation can be handled by an appropriate observation of the integration concept of equidistribution within a spherical approach of locally supported difference wavelets. The objective of this writing, therefore, is to investigate in more detail the specific advantages of both the spherical wavelet theory and the concept of equidistribution on the sphere as essential ingredients in the solution of modern satellite problems such as SST, SGG. More explicitly, the following features are incorporated in this way of thinking about locally supported spherical wavelets corresponding to equidistributed data, namely (i) *rotational symmetry* of scaling and wavelet functions, (ii) *basis property* based on the multiscale modelling of the difference of 'two smoothings' realized by two operations, i.e. dilation and rotation, (iii) *decorrelation* by the probate construction of scale dependent locally supported wavelets (i.e. radial basis functions), (iv) *fast computation* by use of the simple integration technique of equidistribution, (v) *appropriate regularization* of the satellite problem by multiscale approximation. An essential tool is the theory of singular integrals using locally supported kernel functions. Altogether, our locally supported difference wavelet approach based on equidistributed data points is meant as the simplest mathematical realization that models large numbers of satellite data economically and efficiently, to achieve parsimonious representations of rotation-invariant physical quantities such as radial derivatives and potential values of the earth's gravitational field. SST is the technology to be realized by the satellite missions CHAMP (2000), GRACE (2002), while SGG is planned for the future satellite mission GOCE (2004).

1 Definitions and Notation

In this section the notation and the mathematical background that will be used throughout the paper will be introduced briefly.

$\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ denote the set of integers, positive integers, real numbers, respectively. As usual, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}$ be the n-dimensional Euclidean space and let $\Omega_r := \{\xi \in \mathbb{R}^3 \,|\, |\xi| = r\}$, $r \in \mathbb{R}$, $r > 0$, be the sphere of radius r with center in the origin in $\mathbb{R}^3$, where $|x| := \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}$ is the Euclidean norm in $\mathbb{R}^3$ and $x \cdot y := \sum_{i=1}^3 x_i y_i$ is the Euclidean inner product in $\mathbb{R}^3$. The unit sphere Ω_1 will be briefly denoted by Ω.

For points on the sphere Ω_r we introduce spherical polar coordinates as follows: Every vector $x \in \Omega_r$ can be represented with respect to the canonical standard basis in $\mathbb{R}^3$ as $x = r\xi$, $r = |x|$, and $\xi = x/|x| \in \Omega$,

$$\xi = (\xi_1, \xi_2, \xi_3) = \left(\cos(\varphi)\sqrt{1-t^2},\, \sin(\varphi)\sqrt{1-t^2},\, t\right), \quad (\varphi, t) \in [0, 2\pi) \times [-1, 1].$$

Using the coordinate transformation $t = \cos(\vartheta)$, $\vartheta \in [0, \pi]$, one gets the usual polar coordinates $(\varphi, \vartheta) \in [0, 2\pi) \times [0, \pi]$. The surface element of the sphere Ω_r is denoted by $d\omega_r$. In the coordinates (φ, t) it has the representation $d\omega_r(x) = r^2\, d\varphi\, dt$. For the surface element of the unit sphere Ω we write $d\omega$ instead of $d\omega_1$. The surface volume of a measurable subset $\mathcal{D}$ of the sphere Ω_r is denoted by $\|\mathcal{D}\|$. Clearly, $\|\Omega_r\| = 4\pi\, r^2$.

Let $\mathcal{F}(\Omega_r)$ denote the set of all measurable real-valued functions on Ω_r. The subset of all k-times continuously differentiable real-valued functions on Ω_r is $\mathcal{C}^{k}(\Omega_r)$, $k \in \mathbb{N}_0$, and $\mathcal{C}^{\infty}(\Omega_r) := \bigcap_{k=0}^{\infty} \mathcal{C}^{k}(\Omega_r)$. In particular, we let $\mathcal{C}(\Omega_r) := \mathcal{C}^{0}(\Omega_r)$. Define for $F \in \mathcal{C}(\Omega_r)$, and for $F \in \mathcal{F}(\Omega_r)$, respectively,

$$\begin{aligned} \|F\|_{\mathcal{C}(\Omega_r)} &:= \sup_{x \in \Omega_r} |F(x)|, \\ \|F\|_{\mathcal{L}^p(\Omega_r)} &:= \left(\int_{\Omega_r} |F(x)|^p d\omega_r(x)\right)^{1/p} \quad \text{for } 1 \le p < \infty. \end{aligned}$$

(All integrals are understood in the Lebesgue-sense.) It is well-known from functional analysis, that the space $\mathcal{C}(\Omega_r)$ of continuous functions on Ω_r equipped with the supremum norm $\|\cdot\|_{\mathcal{C}(\Omega_r)}$ and the spaces $\mathcal{L}^p(\Omega_r) := \{F \in \mathcal{F}(\Omega_r) \,|\, \|F\|_{\mathcal{L}^p(\Omega_r)} < \infty\}$ equipped with the $\mathcal{L}^p$-norm $\|\cdot\|_{\mathcal{L}^p(\Omega_r)}$, respectively, are Banach spaces. In addition, the space $\mathcal{L}^2(\Omega_r)$ of square-integrable functions on Ω_r with the inner product $(F, G)_{\mathcal{L}^2(\Omega_r)} := \int_{\Omega_r} F(x)\, G(x)\, d\omega_r(x)$ is a real Hilbert space.

Analogously, let $[a, b]$, $a, b \in \mathbb{R}$, $a < b$, be a nonempty interval in $\mathbb{R}$ and let $\mathcal{F}([a, b])$ be the space of all real-valued measurable functions on the interval $[a, b]$. The Banach spaces of continuous, k-times continuously differentiable, infinitely often differentiable, and $\mathcal{L}^p$-integrable functions, $1 \le p < \infty$, on $[a, b]$ are denoted by $\mathcal{C}([a, b])$, $\mathcal{C}^{k}([a, b])$, $\mathcal{C}^{\infty}([a, b])$, and $\mathcal{L}^p([a, b])$, respectively.

Let $\mathcal{D}$ be either a sphere Ω_r or an interval $[a, b]$. As $\mathcal{D}$ is compact, $\mathcal{C}(\mathcal{D}) \subset \mathcal{L}^1(\mathcal{D})$ and $\mathcal{L}^p(\mathcal{D}) \subset \mathcal{L}^1(\mathcal{D})$ for all p, $1 \le p < \infty$. A continuous function $F \in \mathcal{C}(\mathcal{D})$ is said to be Lipschitz-continuous, if there exists a constant C_F, such that $|F(x) - F(y)| \le C_F\, |x - y|$ for all $x, y \in \mathcal{D}$. The constant C_F is called a Lipschitz-constant for F. The support of a function $F \in \mathcal{F}(\mathcal{D})$ is given by $\operatorname{supp}(F) := \overline{\{x \in \mathcal{D} \,|\, F(x) \neq 0\}}$.

In what follows, some facts about spherical harmonics and Legendre polynomials are presented. For more details, the reader is referred to [FrGeSchr], [Mü].

The set of all polynomials in $\mathbb{R}^3$ of degree n is denoted as $\mathrm{Pol}_n(\mathbb{R}^3)$. Let $\mathrm{Harm}_n(\mathbb{R}^3)$ be the space of homogeneous harmonic polynomials of degree n. The space $\mathrm{Harm}_n(\Omega) := \left\{H_n|_{\Omega} \mid H_n \in \mathrm{Harm}_n(\mathbb{R}^3)\right\}$ is called the space of spherical harmonics of degree n. It is a finite-dimensional vector space with dimension $\dim(\mathrm{Harm}_n(\Omega)) = 2n + 1$.

Let $\{Y_{n,j}\}_{j=1,\ldots,2n+1} \subset \mathrm{Harm}_n(\Omega)$ from now on denote a complete $\mathcal{L}^2(\Omega)$-orthonormal system in the vector space $\mathrm{Harm}_n(\Omega)$ for $n \in \mathbb{N}_0$. Then the set $\{Y_{n,j}\} := \bigcup_{n \in \mathbb{N}_0} \{Y_{n,j}\}_{j=1,\ldots,2n+1}$ is a complete $\mathcal{L}^2(\Omega)$-orthonormal system in $\mathcal{L}^2(\Omega)$. In particular, every function $F \in \mathcal{L}^2(\Omega)$ can be represented by means of its Fourier series with respect to $\{Y_{n,j}\}$, i. e.

$$F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} F^{\wedge}(n, j)\, Y_{n,j},$$

with the Fourier coefficients

$$F^\wedge(n,j) := (F, Y_{n,j})_{\mathcal{L}^2(\Omega)} = \int_\Omega F(\eta)\, Y_{n,j}(\eta)\, d\omega(\eta).$$

Furthermore, the span of $\{Y_{n,j}\}$ is dense in $\mathcal{C}(\Omega)$ with respect to the $\|\cdot\|_{\mathcal{C}(\Omega)}$-norm.

The addition theorem establishes a connection between spherical harmonics of degree n and the Legendre polynomial P_n of degree n: Let $\{Y_{n,j}\}_{j=1,\ldots,2n+1}$ be an $\mathcal{L}^2(\Omega)$-orthonormal system in $\mathrm{Harm}_n(\Omega)$, then

$$\sum_{j=1}^{2n+1} Y_{n,j}(\xi)\, Y_{n,j}(\eta) = \frac{2n+1}{4\pi}\, P_n(\xi\cdot\eta), \qquad \xi,\eta\in\Omega,\ n\in\mathbb{N}_0.$$

To $K \in \mathcal{L}^1([-1,1])$ we can associate a function $\widetilde{K} : \Omega\times\Omega \to \mathbb{R}$, defined by $(\xi,\eta)\mapsto \widetilde{K}(\xi,\eta) := K(\xi\cdot\eta)$. Then $\widetilde{K}$ is a so-called radial basis function, it is in $\mathcal{C}(\Omega\times\Omega)$, and $\mathcal{L}^p(\Omega\times\Omega)$, $1\le p<\infty$, if and only if K is in $\mathcal{C}([-1,1])$, and $\mathcal{L}^p([-1,1])$, respectively. Furthermore, for $\xi\in\Omega$ fixed, the function $\eta\mapsto K(\xi\cdot\eta)$, $\eta\in\Omega$, is in $\mathcal{C}(\Omega)$, and $\mathcal{L}^p(\Omega)$, $1\le p<\infty$, if and only if K is in $\mathcal{C}([-1,1])$, and $\mathcal{L}^p([-1,1])$, respectively. The $\mathcal{C}$-norm of $\eta\mapsto K(\xi\cdot\eta)$, and the $\mathcal{L}^p$-norm of $\eta\mapsto K(\xi\cdot\eta)$, respectively, do not depend on $\xi\in\Omega$. This result is obvious for continuous functions, and in the case of $\mathcal{L}^p$-functions it is implied by

$$2\pi \int_{-1}^{1} G(t)\, dt = \int_\Omega G(\xi\cdot\eta)\, d\omega(\eta)$$

for all $G\in\mathcal{L}^1([-1,1])$ and for all $\xi\in\Omega$. Let $\{Y_{n,j}\}$ be a complete orthonormal system in $\mathcal{L}^2(\Omega)$ and $K\in\mathcal{L}^2([-1,1])$. Then $(\xi,\eta)\mapsto K(\xi\cdot\eta)$, $\xi,\eta\in\Omega$, has in the $\mathcal{L}^2$-sense the representation

$$K(\xi\cdot\eta) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} K^\wedge(n)\, Y_{n,j}(\xi)\, Y_{n,j}(\eta),$$

where

$$K^\wedge(n) := 2\pi\int_{-1}^{1} K(t)\, P_n(t)\, dt = 2\pi\, (K, P_n)_{\mathcal{L}^2([-1,1])}.$$

The number $K^\wedge(n)$, $n\in\mathbb{N}_0$, is called n-th Legendre coefficient of K.

Let $F\in\mathcal{L}^2(\Omega)$ and $K\in\mathcal{L}^2([-1,1])$, or $F\in\mathcal{C}(\Omega)$ and $K\in\mathcal{L}^1([-1,1])$. Then

$$(K*F)(\xi) := \int_\Omega K(\xi\cdot\eta)\, F(\eta)\, d\omega(\eta), \quad \xi\in\Omega,$$

is well-defined and is in $\mathcal{L}^2(\Omega)$, and in $\mathcal{C}(\Omega)$, respectively. The function $K*F$ is called (spherical) convolution of K and F. For $K\in\mathcal{L}^2([-1,1])$ and $F\in\mathcal{L}^2(\Omega)$, $K*F$ can be expressed as

$$(K*F)(\xi) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} K^\wedge(n)\, F^\wedge(n,j)\, Y_{n,j}(\xi).$$

An important result in the theory of spherical harmonics is the Funk-Hecke formula: Let K be in $\mathcal{L}^1([-1,1])$ and $n\in\mathbb{N}_0$. Then, for every $Y_n\in\mathrm{Harm}_n(\Omega)$,

$$\int_\Omega K(\xi\cdot\eta)\, Y_n(\eta)\, d\omega(\eta) = K^\wedge(n)\, Y_n(\xi), \quad \xi\in\Omega.$$

2 Approximate Identities on the Sphere

In this section approximate identities for continuous functions, and square-integrable functions, respectively, will be presented, i. e. families of operators $\{I_h\}_{h\in(-1,1)}$, $I_h : \mathcal{X}(\Omega)\to\mathcal{X}(\Omega)$, $F\mapsto I_h(F) := K_h * F$, where $\{K_h\}_{h\in(-1,1)}\subset\mathcal{L}^1([-1,1])$ for $\mathcal{X}(\Omega)=\mathcal{C}(\Omega)$, and $\{K_h\}_{h\in(-1,1)}\subset\mathcal{L}^2([-1,1])$ for $\mathcal{X}(\Omega)=\mathcal{L}^2(\Omega)$, which fulfill $\lim_{h\to 1,\, h<1} \|F - I_h(F)\|_{\mathcal{X}(\Omega)} = 0$. Of special interest are approximate identities generated by so-called ($[h,1]$-)locally supported kernels.

In Subsection 2.1 the terminology of singular integrals, approximate identities, and scaling functions will be explained and several equivalent characterizations of an approximate identity will be given. Further background material can be found in [BeBuPa]. In Subsection 2.2 we shall be concerned with approximate identities generated by so-called $[h,1]$-locally supported kernels i. e. kernels $\{K_h\}_{h\in(-1,1)}$ showing the additional property $\mathrm{supp}(K_h) = [h,1]$ for all $h \in (-1,1)$. Error estimates for $\|F - I_h(F)\|_{\mathcal{C}(\Omega)}$ will be presented in explicit form. Finally two particularly important examples of non-negative $[h,1]$-locally supported kernels that generate an approximate identity in $\mathcal{C}(\Omega)$ (and also in $\mathcal{L}^2(\Omega)$) will be discussed in more detail.

2.1 Singular Integrals, Approximate Identities and Scaling Functions

Definition 2.1 Let $\{K_h\}_{h\in(-1,1)}$ be a family of functions in $\mathcal{L}^1([-1,1])$ or in $\mathcal{L}^2([-1,1])$ satisfying the condition $(K_h)^\wedge(0) = 1$ for all $h \in (-1,1)$. Then the family of bounded linear operators $\{I_h\}_{h\in(-1,1)}$, $I_h : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega)$, $F \mapsto I_h(F)$, given by

$$I_h(F)(\xi) := (K_h * F)(\xi) = \int_\Omega K_h(\xi\cdot\eta)\, F(\eta)\, d\omega(\eta),$$

where $\mathcal{X}(\Omega) = \mathcal{C}(\Omega)$ for $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$ and $\mathcal{X}(\Omega) = \mathcal{L}^2(\Omega)$ for $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2([-1,1])$, is called a *(spherical) singular integral.* The family $\{K_h\}_{h\in(-1,1)}$ is called the *kernel* of the (spherical) singular integral. The singular integral $\{I_h\}_{h\in(-1,1)}$ is said to be an *approximate identity (in $\mathcal{X}(\Omega)$) corresponding to the scaling function* $\{K_h\}_{h\in(-1,1)}$, if the following limit relation holds true:

$$\lim_{\substack{h\to 1,\\ h<1}} \|F - I_h(F)\|_{\mathcal{X}(\Omega)} = 0 \qquad \text{for all } F \in \mathcal{X}(\Omega).$$

Note that the assumption that a kernel $\{K_h\}_{h\in(-1,1)}$ be a scaling function implies that $\{I_h\}_{h\in(-1,1)}$ is an approximate identity. The scaling function $\{K_h\}_{h\in(-1,1)}$ is said to generate the approximate identity $\{I_h\}_{h\in(-1,1)}$. The kernel $\{K_h\}_{h\in(-1,1)}$ is called continuous kernel, $\mathcal{L}^1$-kernel, $\mathcal{L}^2$-kernel, non-negative kernel, etc., if all K_h have this property.

Theorem 2.2 *Let $\{K_h\}_{h\in(-1,1)}$ be a family of functions in $\mathcal{L}^1([-1,1])$ or in $\mathcal{L}^2([-1,1])$, which satisfies $(K_h)^\wedge(0) = 1$ for all $h \in (-1,1)$ and which is $\mathcal{L}^1$-uniformly bounded, i. e.*

$$2\pi \int_{-1}^{1} |K_h(t)|\, dt \le M \qquad \text{for all } h \in (-1,1), \tag{1}$$

with some constant M independent of h. Then the spherical singular integral $\{I_h\}_{h\in(-1,1)}$ defined in Definition 2.1 is an approximate identity (in $\mathcal{X}(\Omega)$) if and only if

$$\lim_{\substack{h\to 1,\\ h<1}} (K_h)^\wedge(n) = 1 \qquad \text{for all } n \in \mathbb{N}_0. \tag{2}$$

Proof: The proof has to be performed separately for the two cases $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$ and $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2([-1,1])$. We start with $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$.

$\Longrightarrow$: By the definition of an approximate identity, $\lim_{h\to 1,\, h<1} \|F - I_h(F)\|_{\mathcal{C}(\Omega)} = 0$ for all $F \in \mathcal{C}(\Omega)$. Particularly, this holds for all spherical harmonics Y_n of degree n. The Funk-Hecke formula implies that $I_h(Y_n)(\xi) = (K_h)^\wedge(n)\, Y_n(\xi)$, $\xi \in \Omega$. Thus,

$$0 = \lim_{\substack{h\to 1,\\ h<1}} \|Y_n - I_h(Y_n)\|_{\mathcal{C}(\Omega)} = \lim_{\substack{h\to 1,\\ h<1}} |1 - (K_h)^\wedge(n)|\, \|Y_n\|_{\mathcal{C}(\Omega)},$$

and (2) follows because of $\|Y_n\|_{\mathcal{C}(\Omega)} \neq 0$ for all spherical harmonics $Y_n \not\equiv 0$ of degree $n \in \mathbb{N}_0$.

$\Longleftarrow$: Let $Y_n \in \mathrm{Harm}_n(\Omega)$, $n \in \mathbb{N}_0$, be arbitrary. We know that $I_h(Y_n)(\xi) = (K_h)^\wedge(n)\, Y_n(\xi)$ for all $\xi \in \Omega$, for all $n \in \mathbb{N}_0$, and all $h \in (-1,1)$. Furthermore, $\lim_{h\to 1,\, h<1} (K_h)^\wedge(n) = 1$ for all $n \in \mathbb{N}_0$. Thus,

$$\lim_{\substack{h\to 1,\\ h<1}} \|Y_n - I_h(Y_n)\|_{\mathcal{C}(\Omega)} = \lim_{\substack{h\to 1,\\ h<1}} |1 - (K_h)^\wedge(n)|\, \|Y_n\|_{\mathcal{C}(\Omega)} = 0.$$

So the assertion $\lim_{h\to 1,\, h<1} \|F - I_h(F)\|_{\mathcal{C}(\Omega)} = 0$ is true for all spherical harmonics Y_n of degree $n \in \mathbb{N}_0$.

Let $F \in \mathcal{C}(\Omega)$ be arbitrary. We have to show, that for every $\varepsilon > 0$ there exists an h_0 so that for every $h \in [h_0, 1)$ the estimate $\|F - I_h(F)\|_{\mathcal{C}(\Omega)} \leq \varepsilon$ holds. This is done in the following way: Let $\varepsilon > 0$ be arbitrary. Choose a complete orthonormal system $\{Y_{n,j}\}$ of spherical harmonics in $\mathcal{L}^2(\Omega)$. Then F can be approximated arbitrarily well (with respect to $\|\cdot\|_{\mathcal{C}(\Omega)}$) by finite linear combinations of the $Y_{n,j}$. Let L_F be such a linear combination, so that $\|F - L_F\|_{\mathcal{C}(\Omega)} \leq \min\{\varepsilon/3, \varepsilon/(3M)\}$, where M is the constant given in (1). By the triangle inequality we get

$$\|F - I_h(F)\|_{\mathcal{C}(\Omega)} \leq \|F - L_F\|_{\mathcal{C}(\Omega)} + \|L_F - I_h(L_F)\|_{\mathcal{C}(\Omega)} + \|I_h(L_F) - I_h(F)\|_{\mathcal{C}(\Omega)}. \tag{3}$$

The first summand in (3) can be estimated by $\varepsilon/3$. For the second summand, we use that

$$L_F(\xi) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} l_{n,j}\, Y_{n,j}(\xi),$$

where only a finite number of the coefficients $l_{n,j}$ is different from zero. By virtue of the linearity of I_h,

$$\begin{aligned} |L_F(\xi) - I_h(L_F)(\xi)| &= \left| \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} l_{n,j} \Big(Y_{n,j}(\xi) - I_h(Y_{n,j})(\xi) \Big) \right| \\ &\leq \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |l_{n,j}|\, |Y_{n,j}(\xi) - I_h(Y_{n,j})(\xi)|. \end{aligned}$$

Let N be the number of coefficients $l_{n,j}$, which are not zero. All these coefficients can be estimated by some constant C which depends only on L_F. Choose h_0 so, that for all $h \in [h_0, 1)$, and for all tuples (n, j) with $l_{n,j} \neq 0$ the norm $\|I_h(Y_{n,j}) - Y_{n,j}\|_{\mathcal{C}(\Omega)} \leq \varepsilon/(3NC)$. Then for all $h \in [h_0, 1)$, $\|L_F - I_h(L_F)\|_{\mathcal{C}(\Omega)} \leq NC(\varepsilon/(3NC)) = \varepsilon/3$. To estimate the last summand in (3), the $\mathcal{L}^1$-uniform boundedness of the kernels K_h is used:

$$\begin{aligned} |I_h(L_F)(\xi) - I_h(F)(\xi)| &= \left| \int_\Omega K_h(\xi\cdot\eta)\, (L_F(\eta) - F(\eta))\, d\omega(\eta) \right| \\ &\leq \int_\Omega |K_h(\xi\cdot\eta)|\, |L_F(\eta) - F(\eta)|\, d\omega(\eta) \\ &\leq \|L_F - F\|_{\mathcal{C}(\Omega)}\, 2\pi \int_{-1}^{1} |K_h(t)|\, dt \\ &\leq (\varepsilon/(3M))M = \varepsilon/3 \end{aligned}$$

Hence, (2) implies $\lim_{h\to 1, h<1} \|F - I_h(F)\|_{\mathcal{C}(\Omega)} = 0$ for all $F \in \mathcal{C}(\Omega)$.

In case of $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2([-1,1])$ the proof can be given as follows:

$\Longrightarrow$: This part of the proof is identical to the one for $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$, only the $\|\cdot\|_{\mathcal{C}(\Omega)}$-norm has to be replaced by the $\|\cdot\|_{\mathcal{L}^2(\Omega)}$-norm.

$\Longleftarrow$: The $\mathcal{L}^1$-uniform boundedness (1) of the functions K_h, $h \in (-1,1)$, and $|P_n(t)| \leq 1$ for all $t \in [-1,1]$ and all $n \in \mathbb{N}_0$ imply that

$$(K_h)^\wedge(n) \leq 2\pi \int_{-1}^{1} |K_h(t)|\, |P_n(t)|\, dt \leq 2\pi \int_{-1}^{1} |K_h(t)|\, dt \leq M,$$

$$(K_h)^\wedge(n) \geq -2\pi \int_{-1}^{1} |K_h(t)|\, |P_n(t)|\, dt \geq -2\pi \int_{-1}^{1} |K_h(t)|\, dt \geq -M.$$

Hence, $(K_h)^\wedge(n) \in [-M, M]$ for all $n \in \mathbb{N}_0$ and all $h \in (-1,1)$, and therefore,

$$\|F - I_h(F)\|^2_{\mathcal{L}^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \Big(1 - (K_h)^\wedge(n)\Big)^2 \Big(F^\wedge(n,j)\Big)^2 \le (M+1)^2 \, \|F\|^2_{\mathcal{L}^2(\Omega)}$$

for all $h \in (-1,1)$ and all $F \in \mathcal{L}^2(\Omega)$. As the upper bound $(M+1)$ of $|1 - (K_h)^\wedge(n)|$ is independent of $h \in (-1,1)$, the limit for $h \to 1$ and the sum may be interchanged. Hence,

$$\lim_{\substack{h\to 1,\\ h<1}} \|F - I_h(F)\|_{\mathcal{L}^2(\Omega)} = \left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \lim_{\substack{h\to 1,\\ h<1}} \Big(1 - (K_h)^\wedge(n)\Big)^2 \Big(F^\wedge(n,j)\Big)^2 \right)^{1/2} = 0$$

for all $F \in \mathcal{L}^2(\Omega)$. □

Restricting our attention to non-negative kernels $\{K_h\}_{h\in(-1,1)}$, i. e. all K_h, $h \in (-1,1)$, satisfy $K_h(t) \ge 0$ for almost all $t \in [-1,1]$, more equivalent characterizations of an approximate identity are deducible. The main advantage of non-negative kernels $\{K_h\}_{h\in(-1,1)}$ is that the property $(K_h)^\wedge(0) = 1$ implies

$$1 = (K_h)^\wedge(0) = 2\pi \int_{-1}^{1} K_h(t)\, dt = 2\pi \int_{-1}^{1} |K_h(t)|\, dt = 2\pi \, \|K\|_{\mathcal{L}^1([-1,1])} \, ,$$

i. e. the $\mathcal{L}^1$-uniformly boundedness condition (1) is valid with the sharp bound $M = 1$.

Theorem 2.3 *Let $\{K_h\}_{h\in(-1,1)}$ be a family of functions in $\mathcal{L}^1([-1,1])$ or in $\mathcal{L}^2([-1,1])$, which satisfy $(K_h)^\wedge(0) = 1$ and which are non-negative. Let $\{I_h\}_{h\in(-1,1)}$ be the spherical singular integral defined in Definition 2.1. Then the following properties are equivalent:*

(i) $\{K_h\}_{h\in(-1,1)}$ *is a non-negative scaling function,*
(ii) $\{I_h\}_{h\in(-1,1)}$ *is an approximate identity,*
(iii) $\lim_{h\to 1,\, h<1} (K_h)^\wedge(n) = 1$ *for all* $n \in \mathbb{N}_0$,
(iv) $\lim_{h\to 1,\, h<1} (K_h)^\wedge(1) = 1$,
(v) $\{K_h\}_{h\in(-1,1)}$ *satisfies the localization property* $\lim_{h\to 1,\, h<1} \int_{-1}^{\delta} K_h(t)\, dt = 0$ *for all* $\delta \in (-1,1)$.

Proof: The following proof is valid for both cases, $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$ and $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2([-1,1])$.

The statements (i) and (ii) are equivalent by definition and the equivalence of (ii) and (iii) was proved in Theorem 2.2. Obviously, (iii) implies (iv). It remains to show, that (v) follows from (iv) and that (v) implies (iii).

(iv) $\Longrightarrow$ (v): Let $\delta \in (-1,1)$ be arbitrary. Because of the non-negativity of K_h,

$$\begin{aligned} 0 \;\le\; \int_{-1}^{\delta} K_h(t)\, dt \;&\le\; \frac{1}{(1-\delta)} \int_{-1}^{\delta} (1-t)\, K_h(t)\, dt \\ &\le\; \frac{1}{(1-\delta)} \int_{-1}^{1} (1-t)\, K_h(t)\, dt \\ &=\; \frac{1}{2\pi}\, \frac{1}{(1-\delta)} \Big((K_h)^\wedge(0) - (K_h)^\wedge(1) \Big). \end{aligned}$$

Taking the limit for $h \to 1$ the localization property follows from (iv).

(v) $\Longrightarrow$ (iii): Property (iii) is equivalent to the following assertion: For every $n \in \mathbb{N}$, and for every $\varepsilon > 0$, there exists $h_0 = h_0(\varepsilon, n) \in (-1,1)$ such that $1 - \varepsilon \le (K_h)^\wedge(n)) \le 1$ for all $h \in [h_0, 1)$. By the non-negativity of K_h and the estimate $|P_n(t)| \le 1$ for all $n \in \mathbb{N}_0$,

$$(K_h)^\wedge(n) = 2\pi \int_{-1}^{1} K_h(t)\, P_n(t)\, dt \le 2\pi \int_{-1}^{1} K_h(t)\, dt = (K_h)^\wedge(0) = 1.$$

Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. For $\delta \in (-1,1)$,

$$(K_h)^\wedge(n) = 2\pi \int_{-1}^{\delta} K_h(t)\, P_n(t)\, dt + 2\pi \int_{\delta}^{1} K_h(t)\, P_n(t)\, dt.$$

As $P_n(1) = 1$, $\delta \in (-1,1)$ can be chosen so close to 1, that $P_n(t) \geq \sqrt{1-(\varepsilon/2)}$ for all $t \in [\delta, 1]$. Thus,

$$(K_h)^\wedge(n) \geq 2\pi \int_{-1}^{\delta} K_h(t)\, P_n(t)\, dt + 2\pi \sqrt{1-(\varepsilon/2)} \int_{\delta}^{1} K_h(t)\, dt. \tag{4}$$

As $|P_n(t)| \leq 1$, for all $\delta \in (-1,1)$

$$-2\pi \int_{-1}^{\delta} K_h(t)\, dt \leq 2\pi \int_{-1}^{\delta} K_h(t)\, P_n(t)\, dt \leq 2\pi \int_{-1}^{\delta} K_h(t)\, dt.$$

Therefore the localization property (v) implies that there exists $h_1 \in (-1,1)$, such that the estimate $2\pi \int_{-1}^{\delta} K_h(t)\, P_n(t)\, dt \geq -\varepsilon/2$ is valid for all $h \in [h_1, 1)$. On the other hand, $(K_h)^\wedge(0) = 1$ for all $h \in (-1,1)$, and the localization property implies

$$\frac{1}{2\pi} = \lim_{\substack{h\to 1,\\ h<1}} \int_{-1}^{1} K_h(t)\, dt = \lim_{\substack{h\to 1,\\ h<1}} \int_{-1}^{\delta} K_h(t)\, dt + \lim_{\substack{h\to 1,\\ h<1}} \int_{\delta}^{1} K_h(t)\, dt = \lim_{\substack{h\to 1,\\ h<1}} \int_{\delta}^{1} K_h(t)\, dt.$$

Hence, there exists $h_2 \in (-1,1)$ such that $2\pi \int_{\delta}^{1} K_h(t)\, dt \geq \sqrt{1-(\varepsilon/2)}$ for all $h \in [h_2, 1)$. Equation (4) implies $1 - \varepsilon \leq (K_h)^\wedge(n) \leq 1$ for all $h \in [h_0, 1)$ where $h_0 := \max\{h_1, h_2\}$. □

Finally, it is worth mentioning that a non-negative scaling function $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$, and $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2([-1,1])$, respectively, which fulfills the assumptions of Theorem 2.3, satisfies the estimate $\|K_h * F\|_{\mathcal{X}(\Omega)} \leq \|F\|_{\mathcal{X}(\Omega)}$ for all $h \in (-1,1)$ and for all $F \in \mathcal{X}(\Omega)$, where $\mathcal{X}(\Omega) = \mathcal{C}(\Omega)$, and $\mathcal{X}(\Omega) = \mathcal{L}^2(\Omega)$, respectively.

2.2 Approximate Identities Generated by $[h,1]$-Locally Supported Scaling Functions

In this subsection approximate identities generated by so-called (non-negative) $[h,1]$-locally supported kernels $\{K_h\}_{h\in(-1,1)}$ are investigated. Due to their non-negativity and local support, i. e. $\mathrm{supp}(K_h) = [h,1]$, such kernels generate an approximate identity, and therefore, are scaling functions. The approximation error $\|F - I_h(F)\|_{\mathcal{C}(\Omega)}$ will be estimated for approximate identities generated by such non-negative $[h,1]$-locally supported scaling functions $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$. Finally we shall present some examples, namely the Haar scaling function, and the smoothed Haar scaling functions. Illustrations of these examples will be shown in Subsection 3.2, together with the figures of the corresponding difference wavelets. We note that all the examples are at least piecewise continuous and consequently also $\mathcal{L}^2$-scaling functions. Non-negative $[h,1]$-locally supported scaling functions have several advantages in wavelet approximation as it will be explained in detail in Section 3 and Section 4 of this paper.

Our considerations start with the definition of $[h,1]$-locally supported kernels.

Definition 2.4 Let $\{K_h\}_{h\in(-1,1)}$ be a family of functions in $\mathcal{L}^1([-1,1])$ or in $\mathcal{L}^2([-1,1])$ that satisfy $(K_h)^\wedge(0) = 1$ for all $h \in (-1,1)$ and $\mathrm{supp}(K_h) = [h,1]$ for all $h \in (-1,1)$. Then $\{K_h\}_{h\in(-1,1)}$ is called an $[h,1]$-*locally supported kernel.*

Next we mention the following important property of non-negative $[h,1]$-locally supported kernels.

Theorem 2.5 *Suppose $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$ or $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2([-1,1])$ is a non-negative $[h,1]$-locally supported kernel. Then the family $\{I_h\}_{h\in(-1,1)}$ defined in Definition 2.1 is an approximate identity corresponding to the $[h,1]$-locally supported non-negative scaling function $\{K_h\}_{h\in(-1,1)}$.*

Proof: The family $\{K_h\}_{h\in(-1,1)}$ satisfies the assumptions of Theorem 2.3 and obeys the localization property because of $\mathrm{supp}(K_h)=[h,1]$. Thus the assertion follows from Theorem 2.3. $\square$

Lemma 2.6 *Let $\{K_h\}_{h\in(-1,1)}\subset\mathcal{L}^1([-1,1])$ be a non-negative $[h,1]$-locally supported scaling function. Suppose that $\{I_h\}_{h\in(-1,1)}$, $I_h:\mathcal{C}(\Omega)\to\mathcal{C}(\Omega)$, $F\mapsto I_h(F):=K_h*F$ is the approximate identity generated by $\{K_h\}_{h\in(-1,1)}$. Assume that F is of class $\mathcal{C}(\Omega)$. Then,*

$$\begin{aligned}|F(\xi)-I_h(F)(\xi)| \quad\leq\quad & \left(2\pi\int_h^{\widetilde{h}} K_h(t)\,dt\right)\sup_{\substack{\eta\in\Omega,\\ h\leq\xi\cdot\eta\leq\widetilde{h}}}|F(\xi)-F(\eta)| \\ &+\left(2\pi\int_{\widetilde{h}}^1 K_h(t)\,dt\right)\sup_{\substack{\eta\in\Omega,\\ \widetilde{h}\leq\xi\cdot\eta\leq 1}}|F(\xi)-F(\eta)|\end{aligned}\tag{5}$$

for every $\widetilde{h}\in[h,1]$ and for all $\xi\in\Omega$. If $F\in\mathcal{C}(\Omega)$ is additionally Lipschitz-continuous with Lipschitz-constant C_F, then

$$\|F-I_h(F)\|_{\mathcal{C}(\Omega)}\leq 2\pi\sqrt{2}\,C_F\left[\left(\int_h^{\widetilde{h}}K_h(t)\,dt\right)(1-h)^{1/2}+\left(\int_{\widetilde{h}}^1 K_h(t)\,dt\right)(1-\widetilde{h})^{1/2}\right]\tag{6}$$

for every $\widetilde{h}\in[h,1]$.

Proof: Let $F\in\mathcal{C}(\Omega)$ be arbitrary. For $\xi\in\Omega$ fixed but arbitrary, $\|K_h(\xi\cdot\eta)\|_{\mathcal{L}^1(\Omega)}=(K_h)^\wedge(0)=1$. This implies

$$\begin{aligned}|F(\xi)-I_h(F)(\xi)| \quad=\quad & \left|\int_{\substack{\eta\in\Omega,\\ h\leq\xi\cdot\eta\leq 1}} K_h(\xi\cdot\eta)\,(F(\xi)-F(\eta))\,d\omega(\eta)\right| \\ \leq\quad & \int_{\substack{\eta\in\Omega,\\ h\leq\xi\cdot\eta\leq\widetilde{h}}} K_h(\xi\cdot\eta)\,|F(\xi)-F(\eta)|\,d\omega(\eta)+\int_{\substack{\eta\in\Omega,\\ \widetilde{h}\leq\xi\cdot\eta\leq 1}} K_h(\xi\cdot\eta)\,|F(\xi)-F(\eta)|\,d\omega(\eta) \\ \leq\quad & \left(2\pi\int_h^{\widetilde{h}}K_h(t)\,dt\right)\sup_{\substack{\eta\in\Omega,\\ h\leq\xi\cdot\eta\leq\widetilde{h}}}|F(\xi)-F(\eta)|+\left(2\pi\int_{\widetilde{h}}^1 K_h(t)\,dt\right)\sup_{\substack{\eta\in\Omega,\\ \widetilde{h}\leq\xi\cdot\eta\leq 1}}|F(\xi)-F(\eta)|\end{aligned}$$

for every $\widetilde{h}\in[h,1]$ and for all $\xi\in\Omega$. This proves (5). To verify (6) note that $|\xi-\eta|^2=(\xi-\eta)\cdot(\xi-\eta)=2(1-\xi\cdot\eta)$ for all $\xi,\eta\in\Omega$. If the function F is additionally Lipschitz-continuous, then $|F(\xi)-F(\eta)|\leq C_F\sqrt{2}\,|1-\xi\cdot\eta|^{1/2}\leq C_F\sqrt{2}\,(1-h)^{1/2}$ for all $\xi,\eta\in\Omega$ satisfying $h\leq\xi\cdot\eta\leq 1$, and $|F(\xi)-F(\eta)|\leq C_F\sqrt{2}\,(1-\widetilde{h})^{1/2}$ for all $\xi,\eta\in\Omega$ satisfying $\widetilde{h}\leq\xi\cdot\eta\leq 1$. Hence, (6) follows from (5). $\square$

The reason for the introduction of the parameter $\widetilde{h}\in[h,1]$ in the previous lemma is the following: In many cases the kernel K_h will assume only very small values on a certain subset $[h,\widetilde{h}]$ of $[h,1]$ and grow extremely fast on $[\widetilde{h},1]$. Therefore the subdivision of the interval $[h,1]$ into $[h,\widetilde{h}]$ and $[\widetilde{h},1]$, $\widetilde{h}\neq 1,h$ chosen suitably, will yield much better results than the cases $\widetilde{h}=1$ or $\widetilde{h}=h$ in estimate (6).

In what follows, we present some examples of non-negative $[h,1]$-locally supported scaling functions. The Legendre coefficients, the supremum norm, and Lipschitz-constants of the examples are calculated, because these constants will be needed for our later considerations (in Section 4 and Section 5).

Example 2.7 The *Haar scaling function* $\{H_h\}_{h\in(-1,1)}\subset\mathcal{L}^2([-1,1])$, $H_h:[-1,1]\to\mathbb{R}$, $t\mapsto H_h(t)$, is given by

$$H_h(t):=\begin{cases}0 & \text{if}\quad t\in[-1,h)\\ \dfrac{1}{2\pi}\,\dfrac{1}{(1-h)} & \text{if}\quad t\in[h,1].\end{cases}$$

Obviously, $H_h(t) \geq 0$ for all $t \in [-1,1]$ and $(H_h)^\wedge(0) = 2\pi \|H_h\|_{\mathcal{L}^1([-1,1])} = 1$ are fulfilled. Thus $\{H_h\}_{h\in(-1,1)}$ generates an approximate identity in $\mathcal{C}(\Omega)$ (and in $\mathcal{L}^2(\Omega)$). Further properties of the Haar scaling function follow in the next example by specialization.

Example 2.8 Let $k \in \mathbb{N}_0$. The *smoothed Haar scaling function* $\{L_h^{(k)}\}_{h\in(-1,1)} \subset \mathcal{C}^{k-1}([-1,1])$ is defined by $L_h^{(k)} : [-1,1] \to \mathbb{R}$, $t \mapsto L_h^{(k)}(t)$,

$$L_h^{(k)}(t) := ((B_h^{(k)})^\wedge(0))^{-1} B_h^{(k)}(t) \quad \text{with} \quad B_h^{(k)}(t) := \begin{cases} 0 & \text{if} \quad t \in [-1,h) \\ \dfrac{(t-h)^k}{(1-h)^k} & \text{if} \quad t \in [h,1]. \end{cases}$$

By definition, $L_h^{(k)}$ is non-negative, has the support $[h,1]$, and satisfies $(L_h^{(k)})^\wedge(0) = 1$. Hence it is a non-negative $[h,1]$-locally supported scaling function. The function $L_h^{(0)}$, $h \in (-1,1)$, coincides with the Haar function H_h. The Legendre coefficients of $B_h^{(k)}$ and, hence, $L_h^{(k)}$, $h \in (-1,1)$, $k \in \mathbb{N}_0$, can be calculated recursively (cf. [FrGeSchr]):

$$(B_h^{(k)})^\wedge(0) = 2\pi \left(\frac{1-h}{k+1}\right) \neq 0, \quad (B_h^{(k)})^\wedge(1) = 2\pi \left(\frac{1-h}{k+1}\right)\left(1 - \frac{1-h}{k+2}\right), \tag{7}$$

$$(B_h^{(k)})^\wedge(n+1) = \left(\frac{2n+1}{n+k+2}\right) h\,(B_h^{(k)})^\wedge(n) + \left(\frac{k+1-n}{n+k+2}\right)(B_h^{(k)})^\wedge(n-1). \tag{8}$$

It can be shown that $|(L_h^{(k)})^\wedge(n)| = O\left([n(1-h)]^{-(3/2)-k}\right)$ for $n \to \infty$. The functions $L_h^{(k)}$, $h \in (-1,1)$, $k \in \mathbb{N}_0$, assume their maximum in $t = 1$. For $k > 2$ the Lipschitz-constant $C_h^{(k)}$ for $L_h^{(k)}$ can be chosen as the maximum of the first derivative, which is also taken in the point $t = 1$. Thus, we obtain

$$\|L_h^{(k)}\|_{\mathcal{C}([-1,1])} = L_h^{(k)}(1) = \frac{1}{2\pi}\frac{(k+1)}{(1-h)}, \qquad k \in \mathbb{N}_0,$$

and

$$C_h^{(k)} := \|(L_h^{(k)})'\|_{\mathcal{C}([-1,1])} = (L_h^{(k)})'(1) = \frac{1}{2\pi}\frac{k(k+1)}{(1-h)^2}, \qquad k \geq 2. \tag{9}$$

The function $L_h^{(0)}$ is constant on its support. Consequently, Equation (9) is also valid for $k = 0$ on $\mathrm{supp}(L_h^{(0)}) = [h,1]$. For $k = 1$ the function $L_h^{(k)}$ is continuous and piecewise linear, thus the Lipschitz-constant $C_h^{(1)}$ can be chosen as the first derivative of $L_h^{(1)}$ on $\mathrm{supp}(L_h^{(1)})$. Hence, Equation (9) is also true for $k = 1$.

3 Spherical Difference Wavelets

A scaling function $\{K_h\}_{h\in(-1,1)}$ generates an approximate identity, which provides nothing else than a sequence of low-pass filters. In this section so-called spherical difference wavelets will be introduced to describe the difference between such low-pass filters. In other words, spherical difference wavelets are understood to act as band-pass filters as the difference of two smoothings.

In Subsection 3.1 the basic definitions will be given and the decomposition and the reconstruction of the approximation $I_h(F) := K_h * F$ of F with spherical difference wavelets will be developed. It will be explained why such a reconstruction of the approximation is not only of theoretical value but is also useful in practical applications. In Subsection 3.2 the spherical difference wavelets will be computed for all examples of $[h,1]$-locally supported non-negative scaling functions known from Subsection 2.2.

3.1 Reconstruction Formula

Definition 3.1 Let $\{I_h\}_{h\in(-1,1)}$ be an approximate identity in $\mathcal{C}(\Omega)$ or $\mathcal{L}^2(\Omega)$, generated by the scaling function $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$, and $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2([-1,1])$, respectively. Let $\{h_j\}_{j\in\mathbb{N}_0} \subset$

$(-1,1]$ be a strict monotonically increasing sequence with $\lim_{j\to\infty} h_j = 1$. Define the sequence $\{T_j\}_{j\in\mathbb{N}_0}$ of bounded linear operators

$$T_j : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega),\ F \mapsto T_j(F) := I_{h_j}(F) = K_{h_j} * F,$$

where $\mathcal{X}(\Omega) = \mathcal{C}(\Omega)$ for $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$, and $\mathcal{X}(\Omega) = \mathcal{L}^2(\Omega)$ for $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2([-1,1])$, respectively. The family $\{\Psi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{L}^1([-1,1])$, and $\{\Psi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{L}^2([-1,1])$, respectively, given by

$$\Psi_j := K_{h_{j+1}} - K_{h_j}$$

is called *spherical difference wavelet (corresponding to the scaling function* $\{K_{h_j}\}_{j\in\mathbb{N}_0}$*)*. Furthermore, define a family $\{R_j\}_{j\in\mathbb{N}_0}$ of bounded linear operators

$$R_j : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega),\ F \mapsto R_j(F) := \Psi_j * F.$$

Note that the spherical difference wavelet $\{\Psi_j\}_{j\in\mathbb{N}_0}$ (corresponding to a scaling function $\{K_h\}_{h\in(-1,1)}$) satisfies $\int_{-1}^{1} \Psi_j(t)\,dt = 0$ for all $j \in \mathbb{N}_0$ because of $(K_h)^\wedge(0) = 1$ for all $h \in (-1,1)$.

Definition 3.2 Let $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1([-1,1])$ or $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2([-1,1])$ be an $[h,1]$-locally supported scaling function, and let $\{h_j\}_{j\in\mathbb{N}_0} \subset (-1,1]$ be a strict monotonically increasing sequence with $\lim_{j\to\infty} h_j = 1$. Then $\{\Psi_j\}_{j\in\mathbb{N}_0}$ is called an $[h,1]$*-locally supported spherical difference wavelet (corresponding to the* $[h,1]$*-locally supported scaling function* $\{K_{h_j}\}_{j\in\mathbb{N}_0}$*)*.

For the remainder of this paper we will call the spherical difference wavelets briefly wavelets, because they are the only type of wavelets regarded in this paper. The next theorem shows, that the low-pass filter T_J, $J \in \mathbb{N}_0$, can be decomposed into a sum of the low-pass filter T_{J_0} and the band-pass filters R_j, $j \in \{J_0, J_0+1, \ldots, J-1\}$ and, thus, be reconstructed as a sum of the latter.

Theorem 3.3 *Let the assumptions and the notation be as in Definition 3.1. Then,*

$$\lim_{J\to\infty} \|F - T_J(F)\|_{\mathcal{X}(\Omega)} = 0 \qquad \textit{for all } F \in \mathcal{X}(\Omega),$$

where

$$T_J(F) = T_{J-1}(F) + R_{J-1}(F) = T_{J_0}(F) + \sum_{j=J_0}^{J-1} R_j(F) \tag{10}$$

for all $J, J_0 \in \mathbb{N}_0$, $0 \le J_0 \le J-1$. *Equation (10) is called a reconstruction of the approximation* $T_J(F)$*. Particularly,*

$$F = T_{J_0}(F) + \sum_{j=J_0}^{\infty} R_j(F) \tag{11}$$

in $\mathcal{X}(\Omega)$*-sense.*

Proof: Equation (10) is a consequence of the definitions of the operators T_j and R_j:

$$T_J(F) = K_{h_J} * F = (K_{h_{J-1}} + \Psi_{J-1}) * F = T_{J-1}(F) + R_{J-1}(F).$$

This proves the first equality. The second equality follows analogously by repeating this process for $T_{J-1}(F), \ldots, T_{J_0+1}(F)$. Equation (11) is a consequence of Equation (10) and the fact that $T_J(F)$ converges to F (in $\mathcal{X}(\Omega)$) for $J \to \infty$. □

Usefulness of the reconstruction (10) in practical applications. Suppose the function $F \in \mathcal{C}(\Omega_\gamma)$ describes some geophysically relevant quantity in a technical problem, which is measured in a huge number of points on a sphere Ω_γ of radius γ around the origin. (Measurements done by satellites surrounding the earth, whose orbits cover approximately the sphere Ω_γ at orbital height, are a typical example of such a situation.) Modelling the function F amounts to gaining a good approximation of F from the measured data. The approximation of F can be performed with the techniques[1] introduced in Section 2, where the evaluation of $(K_h^\gamma * F)(x)$, $x \in \Omega_\gamma$, has to be done by aid of a suitable numerical integration scheme. Some reasons should be listed why it is interesting to use the above reconstruction theorem in the aforementioned applications.

Wavelet thresholding. It might be useful to reconstruct the approximation $T_J(F) = K_{h_J}^\gamma * F$ as the sum of a 'low-frequency approximation' $T_{J_0}(F)$ and the 'meso-frequency approximations' $R_j(F)$, $j = J_0, J_0+1, \dots, J-1$, for doing wavelet thresholding (cf. [FrMiSt]). This is particularly interesting, when the sequences $\{(K_{h_j}^\gamma)^\wedge(n)\}_{j\in\mathbb{N}_0}$ of Legendre coefficients of the generating scaling function $\{K_{h_j}^\gamma\}_{j\in\mathbb{N}_0}$ are strict monotonically increasing for all $n \in \mathbb{N}_0$.

Improving an existing model (locally). Suppose h_0 is chosen such that $T_0(F) = K_{h_0}^\gamma * F$ is the best available model that could be gained from the measured data. If the quantity F is in a certain small area $\mathcal{D} \subset \Omega_\gamma$ less smooth than on the rest of the sphere, then the accuracy of the approximation $T_0(F)$ of F on $\mathcal{D}$ will in general not be as good as on $\Omega_\gamma \setminus \mathcal{D}$. Using an $[\gamma^2 h, \gamma^2]$-locally supported scaling function $\{K_h^\gamma\}_{h\in(-1,1)}$ the integration in $(K_h^\gamma * F)(x)$ is actually an integration over the spherical cap $\Gamma_{\gamma^2 h}(x) := \{w \in \Omega_\gamma \mid \gamma^2 h \le x\cdot w \le \gamma^2\}$. To improve the model of F on $\mathcal{D}$ it is only necessary to make additional measurements in a certain neighborhood of $\mathcal{D}$. With these new data $R_0(F)(x) = (\Psi_0^\gamma * F)(x)$ is calculated for points $x \in \mathcal{D}$ and added to the old approximation $T_0(F)(x)$ in $x \in \mathcal{D}$. (The parameter h_1 has to be chosen in adaptation to the new data and the properties of the numerical integration formula.) $T_1(F)(x) = T_0(F)(x) + R_0(F)(x)$, $x \in \mathcal{D}$, is an improved local model of F on $\mathcal{D}$. The same procedure as described above can be carried out, if there is already an existing model $T_0(F)$ and more data become available on a suitable neighborhood of a subset $\mathcal{D}$. Calculating $R_0(F)(x)$, $x \in \mathcal{D}$, and adding it to $T_0(F)(x)$ delivers locally (on $\mathcal{D}$) a better approximation. In particular, an already existing global model $T_0(F)$ can be improved by calculating and adding $R_0(F)$ if more data become available everywhere on Ω_γ. Furthermore, data of scattered data coverage density can be used.

3.2 Examples of $[h,1]$-Locally Supported Difference Wavelets

In this subsection the spherical difference wavelets are computed for the examples of $[h,1]$-locally supported scaling functions known from Subsection 2.2. In all the examples the same strict monotonically increasing sequence $\{h_j\}_{j\in\mathbb{N}_0}$, $h_j := 1-2^{-j}$, is used. Additionally (an estimate of) the supremum norm $\|\Psi_j\|_{\mathcal{C}([-1,1])}$ and Lipschitz-constants for Ψ_j are deduced explicitly, because these constants are needed in Section 4.

Example 3.4 For the sequence $\{h_j\}_{j\in\mathbb{N}_0}$, $h_j := 1-2^{-j}$, the *Haar scaling function* $\{H_{h_j}\}_{j\in\mathbb{N}_0} \subset \mathcal{L}^2([-1,1])$ is given by

$$H_{h_j}(t) := \begin{cases} 0 & \text{if } \ t \in [-1, 1-2^{-j}) \\ (2\pi)^{-1}\, 2^j & \text{if } \ t \in [1-2^{-j}, 1]. \end{cases}$$

The corresponding *Haar wavelet* is the family $\{\Psi_j^H\}_{j\in\mathbb{N}_0} \subset \mathcal{L}^2([-1,1])$ of functions $\Psi_j^H : [-1,1] \to \mathbb{R}$, $t \mapsto \Psi_j^H(t)$, given by

$$\Psi_j^H(t) := \begin{cases} 0 & \text{if } \ t \in [-1, 1-2^{-j}) \\ -(2\pi)^{-1}\, 2^j & \text{if } \ t \in [1-2^{-j}, 1-2^{-(j+1)}) \\ (2\pi)^{-1}\, 2^j & \text{if } \ t \in [1-2^{-(j+1)}, 1]. \end{cases}$$

Obviously $\|\Psi_j^H\|_{\mathcal{C}([-1,1])} = (2\pi)^{-1}\, 2^j = \|H_{h_j}\|_{\mathcal{C}([-1,1])}$. The function Ψ_j^H is not Lipschitz-continuous on its support, because of the jump at $t = 1-2^{-(j+1)}$, but it is Lipschitz-continuous with Lipschitz-constant $\widetilde{C}_j^{(0)} = 0$ on each subset of its support, where Ψ_j^H is constant, i.e. on the intervals $[1-2^{-j}, 1-2^{-(j+1)})$ and $[1-2^{-(j+1)}, 1]$.

[1] These techniques can be easily adapted to spheres Ω_γ of arbitrary radius γ as it will be sketched briefly at the beginning of Section 5. Referring to the remarks in Section 5, we will denote a scaling function and the corresponding wavelet for the approximation of functions $F \in \mathcal{C}(\Omega_\gamma)$ with an upper index γ.

The Haar wavelet has additionally the property, that

$$\int_{-1}^{1} \Psi_j^H(t)\,\Psi_k^H(t)\,dt = 0, \qquad j,k \in \mathbb{N}_0\,,\ j \neq k, \tag{12}$$

$$\int_{-1}^{1} H_{h_0}(t)\,\Psi_j^H(t)\,dt = 0, \qquad j \in \mathbb{N}_0\,. \tag{13}$$

This implies that $H_{h_0}(\xi\cdot\eta), \Psi_0^H(\xi\cdot\eta), \Psi_1^H(\xi\cdot\eta), \ldots$ as functions of $\eta \in \Omega$, where $\xi \in \Omega$ is kept fixed, are orthogonal. Equations (12) and (13) follow easily from the definition $\Psi_j^H(t) = H_{h_{j+1}}(t) - H_{h_j}(t)$, $t \in [-1,1]$, and calculating the integrals.

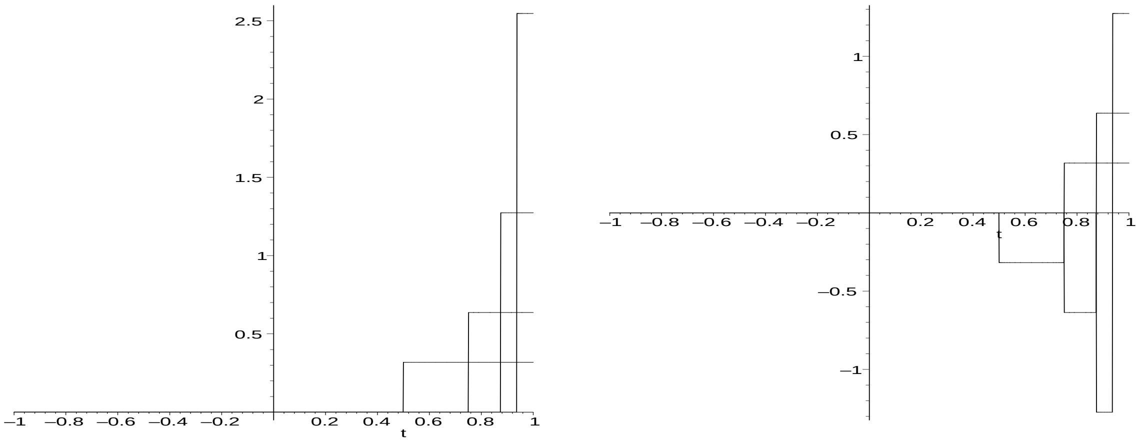

Figure 1: Haar functions for $j = 1,2,3,4$ and the corresponding Haar wavelet functions

Example 3.5 For the sequence $\{h_j\}_{j\in\mathbb{N}_0}$, $h_j := 1-2^{-j}$, the *smoothed Haar scaling functions* $\{L_{h_j}^{(k)}\}_{j\in\mathbb{N}_0} \subset \mathcal{C}^{k-1}([-1,1])$, $k \in \mathbb{N}_0$, are defined by

$$L_{h_j}^{(k)}(t) := \begin{cases} 0 & \text{if} \quad t \in [-1, 1-2^{-j}) \\ (2\pi)^{-1}\,(k+1)\,2^{j(k+1)}(t-1+2^{-j})^k & \text{if} \quad t \in [1-2^{-j}, 1]. \end{cases}$$

The supremum norm and a Lipschitz-constant $C_j^{(k)}$ of $L_{h_j}^{(k)}$ are given by

$$\|L_{h_j}^{(k)}\|_{\mathcal{C}([-1,1])} = \frac{1}{2\pi}\,(k+1)\,2^j, \quad k \in \mathbb{N}_0\,, \qquad \text{and} \qquad C_j^{(k)} := \frac{1}{2\pi}\,k(k+1)\,2^{2j}, \quad k \in \mathbb{N}.$$

The corresponding *smoothed Haar wavelets* are the families $\{\Psi_j^{(k)}\}_{j\in\mathbb{N}_0} \subset \mathcal{C}^{k-1}([-1,1])$, $k \in \mathbb{N}_0$, of functions $\Psi_j^{(k)} : [-1,1] \to \mathbb{R}$, $t \mapsto \Psi_j^{(k)}(t)$, given by

$$\Psi_j^{(k)}(t) := \begin{cases} 0 & \text{if } t \in [-1, 1-2^{-j}) \\ -(2\pi)^{-1}(k+1)2^{j(k+1)}(t-1+2^{-j})^k & \text{if } t \in [1-2^{-j}, 1-2^{-(j+1)}) \\ (2\pi)^{-1}(k+1)2^{j(k+1)}\left[\,2^{k+1}(t-1+2^{-(j+1)})^k - (t-1+2^{-j})^k\,\right] & \text{if } t \in [1-2^{-(j+1)}, 1]. \end{cases}$$

An estimate for the supremum norm of $\Psi_j^{(k)}$ and a Lipschitz-constant $\widetilde{C}_j^{(k)}$ for $\Psi_j^{(k)}$, $k \in \mathbb{N}$, can be obtained by using that $\Psi_j^{(k)}$ and its (piecewise defined) first derivative $(\Psi_j^{(k)})'$ are each the difference of two non-negative functions. Thus, $\|\Psi_j^{(k)}\|_{\mathcal{C}([-1,1])}$, and $\|(\Psi_j^{(k)})'\|_{\mathcal{C}([-1,1])}$ respectively, are not larger than the maximum of the supremum norms of the two non-negative functions. Hence,

$$\|\Psi_j^{(k)}\|_{\mathcal{C}([-1,1])} \leq \|L_{h_{j+1}}^{(k)}\|_{\mathcal{C}([-1,1])} = \frac{1}{2\pi}\,(k+1)\,2^{j+1},$$

$$\widetilde{C}_j^{(k)} := \|(L_{h_{j+1}}^{(k)})'\|_{\mathcal{C}([-1,1])} = (L_{h_{j+1}}^{(k)})'(1) = \frac{1}{2\pi}\, k(k+1)\, 2^{2j+2} = C_{j+1}^{(k)}.$$

$\{\Psi_j^{(0)}\}_{j\in\mathbb{N}_0}$ is the Haar wavelet, which was discussed in Example 3.4 in detail.

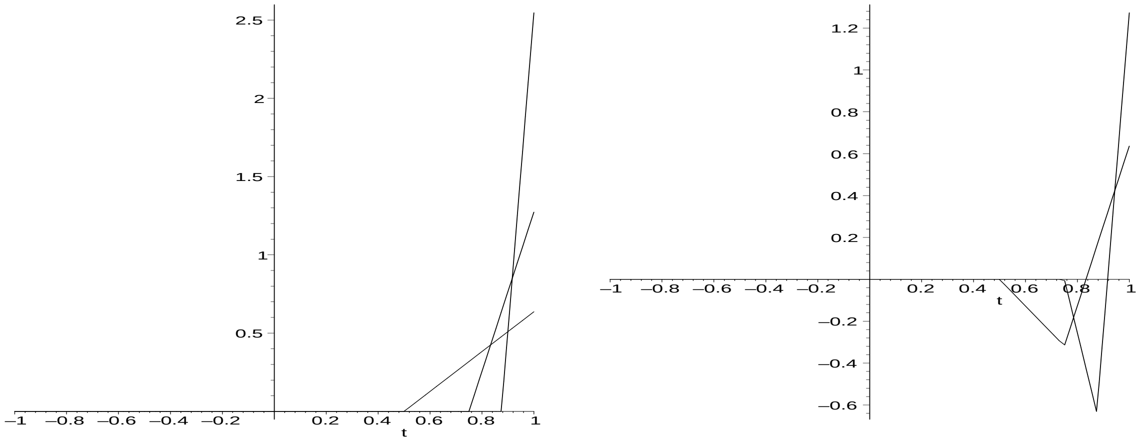

Figure 2: Smoothed Haar functions in case $k = 1$ for $j = 1, 2, 3$ and the corresponding smoothed Haar wavelet functions

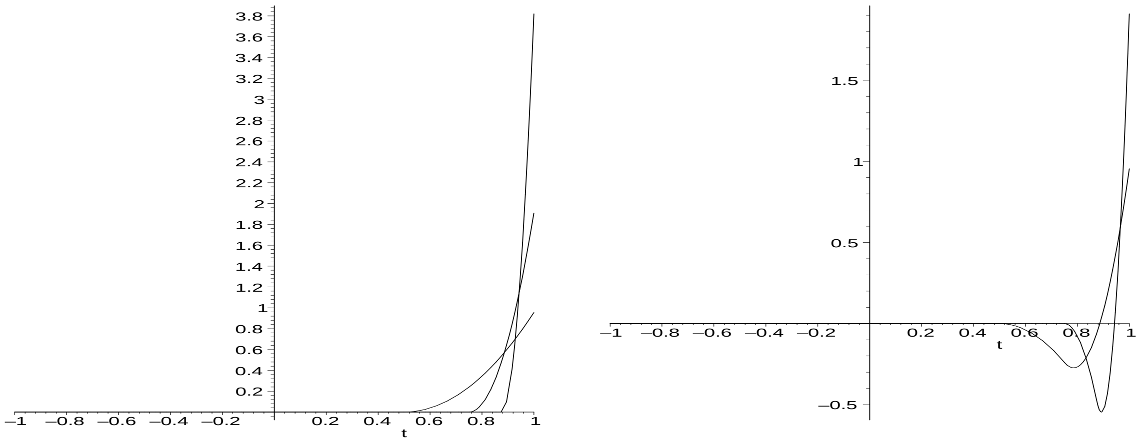

Figure 3: Smoothed Haar functions in case $k = 2$ for $j = 1, 2, 3$ and the corresponding smoothed Haar wavelet functions

4 Numerical Computation of the Approximation

In this section we will focus our attention on the numerical aspects of approximate identities in $\mathcal{C}(\Omega)$ and their wavelet reconstruction. The approximation $T_J(F)$ of $F \in \mathcal{C}(\Omega)$ and its reconstruction from $T_{J_0}(F)$ by means of $R_j(F)$, $j = J_0, J_0+1, \ldots, J-1$, will be discussed from numerical point of view. An approximate integration rule for the evaluation of the convolution integrals $T_j(F)(\xi)$, and $R_j(F)(\xi)$, respectively, will be derived, where the integrands are supposed to be at least continuous. This is no serious restriction as the functions K_h can be chosen continuous and the approximated functions F can be assumed to be continuous in the practical applications we are interested in. As mentioned above, the theory in this paper is meant for satellite problems, where huge amounts of data are given, i. e. F is known at a very large set of points on the (orbital) sphere, and where the data are equidistributed on the (orbital) sphere in the sense of Weyl (see [We]). There are various integration rules for the numerical evaluation of the convolution integrals $(K_{h_j} * F)(\xi)$ and $(\Psi_j * F)(\xi)$, but most of these integration techniques are not applicable to problems, where millions of data occur. In such cases summing up the values of the integrand in the data points with identical integration weights seems to be a suitable way to cope with the large amount of data.

In Subsection 4.1 the approximate integration rule including an error estimate will be formulated and applied to the reconstruction of a continuous function. In Subsection 4.2 the additional advantages of approximate identities generated by an $[h,1]$-locally supported scaling function $\{K_h\}_{h\in(-1,1)}$ will be investigated. Finally, Subsection 4.3 will deal with the implications of these considerations on our examples.

4.1 An Approximate Integration Rule For Large Equidistributed Data Sets

First, some terminology connected with point sets and partitions will be introduced. After that the approximate integration rule, based on Weyl's law of equidistribution (cf. [We]), will be presented including some error estimates. Finally, the numerical computation of $T_J(F)(\xi)$ reconstructed as the sum of $T_{J_0}(F)(\xi)$ and $R_{J_0}(F)(\xi), R_{J_0+1}(F)(\xi)\ldots, R_{J-1}(F)(\xi)$ will be outlined and the approximation error will be estimated. To discuss the efficiency of our approximate integration rule we will determine the number of elementary operations that are needed. Here, an elementary operation consists of one multiplication and one addition.

Definition 4.1 Let $\mathcal{U}\subset\Omega$ be a measurable subset of Ω and let $X_N^{\mathcal{U}} := \{\eta_1^N,\eta_2^N,\ldots,\eta_N^N\}\subset\mathcal{U}$ and $\widetilde{X}_N^{\mathcal{U}} := \{\widetilde{\eta}_1^N,\widetilde{\eta}_2^N,\ldots,\widetilde{\eta}_N^N\}\subset\mathcal{U}$ be two subsets of N points in the set $\mathcal{U}$. $X_N^{\mathcal{U}}$ and $\widetilde{X}_N^{\mathcal{U}}$ are called equivalent, if there exists a permutation p of $\{1,2,\ldots,N\}$ such that $\widetilde{\eta}_{p(j)}^N=\eta_j^N$ for all $j=1,\ldots,N$. An equivalence class with respect to this equivalence relation is called an *ensemble of* $\mathcal{U}$. An ensemble of $\mathcal{U}$ is identified with its elements. (It is a set of N points in $\mathcal{U}$ without any ordering.)

Definition 4.2 Let $X_N^{\mathcal{U}} := \{\eta_1^N,\eta_2^N,\ldots,\eta_N^N\}$ be an ensemble of a measurable subset $\mathcal{U}\subset\Omega$. A set $\mathcal{P}_{X_N^{\mathcal{U}}} := \{\mathcal{U}_{\eta_1^N},\mathcal{U}_{\eta_2^N},\ldots,\mathcal{U}_{\eta_N^N}\}$ is called an *associated partition of* $\mathcal{U}$ *to the ensemble* $X_N^{\mathcal{U}}$, if the following four conditions are satisfied: (i) $\mathcal{U}_{\eta_j^N}$ is measurable and $\|\mathcal{U}_{\eta_j^N}\|>0$ for all $j=1,2,\ldots,N$, (ii) $\eta_j^N\in\mathcal{U}_{\eta_j^N}$ for all $j=1,\ldots,N$, (iii) $\bigcup_{j=1}^N\mathcal{U}_{\eta_j^N}=\mathcal{U}$, and (iv) $\mathcal{U}_{\eta_i^N}\cap\mathcal{U}_{\eta_j^N}=\emptyset$ for all $i,j=1,2,\ldots,N$, $i\neq j$. An associated partition $\mathcal{P}_{X_N^{\mathcal{U}}}$ of $\mathcal{U}$ to the ensemble $X_N^{\mathcal{U}}$ is called *equidistributed*, if

$$\|\mathcal{U}_{\eta_j^N}\| = \int_{\mathcal{U}_{\eta_j^N}} d\omega(\eta) = \frac{\|\mathcal{U}\|}{N} \qquad \text{for all } j\in\{1,\ldots,N\}.$$

The set of all equidistributed partitions $\mathcal{P}_{X_N^{\mathcal{U}}}$ of $\mathcal{U}$ to $X_N^{\mathcal{U}}$ is denoted by $\Pi_{X_N^{\mathcal{U}}}$. The *partition size of the ensemble* $X_N^{\mathcal{U}}$ *of* $\mathcal{U}$ is defined by

$$\sigma(X_N^{\mathcal{U}}) := \inf_{\mathcal{P}_{X_N^{\mathcal{U}}}=\{\mathcal{U}_{\eta_1^N},\ldots,\mathcal{U}_{\eta_N^N}\}\in\Pi_{X_N^{\mathcal{U}}}} \ \max_{j=1,\ldots,N} \ \sup_{\xi\in\mathcal{U}_{\eta_j^N}} |\xi-\eta_j^N|.$$

Theorem 4.3 *Let $\mathcal{U}$ be a measurable subset of Ω, let $X_N^{\mathcal{U}} := \{\eta_1^N,\ldots,\eta_N^N\}\subset\mathcal{U}$ be an ensemble of $\mathcal{U}$, and let $\mathcal{P}_{X_N^{\mathcal{U}}} := \{\mathcal{U}_{\eta_1^N},\ldots,\mathcal{U}_{\eta_N^N}\}$ be an arbitrary equidistributed, associated partition of $\mathcal{U}$ to the ensemble $X_N^{\mathcal{U}}$. Suppose that G is a continuous, real valued function on $\mathcal{U}$ which is bounded on $\mathcal{U}$. Then*

$$\left|\frac{1}{\|\mathcal{U}\|}\int_{\mathcal{U}} G(\eta)d\omega(\eta) - \frac{1}{N}\sum_{k=1}^N G(\eta_k^N)\right| \leq \inf_{\mathcal{P}_{X_N^{\mathcal{U}}}\in\Pi_{X_N^{\mathcal{U}}}} \ \max_{k=1,\ldots,N} \ \sup_{\eta\in\mathcal{U}_{\eta_k^N}} |G(\eta)-G(\eta_k^N)|. \tag{14}$$

Proof: The definition of an equidistributed associated partition of $\mathcal{U}$ to $X_N^{\mathcal{U}}$ implies that

$$\frac{1}{\|\mathcal{U}\|}\int_{\mathcal{U}} G(\eta)\,d\omega(\eta) - \frac{1}{N}\sum_{k=1}^N G(\eta_k^N) = \frac{1}{\|\mathcal{U}\|}\sum_{k=1}^N \int_{\mathcal{U}_{\eta_k^N}} \Big(G(\eta)-G(\eta_k^N)\Big)\,d\omega(\eta) \tag{15}$$

holds true. The estimate (14) follows from (15) by estimating the integrals by the supremum of the integrand multiplied with $\|\mathcal{U}_{\eta_j^N}\| = \|\mathcal{U}\|/N$, and by taking the infimum over all equidistributed associated partitions $\mathcal{P}_{X_N^{\mathcal{U}}}$ of $\mathcal{U}$ to $X_N^{\mathcal{U}}$. □

Corollary 4.4 *Let $\mathcal{U} \subset \Omega$ be a measurable subset. Suppose that $\{N_j\}_{j\in\mathbb{N}_0} \subset \mathbb{N}$ is a strict monotonically increasing sequence. Assume that $\{X_{N_j}^{\mathcal{U}}\}_{j\in\mathbb{N}_0}$ is a sequence of ensembles $X_{N_j}^{\mathcal{U}} := \{\eta_1^{N_j}, \eta_2^{N_j}, \ldots, \eta_{N_j}^{N_j}\}$ of $\mathcal{U}$ satisfying $\lim_{j\to\infty} \sigma(X_{N_j}^{\mathcal{U}}) = 0$. Then, for $G \in \mathcal{C}(\overline{\mathcal{U}})$,*

$$\lim_{j\to\infty} \left| \frac{1}{\|\mathcal{U}\|} \int_{\mathcal{U}} G(\eta)\, d\omega(\eta) - \frac{1}{N_j} \sum_{k=1}^{N_j} G(\eta_k^{N_j}) \right| = 0. \tag{16}$$

If $G \in \mathcal{C}(\overline{\mathcal{U}})$ is additionally Lipschitz-continuous with Lipschitz-constant C_G, then

$$\left| \frac{1}{\|\mathcal{U}\|} \int_{\mathcal{U}} G(\eta)\, d\omega(\eta) - \frac{1}{N_j} \sum_{k=1}^{N_j} G(\eta_k^{N_j}) \right| \le C_G\, \sigma(X_{N_j}^{\mathcal{U}}).$$

Proof: For a Lipschitz-continuous function G the statement is a consequence of the estimate (14) in Theorem 4.3. If G is assumed to be continuous only, Equation (16) follows from the estimate (14) in Theorem 4.3 and the fact that G is uniformly continuous on $\overline{\mathcal{U}}$. □

Corollary 4.4 shows that the mean value of the function values of a continuous function $G \in \mathcal{C}(\overline{\mathcal{U}})$ in the points of a suitable ensemble $X_N^{\mathcal{U}}$ of $\mathcal{U} \subset \Omega$ yields an approximation of the integral $\frac{1}{\|\mathcal{U}\|} \int_{\mathcal{U}} G(\eta)\, d\omega(\eta)$, i.e. an *approximate integration rule for continuous integrands* $G \in \mathcal{C}(\overline{\mathcal{U}})$:

$$\frac{1}{\|\mathcal{U}\|} \int_{\mathcal{U}} G(\eta)\, d\omega(\eta) \;\approx\; \frac{1}{N} \sum_{k=1}^{N} G(\eta_k^N). \tag{17}$$

This approximate integration rule is useful for very large and (in the sense of Weyl) equidistributed point sets, because all the data can be used and the number of elementary operations is just N for a dataset $\{(\eta_1^N, G(\eta_1^N)), (\eta_2^N, G(\eta_2^N)), \ldots, (\eta_N^N, G(\eta_N^N))\}$.

Now we are able to apply this approximate integration rule for the evaluation of the approximation $T_J(F)$, $F \in \mathcal{C}(\Omega)$, as indicated by Theorem 3.3.

Theorem 4.5 *Assume that $\{N_j\}_{j\in\mathbb{N}_0} \subset \mathbb{N}$ is a strict monotonically increasing sequence. Let $\{X_{N_j}^{\Omega}\}_{j\in\mathbb{N}_0}$, $X_{N_j}^{\Omega} := \{\eta_1^{N_j}, \eta_2^{N_j}, \ldots, \eta_{N_j}^{N_j}\}$, be a sequence of ensembles of Ω that is hierarchical, i. e. $\eta_i^{N_j} = \eta_i^{N_{j+1}}$ for all $i = 1, 2, \ldots, N_j$ and for all $j \in \mathbb{N}_0$, and that satisfies $\lim_{j\to\infty} \sigma(X_{N_j}^{\Omega}) = 0$. Let $\{K_h\}_{h\in(-1,1)} \subset \mathcal{C}([-1,1])$ be a continuous scaling function. Suppose $\{h_j\}_{j\in\mathbb{N}_0} \subset (-1,1]$ is a strict monotonically increasing sequence with $\lim_{j\to\infty} h_j = 1$. Let T_j and R_j, $j \in \mathbb{N}_0$, denote the operators defined in Definition 3.1 corresponding to the sequence $\{h_j\}_{j\in\mathbb{N}_0}$ and the scaling function $\{K_h\}_{h\in(-1,1)}$. Denote $\mathcal{U}_j(\xi) := \overline{\{\eta \in \Omega \,|\, K_{h_j}(\xi\cdot\eta) \neq 0\}}$, $\widetilde{\mathcal{U}}_j(\xi) := \overline{\{\eta \in \Omega \,|\, \Psi_j(\xi\cdot\eta) \neq 0\}}$, $j \in \mathbb{N}_0$, and define $\widehat{X}_{N_j}^{\mathcal{U}} := X_{N_j}^{\Omega} \cap \mathcal{U}$, $j \in \mathbb{N}_0$, for an arbitrary measurable subset $\mathcal{U}$ of Ω and regard $\widehat{X}_{N_j}^{\mathcal{U}}$ as an ensemble of $\mathcal{U}$. Suppose F is in $\mathcal{C}(\Omega)$. Then $T_j(F)$, and $R_j(F)$, $j \in \mathbb{N}_0$, respectively, can be evaluated approximately by the formulae*[2]

$$T_j(F)(\xi) \;\approx\; \frac{\|\mathcal{U}_j(\xi)\|}{\#(\widehat{X}_{N_j}^{\mathcal{U}_j(\xi)})} \sum_{\zeta\in\widehat{X}_{N_j}^{\mathcal{U}_j(\xi)}} F(\zeta)\, K_{h_j}(\xi\cdot\zeta), \tag{18}$$

$$R_j(F)(\xi) \;\approx\; \frac{\|\widetilde{\mathcal{U}}_j(\xi)\|}{\#(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)})} \sum_{\zeta\in\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)}} F(\zeta)\, \Psi_j(\xi\cdot\zeta) \tag{19}$$

[2] If Y is a finite point set then $\#(Y)$ denotes the number of points in Y.

for all $\xi \in \Omega$. An arbitrary prescribed accuracy of the right-hand side of (18) and (19) can be obtained uniformly in $\xi \in \Omega$ by choosing the sequence $\{X_{N_j}^{\Omega}\}_{j\in\mathbb{N}_0}$ in an appropriate way. The function $F \in \mathcal{C}(\Omega)$ can be approximated by $F_J : \Omega \to \mathbb{R}$, $\xi \mapsto F_J(\xi)$, as follows:

$$F_J(\xi) := \frac{\|\mathcal{U}_{J_0}(\xi)\|}{\#(\widehat{X}_{N_{J_0}}^{\mathcal{U}_{J_0}(\xi)})} \sum_{\zeta\in\widehat{X}_{N_{J_0}}^{\mathcal{U}_{J_0}(\xi)}} F(\zeta)\, K_{h_{J_0}}(\xi\cdot\zeta) + \sum_{j=J_0}^{J-1} \frac{\|\widetilde{\mathcal{U}}_j(\xi)\|}{\#(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)})} \sum_{\zeta\in\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)}} F(\zeta)\, \Psi_j(\xi\cdot\zeta), \tag{20}$$

$J > J_0$. If F and the functions K_h are additionally Lipschitz-continuous, the error of the approximation F_J can be estimated as follows:

$$\begin{aligned} \|F - F_J\|_{\mathcal{C}(\Omega)} \;\le\; & \|F - T_J(F)\|_{\mathcal{C}(\Omega)} + \|\mathcal{U}_{J_0}(\xi)\| \left(C_{J_0}\, \|F\|_{\mathcal{C}(\Omega)} + C_F\, \|K_{h_{J_0}}\|_{\mathcal{C}([-1,1])}\right) \sigma(\widehat{X}_{N_{J_0}}^{\mathcal{U}_{J_0}(\xi)}) \\ & + \sum_{j=J_0}^{J-1} \|\widetilde{\mathcal{U}}_j(\xi)\| \left(\widetilde{C}_j\, \|F\|_{\mathcal{C}(\Omega)} + C_F\, \|\Psi_j\|_{\mathcal{C}([-1,1])}\right) \sigma(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)}), \end{aligned} \tag{21}$$

where C_F, C_{J_0}, and $\widetilde{C}_j$, $j = J_0, J_0+1, \ldots, J-1$, respectively, are Lipschitz-constants for the functions F, $K_{h_{J_0}}$, and Ψ_j, $j = J_0, J_0+1, \ldots, J-1$, respectively.

Proof: The approximation formulae (18) and (19) are immediate consequences of the numerical integration rule (17). That every prescribed accuracy of the right-hand side of (18) can be obtained uniformly in $\xi \in \Omega$ by choosing the sequence $\{X_{N_j}^{\Omega}\}_{j\in\mathbb{N}_0}$ in an appropriate way follows from the uniform continuity of F and K_{h_j} and from the estimate

$$\begin{aligned} |K_{h_j}(\xi\cdot\eta)\, F(\eta) - K_{h_j}(\xi\cdot\zeta)\, F(\zeta)| \;\le\; & \|K_{h_j}\|_{\mathcal{C}([-1,1])}\, |F(\eta) - F(\zeta)| \\ & + \|F\|_{\mathcal{C}(\Omega)}\, |K_{h_j}(\xi\cdot\eta) - K_{h_j}(\xi\cdot\zeta)| \end{aligned}$$

for all $\xi \in \Omega$. The argumentation for (19) is similar. These formulae combined with Theorem 3.3 imply that $F_J(\xi)$ is a suitable approximation for $T_J(F)(\xi)$ and consequently for $F(\xi)$, $\xi \in \Omega$, provided the sequence $\{X_{N_j}^{\Omega}\}_{j\in\mathbb{N}_0}$ is chosen in adaptation to the growth of the functions K_{h_j}.

To prove the estimate of the approximation error in case F, K_h, and Ψ_j, $j \in \mathbb{N}_0$ are Lipschitz-continuous we first determine Lipschitz-constants for the integrands in $T_j(F)(\xi)$ and $R_j(F)(\xi)$, $\xi \in \Omega$ fixed but arbitrary:

$$\begin{aligned} |F(\eta)\, K_{h_j}(\xi\cdot\eta) - F(\zeta)\, K_{h_j}(\xi\cdot\zeta)| &\le \left(C_j\, \|F\|_{\mathcal{C}(\Omega)} + C_F\, \|K_{h_j}\|_{\mathcal{C}([-1,1])}\right) |\eta - \zeta|, \\ |F(\eta)\, \Psi_j(\xi\cdot\eta) - F(\zeta)\, \Psi_j(\xi\cdot\zeta)| &\le \left(\widetilde{C}_j\, \|F\|_{\mathcal{C}(\Omega)} + C_F\, \|\Psi_j\|_{\mathcal{C}([-1,1])}\right) |\eta - \zeta| \end{aligned}$$

for all $\eta, \zeta \in \Omega$, where C_F, C_j, and $\widetilde{C}_j$ respectively, are the Lipschitz-constants of the functions F, K_{h_j}, and Ψ_j respectively. The triangle inequality yields

$$|F(\xi) - F_J(\xi)| \le |F(\xi) - T_J(F)(\xi)| + |T_J(F)(\xi) - F_J(\xi)|.$$

The second term can be estimated with the help of the triangle inequality, the reconstruction theorem, and Corollary 4.4.

$$\begin{aligned} |T_J(F)(\xi) - F_J(\xi)| \;\le\; & \left| T_{J_0}(F)(\xi) - \frac{\|\mathcal{U}_{J_0}(\xi)\|}{\#(\widehat{X}_{N_{J_0}}^{\mathcal{U}_{J_0}(\xi)})} \sum_{\zeta\in\widehat{X}_{N_{J_0}}^{\mathcal{U}_{J_0}(\xi)}} F(\zeta)\, K_{h_{J_0}}(\xi\cdot\zeta) \right| \\ & + \sum_{j=J_0}^{J-1} \left| R_j(F)(\xi) - \frac{\|\widetilde{\mathcal{U}}_j(\xi)\|}{\#(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)})} \sum_{\zeta\in\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)}} F(\zeta)\, \Psi_j(\xi\cdot\zeta) \right| \\ \le\; & \|\mathcal{U}_{J_0}(\xi)\| \left(C_{J_0}\, \|F\|_{\mathcal{C}(\Omega)} + C_F\, \|K_{h_{J_0}}\|_{\mathcal{C}([-1,1])}\right) \sigma(\widehat{X}_{N_{J_0}}^{\mathcal{U}_{J_0}(\xi)}) \\ & + \sum_{j=J_0}^{J-1} \|\widetilde{\mathcal{U}}_j(\xi)\| \left(\widetilde{C}_j\, \|F\|_{\mathcal{C}(\Omega)} + C_F\, \|\Psi_j\|_{\mathcal{C}([-1,1])}\right) \sigma(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)}) \end{aligned}$$

□

The estimate (21) in Theorem 4.5 implies that an approximation F_J with ε-accuracy can be obtained, if J is chosen so large, that $\|F - T_J(F)\|_{\mathcal{C}(\Omega)} \leq \varepsilon/2$ and if the hierarchical sequence $\{X_{N_j}^{\Omega}\}_{j\in\mathbb{N}_0}$ is chosen such that the $\sigma(X_{N_j}^{\Omega})$, $j = J_0, \ldots, J$, get so small, that each of the other summands in (21) becomes smaller than $\varepsilon/(2(J - J_0 + 1))$. The same can be done in the case of continuous (but not Lipschitz-continuous) integrands, but in this case the theorem yields no quantitative error estimate. The partition sizes $\sigma(X_{N_j}^{\Omega})$, $j = J_0, \ldots, J$, that are needed for a prescribed accuracy depend on the growth behavior of the kernel and on the undulations of the integrand. Generally it can be said that, the larger the index j the faster the growth of the kernel K_{h_j}, and consequently a smaller partition size, i. e. more discretization points, is needed.

In practical applications a function F will be known on a large (in the sense of Weyl) equidistributed point set $X_{N_J}^{\Omega}$. The continuous scaling function $\{K_h\}_{h\in(-1,1)}$, as well as the sequence $\{h_j\}_{j\in\mathbb{N}_0}$ and the hierarchical point sets $X_{N_j}^{\Omega} \subset X_{N_J}^{\Omega}$, $j = J_0, \ldots, J-1$, have then to be chosen in adaptation to the problem and the given dataset. It is desirable to choose the numbers N_j so that they satisfy the relations $N_j \approx N_{j+1}/2$ for $J_0 \leq j \leq J-1$, because in this case the approximation $T_J(F)$ of F can be reconstructed with at most $2N_J$ elementary operations.

One difficulty in this approach is the choice of the hierarchical point sets $X_{N_j}^{\Omega}$, $j = J_0, \ldots, J-1$, from the given set of points $X_{N_J}^{\Omega}$, where the function F is known. It is by no means a trivial problem to find a method for creating a suitable hierarchical sequence of point sets without a time-consuming strategy for the selection of the points. The authors do not want to investigate this problem further in this paper. Some considerations can be found in [Br] and [Gö].

4.2 Improvements for Locally Supported Kernels

All results from the previous section can be applied to our examples of approximate identities generated by non-negative $[h, 1]$-locally supported scaling functions. In this case the number of needed elementary operations is even much smaller because of the local support of the integrand in the convolution integrals. This enables the calculation of F_J in Theorem 4.5 for $\{h_j\}_{j\in\mathbb{N}_0}$ chosen as $h_j := 1-2^{-j}$ with approximately $2^{-J_0+1}N_J$ elementary operations (for N_J given data points) without hierarchical point sets, as it will be explained in detail below.

In the examples of non-negative $[h, 1]$-locally supported scaling functions (Subsections 2.2 and 3.2) the sequence $\{h_j\}_{j\in\mathbb{N}_0}$, given by $h_j := 1 - 2^j$, was chosen. Thus, $\mathcal{U}_j(\xi) = \Gamma_{h_j}(\xi)$, and $\widetilde{\mathcal{U}}_j(\xi) = \Gamma_{h_j}(\xi)$ in Theorem 4.5, where $\Gamma_h(\xi) := \{\eta \in \Omega \,|\, h \leq \xi \cdot \eta \leq 1\}$ is the spherical cap with center ξ and size $1 - h$. Obviously $\|\Gamma_h(\xi)\| = 2\pi\,(1 - h)$, hence, $\#(\widehat{X}_{N_j}^{\mathcal{U}_j(\xi)}) \approx 2^{-j-1}\#(X_{N_j}^{\Omega})$, and $\#(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)}) \approx 2^{-j-1}\#(X_{N_{j+1}}^{\Omega})$, $j = J_0, \ldots, J$. This means that the total amount of elementary operations in Theorem 4.5 is approximately

$$2^{-J_0-1}\#(X_{N_{J_0}}^{\Omega}) + \sum_{j=J_0}^{J-1} 2^{-j-1}\#(X_{N_{j+1}}^{\Omega}) = 2^{-J_0-1}N_{J_0} + \sum_{j=J_0}^{J-1} 2^{-j-1}N_{j+1} \tag{22}$$

in case of an $[h, 1]$-locally supported scaling function and the sequence $\{h_j\}_{j\in\mathbb{N}_0}$ given by $h_j := 1 - 2^j$.

To use these advantage in practical applications it is necessary to find a quick strategy to decide whether a point $\eta \in \Omega$ lies in the the spherical cap $\Gamma_h(\xi)$. We will not elaborate on this point here further. Some considerations on a similar problem can be found in [FrGlSchr].

Using such a strategy the approximation F_J given in Theorem 4.5 can also be calculated with approximately $2^{-J_0+1}N_J$ elementary operations without the use of hierarchical point sets for $[h, 1]$-locally supported kernels and the sequence $\{h_j\}_{j\in\mathbb{N}_0}$, $h_j := 1 - 2^j$: To achieve this, the different sets $X_{N_j}^{\Omega}$ in Theorem 4.5 are all replaced by $X_{N_J}^{\Omega}$, and the formula (22) yields at most $2^{-J_0+1}N_J$ elementary operations. This means on the one hand that some of the convolution integrals are calculated with unnecessary accuracy, but on the other hand the problem of the choice of the hierarchical point sets does no longer occur. Additionally the error estimate gets much simpler.

4.3 Error Discussion of the Examples

The estimates of the approximation error in case of hierarchical point sets given in Theorem 4.5 will be applied to the approximate identities generated by the smoothed Haar scaling functions ($k \geq 1$). The case of the Haar scaling function will be treated separately because of the discontinuity.

In this subsection we make the following assumptions: Let $\{N_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ be a strict monotonically increasing sequence, and let $\{X^{\Omega}_{N_j}\}_{j\in\mathbb{N}}$, where $X^{\Omega}_{N_j} := \{\eta_1^{N_j}, \eta_2^{N_j}, \ldots, \eta_{N_j}^{N_j}\}$, be a hierarchical sequence of ensembles of Ω, such that $\lim_{j\to\infty} \sigma(X^{\Omega}_{N_j}) = 0$. Moreover, the notation from Theorem 4.5 will be adopted.

Lemma 4.6 *Let $\{L_h^{(k)}\}_{h\in(-1,1)}$, $k \in \mathbb{N}$, be the smoothed Haar scaling functions introduced in Example 2.8, and define $\{h_j\}_{j\in\mathbb{N}}$ as usual by $h_j := 1 - 2^{-j}$. Assume that $F \in \mathcal{C}(\Omega)$ is a Lipschitz-continuous function with Lipschitz-constant C_F. Then the error of the approximation $F_J^{(k)}$ of F defined in Theorem 4.5 by (20) can be estimated by*

$$\begin{aligned}\|F - F_J^{(k)}\|_{\mathcal{C}(\Omega)} \;\leq\; & \left(\sum_{j=J_0-1}^{J-1} (k+1)\left[2^{j+2}\,k\,\|F\|_{\mathcal{C}(\Omega)} + 2\,C_F\right]\sigma(\widehat{X}^{\widetilde{\mathcal{U}}_j(\xi)}_{N_{j+1}})\right) + \sqrt{2}\,C_F \\ & \min_{\widetilde{h}\in[1-2^{-J},1]}\left[2^{J(k+(1/2))}(\widetilde{h}-(1-2^{-J}))^{k+1} + \left(1 - 2^{J(k+1)}\,(\widetilde{h}-(1-2^{-J}))^{k+1}\right)(1-\widetilde{h})^{1/2}\right],\end{aligned}$$

where $\widetilde{\mathcal{U}}_j(\xi) := \Gamma_{h_j}(\xi)$ for $j = J_0, \ldots, J-1$, $\widetilde{\mathcal{U}}_{J_0-1}(\xi) := \Gamma_{h_{J_0}}(\xi)$, and $\Gamma_h(\xi) := \{\eta \in \Omega \,|\, h \leq \xi\cdot\eta \leq 1\}$.

The second summand in the estimate of the approximation error is the estimate of $\|F - T_J(F)\|_{\mathcal{C}(\Omega)}$ developed in Lemma 2.6.

Lemma 4.7 *Let $\{H_h\}_{h\in(-1,1)}$ be the Haar scaling function introduced in Example 2.7, and define $\{h_j\}_{j\in\mathbb{N}}$ by $h_j := 1 - 2^{-j}$. Assume that $F \in \mathcal{C}(\Omega)$ is a Lipschitz-continuous function with Lipschitz-constant C_F. Then the error of the approximation $F_J^H : \Omega \to \mathbb{R}$, $\xi \mapsto F_J^H(\xi)$ of F, defined by*

$$\begin{aligned}F_J^H(\xi) \;:=\; & \frac{1}{\#(\widehat{X}^{\widetilde{\mathcal{U}}_{J_0}(\xi)}_{N_{J_0}})}\sum_{\zeta\in\widehat{X}^{\widetilde{\mathcal{U}}_{J_0}(\xi)}_{N_{J_0}}} F(\zeta) \\ & + \frac{1}{2}\sum_{j=J_0}^{J-1}\left(\frac{1}{\#(\widehat{X}^{\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}})}\sum_{\zeta\in\widehat{X}^{\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}}} F(\zeta) \;-\; \frac{1}{\#(\widehat{X}^{\widetilde{\mathcal{U}}_j(\xi)\setminus\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}})}\sum_{\zeta\in\widehat{X}^{\widetilde{\mathcal{U}}_j(\xi)\setminus\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}}} F(\zeta)\right),\end{aligned}$$

can be estimated by

$$\begin{aligned}\|F - F_J^H\|_{\mathcal{C}(\Omega)} \;\leq\; & C_F\left(\sigma(\widehat{X}^{\widetilde{\mathcal{U}}_{J_0}(\xi)}_{N_{J_0}}) + \frac{1}{2}\sum_{j=J_0}^{J-1}\left[\sigma(\widehat{X}^{\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}}) + \sigma(\widehat{X}^{\widetilde{\mathcal{U}}_j(\xi)\setminus\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}})\right]\right) \\ & + \min_{\widetilde{h}\in[1-2^{-J},1]}\sqrt{2}\,C_F\left[2^{J/2}\,(\widetilde{h}-(1-2^{-J})) + 2^J\,(1-\widetilde{h})^{3/2}\right],\end{aligned}$$

where $\widetilde{\mathcal{U}}_j(\xi) := \Gamma_{h_j}(\xi)$ for $j = J_0, \ldots, J$, and $\Gamma_h(\xi) := \{\eta \in \Omega \,|\, h \leq \xi\cdot\eta \leq 1\}$.

Proof: $F_J^H(\xi)$ can also be written in the following form

$$\begin{aligned}F_J^H(\xi) = & \frac{\|\widetilde{\mathcal{U}}_{J_0}(\xi)\|}{\#(\widehat{X}^{\widetilde{\mathcal{U}}_{J_0}(\xi)}_{N_{J_0}})}\sum_{\zeta\in\widehat{X}^{\widetilde{\mathcal{U}}_{J_0}(\xi)}_{N_{J_0}}} F(\zeta)\,H_{h_{J_0}}(\xi\cdot\zeta) \\ & + \sum_{j=J_0}^{J-1}\left(\frac{\|\widetilde{\mathcal{U}}_{j+1}(\xi)\|}{\#(\widehat{X}^{\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}})}\sum_{\zeta\in\widehat{X}^{\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}}} F(\zeta)\,\Psi_j^H(\xi\cdot\zeta) + \frac{\|\widetilde{\mathcal{U}}_j(\xi)\setminus\widetilde{\mathcal{U}}_{j+1}(\xi)\|}{\#(\widehat{X}^{\widetilde{\mathcal{U}}_j(\xi)\setminus\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}})}\sum_{\zeta\in\widehat{X}^{\widetilde{\mathcal{U}}_j(\xi)\setminus\widetilde{\mathcal{U}}_{j+1}(\xi)}_{N_{j+1}}} F(\zeta)\,\Psi_j^H(\xi\cdot\zeta)\right).\end{aligned}$$

Hence, F_J^H is a suitable approximation for F according to Theorem 4.3. To verify the error estimate we proceed similarly as in the proof of Theorem 4.5. By the triangle inequality we find

$$|F(\xi) - F_J^H(\xi)| \leq |F(\xi) - T_J^H(F)(\xi)| + |T_J^H(F)(\xi) - F_J^H(\xi)|.$$

The estimate of the first term is a consequence of Lemma 2.6. Applying the triangle inequality to the second term yields

$$\begin{aligned}
|T_J^H(F)(\xi) - F_J^H(\xi)| \;\leq\; & \left| T_{J_0}^H(F)(\xi) - \frac{\|\widetilde{\mathcal{U}}_{J_0}(\xi)\|}{\#(\widehat{X}_{N_{J_0}}^{\widetilde{\mathcal{U}}_{J_0}(\xi)})} \sum_{\zeta \in \widehat{X}_{N_{J_0}}^{\widetilde{\mathcal{U}}_{J_0}(\xi)}} F(\zeta)\, H_{h_{J_0}}(\xi \cdot \zeta) \right| \\
& + \sum_{j=J_0}^{J-1} \Bigg(\left| \int_{\widetilde{\mathcal{U}}_{j+1}(\xi)} F(\eta)\, \Psi_j^H(\xi \cdot \eta)\, d\omega(\eta) - \frac{\|\widetilde{\mathcal{U}}_{j+1}(\xi)\|}{\#(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_{j+1}(\xi)})} \sum_{\zeta \in \widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_{j+1}(\xi)}} F(\zeta)\, \Psi_j^H(\xi \cdot \zeta) \right| \\
& + \left| \int_{\widetilde{\mathcal{U}}_j(\xi) \setminus \widetilde{\mathcal{U}}_{j+1}(\xi)} F(\eta)\, \Psi_j^H(\xi \cdot \eta)\, d\omega(\eta) - \frac{\|\widetilde{\mathcal{U}}_j(\xi) \setminus \widetilde{\mathcal{U}}_{j+1}(\xi)\|}{\#(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)\setminus\widetilde{\mathcal{U}}_{j+1}(\xi)})} \sum_{\zeta \in \widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)\setminus\widetilde{\mathcal{U}}_{j+1}(\xi)}} F(\zeta)\, \Psi_j^H(\xi \cdot \zeta) \right| \Bigg).
\end{aligned}$$

Note that $\|\widetilde{\mathcal{U}}_j(\xi) \setminus \widetilde{\mathcal{U}}_{j+1}(\xi)\| = \|\widetilde{\mathcal{U}}_{j+1}(\xi)\|$ because of the definition of the sequence $\{h_j\}_{j\in\mathbb{N}}$ and that the function H_{J_0} as well as the wavelet functions Ψ_j^H are constant in each term. Hence these constants can be put outside the absolute value and each term can be estimated by applying Corollary 4.4.

$$\begin{aligned}
|T_J^H(F)(\xi) - F_J^H(\xi)| \;\leq\; & \frac{\|\widetilde{\mathcal{U}}_{J_0}(\xi)\|}{2\pi\,(1-h_{J_0})} C_F\, \sigma(\widehat{X}_{N_{J_0}}^{\widetilde{\mathcal{U}}_{J_0}(\xi)}) \\
& + \sum_{j=J_0}^{J-1} \frac{\|\widetilde{\mathcal{U}}_{j+1}(\xi)\|}{2\pi\,(1-h_j)} C_F \left(\sigma(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_{j+1}(\xi)}) + \sigma(\widehat{X}_{N_{j+1}}^{\widetilde{\mathcal{U}}_j(\xi)\setminus\widetilde{\mathcal{U}}_{j+1}(\xi)}) \right)
\end{aligned}$$

Inserting $\|\widetilde{\mathcal{U}}_j(\xi)\| = 2\pi\,(1-h_j) = 2\pi\, 2^{-j}$ for $j = J_0, \ldots, J$ yields the desired assertion. □

5 Multiscale Inversion of Pseudodifferential Equations of Modern Satellite Geodesy

Finally our purpose is to apply our theory of $[h,1]$-locally supported difference wavelets to pseudodifferential equations of modern satellite technology. Therefore, we need to transfer the theory developed in Sections 2 to 4 from the unit sphere Ω to an arbitrary sphere Ω_r of radius r. This is a straightforward procedure, and we will just sum up briefly the results.

Let $\{Y_{n,j}\}$ be a complete orthonormal system in $\mathcal{L}^2(\Omega)$ and let Ω_r be the sphere of radius r with center in the origin. Then the set $\{Y_{n,j}^r\}$ given by $Y_{n,j}^r(x) := \frac{1}{r} Y_{n,j}(x/r)$, $x \in \Omega_r$, $n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$, is a complete orthonormal system in $\mathcal{L}^2(\Omega_r)$. The Fourier coefficients of a function $F \in \mathcal{L}^2(\Omega_r)$ with respect to this system will be denoted by $F_r^\wedge(n,j) := (F, Y_{n,j}^r)_{\mathcal{L}^2(\Omega_r)}$. For $K \in \mathcal{L}^1([-r^2, r^2])$, the number $K^\wedge(n) := 2\pi \int_{-r^2}^{r^2} K(s) P_n(s/r^2)\, ds$, $n \in \mathbb{N}_0$, is called n-th (generalized) Legendre coefficient.

The theory of approximate identities extends to Ω_r in the following way. A family $\{K_h^r\}_{h\in(-1,1)}$ in $\mathcal{L}^1([-r^2, r^2])$ or in $\mathcal{L}^2([-r^2, r^2])$ that satisfies $(K_h^r)^\wedge(0) = 1$ for all $h \in (-1,1)$ is called a scaling function (i.e. generates an approximate identity in $\mathcal{X}(\Omega_r)$) if $\lim_{h\to 1,\, h<1} \|F - \int_{\Omega_r} F(y)\, K_h^r(x\cdot y)\, d\omega_r(y)\|_{\mathcal{X}(\Omega_r)} = 0$ for all $F \in \mathcal{X}(\Omega_r)$, where $\mathcal{X}(\Omega_r) = \mathcal{C}(\Omega_r)$, and $\mathcal{X}(\Omega_r) = \mathcal{L}^2(\Omega_r)$, respectively. There is an isomorphism from the set of all $\mathcal{L}^j$-scaling functions $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^j([-1,1])$, $j = 1, 2$, (for the unit sphere Ω) onto the set of all scaling functions $\{K_h^r\}_{h\in(-1,1)} \subset \mathcal{L}^j([-r^2, r^2])$ (for the sphere Ω_r) given by

$$K_h^r(s) := \frac{1}{r^2} K_h(s/r^2), \qquad s \in [-r^2, r^2]. \tag{23}$$

The Legendre coefficients of the image $\{K_h^r\}_{h\in(-1,1)}$ of $\{K_h\}_{h\in(-1,1)}$ are the same as those of $\{K_h\}_{h\in(-1,1)}$, i.e. $(K_h^r)^\wedge(n) = (K_h)^\wedge(n)$ for all $n \in \mathbb{N}_0$. Therefore, we will from now on regard a scaling function $\{K_h^r\}_{h\in(-1,1)}$ (for Ω_r) as specified via Equation (23) by some scaling function $\{K_h\}_{h\in(-1,1)}$ for the unit sphere, and we will denote the Legendre coefficients of K_h^r also by $(K_h)^\wedge(n)$. An $[h,1]$-locally supported scaling function $\{K_h\}_{h\in(-1,1)}$ (for the unit sphere Ω) is mapped onto an $[r^2h, r^2]$-locally supported scaling function $\{K_h^r\}_{h\in(-1,1)}$ for Ω_r, i.e. $\mathrm{supp}(K_h^r) = [r^2h, r^2]$ for all $h \in (-1,1)$. We mention that $\{K_h^r\}_{h\in(-1,1)} \subset \mathcal{L}^2([-r^2, r^2])$ regarded as radial basis function has the representation

$$K_h^r(x\cdot y) = \sum_{n=0}^{\infty} (K_h)^\wedge(n) \sum_{k=1}^{2n+1} Y_{n,k}^r(x)\, Y_{n,k}^r(y), \qquad x, y \in \Omega_r.$$

The spherical difference wavelets corresponding to a scaling function $\{K_h^r\}_{h\in(-1,1)}$ are introduced in an analogous manner, and the numerical results presented in Section 4 are completely applicable with the necessary slight changes in formalism.

5.1 The Operator Equation

Suppose, as it will be done in modern satellite geodesy, that $\Lambda : \mathcal{L}^2(\Omega_R) \to \mathcal{L}^2(\Omega_\gamma)$, $R < \gamma$, is a linear (rotation-invariant) pseudodifferential operator given by

$$\Lambda F = \sum_{n=0}^{\infty} \Lambda^\wedge(n) \sum_{k=1}^{2n+1} F_R^\wedge(n,k)\, Y_{n,k}^\gamma, \qquad F \in \mathcal{L}^2(\Omega_R),$$

with the following additional properties:

(i) $(\Lambda)^\wedge(n) \neq 0$ for all $n \in \mathbb{N}_0$,

(ii) $\sum\limits_{n=0}^{\infty} (2n+1)\, (\Lambda^\wedge(n))^2 < \infty$.

The sequence $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ is called the 'symbol' of Λ, it is in the language of functional analysis the system of singular values of Λ. Under these assumptions it is clear that Λ represents an injective, bounded, compact linear operator with $\overline{\mathrm{im}(\Lambda)} = \mathcal{L}^2(\Omega_\gamma)$. The image $\mathrm{im}(\Lambda)$ of Λ is equal to the Sobolev-like subspace $\Lambda(\mathcal{L}^2(\Omega_R)) = \mathcal{H}(\{(\Lambda^\wedge(n))^{-1}\}; \Omega_\gamma)$ of $\mathcal{L}^2(\Omega_\gamma)$ (for more notational details the reader is referred to [FrGeSchr]). Therefore, the theory of inverse problems (see, for example, [Lo]) tells us that Λ^{-1} is not bounded on $\mathrm{im}(\Lambda)$.

Let $\{K_h^\gamma\}_{h\in(-1,1)}$ be a (piecewise) continuous $[\gamma^2 h, \gamma^2]$-locally supported scaling function. Furthermore, suppose that $\{h_j\}_{j\in\mathbb{N}_0} \subset (-1,1]$ is a strict monotonically increasing sequence satisfying $\lim_{j\to\infty} h_j = 1$. Consider a multiscale approximation G_J of a function $G \in \mathcal{L}^2(\Omega_\gamma)$ given (analogously to Theorem 4.5) by

$$G_J(x) := \frac{4\pi\,\gamma^2}{N_{J_0}} \sum_{k=1}^{N_{J_0}} G(w_k^{N_{J_0}})\, K_{h_{J_0}}^\gamma(x\cdot w_k^{N_{J_0}}) + \sum_{j=J_0}^{J-1} \frac{4\pi\,\gamma^2}{N_{j+1}} \sum_{k=1}^{N_{j+1}} G(w_k^{N_{j+1}})\, \Psi_j^\gamma(x\cdot w_k^{N_{j+1}}), \tag{24}$$

$x \in \Omega_\gamma$, corresponding to the data $\{(w_1^{N_J}, G(w_1^{N_J})), \dots, (w_{N_J}^{N_J}, G(w_{N_J}^{N_J}))\}$, where the point set $X_{N_J}^{\Omega_\gamma} = \{w_1^{N_J}, \dots, w_{N_J}^{N_J}\} \subset \Omega_\gamma$ is equidistributed (in the sense of Weyl) and $\{X_{N_j}^{\Omega_\gamma}\}_{j=J_0,\dots,J}$ forms a suitable hierarchical sequence of subsets $X_{N_j}^{\Omega_\gamma} = \{w_1^{N_j}, \dots, w_{N_j}^{N_j}\}$ of $X_{N_J}^{\Omega_\gamma}$. In Equation (24), we do not make use of the local support of the integrand (as in Theorem 4.5), because the support depends on $x \in \Omega_\gamma$. Instead, we need an approximation of G as linear combination of the functions $K_{h_{J_0}}^\gamma(x\cdot w_k^{N_{J_0}})$, $k = 1, 2, \dots, N_{J_0}$, $\Psi_j^\gamma(x\cdot w_k^{N_{j+1}})$, $k = 1, 2, \dots, N_{j+1}$, $j = J_0, J_0+1, \dots, J-1$, that is valid for all $x \in \Omega_\gamma$.

It is known that the problem

$$\Lambda F_J = G_J, \qquad F_J \in \mathcal{L}^2(\Omega_R), \tag{25}$$

is solvable if and only if G_J is a member of $\mathrm{im}(\Lambda)$, i.e. G_J has to satisfy the spectral condition

$$\sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \left(\frac{(G_J)_\gamma^\wedge(n,k)}{\Lambda^\wedge(n)} \right)^2 < \infty. \tag{26}$$

In the approach presented in this paper the last condition, of course, is a restriction on the (piecewise) continuous $[h\,\gamma^2, \gamma^2]$-locally supported scaling function $\{K_h^\gamma\}_{h\in(-1,1)}$. To be more specific, the operator equation (25) is uniquely solvable corresponding to the left hand side (24) by the function

$$F_J(y) = \frac{4\pi\,\gamma^2}{N_{J_0}} \sum_{k=1}^{N_{J_0}} G(w_k^{N_{J_0}})\,(\Lambda^{-1}K_{h_{J_0}}^\gamma)(y\cdot w_k^{N_{J_0}}) + \sum_{j=J_0}^{J-1} \frac{4\pi\,\gamma^2}{N_{j+1}} \sum_{k=1}^{N_{j+1}} G(w_k^{N_{j+1}})\,(\Lambda^{-1}\Psi_j^\gamma)(y\cdot w_k^{N_{j+1}}), \quad (27)$$

$y\in\Omega_R$, (where $\Lambda^{-1}K_{h_{J_0}}^\gamma$ is meant in the sense that Λ_x^{-1} is applied to $K_{h_{J_0}}^\gamma(x\cdot w_k^{N_{J_0}})$ regarded as function of x, and analogously in the case $\Lambda^{-1}\Psi_j^\gamma$) if and only if

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi\,\gamma^2}\left(\frac{(K_{h_j})^\wedge(n)}{\Lambda^\wedge(n)}\right)^2 < \infty, \qquad j = J_0,\ldots,J. \quad (28)$$

As significant examples of an operator equation (25) we mention two important methods of modern satellite technology, namely satellite-to-satellite tracking (SST), and satellite gravity gradiometry (SGG).

5.2 Satellite Problems

The problem of determining the earth's gravitational potential F_J on the 'earth's sphere' Ω_R (with radius R) from satellite data G_J at the 'orbital sphere' Ω_γ (with radius $\gamma > R$) can be formulated by an operator equation $\Lambda F_J = G_J$, where the symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ of the operator Λ is given by

$$\Lambda^\wedge(n) = \begin{cases} \left(\frac{R}{\gamma}\right)^n \frac{n+1}{\gamma}, \quad n = 0,1,\ldots & \text{in case of SST} \\ \left(\frac{R}{\gamma}\right)^n \frac{(n+1)(n+2)}{\gamma^2}, \quad n = 0,1,\ldots & \text{in case of SGG} \end{cases}$$

(see [Fr]). Obviously, the properties (i) and (ii) are satisfied by the operator Λ. The solvability condition (28) generally is not fulfilled by a (piecewise) continuous $[h\,\gamma^2,\gamma^2]$-locally supported scaling function $\{K_h^\gamma\}_{h\in(-1,1)}$. Consequently, F_J cannot be calculated by application of Λ^{-1} to the formula (24). Instead we have to 'regularize', i.e. Equation (27) has to be replaced by the formula

$$F_J^{\mathrm{reg}}(y) = \frac{4\pi\,\gamma^2}{N_{J_0}} \sum_{k=1}^{N_{J_0}} G(w_k^{N_{J_0}})\,(\Lambda^{-1}K_{h_{J_0}}^{\mathrm{reg}})(y\cdot w_k^{N_{J_0}}) + \sum_{j=J_0}^{J-1} \frac{4\pi\,\gamma^2}{N_{j+1}} \sum_{k=1}^{N_{j+1}} G(w_k^{N_{j+1}})\,(\Lambda^{-1}\Psi_j^{\mathrm{reg}})(y\cdot w_k^{N_{j+1}}),$$

$y\in\Omega_R$, where the family $\{\Lambda^{-1}K_{h_j}^{\mathrm{reg}}\}_{j\in\mathbb{N}_0}$ is the so-called regularization scaling function, defined by

$$(\Lambda^{-1}K_{h_j}^{\mathrm{reg}})(y\cdot w) = \sum_{n=0}^{\infty}(\Lambda^{-1}K_{h_j}^{\mathrm{reg}})^\wedge(n) \sum_{k=1}^{2n+1} Y_{n,k}^R(y)\,Y_{n,k}^\gamma(w), \qquad y\in\Omega_R,\ w\in\Omega_\gamma,$$

and $\{\Lambda^{-1}\Psi_j^{\mathrm{reg}}\}_{j\in\mathbb{N}_0}$ is the associated regularization difference wavelet, which is defined by $\Lambda^{-1}\Psi_j^{\mathrm{reg}} := \Lambda^{-1}K_{h_{j+1}}^{\mathrm{reg}} - \Lambda^{-1}K_{h_j}^{\mathrm{reg}}$, $j\in\mathbb{N}_0$. By construction $\Lambda^{-1}K_{h_j}^{\mathrm{reg}}$ is rotation-invariant.

Typically truncated singular value decomposition may be used as regularization procedure, i.e.

$$(\Lambda^{-1}K_{h_j}^{\mathrm{reg}})^\wedge(n) := \begin{cases} (K_{h_j})^\wedge(n)(\Lambda^\wedge(n))^{-1} & \text{if} \quad n = 0,1,\ldots,L_j \\ 0 & \text{if} \quad n = L_j+1, L_j+2,\ldots, \end{cases} \quad (29)$$

$j\in\mathbb{N}_0$, with a sequence of integers $\{L_j\}_{j\in\mathbb{N}_0}$ satisfying $L_j < L_{j+1}$ for all $j\in\mathbb{N}_0$.

In the case of smoothed Haar scaling functions, all $(\Lambda^{-1}K_{h_j}^{\mathrm{reg}})(y\cdot w_k^{N_{j+1}})$, $(\Lambda^{-1}K_{h_j}^{\mathrm{reg}})(y\cdot w_k^{N_j})$, $y\in\Omega_R$, can be calculated recursively from their Legendre expansions using the recursion relations (7), (8).

5.3 Regularization by Truncated Singular Value Decomposition Wavelets

Finally a few words shall be made about regularization by truncated singular value decomposition regularization wavelets (cf. [Fr], [Schn]). For that purpose we start with the sequence of symbols $\{\{(\Lambda^{-1}K^{\mathrm{reg}}_{h_j})^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ given by (29) for a sequence $\{L_j\}_{j\in\mathbb{N}_0}$ satisfying $L_j < L_{j+1}$ for all $j \in \mathbb{N}_0$ and $\lim_{j\to\infty} L_j = \infty$. Moreover, we consider the rotation-invariant pseudodifferential operators $\{T_j\}_{j=J_0,J_0+1,\ldots}$, $T_j : \mathcal{L}^2(\Omega_\gamma) \to \mathcal{L}^2(\Omega_R)$, given by

$$T_j G = \sum_{n=0}^{L_j} (\Lambda^{-1}K^{\mathrm{reg}}_{h_j})^\wedge(n) \sum_{k=1}^{2n+1} G^\wedge_\gamma(n,k)\, Y^R_{n,k}. \tag{30}$$

(The function F^{reg}_J given in the last subsection is a numerical approximation of $T_J G$.) The condition

$$\sup_n |(\Lambda^{-1}K^{\mathrm{reg}}_{h_j})^\wedge(n)| = C(h_j) < \infty,$$

implies that

$$\sup_{\substack{G\in\mathcal{L}^2(\Omega_\gamma),\\ G\neq 0}} \frac{\|T_jG\|_{\mathcal{L}^2(\Omega_R)}}{\|G\|_{\mathcal{L}^2(\Omega_\gamma)}} = \sup_{\substack{G\in\mathcal{L}^2(\Omega_\gamma),\\ G\neq 0}} \frac{\left(\sum\limits_{n=0}^{\infty}\sum\limits_{k=1}^{2n+1} \Big((\Lambda^{-1}K^{\mathrm{reg}}_{h_j})^\wedge(n)\Big)^2 \left(G^\wedge_\gamma(n,k)\right)^2\right)^{1/2}}{\left(\sum\limits_{n=0}^{\infty}\sum\limits_{k=1}^{2n+1} \left(G^\wedge_\gamma(n,k)\right)^2\right)^{1/2}} \leq C(h_j) < \infty.$$

In other words, T_j is bounded on $\mathcal{L}^2(\Omega_\gamma)$ for all $j = J_0, J_0+1, \ldots$. For $G \in \mathrm{im}(\Lambda)$ we have

$$\lim_{j\to\infty} \|\Lambda^{-1}G - T_jG\|^2_{\mathcal{L}^2(\Omega_R)} = \lim_{j\to\infty}\left(\sum_{n=0}^{\infty}\Big(1-(K^{\mathrm{trunc}}_{h_j})^\wedge(n)\Big)^2 \left(\Lambda^\wedge(n)\right)^{-2} \left(G^\wedge_\gamma(n,k)\right)^2\right),$$

where $\{K^{\mathrm{trunc}}_{h_j}\}_{j\in\mathbb{N}_0}$ denotes the truncated kernel $\{K_{h_j}\}_{j\in\mathbb{N}_0}$, more precisely

$$(K^{\mathrm{trunc}}_{h_j})^\wedge(n) = \begin{cases} (K_{h_j})^\wedge(n) & \text{if} \quad n = 0,1,\ldots,L_j \\ 0 & \text{if} \quad n = L_j+1, L_j+2, \ldots, \end{cases}$$

$j \in \mathbb{N}_0$. In order to interchange the limit and the infinite sum we observe that

$$\Big(1-(K^{\mathrm{trunc}}_{h_j})^\wedge(n)\Big)^2 \leq C$$

uniformly in j and n for all scaling functions $\{K_h\}_{h\in(-1,1)}$ that satisfy the assumptions of Theorem 2.2. Especially, the limit and infinite sum may be interchanged for all our examples. Consequently, because of $\lim_{j\to\infty}(K_{h_j})^\wedge(n) = 1$ for all $n \in \mathbb{N}_0$ we obtain

$$\lim_{j\to\infty} \|\Lambda^{-1}G - T_jG\|_{\mathcal{L}^2(\Omega_R)} = 0.$$

Summarizing our results we are therefore led to the following conclusion.

Theorem 5.1 *The sequence $\{T_j\}_{j=J_0,J_0+1,\ldots}$, $T_j : \mathcal{L}^2(\Omega_\gamma) \to \mathcal{L}^2(\Omega_R)$, as defined by (30), is a regularization of Λ^{-1} in the following sense:*

(i) T_j is bounded on $\mathcal{L}^2(\Omega_\gamma)$ for all $j = J_0, J_0+1, \ldots$,

(ii) $\lim\limits_{j\to\infty} T_jG = \Lambda^{-1}G$ provided that $G \in \mathrm{im}(\Lambda)$.

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