A Fuzzy Programming Approach to Multicriteria Facility Location Problems

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Abstract

Facility Location Problems are concerned with the optimal location of one or several new facilities, with respect to a set of existing ones. The objectives involve the distance between new and existing facilities, usually a weighted sum or weighted maximum. Since the various stakeholders (decision makers) will have different opinions of the importance of the existing facilities, a multicriteria problem with several sets of weights, and thus several objectives, arises. In our approach, we assume the decision makers to make only fuzzy comparisons of the different existing facilities. A geometric mean method is used to obtain the fuzzy weights for each facility and each decision maker. The resulting multicriteria facility location problem is solved using fuzzy techniques again. We prove that the final compromise solution is weakly Pareto optimal and Pareto optimal, if it is unique, or under certain assumptions on the estimates of the Nadir point. A numerical example is considered to illustrate the methodology.

Keywords: Location theory, Multicriteria optimization, Fuzzy Programming, Triangular fuzzy number, Linear membership function.

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1 Introduction

The development of location theory has started with the optimal location of a single facility in the plane $\Re^2$, with respect to a set of existing facilities. This problem was first described in the 17th century by Fermat. In facility location literature, the two most important classes of problems that arise are the weighted sum and the weighted maximum of distances (minisum or median and minimax or center objective function). In realistic situations it is often not possible to consider only one criterion of the median (or Weber) or center (or Weber-Rawls) type.
Since the different decision makers involved in a locational decision will have different opinions of the importance of the existing facilities, a multicriteria problem with several sets of weights, and thus several objectives, arises. In such a multicriteria location problem a compromise location has to be found: For a multicriteria optimization problem, usually there are no "optimal" solutions as in the case of single criteria problems, but only preferred or compromise solutions are available. The preferred solution must be Pareto optimal (also called an efficient or a non-dominated solution).
Multicriteria facility location problems have received increasing attention in recent years, contributing to and applying theoretical results of multicriteria mathematical programming. Planar location problems with multiple objectives have been considered, among others, in [31], [8], [13], [22], [24], [15], [17], [18], [25],[19]. For an overview see also [11]. Parametric location problems which are closely related to multiple objective locations problems are discussed in [6].
Multiple criteria location problems with fuzzy approaches have been discussed by some researchers. Bhattacharya et al. have presented a fuzzy goal programming model for locating a single new facility under three criteria and locating multiple new facilities under two objectives in a plane bounded by a convex polygon in [2] and [3]. In [4] multiple facility minisum location problems under area restrictions were considered. Here the authors assume that the cost per unit distance is not known exactly, and is variable, unreliable and imprecise. They have considered flat fuzzy numbers to represent the cost per unit distance. In [5] they have also presented an interactive fuzzy goal programming model for locating multiple facility on a plane bounded by a convex polygon under multiple criteria. In [26] the authors examined a multiple criteria network location problem and used the fuzzy set theoretic approach to obtain a preferred solution among the nondominated solutions. However, incorrect mathematical formulations and inconsistencies in [26] were pointed out in [9]. In none of these papers the case where the decision makers have different opinions of importance (or relative strength) of the set of existing locations has been considered.

In this paper we consider multicriteria single facility location problems, where decision makers, i.e. all persons involved in the locational decision may have different opinions of importance of the set of existing locations $Ex = \{Ex_1, \ldots, Ex_M\}$ in the plane. These opinions represent their view of the relative strength of the new location at that point. These relative strengths are linguistic (e.g. "approximately", "slightly less", "at most as important") or fuzzy in nature rather than crisp values. This situation arises when decision makers are asked to give his or her personal view to locate a single new facility in the plane. In this situation decision makers compare one existing location to another keeping some criteria in his or her mind to locate the single new facility.

The main aim of this paper is twofold. Firstly we present a method to estimate the fuzzy weights of the set of existing locations for each decision maker. The importance (or relative strength) of the existing locations are given by the decision makers (various stakeholders) and are assumed to be fuzzy (triangular fuzzy numbers, see [20]). Secondly we present a method to find an "optimal compromise" solution of the fuzzy multicriteria location problem. Arising from determination of the fuzzy weights. To obtain the best compromise solution among the (weakly) Pareto optimal solutions, we use the concept and technique of fuzzy set theory (see [32] or [33]).

In our approach, we first formulate a fuzzy pairwise comparison matrix by comparing the existing locations with one another. To represent the importance (or relative strength) of the existing locations, we use Saaty's ratio scale 1 to 9 (see [27, 28]) in a fuzzy environment. In order to extract the fuzzy weights from the this pairwise comparison matrix we use the geometric mean technique (see [7]). The resulting weights are again fuzzy. Then we utilize the index of optimism $\lambda \in [0, 1]$ to convert these fuzzy weights to crisp numbers. Finally we compute normalized weights which are crisp again. Assigning these normalized weights to the existing locations, we use a fuzzy technique with a linear membership function to solve the multicriteria single facility location problem.

The rest of the paper is organized as follows. In Section 2 a mathematical model of the problem is presented. Determination of fuzzy weights is explained in Section 3. In Section 4, a fuzzy programming technique is proposed to solve the problem. Pareto optimality of the compromise solution obtained by the method of Section 4 is discussed in Section 5. A numerical example with rectilinear (l_1) and Euclidean (l_2) distance and two objectives is given in Section 6 to illustrate the proposed methodology. Finally, some conclusions and an outlook to further research are given in Section 7.

2 Fuzzy Multiobjective Location Problems

Let $Ex = \{Ex_1, \ldots, Ex_M\}$ be a set of existing locations in the plane, where $Ex_m = (a_{m1}, a_{m2})$ for all $m \in \mathcal{M} := \{1, \ldots, M\}$. Furthermore let $X = (x_1, x_2)$ be the single new facility to be located. Let $S \subset I\!R^2$ be the set of all feasible solutions, i.e. all possible locations for the new facility.

By $\tilde{w}_m^q$ we denote the fuzzy weight of the existing facility Ex_m assigned by decision maker q. If $d_m(Ex_m, X)$ is a convex distance between the existing location Ex_m and the new facility X we consider Q objective functions f_q, where each f_q is either

$$f_q(X) := \sum_{m=1}^{M} \tilde{w}_m^q d_m(Ex_m, X) \tag{1}$$

or

$$f_q(X) := \max_{m \in M} \tilde{w}_m^q d_m(Ex_m, X). \tag{2}$$

In (1) and (2) $d_m(Ex_m, X)$ means the distance between the points Ex_m and X, where we allow different kinds of distances for different existing facilities (to account e.g. for different accesibility of existing facilities). We only require that d_m satisfies the axioms of a metric for all m.

For each $q \in Q$, $\min_{x \in S} f_q(X)$ is either a single objective planar median or center location problem with fuzzy weights.

3 Determination of Fuzzy Weights

In this section, we present a method to obtain the fuzzy weights $\tilde{w}_m^q$ in (1) and (2) using Saaty's 9 point ratio scale in a fuzzy context. In order to do so, we have to introduce some basics of fuzzy theory.

A triangular fuzzy number $\tilde{N}$ can be defined as a triplet (l, m, n). Here l and n stand for the lower and upper value of $\tilde{N}$, and m denotes the modal value. The membership function of the fuzzy number $\tilde{N}$ is defined as $\mu_{\tilde{N}}(x)$.

$$\mu_{\tilde{N}}(x) = \begin{cases} 0 & if \quad x \leq l \\ \frac{x-l}{m-l} & if \quad l < x \leq m \\ \frac{n-x}{n-m} & if \quad m < x < n \\ 0 & if \quad x \geq n \end{cases} . \tag{3}$$

Algebraic operations on triangular fuzzy number are defined now. Consider two triangular fuzzy numbers $\tilde{N}_1 = (l_1, m_1, n_1)$ and $\tilde{N}_2 = (l_2, m_2, n_2)$. Then addition, multiplication and inverse can be defined as follows.

1. Addition

$$\tilde{N}_1 \oplus \tilde{N}_2 = (l_1, m_1, n_1) \oplus (l_2, m_2, n_2) = (l_1 + l_2, m_1 + m_2, n_1 + n_2)$$

2. Multiplication

$$\tilde{N}_1 \odot \tilde{N}_2 = (l_1, m_1, n_1) \odot (l_2, m_2, n_2) = (l_1 l_2, m_1 m_2, n_1 n_2)$$

3. Inverse

$$\tilde{N}_1^{-1} = (l_1, m_1, n_1)^{-1} = (\frac{1}{n_1}, \frac{1}{m_1}, \frac{1}{l_1})$$

Each decision maker is asked to perform a pairwise comparison of the M existing facilities. We use a nine-point ratio scale for this purpose (see [28],[27]). Decision makers are asked to compare existing facilities pairwise, answering questions "How much more important is facility j as compared to facility i?". Ratios are expressed as numbers between 1 and 9, where 1 means equally important and 9 means absolutely more important, with intermediate grades weakly (3), strongly (5), and very strongly (7) more important.
However, we expect these judgements to be fuzzy in nature, i.e. instead of the crisp numbers $\{1, \ldots, 9\}$ we rather use fuzzy ratios $\{\tilde{1}, \ldots, \tilde{9}\}$. Therefore a judgement matrix $\tilde{A}$, the entries of which are triangular fuzzy numbers $\tilde{a}_{ij} = (\alpha_{ij}, \beta_{ij}, \gamma_{ij})$, is obtained. So the decision makers are asked to provide the modal value as an estimate of the ratio of importance, but also a left and right spread as bounds. Note that the entries on the diagonal of the matrix must be crisp and equal to one. These fuzzy numbers indicate the relative strength of the M existing facilities. Therefore

$$\tilde{A}_q = \begin{bmatrix} 1 & \tilde{a}_{12} & \tilde{a}_{13} & \ldots & \tilde{a}_{1(M-1)} & \tilde{a}_{1M} \\ \frac{1}{\tilde{a}_{21}} & 1 & \tilde{a}_{23} & \ldots & \tilde{a}_{2(M-1)} & \tilde{a}_{2M} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{\tilde{a}_{(M-1)1}} & \frac{1}{\tilde{a}_{(M-1)2}} & \frac{1}{\tilde{a}_{(M-1)3}} & \ldots & 1 & \tilde{a}_{(M-1)2} \\ \frac{1}{\tilde{a}_{M1}} & \frac{1}{\tilde{a}_{M2}} & \frac{1}{\tilde{a}_{M3}} & \ldots & \frac{1}{\tilde{a}_{M(M-1)}} & 1 \end{bmatrix}, \qquad (4)$$

where

$$\tilde{a}_{ij} \begin{cases} \in \{\tilde{1}, \ldots, \tilde{9}\} & \text{if } i < j \\ 1 & \text{if } i = j \\ \in \{\tilde{1}^{-1}, \ldots, \tilde{9}^{-1}\} & \text{if } i > j \end{cases}$$

is a fuzzy pairwise comparison matrix for the qth decision maker.
From the fuzzy judgement matrix $\tilde{A}_q$ the fuzzy weight $\tilde{w}_i^q = (\varepsilon_i^q, \xi_i^q, \eta_i^q)$ for facility i according to the judgement of decision maker q can be computed using a geometric mean

technique. Usually an eigenvector is chosen to define the weights of the different facilities. In the fuzzy context, instead of computing a fuzzy eigenvector, [7] proposed a geometric mean technique. This method is easier to apply and can be well justified. For details we refer to [7].

Define $\alpha_i = [\prod_{j=1}^{m} \alpha_{ij}]^{\frac{1}{m}}$ and $\alpha = \sum_{i=1}^{m} \alpha_i$. Similarly, let $\beta_i = [\prod_{j=1}^{m} \beta_{ij}]^{\frac{1}{m}}$ and $\beta = \sum_{i=1}^{m} \beta_i$ and also $\gamma_i = [\prod_{j=1}^{m} \gamma_{ij}]^{\frac{1}{m}}$ and $\gamma = \sum_{i=1}^{m} \gamma_i$. The fuzzy weights $\tilde{w}_i^q = (\varepsilon_i^q, \xi_i^q, \eta_i^q)$ are determined by $(\alpha_i \gamma^{-1}, \beta_i \beta^{-1}, \gamma_i \alpha^{-1})$. Note that the fuzzy weights $\tilde{w}_i^q$ are correct triangular fuzzy numbers again, in the sense that the lower value ε_i^q is less than the modal value ξ_i^q, which is less than the upper value η_i^q.

At this stage a fuzzy multiobjective location problem

$$\min_{X \in S} f_q(X), \tag{5}$$

with objectives of median type (1) and/or of center type (2) is obtained.

We will now use the index of optimism $\lambda \in [0, 1]$ introduced in [10] to convert the fuzzy weights into crisp weights. A larger index indicates a higher degree of optimism and will imply that the crisp weight w_i^q related to $\tilde{w}_i^q$ is closer to the upper value η_i^q. The fuzzy weights with index of optimism are defined as

$$w_i^q = \lambda \eta_i^q + (1 - \lambda) \varepsilon_i^q, \ \lambda \in [0, 1].$$

The crisp values w_i^q can now easily be normalized, such that $\sum_{i=1}^{M} w_i^q = 1$ for all $q = 1, \ldots, Q$. Then a multiobjective location problem

$$\min_{X \in S} (f_1(X), \ldots, f_Q(X)) \tag{6}$$

is obtained.

4 A Fuzzy Technique to Solve Multicriteria Single Facility Location Problems

For this approach, we need estimates of the lowest and highest value each of the objectives can attain for Pareto optimal solutions of (6). This can be done by solving Q single objective single facility location problems

$$\min_{X \in S} f_q(X) \tag{7}$$

for each $q \in \{1, \ldots, Q\}$, taking one criterion at a time. Here the estimated crisp weights w_i^q appear as weights in f_q.

Now, if $X^{(q)}$ is an optimal solution of the q-th single objective problem (7), $X^{(q)}$ can be evaluated for all objectives. According to each solution and value for every objective, a pay-off matrix with entries $f_{pq} = f_q(X^{(p)}),\ q, p = 1, \ldots, Q$ can be formulated as follows.

$$\begin{array}{c|cccc} & f_1(X) & f_2(X) & \ldots & f_Q(X) \\ \hline & & & & \\ X^{(1)} & f_{11} & f_{12} & \ldots & f_{1Q} \\ X^{(2)} & f_{21} & f_{22} & \ldots & f_{2Q} \\ \vdots & \ldots & \ldots & \ldots & \ldots \\ X^{(Q)} & f_{Q1} & f_{Q2} & \ldots & f_{QQ} \end{array} \tag{8}$$

The fuzzy technique to solve the multicriteria location problem uses the results from the payoff table to define a linear membership function for each of the criteria. With these membership functions and a max-min operation, the problem is converted to a single objective optimization problem.

From the pay-off table, we find estimates of the upper (U_q) and lower (L_q) values for each criterion corresponding to the set of Pareto solutions of (6), $U_q = \max(f_{1q}, f_{2q}, \ldots, f_{Qq})$ and $L_q = f_{qq}$, $q = 1, 2, \ldots, Q$. Now we can use well known techniques of fuzzy programming to formulate the problem as a maximum achievment problem.

Using U_q and L_q, a linear membership function $\mu^L_{f_q}$ (see [33]) for each of the Q criteria is obtained, by

$$\mu^L_{f_q}(x) = \begin{cases} 1 & \text{if } f_q \leq L_q \\ \frac{U_q - f_q}{U_q - L_q} & \text{if } L_q < f_q < U_q \\ 0 & \text{if } f_q \geq U_q \end{cases} . \tag{9}$$

Using the max-min operator (see [1]) the above problem can be converted to a single criterion problem, (10) - (13), namely

$$\max R, \tag{10}$$

subject to the constraints

$$f_q(X) + R(U_q - L_q) \leq U_q,\ q = 1, \ldots, Q \tag{11}$$

$$R \in [0, 1] \tag{12}$$

$$X \in S. \tag{13}$$

The final problem obtained is a convex programming problem (provided that S is convex) in variables X and R, which can readily be solved using existing algorithms and software. We note that solving the problem with different values of the index of optimism λ, several sets of weights can be obtained. Therefore several possible compromise solutions will be available.
In the next section we will discuss Pareto otpimality of the final compromise solution obtained from solving (10) - (13) for the multiobjective problem (6).

5 Pareto Optimality of the Final Compromise Solution

Let us assume that S is a closed and bounded subset of $I\!R^2$. This is not a very strong assumption. Usually the set of Pareto optimal solutions of a location problem can be shown to be restricted to such a set, which can be computed from the coordinates of the existing facilities. For details see the references mentioned in the introduction.
Let S_{Par} denote the set of Pareto optimal solutions of the multicriteria location problem $\min_{X \in S}(f_1(X), \ldots, f_Q(X))$,

$$S_{Par} := \{X \in S : \nexists X' \in S \text{ s.t. } f_q(X') \leq f_q(X) \; q = 1, \ldots, Q \;\text{ and } f(X') \neq f(X)\}.$$

Furthermore we will use the set of weakly Pareto optimal solutions. A point $X \in S$ is weakly Pareto optimal, if there is no $X' \in S$ such that $f_q(X') < f_q(X)$ for all $q = 1, \ldots, Q$. We denote the set of weakly Pareto optimal solutions of (6) by S_{w-Par}. The compactness assumption and the continuity of the objectives f_q guarantee that S_{Par} and S_{w-Par} are nonempty, see e.g. [29]. We obtain the following result.

Theorem 1 1. *An optimal solution X^* of (10) - (13) is weakly Pareto optimal for the multicriteria location problem (6).*

2. *The problem (10) - (13) has at least one optimal solution, which is Pareto optimal for (6).*

Proof:

1. Assume the contrary. Then there is an $X \in S$ such that $f_q(X) < f_q(X^*)$ $\forall q = 1, \ldots, Q$. Let R^* be the optimal solution value of problem (10) - (13). Then

$$f_q(X) + R^*(U_q - L_q) < f_q(X^*) + R^*(U_q - L_q) \leq U_q, \; \forall q = 1, \ldots, Q.$$

Therefore there exists an $R > R^*$ and an $i \in \{1, \dots, Q\}$ such that

$$f_i(X) + R(U_i - L_i) = U_i,$$

$$f_q(X) + R(U_q - L_q) \leq U_q,\ q \neq i.$$

This contradicts the fact that X^* is an optimal solution of (10) – (13).

2. Suppose X^* is not a Pareto optimal solution. Then there exists an $X \in S_{Par}$ such that $f_q(X) \leq f_q(X^*)$ $\forall q = 1, \dots, Q$ and $f_i(X) < f_i(X^*)$, for some i. Therefore

$$f_q(X) + R^*(U_q - L_q) \leq U_q\ \forall q = 1, \dots, Q.$$

Because X^* is an optimal solution of (10) - (13), so is X. Thus the claim is proved.

□

Note that Theorem 1 implies that whenever the solution of Problem (10) - (13) is unique, then it is also Pareto optimal.

For problems with only two objectives, we can strengthen the result of Theorem 1. To do so we have to introduce some further notation. Since all objectives are continuous they attain their minima on S and we can define $y_q^0 := \min_{X \in S} f_q(X)$. Then $y^0 := (y_1^0, \dots, y_Q^0)$ is called the ideal point of the multicriteria optimization problem (6). The ideal point is a lower bound on all objective values. An upper bound on the objective values for the Pareto set is given by the Nadir point y^N, where

$$y_q^N := \max_{X \in S_{Par}} f_q(X).$$

We assume that $y_q^0 < y_q^N$ for all q, because otherwise the qth objective is irrelevant for the optimization. Concerning the pay-off matrix mentioned in Section 4 we observe that $L_q = y_q^0$. However, $U_q = y_q^N$ is not necessarily true, because, U_q may over- or underestimate y_q^N, see e.g. [21] for an example.

Under the general asumption that $|S_{Par}| \geq 2$ (otherwise $y^N = y^0$ and the objectives are not conflicting) and that S is convex, we obtain Theorem 2.

Theorem 2 *Assume that S is convex, and that $U_q \leq y_q^N$ for $q = 1, 2$. Furthermore assume that U is a feasible estimate of y^N, i.e. there is at least one $X \in S$ such that $f_q(X) < U_q$ for $q = 1, \dots, 2$. Then an optimal solution X^* of (10) - (13) is Pareto optimal for (6).*

Proof:

Let X^* be an optimal solution of (10) - (13), i.e. X^* solves

$$\max_{X \in S} \min_{q=1,\ldots,Q} \frac{U_q - f_q(X)}{U_q - L_q},$$

or, if we use the constants $\alpha_q := (U_q - L_q)^{-1} > 0$,

$$\max_{X \in S} \min_{q=1,\ldots,Q} \alpha_q (U_q - f_q(X)).$$

From Theorem 1 we know that X^* is weakly Pareto optimal. Therefore we will look at S_{w-Par} in more detail. We use a result from [23], which states that

$$S_{w-Par} = \mathrm{Opt}(f_1) \cup \mathrm{Opt}(f_2) \cup S_{Par}, \tag{14}$$

where $\mathrm{Opt}(f_q)$ denotes the set of optimal solutions of

$$\min_{X \in S} f_q(X).$$

From (14) we have a partition of S_{Par} as follows:

$$\begin{aligned} S_{w-Par} &= (\mathrm{Opt}(f_1) \cup \mathrm{Opt}(f_2)) \cup (S_{Par} \setminus (\mathrm{Opt}(f_1) \cup \mathrm{Opt}(f_2))) \\ &=: S_1 \cup S_2 \end{aligned}$$

Note that S_1 and S_2 are disjoint and that due to the general asumption that no ideal solutions X satisfying $f(X) = y^0$ exist, $|S_{Par}| \geq 2$, and due to the connectedness of S_{Par} (see [30]), we know that $S_2 \neq \emptyset$.

Now we show that for all $X^1 \in S_1$ $f_q(X^1) \geq y_q^N$ for $q = 1$ or $q = 2$ and that for all $X^2 \in S_2$ $f_q(X^2) < y_q^N$ for $q = 1$ and $q = 2$. Then

$$\begin{aligned} \min_{q=1,2} \{\alpha_q (U_q - f_q(X^1))\} &\leq \min_{q=1,2} \{\alpha_q (y_q^N - f_q(X^1))\} \leq 0 \\ &< \min_{q=1,2} \{\alpha_q (y_q^N - f_q(X^2))\} \end{aligned} \tag{15}$$

for all $X^1 \in S_1$ and all $X^2 \in S_2$.

1. Let $X^1 \in S_1$. By definition either $f_1(X^1) = \min_{X \in S} f_1(X)$ or $f_2(X^1) = \min_{X \in S} f_2(X)$. Without loss of generality we only consider the first case and show that $f_1(X^1) = \min_{X \in S} f_1(X)$ implies $f_2(X^1) \geq y_2^N$. Suppose that this would not be true and let X^1 be such that $f_2(X^1) < y_2^N$. Now take any $X \in S_{Par}, X \notin \mathrm{Opt}(f_1)$. By the choice of X^1 we have $f_1(X) \geq f_1(X^1)$ and therefore $f_2(X) < f_2(X^1) < y_2^N$. This would imply $\max_{X \in S_{Par}} f_2(X) \leq f_2(X^1) < y_2^N$, which contradicts the definition of y^N.

2. Next let $X^2 \in S_2$. Thus $X^2 \in S_{Par}$ and $f_q(X^2) \leq y_q^N$ by definition. Note that $f_q(X^2) = y_q^N$ would imply $X^2 \in S_1$. Indeed, if e.g. $f_1(X^2) = y_1^N$ this yields $f_1(X) \leq f_1(X^2)$ for all $X \in S_{Par}$, therefore $f_2(X) \geq f_2(X^1)$ for all $X \in S_{Par}$. Thus we would imply $f_2(X^1) = \min_{X \in S} f_2(X)$. So $f_q(X^2) < y_q^N$ as claimed.

We have now proved the theorem for $U = y^N$. In the general case $U_q \leq y_q^N$ we can use the fact that U is a feasible estimate for y^N. Therefore there exists $X \in S$ such that $f_q(X) < U_q \leq y_q^N$, $q = 1, 2$. Therefore there exists $X' \in S_{Par}$ with $f_q(X') < y_q^N$, and analogous to 2. above, all such X' are in S_2. The fact that $\min_{q=1,2}\{\alpha_q(U_q - f_q(X')\} > 0$ together with the first and second inequality in (15) completes the proof. □

Remark 1 *The proof of Theorem 2 actually shows that a solution of (10) - (13) is properly Pareto optimal (see [14]). This is due to the fact that S_{Par} is the closure of the set of properly Pareto optimal solutions (see e.g. [29]), and in defining S_2 we just omitted the two boundary points of S_{Par}, cf. Figure 1.*

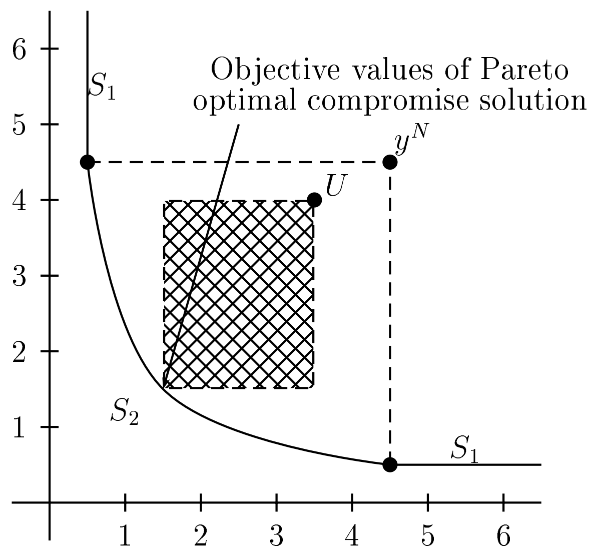

Figure 1: Illustration of Theorem 2

Theorem 2 indicates, that it is desirable to have estimates of the upper bounds of objectives that are less or equal than the Nadir values. The condition $U_q \leq y_q^N$ of Theorem 2 can be satisfied for $Q = 2$ if the pay-off table is replaced by the solution of lexicographic optimization problems

$$\text{lexmin}_{X \in S}(f_1(X), f_2(X))$$

and

$$\text{lexmin}_{X \in S}(f_2(X), f_1(X)).$$

With $\text{Opt}(f_q) := \{X^* \in S : f_q(X^*) = \min_{X \in S} f_q(X)\}$ we define $L_q = \min_{X \in S} f_q(X)$ and $U_q := min_{X \in Opt_j} f_j(X), q \neq j, q = 1, 2$. Then $y^0 = (L_1, L_2)$ and $y^N = (U_1, U_2)$.
For the general case of $Q \geq 3$ criteria unfortunately no procedure is known to determine y^N (see [21]). In this case, we can get lower estimates $\bar{U}_q \leq y_q^N$ of the Nadir values from payoff table obtained from the solution of problems

$$\min_{X \in S} f_q(X) + \sum_{i \neq q} \epsilon_i f_i(X) \tag{16}$$

for each q, where $\epsilon_i > 0$ are small, e.g. $\epsilon_i = \frac{1}{Q^2}$. This is due to the fact that an optimal solution of (16) is Pareto optimal for the multiobjective location problem.

6 Numerical Example

In this example we consider two criteria one of which is a median objective and one of which is a center objective.
Let $Ex_1 = (1, 1)$, $Ex_2 = (2, 3)$, $Ex_3 = (4, 2)$ be the coordinates of the three existing locations in the plane. The relative strength of the existing locations is presented as a comparison matrix, where the elements of the matrix are triangular fuzzy number with left and right spread. In this example we consider $\tilde{2} = (1, 2, 4)$ and $\tilde{4} = (2, 4, 6)$.
The comparison matrix for the existing locations is as follows:

$$\tilde{A}_1 = \tilde{A}_2 = \begin{bmatrix} 1 & (1, 2, 4) & (2, 4, 6) \\ (\frac{1}{4}, \frac{1}{2}, 1) & 1 & (1, 2, 4) \\ (\frac{1}{6}, \frac{1}{4}, \frac{1}{2}) & (\frac{1}{4}, \frac{1}{2}, 1) & 1 \end{bmatrix} \tag{17}$$

For convenience, we used the same comparison matrix (17) to obtain the weights for both objectives, one of which is a center and one of which is a median objective. Using the geometric mean technique discussed in Section 3, we got the following fuzzy weights for the three locations:

$$(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) = \begin{pmatrix} (0.24, & 0.57, & 1.29) \\ (0.12, & 0.29, & 0.71) \\ (0.07, & 0.14, & 0.35) \end{pmatrix}.$$

Using an index of optimism $\lambda = 0.5$ we get the normalized crisp weights

$$(w_1, w_2, w_3) \approx (0.55, 0.30, 0.15).$$

Using the fuzzy programming technique of Section 4 the optimal compromise location of the above fuzzy multicriteria facility location problem with rectilinear distance $d_m(Ex_m, X) = l_1(Ex_m, X)$ and Euclidean distance $d_m(Ex_m, X) = l_2(Ex_m, X)$ is shown in Table 1. We have used AMPL (A Modelling Language for Mathematical Programming, see [12]) and LOLA (Library of Location Algorithms, see [16]), to solve the single objective location problems. AMPL has been used to model and solve the single objective problem (10) - (13).

Distance	Location X^*	Objectives	R
l_1	$(1.2647, 1.2647)$	$(1.5529, 0.7412)$	$R = 0.49$
l_2	$(1.2817, 1.3714)$	$(1.2103, 0.5342)$	$R = 0.55$

Table 1: Optimal Solutions of the Numerical Example

Both solutions are indeed Pareto optimal. The Example is shown in Figure 2.

7 Conclusions

In this paper we have considered multicriteria facility location problems, where the decision makers have different opinions of the importance of the set of existing facilities. Decision makers' opinions are collected in fuzzy comparison matrices, from which fuzzy weights are obtained by a geormetric mean method. Using an index of optimism the weights of the existing locations are extracted. We proved that the final compromise solution is weakly Pareto optimal and Pareto optimal, if it is unique or the decision space is convex and overestimation of Nadir values is avoided. Our method takes into account the difficulties of people to express clear and crisp judgements, thus making more realistic models of location problems available. The proposed solution technique guarntees that a best compromise solution is found. Further extensions of our concepts to multifacility problems and problems on networks are under research.

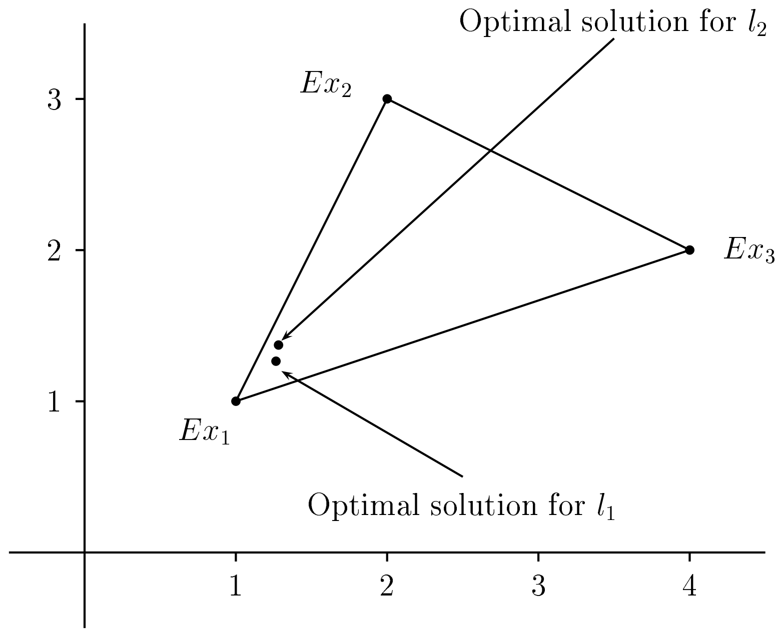

Figure 2: Existing Facilities and Compromise Lcations

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