Multicriteria Ordered Weber Problems

Stefan Nickel * Justo Puerto † Antonio M. Rodríguez-Chía ‡
Ansgar Weißler*

April 29, 1999

Abstract

In this paper we deal with the determination of the whole set of Pareto-solutions of location problems with respect to Q general criteria. These criteria include median, center or cent-dian objective functions as particular instances. The paper characterizes the set of Pareto-solutions of all these multicriteria problems. An efficient algorithm for the planar case is developed and its complexity is established. Extensions to higher dimensions as well as to the non-convex case are also considered. The proposed approach is more general than the previously published approaches to multi-criteria location problems and includes almost all of them as particular instances.

Keywords: Location Theory, Multi-criteria Optimization, Algebraic Optimization, Geometrical Algorithms

1 Introduction

In the process of locating a new facility usually more than one decision maker is involved. This is due to the fact that typically the cost connected to the decision is relatively high. Of course, different persons may (or will) have different (conflicting) objectives. On other occasions, different scenarios must be compared in order to be implemented, or simply uncertainty in the parameters leads to consider different replications of the objective function. If only one objective has to be taken into account a broad range of models is available in the literature (see Chapter 11 in [Dre95]). In contrast to that only a few papers looked at (more realistic) models for facility location, where more than one objective is involved (see [FP95, HN96]). One of the main deficiencies of the existing approaches is that only a few number (in most papers 1) of different types of objectives can be considered and solution approaches depend very much on a specific chosen metric. Also a detailed complexity analysis is missing in most of the papers.

On the other hand there is a clear need for flexible models where the complexity status is known, since these are prerequisites for a successful implementation of a decision support

*Fachbereich Mathematik. Universität Kaiserslautern.
†Facultad de Matemáticas. Universidad de Sevilla. C/Tarfia s/n, 41012 Sevilla, Spain.
‡Facultad de Ciencias del Mar. Universidad de Cádiz. Poligono Rio San Pedro. Puerto Real, Cádiz, Spain.

system for location planning which can really be used by decision-makers. In this paper we present a model for continuous multi-criteria location problems which fulfills the requirement of flexibility with respect to the choice of objective functions. To this end we present a new type of objective function (called ordered Weber function), developed in [PF95, RCNPF96, PACFP98], which includes most of the classical location objective functions as special cases, like for e.g. the Weber objective, the center objective, the cent-dian objective and the Weber objective with positive and negative weights.

Additionally, we allow the use of polyhedral gauges as distance functions in each objective function. It should be mentioned that by the polyhedral gauge approach we are able to approximate every gauge (see [Val64, WW85]). The outline of the rest of the paper is as follows :

In Section 2 the problem is formally introduced and basic tools and definitions are presented. Section 3 is devoted to the bicriteria case in the plane, while Sections 4 and 5 extend these results to the general planar Q-criteria case. In Section 6 generalizations looking at the non-convex and at the n-dimensional case are discussed. The paper ends with some conclusions and an outlook to future research. Throughout the paper we keep track of the complexity of the presented algorithms.

2 Basic Tools and Definitions

First we restate some definitions which are needed throughout the paper.

Denote the set of demand points by $A := \{a_1, \ldots, a_M\}$. Let $B_i \subset \mathbb{R}^n$ be a compact, convex set containing the origin in its interior, for $i \in \mathcal{M} := \{1, 2, \ldots, M\}$. The gauge with respect to B_i is defined as

$$\gamma_i \; : \; \mathbb{R}^n \; \to \; \mathbb{R} \quad , \quad \gamma_i(x) := \inf\{\, r > 0 \; : \; x \in rB_i \,\} \tag{1}$$

the polar set B_i° of B_i is given by

$$B_i^o := \{\, p \in \mathbb{R}^n \; : \; \langle p, x\rangle \leq 1 \quad \forall\, x \in B_i \,\} \tag{2}$$

the normal cone to B_i at x is given by

$$N(B_i, x) := \{p \in \mathbb{R}^n \; : \; \langle p, y - x\rangle \leq 0 \quad \forall\, y \in B_i\} \tag{3}$$

and the boundary of B_i is denoted by $bd(B_i)$.

The case we will mainly consider in this paper is where each γ_i with $i \in \mathcal{M}$ is a polyhedral gauge, which means B_i is a convex polytope with extreme points $Ext(B_i) := \{e^i_1, \ldots, e^i_{G_i}\}$. Let $G_{\max} := \max\{G_i \; : \; i \in \mathcal{M}\}$. In this case we define fundamental directions $d^i_1, \ldots, d^i_{G_i}$ as the half-lines defined by 0 and $e^i_1, \ldots, e^i_{G_i}$. Further, we define Γ^i_g as the cone generated by d^i_g and d^i_{g+1} (fundamental directions of B_i) where $d^i_{G_i+1} := d^i_1$.

Let $\pi = (p_i)_{i\in\mathcal{M}}$ be a family of elements of $\mathbb{R}^n$ such that $p_i \in B_i^o$ for each $i \in \mathcal{M}$ and let $\mathrm{C}_\pi = \bigcap_{i\in\mathcal{M}}(a_i + N(B_i^o, p_i))$. A nonempty convex set C is called an elementary convex set if there exists a family π such that $\mathrm{C}_\pi = \mathrm{C}$.

It should be noted that if the unit balls are polytopes we can obtain the elementary convex sets as intersection of cones generated by fundamental directions of these balls pointed at each demand point. Therefore each elementary convex set is a polyhedron whose vertices are called intersection points (see Figure 1). Finally, in the case of $\mathbb{R}^2$ there exists an upper bound on

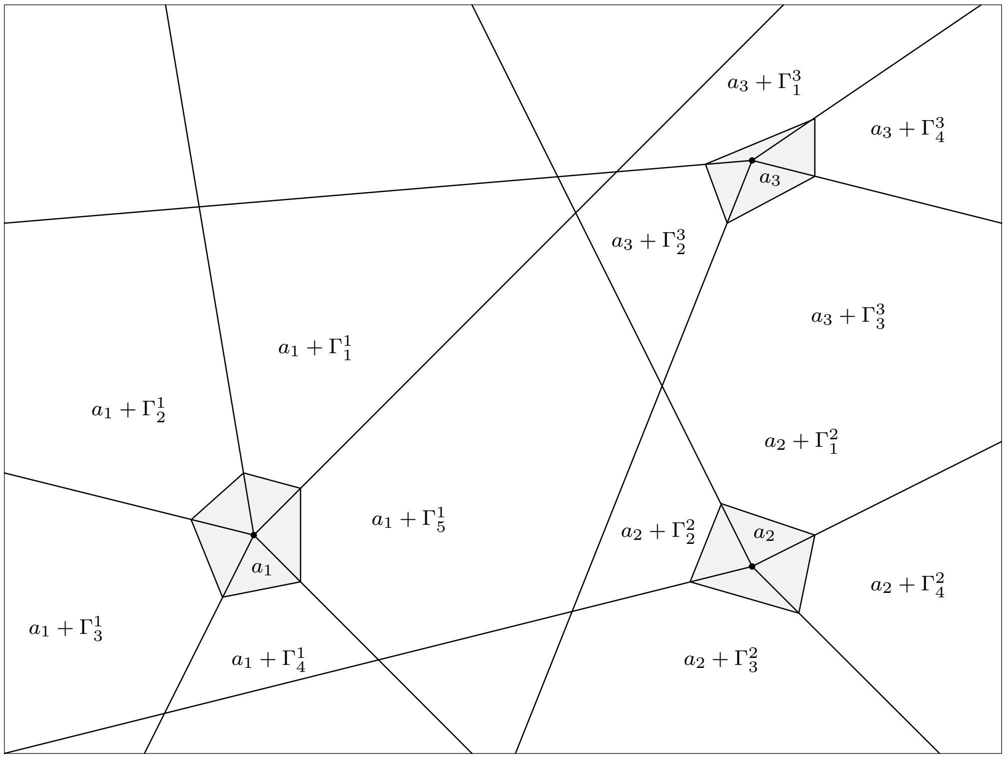

Figure 1: Existing facilities a_i , $i \in \mathcal{M}$, and their unit balls B_i

the number of elementary convex sets which is $O((MG_{\max})^2)$. [DM85] proved that the Weber problem is linear in each elementary convex set. Therefore, if we consider polyhedral gauges, there always exists an optimal solution to the Weber problem in the set of intersection points.

2.1 A General Approach : The Ordered Weber Problem

In this section we present a general location model, the ordered Weber problem, introduced by [PF95] and later elaborated for the polyhedral case by [RCNPF96].

Consider the set of demand points $A = \{a_1, \dots, a_M\}$, the corresponding gauges $\gamma_i(\cdot)$, $i \in \mathcal{M}$, and two sets of nonnegative scalars $\Omega := \{\omega_1, \dots, \omega_M\}$ and $\Lambda := \{\lambda_1, \dots \lambda_M\}$, where the element ω_i , $i \in \mathcal{M}$, is the weight of the importance given to the existing facility a_i , $i \in \mathcal{M}$, and the elements of Λ allow to choose among different kinds of objective functions. Given a permutation σ of $\mathcal{M}$ verifying

$$\omega_{\sigma(1)}\gamma_{\sigma(1)}(x - a_{\sigma(1)}) \leq \omega_{\sigma(2)}\gamma_{\sigma(2)}(x - a_{\sigma(2)}) \leq \ldots \leq \omega_{\sigma(M)}\gamma_{\sigma(M)}(x - a_{\sigma(M)})$$

we define $\gamma(x - A)_{(i)} := \omega_{\sigma(i)}\gamma_{\sigma(i)}(x - a_{\sigma(i)})$. The ordered Weber problem is given by the following formulation

$$\min_{x \in \mathbb{R}^n} \; F(x) = \sum_{i=1}^{M} \lambda_i \gamma(x - A)_{(i)} \tag{4}$$

The set of optimal solutions of this problem is called $\mathcal{X}^*(F)$ or simply $\mathcal{X}^*$ if this is possible without causing confusion. This objective function looks very much like the objective function of the classical Weber problem, but in fact this function is point-wise defined and in general not convex as the following example shows.

Example 2.1 *Consider two demand points $a_1 = (0, 0)$ and $a_2 = (10, 5)$, weights $\lambda_1 = 100$ and $\lambda_2 = 1$ with l_1-norm and $\omega_1 = \omega_2 = 1$. We obtain two optimal solutions to Problem (4), lying in each demand point. Therefore the objective function is not convex since we have a non-convex optimal solution set (see Figure 2). .*

$$\begin{aligned} F(a_1) &= 100 \times 0 + 1 \times 15 = 15 \\ F(a_2) &= 100 \times 0 + 1 \times 15 = 15 \\ F(\frac{1}{2}(a_1 + a_2)) &= 100 \times 7.5 + 1 \times 7.5 = 757.5 \end{aligned}$$

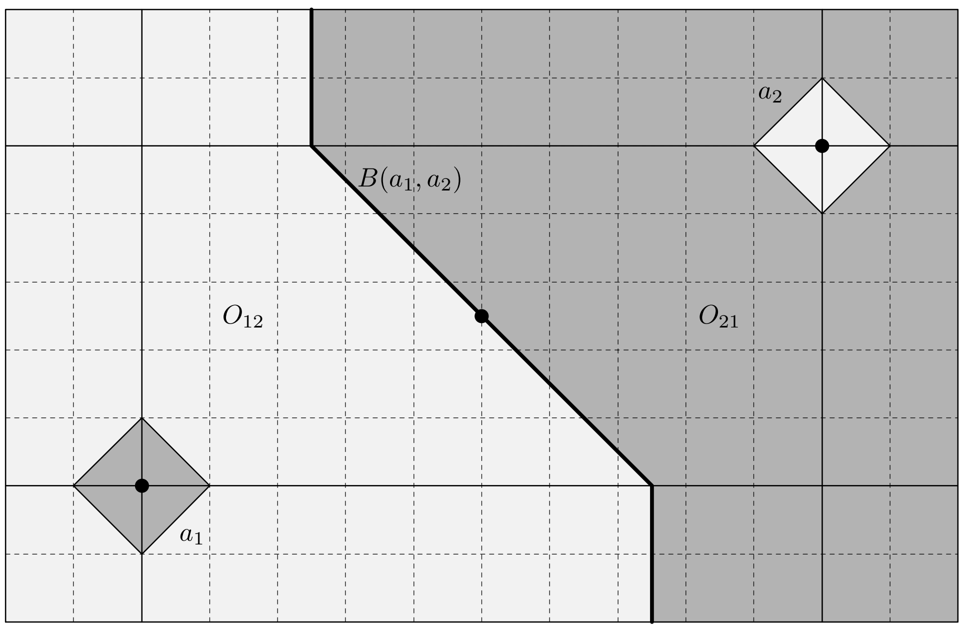

Figure 2: Illustration to Example 2.1

Nevertheless, if we assume that the weights are in increasing order we obtain that the objective function is convex (see [PF95] for more details).

In spite of its difficulty, the study of this model is very important because it provides a quite general framework to deal with continuous location problems, as the Theorem 2.1 shows. To describe the different types of location problems we use a 5-Position classification scheme Pos1/Pos2/Pos3/Pos4/Pos5 , which allows us to indicate the number of new facilities in (Pos1) and the type of the problem as planar ($\mathbb{R}^2$), network-based ($\mathcal{N}$), discrete (D) etc. in (Pos2). Any assumptions and restrictions such as $w_m = 1$ for all $m \in \mathcal{M}$, etc. are given in (Pos3). The type of distance function such as l_p, general distance function d, etc. is contained in (Pos4), and the type of objective function such as $\sum$ for the classical Weber function, max for the center function, CD_ω for the cent-dian function, $\sum_{ord}$ for the ordered Weber function etc. appears in (Pos5). For more details see [HN98].

The next result demonstrates how the classical location objective functions are related to the ordered Weber function. Although this result is already known (see [NP99]) we include the proof for the sake of readability.

Theorem 2.1

1. *The classical Weber problem* $1/\mathbb{R}^n/\bullet/\gamma_i/\sum$ *is equivalent to the ordered Weber problem* $1/\mathbb{R}^n/\lambda_1 = \ldots = \lambda_M = 1/\gamma_i/\sum_{\text{ord}}$.
2. *The center problem* $1/\mathbb{R}^n/\bullet/\gamma_i/\max$ *is equivalent to the ordered Weber problem* $1/\mathbb{R}^n/\lambda_1 = \ldots = \lambda_{M-1} = 0 \ \wedge \ \lambda_M = 1/\gamma_i/\sum_{\text{ord}}$.
3. *The cent-dian problem* $1/\mathbb{R}^n/\bullet/\gamma_i/CD_\omega$ *is equivalent to the ordered Weber problem* $1/\mathbb{R}^n/\lambda_1 = \ldots = \lambda_{M-1} = \omega \ \wedge \ \lambda_M = 1/\gamma_i/\sum_{\text{ord}}$.

Proof:

1. $F(x) = \sum_{i=1}^{M} \lambda_i \gamma(x-A)_{(i)} = \sum_{i=1}^{M} \gamma(x-A)_{(i)} = \sum_{i=1}^{M} \omega_i \gamma(x-a_i)$
2. $F(x) = \sum_{i=1}^{M} \lambda_i \gamma(x-A)_{(i)} = \gamma(x-A)_{(M)} = \max_{i=1,\ldots,M} \{\omega_i \gamma(x-a_i)\}$
3. Analogous to 1. and 2.

■

It should be noted that the computation of $F(x)$ is not a trivial task. We do not have an explicit formula of F in $\mathbb{R}^n$, because we have different expressions for F depending on the order in the sequence of distances. Anyway, F behaves as the classical Weber function in a region where the order does not change. To this end, we use the concept of ordered regions.

The set $B(a_i, a_j)$ for $i \neq j$ consisting of points

$$\{x \in \mathbb{R}^n \,:\, \omega_i \gamma_i(x-a_i) = \omega_j \gamma_j(x-a_j)\}$$

is called the bisector of a_i and a_j with respect to (ω_i, γ_i) and (ω_j, γ_j). Note that $B(a_i, a_j) = B(a_j, a_i)$. Given a permutation σ on the index set $\mathcal{M}$ the ordered region O_σ consists of the points (see Figure 3)

$$O_\sigma := \{x \in \mathbb{R}^n \,:\, \omega_{\sigma(1)}\gamma_{\sigma(1)}(x-a_{\sigma(1)}) \leq \ldots \leq \omega_{\sigma(M)}\gamma_{\sigma(M)}(x-a_{\sigma(M)})\,\} \tag{5}$$

Note that the concept of ordered regions can be seen as an extension to classical Voronoi diagrams. The set we obtain as intersection of an elementary convex set and an ordered region is called a generalized elementary convex set. The vertices of the generalized elementary convex set are called generalized intersection points and the set containing all the generalized intersection points is denoted by $\mathcal{GIP}$. The full dimensional generalized elementary convex sets are called cells. The set of all cells is denoted by $\mathcal{C}$.

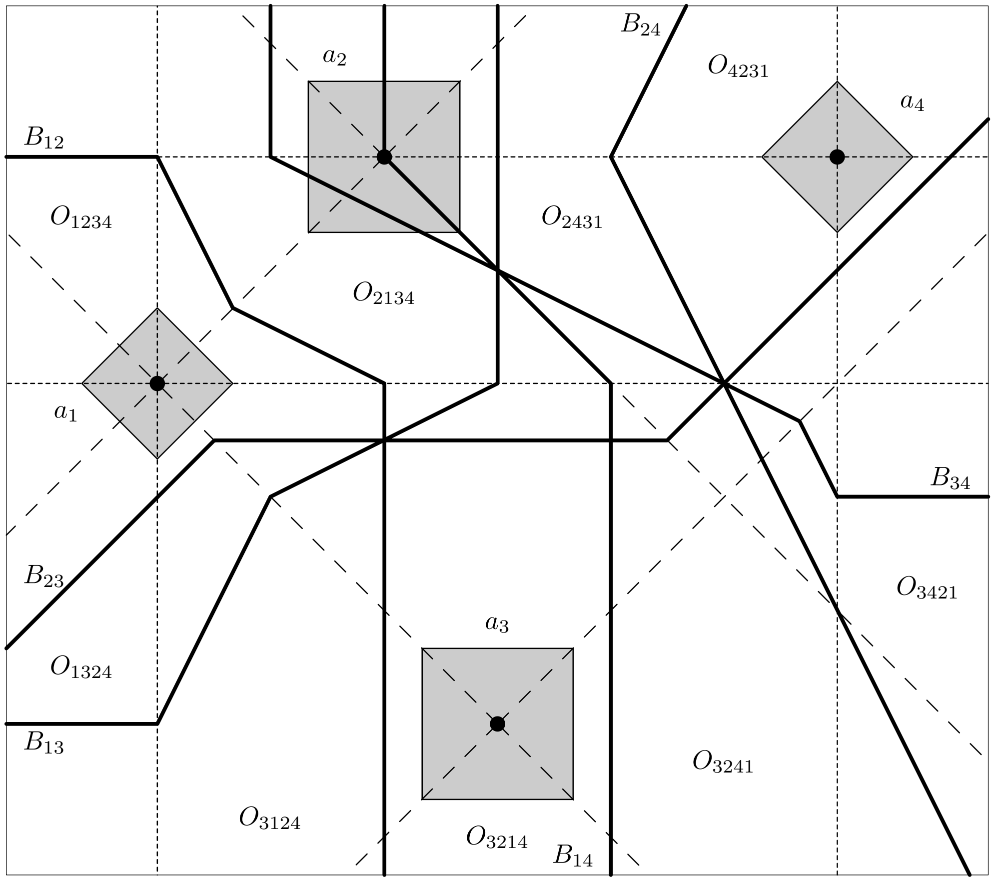

Figure 3: Bisector lines and ordered regions generated by 4 existing facilities $a_1, \ldots, a_4$ associated with the l_1-norm respectively the l_∞-norm for $\Omega := \{1, 1, 1, 1\}$

[PF95] obtained that the objective function of the Ordered Weber problem is linear in each generalized elementary convex set. Therefore there exists an optimal solution of the Ordered Weber problem in $\mathcal{GIP}$. In the case of polyhedral gauges with at most $G_{\max}$ fundamental directions, [RCNPF96] obtained an upper bound of the number of generalized elementary convex sets which in $\mathbb{R}^2$ is $O(M^4 G^2_{max})$. For the sake of readability the proof of this bound can be found in the appendix at the end of the paper.

2.2 Multi-criteria Problems and Level Sets

Let $F^1, \ldots, F^Q$ be functions from $\mathbb{R}^n$ to $\mathbb{R}$. If we want to optimize simultaneously all these objective functions we get points in a Q-dimensional objective space and we do not have the canonical order of $\mathbb{R}$ anymore. In this section, we introduce two different orderings of $\mathbb{R}^Q$, the lexicographic and Pareto ordering, which can be given in the following way. Let $z, \bar{z} \in \mathbb{R}^Q$ and $\mathcal{Q} := \{1, \ldots, Q\}$, then

$$z \leq_{\text{lex}} \bar{z} \quad :\Leftrightarrow \quad z = \bar{z} \text{ or } z_q < \bar{z}_q \text{ for } q := \min\{i \in \mathcal{Q} \,:\, z_i \neq \bar{z}_i\}$$

$$z \preceq \bar{z} \quad :\Leftrightarrow \quad z_q \leq \bar{z}_q \quad \forall q \in \mathcal{Q} \text{ and } z_p < \bar{z}_p \text{ for some } p \in \mathcal{Q} \,.$$

A point $x \in \mathbb{R}^n$ is called a lexicographic location (or lex-optimal) if there exists a permutation π of the set $\mathcal{Q}$ such that

$$(F^{\pi(1)}(x), F^{\pi(2)}(x), \dots, F^{\pi(Q)}(x)) \leq_{\text{lex}} (F^{\pi(1)}(y), F^{\pi(2)}(y), \dots, F^{\pi(Q)}(y)) \quad \forall\, y \in \mathbb{R}^n .$$

We denote the set of lexicographic solutions by $\mathcal{X}^*_{\text{lex}}\left(F^1, \dots, F^Q\right)$ or simply by $\mathcal{X}^*_{\text{lex}}$ if this is possible without causing confusion. The set of lexicographic solutions with respect to a fixed permutation π is denoted by $\mathcal{X}^*_{\pi(1),\dots,\pi(Q)}$.

On the other hand a point $x \in \mathbb{R}^n$ is called a Pareto location (or Pareto optimal) if there exists no $y \in \mathbb{R}^n$ such that

$$(F^1(y), F^2(y), \dots, F^Q(y)) \preceq (F^1(x), F^2(x), \dots, F^Q(x))$$

We denote the set of Pareto solutions by $\mathcal{X}^*_{\text{Par}}\left(F^1, \dots, F^Q\right)$ or simply by $\mathcal{X}^*_{\text{Par}}$ if this is possible without causing confusion. Note that $\mathcal{X}^*_{\text{lex}} \subseteq \mathcal{X}^*_{\text{Par}}$.

For technical reasons we will also use the concept of weak Pareto optimality and strict Pareto optimality. A point $x \in \mathbb{R}^n$ is called a weak Pareto location (or weakly Pareto optimal) if there exists no $y \in \mathbb{R}^n$ such that

$$F^q(y) < F^q(x) \quad \forall\, q \in \mathcal{Q} .$$

We denote the set of weak Pareto solutions by $\mathcal{X}^*_{\text{w-Par}}\left(F^1, \dots, F^Q\right)$ or simply by $\mathcal{X}^*_{\text{w-Par}}$ if this is possible without causing confusion. A point $x \in \mathbb{R}^n$ is called a strict Pareto location (or strictly Pareto optimal) if there exists no $y \in \mathbb{R}^n$ such that

$$F^q(y) \leq F^q(x) \quad \forall\, q \in \mathcal{Q} .$$

We denote the set of strict Pareto solutions by $\mathcal{X}^*_{\text{s-Par}}\left(F^1, \dots, F^Q\right)$ or simply by $\mathcal{X}^*_{\text{w-Par}}$ if this is possible without causing confusion. Note that $\mathcal{X}^*_{\text{s-Par}} \subseteq \mathcal{X}^*_{\text{Par}} \subseteq \mathcal{X}^*_{\text{w-Par}}$.

In order to obtain a geometrical characterization of a Pareto solution we use the concept of level sets.

For a function $F : \mathbb{R}^n \to \mathbb{R}$ the level set for a value $z \in \mathbb{R}$ is given by

$$L_{\leq}(F, z) := \{x \in \mathbb{R}^n \;:\; F(x) \leq z\}$$

and the level curve for a value $z \in \mathbb{R}^n$ is given by

$$L_{=}(F, z) := \{x \in \mathbb{R}^n \;:\; F(x) = z\}$$

Using the level sets and level curves [HN96] obtained the following characterizations:

$$x \in \mathcal{X}^*_{\text{w-Par}}\left(F^1, \dots, F^Q\right) \;\Leftrightarrow\; \bigcap_{q=1}^{Q} L_{<}(F^q, F^q(x)) = \emptyset \tag{6}$$

$$x \in \mathcal{X}^*_{\text{Par}}\left(F^1, \dots, F^Q\right) \;\Leftrightarrow\; \bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(x)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(x)) \tag{7}$$

$$x \in \mathcal{X}^*_{\text{s-Par}}\left(F^1, \dots, F^Q\right) \;\Leftrightarrow\; \bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(x)) = \{x\} \tag{8}$$

Finally [War83] proved, that the set $\mathcal{X}^*_{\text{Par}}$ is connected, provided that the objective functions are convex.

3 Bicriteria Ordered Weber Problems

3.1 Properties of $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/2 - (\sum_{\mathrm{ord}})_{\mathrm{Par}}$

In this and the following sections we restrict ourselves to the plane in order to use the geometric properties of the $\mathbb{R}^2$ to develop efficient algorithms. Moreover, we first consider the bicriteria case, since - as will be seen later - it is the basis for solving the Q-criteria case.

To this end, in this section we are looking for the Pareto solutions of the following vector optimization problem in $\mathbb{R}^2$

$$\min_{x\in\mathbb{R}^2} \begin{pmatrix} F^1(x) := \sum_{i=1}^{M} \lambda_i^1 \gamma^1 (x-A)_{(i)} \\ F^2(x) := \sum_{i=1}^{M} \lambda_i^2 \gamma^2 (x-A)_{(i)} \end{pmatrix}$$

where the weights λ_i^q are in increasing order with respect to the index i for each $q = 1, 2$, that is,

$$\lambda_1^q \leq \lambda_2^q \leq \ldots \leq \lambda_M^q \,, \quad q = 1, 2$$

and $\gamma^q(x-A)_{(i)}$ depends on the set Ω^q of importance given to the existing facilities by the q-th criterion, $q = 1, 2$. Therefore the previous vector optimization problem is convex, as was discussed in Section 2. In the classification scheme we use this problem is denoted by $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/2 - (\sum_{ord})_{par}$.

Note that in a multi-criteria setting each objective function F^q, $q \in \mathcal{Q}$, generates its own set of bisector lines. Therefore in the multi-criteria case the generalized elementary convex sets are generated by all the fundamental directions d_g^i, $i = 1, \ldots, M$, $g = 1 \ldots G_i$, and the bisector lines $B^q(a_i, a_j)$, $q \in \mathcal{Q}$.

We are able to give a geometrical characterization of the set $\mathcal{X}^*_{\mathrm{Par}}$ by the following theorem.

Theorem 3.1 *$\mathcal{X}^*_{\mathrm{Par}}(F^1, F^2)$ is a connected chain from $\mathcal{X}^*(F^1)$ to $\mathcal{X}^*(F^2)$ consisting of facets or vertices of cells or complete cells.*

Proof:

First of all, we know that $\mathcal{X}^*_{\mathrm{Par}} \neq \emptyset$, so we can choose $x \in \mathcal{X}^*_{\mathrm{Par}}$. There exists at least one cell $\mathrm{C} \in \mathcal{C}$ with $x \in \mathrm{C}$. Hence three cases can occur:

1. $x \in int(\mathrm{C})$: Since $x \in \mathcal{X}^*_{\mathrm{Par}}$ we obtain

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(x)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(x))$$

and by linearity of the ordered Weber problem in each cell we have

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(y)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(y)) \quad \forall\, y \in \mathrm{C}$$

which means $y \in \mathcal{X}^*_{\mathrm{Par}} \;\forall\, y \in \mathrm{C}$, hence $\mathrm{C} \subseteq \mathcal{X}^*_{\mathrm{Par}}$

2. $x \in \overline{ab} := conv\{a, b\} \subset bd(\mathrm{C})$ and $a, b \in Ext(\mathrm{C})$. We can choose $y \in int(\mathrm{C})$ and 2 cases can occur:

   (a) $y \in \mathcal{X}^*_{\text{Par}}$. Hence we can continue as in Case 1.

   (b) $y \notin \mathcal{X}^*_{\text{Par}}$. Therefore using the linearity we obtain first

   $$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(z)) \neq \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(z)) \quad \forall\, z \in int(\mathrm{C})$$

   and second, we have

   $$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(z)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(z)) \quad \forall\, z \in \overline{ab}$$

   since $x \in \mathcal{X}^*_{\text{Par}}$. Therefore we have that $\mathrm{C} \not\subseteq \mathcal{X}^*_{\text{Par}}$ and $\overline{ab} \subseteq \mathcal{X}^*_{\text{Par}}$.

3. $x \in Ext(\mathrm{C})$. We can choose $y \in int(\mathrm{C})$ and two cases can occur

   (a) If $y \in \mathcal{X}^*_{\text{Par}}$, we can continue as in Case 1.

   (b) If $y \notin \mathcal{X}^*_{\text{Par}}$, we choose $z_1, z_2 \in Ext(\mathrm{C})$ such that $\overline{xz_1}, \overline{xz_2}$ are faces of C,

      i. If z_1 or z_2 are in $\mathcal{X}^*_{\text{Par}}$, we can continue as in Case 2.

      ii. If z_1 and z_2 are not in $\mathcal{X}^*_{\text{Par}}$, then using the linearity in the same way as before we obtain that $(\mathrm{C} \setminus \{x\}) \cap \mathcal{X}^*_{\text{Par}} = \emptyset$

Hence, we obtain that the set of Pareto solutions consists of complete cells, complete faces and vertices of these cells. Since we know that the set $\mathcal{X}^*_{\text{Par}}$ is connected, the proof is completed. ■

3.2 An algorithm for solving $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/2 - (\Sigma_{ord})_{par}$

The idea of the bicriteria algorithm is to start in a vertex x of the cell structure which belongs to $\mathcal{X}^*_{\text{Par}}$, say $x \in \mathcal{X}^*_{1,2}$. Then using the connectivity of $\mathcal{X}^*_{\text{Par}}$ the algorithm proceeds moving from vertex x to another Pareto optimal vertex y of the cell structure which is connected with the previous one by an elementary convex set. This procedure is repeated until the end of the chain in $\mathcal{X}^*_{2,1}$ is reached.

By the linearity of the level sets in each cell we can distinguish the following disjoint cases, if $x \in \mathcal{X}^*_{\text{Par}}$:

A : $\mathrm{C} \subseteq \mathcal{X}^*_{\text{Par}}$, i.e. C is contained in the chain.

B : $\overline{xy}$ and $\overline{xz}$ are candidates for $\mathcal{X}^*_{\text{Par}}$ and $int(C) \not\subset \mathcal{X}^*_{\text{Par}}$.

C : $\overline{xy}$ is candidate for $\mathcal{X}^*_{\text{Par}}$ and $\overline{xz}$ is not contained in $\mathcal{X}^*_{\text{Par}}$.

D : $\overline{xz}$ is candidate for $\mathcal{X}^*_{\text{Par}}$ and $\overline{xy}$ is not contained in $\mathcal{X}^*_{\text{Par}}$.

E : Neither $\overline{xy}$ nor $\overline{xz}$ are contained in $\mathcal{X}^*_{\text{Par}}$.

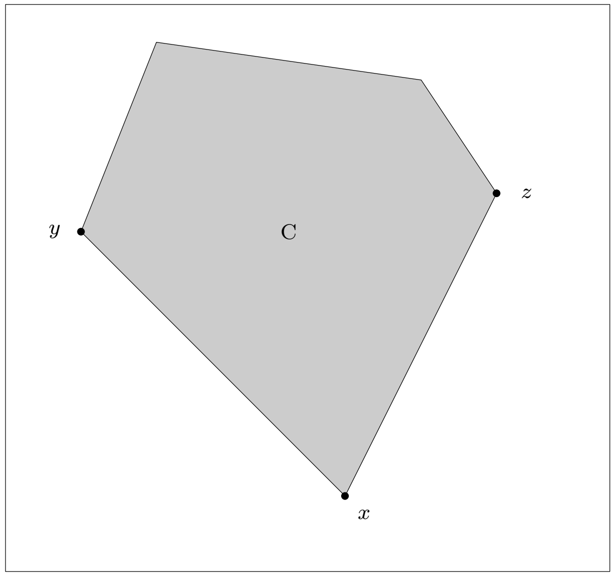

Figure 4: Illustration to Lemma 3.2 : $y, x, z \in Ext(\mathrm{C})$ in counterclockwise order

We denote by $\mathbf{sit}(\mathrm{C}, x)$ the situation appearing in cell C according to the extreme point x of C.

Lemma 3.1 *Let* $\mathrm{C}_1, \ldots, \mathrm{C}_{P_x}$ *be the cells containing the intersection point* x *, considered in counterclockwise order, and* $y_1, \ldots, y_{P_x}$ *the intersection points adjacent to* x *, considered in counterclockwise order, (see Figure 5). If* $x \in \mathcal{X}^*_{\mathrm{Par}}$ *and* $i \in \{1, \ldots, P_x\}$*, then the following holds :*

$$\overline{xy_{i+1}} \subseteq \mathcal{X}^*_{\mathrm{Par}} \iff \left\{ \begin{array}{ll} & \mathbf{sit}(\mathrm{C}_i, x) = \mathbf{A} \\ or & \mathbf{sit}(\mathrm{C}_{i+1}, x) = \mathbf{A} \\ or & \left\{ \begin{array}{ll} & \mathbf{sit}(\mathrm{C}_i, x) \in \{\mathbf{B}, \mathbf{C}\} \\ and & \mathbf{sit}(\mathrm{C}_{i+1}, x) \in \{\mathbf{B}, \mathbf{D}\} \end{array} \right\} \end{array} \right\}$$

Proof:

The proof of this lemma follows directly from the exhaustive case analysis in the appendix.

■

Applying these two results we describe the following algorithm.

ALGORITHM 3.1

(Solving $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/2 - (\sum_{ord})_{par}$ *.)*

Input:

1. *Demand points* $a_i \in \mathbb{R}^2$ *,* $i \in \mathcal{M}$ *.*
2. *Weights* λ_i^q *,* $i \in \mathcal{M}$ *,* $q = 1, 2$ *satisfying* $0 \leq \lambda_1^q \leq \ldots \leq \lambda_M^q$ *for* $q = 1, 2$ *.*
3. *Weights* ω_i^q *,* $i \in \mathcal{M}$ *,* $q = 1, 2$ *satisfying* $\omega_i^q \geq 0$ *for* $i \in \mathcal{M}$ *,* $q = 1, 2$ *.*

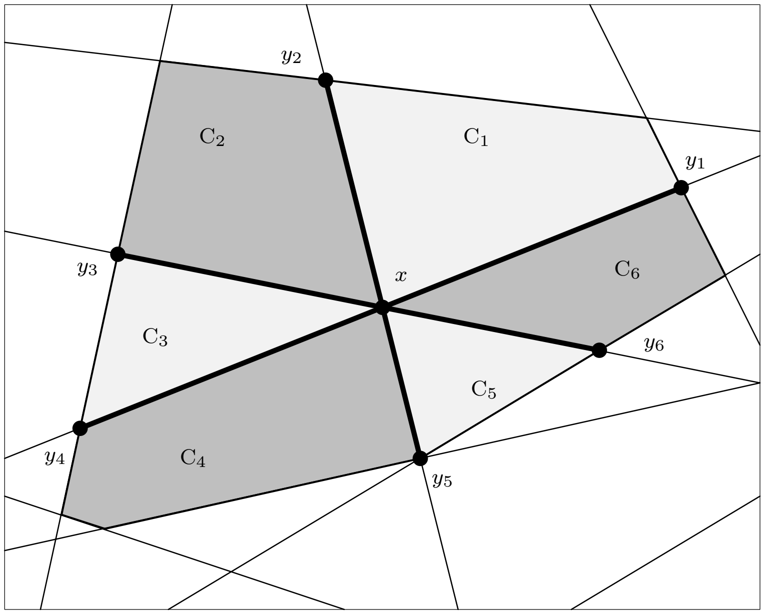

Figure 5: Illustration to Lemma 3.1 with $P_x = 6$

4. *Polyhedral gauges* $\gamma_i : \mathbb{R}^2 \to \mathbb{R}$, $i \in \mathcal{M}$.

Output:

1. $\mathcal{X}^*_{\text{Par}}(F^1, F^2)$.

Steps:

1. *Computation of the planar graph generated by the cells.*
2. *Compute the two sets of lexicographical solutions* $\mathcal{X}^*_{1,2}$, $\mathcal{X}^*_{2,1}$.
3. *IF* $\mathcal{X}^*_{1,2} \cap \mathcal{X}^*_{2,1} \neq \emptyset$
4. *THEN (*$\star$ *trivial case :* $\mathcal{X}^*(F^1) \cap \mathcal{X}^*(F^2) \neq \emptyset$ $\star$*)* $\mathcal{X}^*_{\text{Par}} := co\{\mathcal{X}^*_{1,2}\}$
5. *ELSE (*$\star$ *non trivial case :* $\mathcal{X}^*(F^1) \cap \mathcal{X}^*(F^2) = \emptyset$ $\star$*)*
6. $\mathcal{X}^*_{\text{Par}} := \mathcal{X}^*_{1,2} \cup \mathcal{X}^*_{2,1}$
7. *Choose* $x \in \mathcal{X}^*_{1,2} \cap \mathcal{GIP}$ *and* $i := 0$.
8. *WHILE* $x \notin \mathcal{X}^*_{2,1}$ *DO*
9. *BEGIN*
10. *REPEAT*
11. $i := i + 1$

IF $i > P_x$ *THEN* $i := i - P_x$

UNTIL $\mathbf{sit}(\mathrm{C}_i, x) := \mathbf{A}$ OR ($\mathbf{sit}(\mathrm{C}_i, x) \in \{\mathbf{B,C}\}$ AND $\mathbf{sit}(\mathrm{C}_{i+1}, x) \in \{\mathbf{B,D}\}$)

IF $\mathbf{sit}(\mathrm{C}_i, x) := \mathbf{A}$

THEN (⋆ *We have found a bounded cell.* ⋆) $\mathcal{X}^*_{\mathrm{Par}} := \mathcal{X}^*_{\mathrm{Par}} \cup \mathrm{C}_i$

ELSE (⋆ *We have found a bounded face.* ⋆) $\mathcal{X}^*_{\mathrm{Par}} := \mathcal{X}^*_{\mathrm{Par}} \cup \overline{xy_i}$

$temp := x$

$x := y_i$

$i := i_x(temp) - 1$ (⋆ Where $i_x(temp)$ is the index of $temp$ in the list of adjacent generalized intersection points to the generalized intersection point x. ⋆).

END ■

[BO79] proved that the computation of a planar graph induced by n line segments in the plane, can be computed in $O((n+s) log\, n)$ time, where s is the number of intersection points of the line segments. [BO79] method only works for line segments. Therefore, we need to assure in our problem that we can replace half-lines by line segments. The following lemma shows that we only have to look for the solutions in a bounded region. Hence all half-lines defining the cells are transformed into segments.

Lemma 3.2 *For* $x \in \mathcal{X}^*_{\mathrm{Par}}\left(F^1, \ldots, F^Q\right)$ *holds*

$$l_2(x) \leq 2 \max_{m \in \mathcal{M}} \left\{ l_2(a_m) \frac{r^m_{\max}}{r^m_{\min}} \right\} =: R$$

with

$$r^m_{\max} := \max_{i=1,\ldots,G_m} \{l_2(e^m_i)\} \ \forall\, m \in \mathcal{M}$$

and

$$r^m_{\min} := \min_{i=1,\ldots,G_m} \left\{ l_2 \left(\frac{\langle e^m_{i+1}, e^m_{i+1} - e^m_i \rangle}{\langle e^m_{i+1} - e^m_i, e^m_{i+1} - e^m_i \rangle} e^m_i - \frac{\langle e^m_i, e^m_{i+1} - e^m_i \rangle}{\langle e^m_{i+1} - e^m_i, e^m_{i+1} - e^m_i \rangle} e^m_{i+1} \right) \right\} \ \forall\, m \in \mathcal{M} \ ,$$

i. e.

$$\mathcal{X}^*_{\mathrm{Par}}\left(F^1, \ldots, F^Q\right) \subseteq B(0, R) \, .$$

Proof:

Indeed, we know that $\mathcal{X}^*_{\mathrm{Par}}\left(F^1, \ldots, F^Q\right) \subseteq \mathcal{X}^*_{\mathrm{w-Par}}\left(F^1, \ldots, F^Q)\right)$. We show in the following

$$\mathcal{X}^*_{\mathrm{w-Par}}\left(F^1, \ldots, F^Q\right) \subseteq \mathcal{X}^*_{\mathrm{w-Par}}\left(\gamma_1(. - a_1), \ldots, \gamma_M(. - a_M)\right) \, .$$

Let $x \in \mathbb{R}^n \setminus \mathcal{X}^*_{\text{w-Par}}\left(\gamma_1(.-a_1),\ldots,\gamma_M(.-a_M)\right)$. Then, there exists $y \in \mathbb{R}^n$ such that $\gamma_m(y) < \gamma_m(x)$ for all $m \in \mathcal{M}$. For any $0 \le \lambda_1 \le \ldots \le \lambda_M$ the function

$$\begin{array}{ccc} \mathbb{R}^M_+ & \to & \mathbb{R} \\ x & \mapsto & \sum_{m=1}^M \lambda_m x_{\sigma(m)} \end{array}$$

with a permutation σ of $\mathcal{M}$ such that $x_{\sigma(1)} \le \ldots \le x_{\sigma(M)}$, is non-decreasing. Then $F^q(y) < F^q(x)$ for all $1 \le q \le Q$. This means $x \in \mathbb{R}^n \setminus \mathcal{X}^*_{\text{w-Par}}\left(F^1,\ldots,F^Q\right)$.

Now it is sufficient to prove the claim for

$$x \in \mathcal{X}^*_{\text{w-Par}}\left(\gamma_1(.-a_1),\ldots,\gamma_M(.-a_M)\right).$$

Let $x \in \mathcal{X}^*_{\text{w-Par}}\left(\gamma_1(.-a_1),\ldots,\gamma_M(.-a_M)\right)$. Then there exists no point $y \in \mathbb{R}^2$ which strictly dominates x. That means, especially for $y = 0$, there exists an index $m_x \in \mathcal{M}$ with

$$\gamma_{m_x}(x - a_{m_x}) \le \gamma_{m_x}(0 - a_{m_x}) = \gamma_{m_x}(-a_{m_x}) \quad .$$

Using the triangular inequality this means

$$\gamma_{m_x}(x) = \gamma_{m_x}(a_{m_x} + x - a_{m_x}) \le \gamma_{m_x}(a_{m_x}) + \gamma_{m_x}(x - a_{m_x}) \le \gamma_{m_x}(a_{m_x}) + \gamma_{m_x}(-a_{m_x})$$

By elementary calculations using the law of sines we have

$$l_2(x) \le \gamma_{m_x}(x) \cdot r^{m_x}_{\text{max}} \qquad \text{and} \qquad \gamma_{m_x}(\pm a_{m_x}) \le \frac{l_2(\pm a_{m_x})}{r^{m_x}_{\text{min}}} \quad .$$

Summarizing the previous estimations we obtain

$$\begin{aligned} l_2(x) &\le \gamma_{m_x}(x) \cdot r^{m_x}_{\text{max}} \\ &\le \Big(\gamma_{m_x}(a_{m_x}) + \gamma_{m_x}(-a_{m_x})\Big) r^{m_x}_{\text{max}} \\ &\le \Big(l_2(a_{m_x}) + l_2(-a_{m_x})\Big) \frac{r^{m_x}_{\text{max}}}{r^{m_x}_{\text{min}}} \\ &\le 2\, l_2(a_{m_x}) \frac{r^{m_x}_{\text{max}}}{r^{m_x}_{\text{min}}} \end{aligned}$$

The right side of the previous inequality depends on $m_x \in \mathcal{M}$. To avoid this we have to consider the maximum over all indices $m \in \mathcal{M}$ which leads to the inequality:

$$l_2(x) \le 2 \max_{m \in \mathcal{M}} \left\{ l_2(a_m) \frac{r^m_{\text{max}}}{r^m_{\text{min}}} \right\}$$

■

This lemma implies that in the case of the ordered Weber problem the computation of the planar graph generated by the fundamental directions and bisector lines can be done in $O((M^2 G_{\text{max}} + M^4 G^2_{\text{max}}) \log(M^2 G_{\text{max}})) = O(M^4 G^2_{\text{max}} \log(M G_{\text{max}}))$.

The evaluation of the ordered Weber function for one point needs $O(M \log(M G_{\text{max}}))$, therefore we obtain $O(M^5 G^2_{\text{max}} \log(M G_{\text{max}}))$ for the computation of lexicographic solutions. At the end, the complexity for computing the chain is $O(M^5 G^2_{\text{max}} \log(M G_{\text{max}}))$, since we have to consider at most $O(M^4 G^2_{max})$ cells and the determination of **sit**$(.,.)$ can be done in $O(M \log(M G_{max}))$. The overall complexity is $O(M^5 G^2_{\text{max}} \log(M G_{\text{max}}))$.

4 The 3-Criteria Case

In this section we turn to the 3-criteria case and we develop an efficient algorithm for computing $\mathcal{X}^*_{\mathrm{Par}}(F^1, F^2, F^3)$ using the results of the bicriteria case.

The first lemma establishes some useful geometric relations.

Lemma 4.1

1. *For a convex function $F : \mathbb{R}^n \to \mathbb{R}$ and a real value $z \in \mathbb{R}$ holds*
$$L_<(F,z) = \mathrm{int}(L_<(F,z)) \subseteq \mathrm{int}(L_\leq(F,z)) \subseteq \mathrm{ri}(L_\leq(F,z)) \quad . \tag{9}$$

2. *The tangent cone to a closed convex set $C \subseteq \mathbb{R}^n$ at a point $x \in C$ is the closure of the cone generated by $C - \{x\}$, i. e.*
$$\begin{aligned} T_C(x) &= \overline{\mathrm{cone}}(C - \{x\}) = \mathrm{cl}(\mathbb{R}^+(C - \{x\})) \\ &= \mathrm{cl}\,\{\, d \in \mathbb{R}^n : d = \lambda(y-x), y \in C, \lambda \geq 0 \,\} \quad . \end{aligned} \tag{10}$$
and especially
$$C - \{x\} \subseteq T_C(x) \quad \text{or} \quad C \subseteq T_C(x) + \{x\} \tag{11}$$

3. *For two nonempty closed convex sets $C_1, C_2 \in \mathbb{R}^n$ and $x \in C_1 \cap C_2$ holds*
$$T_{C_1 \cap C_2}(x) \subseteq T_{C_1}(x) \cap T_{C_2}(x) \quad . \tag{12}$$
If in addition $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$ *holds, then*
$$T_{C_1 \cap C_2}(x) = T_{C_1}(x) \cap T_{C_2}(x) \quad . \tag{13}$$

Proof:

1. The equality follows from the continuity of F. The first inclusion follows directly from the definitions of level sets and strict level sets, whereas the second inclusion follows from the definitions of the interior and the relative interior.

2. See [HUL93], Chapter III, Proposition 5.2.1 .

3. See [HUL93], Chapter III, Propositions 5.3.1 and 5.3.2 .

■

We denote by
$$C_\infty(\mathbb{R}^+_0, \mathbb{R}^2) := \left\{ \varphi \,|\, \varphi : \mathbb{R}^+_0 \to \mathbb{R}^2, \varphi \,\text{continuous}, \lim_{t\to\infty} l_2((\varphi(t)) = +\infty \right\} \quad . \tag{14}$$
the set of continuous curves, which map the set of non-negative numbers $\mathbb{R}^+_0 := [0, \infty)$ into the two-dimensional space $\mathbb{R}^2$ and whose image $\varphi(\mathbb{R}^+_0)$ is an unbounded set in $\mathbb{R}^2$.

For a set $C \subseteq \mathbb{R}^2$ we define the enclosure of C by
$$\begin{aligned} \mathrm{encl}\,(C) \;:=\; \Big\{ & x \in \mathbb{R}^2 : \exists\, \varepsilon > 0 \text{ with } B(x,\varepsilon) \cap C = \emptyset\,, \exists\, t_\varphi \in [0,\infty) \text{ with} \\ & \varphi(t_\varphi) \in C \text{ for all } \varphi \in C_\infty(\mathbb{R}^+_0, \mathbb{R}^2) \text{ with } \varphi(0) = x \Big\} \quad . \end{aligned} \tag{15}$$

Notice that $C \cap \mathrm{encl}\,(C) = \emptyset$. Informally spoken $\mathrm{encl}\,(C)$ contains all the points which are surrounded by C, but do not belong itself to C.

Lemma 4.2
If $x \in \mathbb{R}^n$ is dominated by $y \in \mathbb{R}^n$ with respect to strict Pareto optimality , then $z_\lambda := x + \lambda(x-y) \in \mathbb{R}^n$ with $\lambda \geq 0$ is dominated by x with respect to strict Pareto optimality.

Proof:
From $z_\lambda := x + \lambda(x-y), \lambda \geq 0$ follows $x = \frac{1}{1+\lambda} z_\lambda + \frac{\lambda}{1+\lambda} y$ with $\lambda \geq 0$.
Since y dominates x with respect to strict Pareto optimality and from the convexity of $F^1, \ldots, F^Q$

we obtain $\quad F^q(x) \;\leq\; \frac{1}{1+\lambda} F^q(z_\lambda) + \frac{\lambda}{1+\lambda} F^q(y) \;<\; \frac{1}{1+\lambda} F^q(z_\lambda) + \frac{\lambda}{1+\lambda} F^q(x) \quad$ for all $q \in \mathcal{Q}$,

which implies $\quad (1+\lambda) F^q(x) \;<\; F^q(z_\lambda) + \lambda F^q(x) \quad$ for all $q \in \mathcal{Q}$

respectively $\quad F^q(x) \;<\; F^q(z_\lambda) \quad$ for all $q \in \mathcal{Q}$,

i. e. x dominates z_λ with respect to strict Pareto optimality. ■

We denote the union of the bicriteria chains including the 1-criterion solutions by

$$\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right) := \bigcup_{q=1}^{3} \mathcal{X}^*(F^q) \cup \bigcup_{q=1}^{2} \bigcup_{p=q+1}^{3} \mathcal{X}^*_{\text{Par}}\left(F^p, F^q\right) \quad . \tag{16}$$

We use the abbreviation *gen* since this set will *generate* the set $\mathcal{X}^*_{\text{Par}}\left(F^1, F^2, F^3\right)$. The next

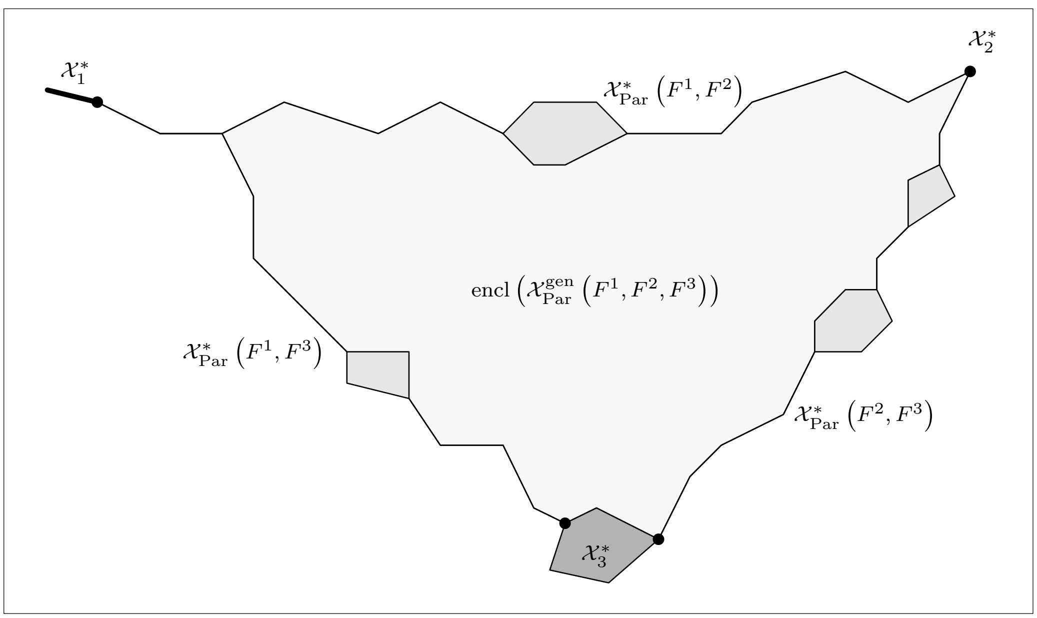

Figure 6: The enclosure of $\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right)$

lemmata give detailed geometric descriptions of parts of the Pareto solution which are needed to build up $\mathcal{X}^*_{\text{Par}}\left(F^1, F^2, F^3\right)$. We will also learn more about the part of the plane which is crossed by the Pareto chain.

Lemma 4.3

$$\text{cl}\left(\text{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right)\right)\right) \subseteq \mathcal{X}^*_{\text{s-Par}}\left(F^1, F^2, F^3\right)$$

Proof:
Let $x \in \operatorname{encl}(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3))$. Assume $x \notin \mathcal{X}^{*}_{\text{s-Par}}(F^1, F^2, F^3)$. Then there exists a point $y \in \mathbb{R}^2$ which dominates x with respect to strict Pareto optimality. Consider the curve

$$\varphi : \mathbb{R}^{+}_{0} \to \mathbb{R}^2 \, , \, t \mapsto x + t(x-y) \quad .$$

Obviously φ is continuous and fulfills $\lim_{t\to\infty} l_2((\varphi(t)) = +\infty$, i. e. $\varphi \in C_\infty(\mathbb{R}^{+}_{0}, \mathbb{R}^2)$. Moreover $\varphi(0) = x$. Since $x \in \operatorname{encl}(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3))$, there exists $t \in [0, \infty)$ with

$$z_t := \varphi(t) \in \mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right) \quad . \tag{17}$$

By Lemma 4.2 we have $F^q(x) \leq F^q(z_t)$ for all $q \in \mathcal{Q}$. Hence we can continue with the following case analysis with respect to (17):

Case 1: $z_t \in \mathcal{X}^*(F^q)$ for some $q \in \{1,2,3\}$:
$\Rightarrow \; x \in \mathcal{X}^*(F^q) \;\Rightarrow\; x \in \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3) \;\Rightarrow$ Contradiction !

Case 2: $z_t \in \mathcal{X}^{*}_{\text{Par}}(F^p, F^q)$ for some $p, q \in \{1,2,3\}, p < q$:
$\Rightarrow \; x \in \mathcal{X}^{*}_{\text{Par}}(F^p, F^q) \;\Rightarrow\; x \in \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3) \;\Rightarrow$ Contradiction !

Therefore we have $x \in \mathcal{X}^{*}_{\text{s-Par}}(F^1, F^2, F^3)$, i. e. $\operatorname{encl}(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3)) \subseteq \mathcal{X}^{*}_{\text{s-Par}}(F^1, F^2, F^3)$. Since $\mathcal{X}^{*}_{\text{s-Par}}(F^1, F^2, F^3)$ is closed (see [Whi82], Chapter 4, Theorem 27) we obtain

$$\operatorname{cl}\left(\operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1, F^2, F^3\right)\right)\right) \subseteq \operatorname{cl}\left(\mathcal{X}^{*}_{\text{s-Par}}\left(F^1, F^2, F^3\right)\right) = \mathcal{X}^{*}_{\text{s-Par}}\left(F^1, F^2, F^3\right) \quad .$$

∎

For $q \in \mathcal{Q}$ we use the abbreviations

$$L^q_{\leq}(x) := L_{\leq}(F^q, F^q(x)) \tag{18}$$

and

$$L^q_{<}(x) := L_{<}(F^q, F^q(x)) \tag{19}$$

and

$$T^q_{\leq}(x) := T_{L^q_{\leq}(x)}(x) := T_{L_{\leq}(F^q, F^q(x))}(x) \quad . \tag{20}$$

For $p, q \in \mathcal{Q}$ we use the abbreviations

$$L^{p\cap q}_{\leq}(x) := L^p_{\leq}(x) \cap L^q_{\leq}(x) := L_{\leq}(F^p, F^p(x)) \cap L_{\leq}(F^q, F^q(x)) \tag{21}$$

and

$$L^{p\cap q}_{<}(x) := L^p_{<}(x) \cap L^q_{<}(x) := L_{<}(F^p, F^p(x)) \cap L_{<}(F^q, F^q(x)) \tag{22}$$

and

$$T^{p\cap q}_{\leq}(x) := T_{L^{p\cap q}_{\leq}(x)}(x) := T_{L^p_{\leq}(x) \cap L^q_{\leq}(x)}(x) := T_{L_{\leq}(F^p, F^p(x)) \cap L_{\leq}(F^q, F^q(x))}(x) \quad . \tag{23}$$

Lemma 4.4
If the assumptions

$$\bigcap_{q=1}^{3} L^q_{<}(x) = \emptyset \tag{24}$$

and

$$L_<^{p\cap q}(x) \neq \emptyset \;\forall\, p,q \in \{1,2,3\}, p<q \tag{25}$$

are fulfilled, then

$$\bigcap_{q=1}^{3} L_\leq^q(x) = \{x\} \quad , \tag{26}$$

$$\{x\} + \bigcap_{q=1}^{3} T_\leq^q(x) = \{x\} \tag{27}$$

and

$$\Big(\{x\} \;-\; T_\leq^{p\cap q}(x)\Big) \;\cap\; \mathcal{X}^*_{\mathrm{w-Par}}\,(F^p, F^q) \;=\; \emptyset \;\forall\, p,q \in \{1,2,3\}, p<q \quad . \tag{28}$$

Proof:
From (25) we can especially conclude $L_<^q(x) \neq \emptyset, L_<^q(x) = \mathrm{int}(L_\leq^q(x)), L_=^q(x) = \mathrm{bd}(L_\leq^q(x))$ for all $q \in \{1,2,3\}$. Obviously $x \in \bigcap_{q=1}^3 L_\leq^q(x)$. Assume there exists $y \in \bigcap_{q=1}^3 L_\leq^q(x) \setminus \{x\}$, i. e. $y \in L_\leq^q(x) \setminus \{x\}$ for all $q \in \{1,2,3\}$. We can distinguish the following cases:

Case 1: $y \in L_<^p(x)$ for some $p \in \{1,2,3\}$:
Then there exists $\varepsilon > 0$ with $B(x,\varepsilon) \subseteq L_<^p(x)$. By (25) we can choose $u \in L_<^{r\cap s}(x), r,s \in \{1,2,3\} \setminus \{p\}, r<s$. Now we choose $\lambda \in (0,1)$, such that

$$z_\lambda := \lambda u + (1-\lambda) y \in B(y,\varepsilon) \subseteq L_<^p(x) \quad .$$

For $q = r,s$ we obtain by the convexity of F^q

$$\begin{aligned} F^q(z_\lambda) \;&\leq\; \lambda F^q(u) + (1-\lambda) F^q(y) \\ &<\; \lambda F^q(x) + (1-\lambda) F^q(x) \;=\; F^q(x) \quad , \end{aligned}$$

i. e. $z_\lambda \in L_<^q(x)$ and hence $z_\lambda \in \bigcap_{q=1}^3 L_<^q(x)$. Contradiction to (24).

Case 2: $y \in L_=^q(x)$ for all $q \in \{1,2,3\}$:
By the convexity of F^q we have $\overline{xy} \subseteq L_\leq^q(x)$ for some $q \in \{1,2,3\}$. Hence we distinguish again two cases:

Case 2.1: $\overline{xy} \not\subseteq L_=^p(x)$ for some $p \in \{1,2,3\}$:
Then $\mathrm{ri}(\overline{xy}) \subseteq L_<^p(x)$. We choose $\tilde{y} \in \mathrm{ri}(\overline{xy})$ and the result follows by **Case 1** for $y := \tilde{y}$.

Case 2.2: $\overline{xy} \subseteq L_=^q(x)$ for all $q \in \{1,2,3\}$:
Now we consider the two half-spaces generated by the carrier line of the segment $[x,y]$. Hence by (25) the level sets $L_\leq^q(x), q \in \{1,2,3\}$, ly in the same half-space. But then $\bigcap_{q=1}^3 L_<^q(x) \neq \emptyset$ which contradicts (24).

Therefore (26) is proven.

Instead of (27) we prove $\bigcap_{q=1}^3 T_\leq^q(x) = \{0\}$. Assume there exists $y \neq 0$ with $y \in \bigcap_{q=1}^3 T_\leq^q(x)$, i. e. $y \in T_\leq^q(x)$ for $q \in \{1,2,3\}$. By $L_<^q(x) \neq \emptyset$ the level set $L_\leq^q(x)$ is a non-degenerate polyhedron in $\mathbb{R}^2$ and there exists $\lambda_q > 0$, such that $x + \lambda_q y \in L_\leq^q(x)$ for $q \in \{1,2,3\}$ (compare with Figure 7). We define $\lambda := \min\{\lambda_1, \lambda_2, \lambda_3\} > 0$. Using $x \in L_\leq^q(x)$ and the convexity of $L_\leq^q(x)$, we obtain $x + \lambda y \in [x, x+\lambda_q y] \subseteq L_\leq^q(x)$ for $q \in \{1,2,3\}$. Finally we obtain $x + \lambda y \in \bigcap_{q=1}^3 L_\leq^q(x)$, which contradicts (26). Therefore (27) is proven.

Let $y \in \left(\{x\} - T_{\leq}^{p\cap q}(x)\right)$, then there exists $u \in T_{\leq}^{p\cap q}(x)$ with $y = x - u$. Since $u \in T_{\leq}^{p\cap q}(x)$ there exists $v \in L_{\leq}^{p}(x) \cap L_{\leq}^{q}(x)$ and $\lambda \geq 0$ with $u = \lambda(v - x)$ (compare with Figure 7). Therefore we have

$$y = x - \lambda(v - x) \text{ respectively } x = \frac{1}{1+\lambda} y + \frac{\lambda}{1+\lambda} v \quad .$$

For $r = p, q$ we obtain now

$$\begin{aligned} F^r(x) &\leq \frac{1}{1+\lambda} F^r(y) + \frac{\lambda}{1+\lambda} F^r(v)\,, \text{ since } F^r \text{ is convex} \\ &\leq \frac{1}{1+\lambda} F^r(y) + \frac{\lambda}{1+\lambda} F^r(x)\,, \text{ since } v \in L_{\leq}^{p\cap q}(x) \subseteq L_{\leq}^{r}(x) \end{aligned}$$

which implies

$$(1+\lambda) F^r(x) \leq F^r(y) + \lambda F^r(x)$$

respectively

$$F^r(x) \leq F^r(y)\,.$$

By (6) we obtain $x \notin \mathcal{X}^*_{\mathrm{w-Par}}(F^p, F^q)$ and hence $y \notin \mathcal{X}^*_{\mathrm{w-Par}}(F^p, F^q)$. Therefore (28) is proven.

■

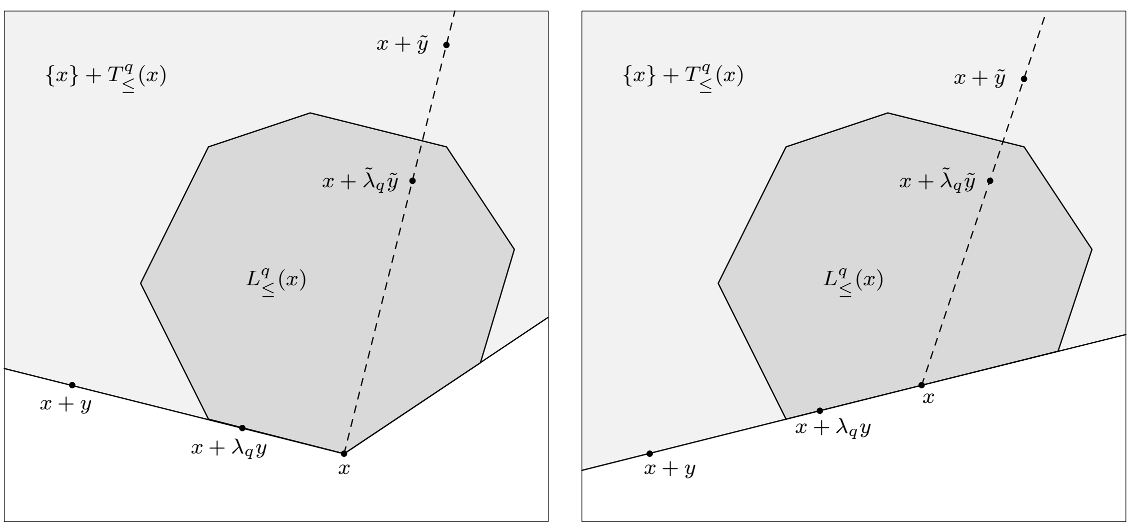

Figure 7: Illustration to the proof of Lemma 4.4

Lemma 4.5
If $L_{<}^{p\cap q}(x) \neq \emptyset$ *for some* $p, q \in \{1, 2, 3\}, p < q$*, then*

$$L_{<}^{p\cap q}(x) \cap \mathcal{X}^*_{\mathrm{w-Par}}(F^p, F^q) \neq \emptyset \quad . \tag{29}$$

Proof:
By $L_{<}^{p\cap q}(x) \neq \emptyset$ we have $x \notin \mathcal{X}^*_{\mathrm{w-Par}}(F^p, F^q)$. Therefore, without loss of generality, there exists a $y \in \mathcal{X}^*_{\mathrm{w-Par}}(F^p, F^q)$ that dominates x, i. e. $F^p(y) < F^p(x)$ and $F^q(y) < F^q(x)$. Hence

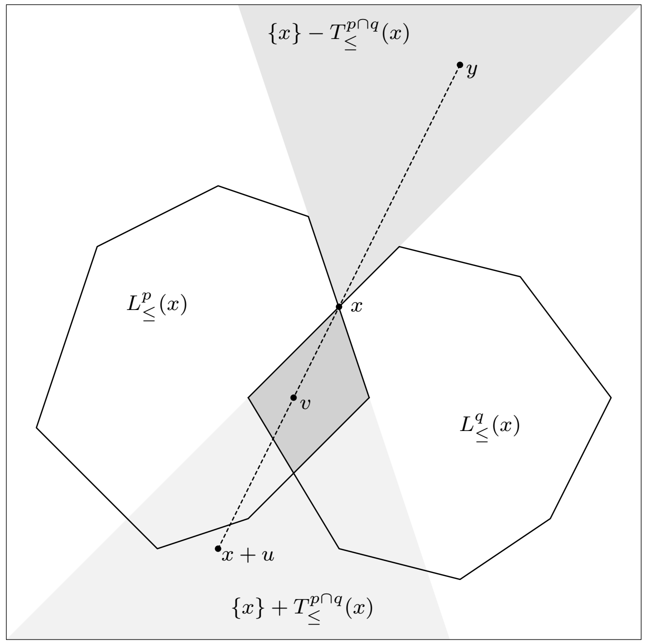

Figure 8: Illustration to the proof of Lemma 4.4

$y \in L_{<}^{p\cap q}(x)$, that means $L_{<}^{p\cap q}(x) \cap \mathcal{X}_{\mathrm{w-Par}}^{*}(F^p, F^q) \neq \emptyset$. ■

The curve $\varphi \in C_\infty(\mathbb{R}_0^+, \mathbb{R}^2)$ with $\varphi(0) = x$ **separates** the sets C and D with respect to the convex cone Γ pointed at x, if $C, D \subseteq \Gamma$ and there is no continuous curve $\zeta \in C([0,1], \mathbb{R}^2)$ with $\zeta([0,1]) \subseteq \Gamma, \zeta(0) \in C, \zeta(1) \in D$ verifying $\varphi(\mathbb{R}_0^+) \cap \zeta([0,1]) \neq \emptyset$ (see Figure 9).

Lemma 4.6

$$\mathcal{X}_{\mathrm{w-Par}}^{*}\left(F^1, F^2, F^3\right) \subseteq \mathcal{X}_{\mathrm{Par}}^{\mathrm{gen}}\left(F^1, F^2, F^3\right) \cup \mathrm{encl}\left(\mathcal{X}_{\mathrm{Par}}^{\mathrm{gen}}\left(F^1, F^2, F^3\right)\right)$$

Proof:
Let $x \in \mathcal{X}_{\mathrm{w-Par}}^{*}(F^1, F^2, F^3)$. Then we have by (6) $\bigcap_{q=1}^{3} L_{<}^{q}(x) = \emptyset$ and we can distinguish the following cases.

If $L_{<}^{p\cap q}(x) = \emptyset$ for some $p, q \in \{1,2,3\}, p < q$, then

$$x \in \mathcal{X}_{\mathrm{w-Par}}^{*}(F^p, F^q) = \mathcal{X}^*(F^p) \cup \mathcal{X}_{\mathrm{Par}}^{*}(F^p, F^q) \cup \mathcal{X}^*(F^q) \subseteq \mathcal{X}_{\mathrm{Par}}^{\mathrm{gen}}\left(F^1, F^2, F^3\right)$$

and we are done.

On the other hand, if $L_{<}^{p\cap q}(x) \neq \emptyset$ for all $p, q \in \{1,2,3\}, p < q$, we will prove $x \in \mathrm{encl}(\mathcal{X}_{\mathrm{Par}}^{\mathrm{gen}}(F^1, F^2, F^3))$.

We have by (6) $x \notin \mathcal{X}_{\mathrm{w-Par}}^{*}(F^p, F^q)$ for all $p, q \in \{1,2,3\}, p < q$, and hence $x \notin \mathcal{X}_{\mathrm{Par}}^{\mathrm{gen}}(F^1, F^2, F^3)$. Since $\mathcal{X}_{\mathrm{w-Par}}^{*}(F^p, F^q)$ is closed (see [Whi82], Chapter 4, Theorem 27) for all $p, q \in \{1,2,3\}, p < q$, the set $\mathcal{X}_{\mathrm{Par}}^{\mathrm{gen}}(F^1, F^2, F^3)$ is also closed. Hence there exists $\varepsilon > 0$ with $B(x, \varepsilon) \cap \mathcal{X}_{\mathrm{Par}}^{\mathrm{gen}}(F^1, F^2, F^3) = \emptyset$.

Moreover we have to prove that all curves $\varphi \in C_\infty(\mathbb{R}_0^+, \mathbb{R}^2)$ with $\varphi(0) = x$ fulfill $\varphi(\mathbb{R}_0^+) \cap \mathcal{X}_{\mathrm{Par}}^{\mathrm{gen}}(F^1, F^2, F^3) \neq \emptyset$.

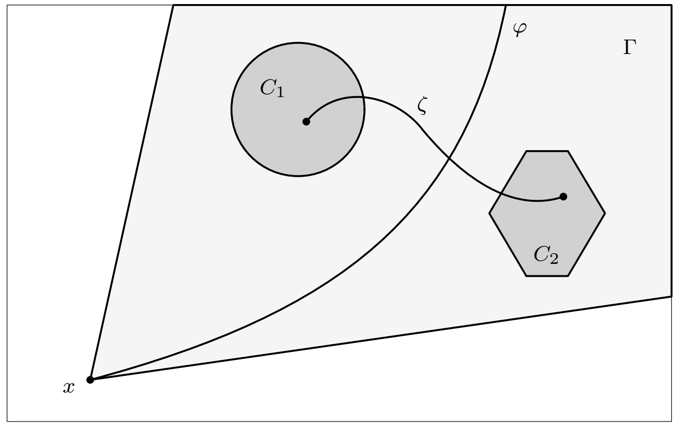

Figure 9: φ separates C_1 and C_2 with respect to Γ

By (29) we have

$$C^{p\cap q}(x) := L_<^{p\cap q}(x) \cap \mathcal{X}^*_{\text{w}-\text{Par}}(F^p, F^q) \neq \emptyset \tag{30}$$

and by (30), (22) and (11) we have

$$C^{p\cap q}(x) \subseteq L_<^{p\cap q}(x) \subseteq \begin{cases} L_<^p(x) \subseteq L_\leq^p(x) \subseteq \{x\} + T_\leq^p(x) \\ L_<^q(x) \subseteq L_\leq^q(x) \subseteq \{x\} + T_\leq^q(x) \end{cases} \tag{31}$$

for all $p, q \in \{1, 2, 3\}, p < q$.

Moreover we have $\mathcal{X}^*(F^q) \subseteq L_\leq^q(x) \subseteq \{x\} + T_\leq^q(x)$ for all $q \in \{1, 2, 3\}$ by (11). In addition we have

$$\begin{aligned} \mathcal{X}^*(F^p) \cup C^{p\cap q}(x) &\subseteq \mathcal{X}^*(F^p) \cup \mathcal{X}^*_{\text{w}-\text{Par}}(F^p, F^q) && \text{by (30)} \\ &= \mathcal{X}^*_{\text{w}-\text{Par}}(F^p, F^q) \\ &\subseteq \mathbb{R}^2 \setminus \left(\{x\} - T_\leq^{p\cap q}(x)\right) && \text{by (28)} \end{aligned}$$

for all $p, q \in \{1, 2, 3\}, p < q$.

Now let $\varphi \in C_\infty(\mathbb{R}_0^+, \mathbb{R}^2)$ with $\varphi(0) = x$.

If $\varphi(\mathbb{R}_0^+) \cap C^{p\cap q}(x) \neq \emptyset$ for some $p, q \in \{1, 2, 3\}, p < q$, we have

$$\begin{aligned} \emptyset &\neq \varphi(\mathbb{R}_0^+) \cap \mathcal{X}^*_{\text{w}-\text{Par}}(F^p, F^q) && \text{by (30)} \\ &\subseteq \varphi(\mathbb{R}_0^+) \cap \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3) && \text{by (16)} \end{aligned}$$

and we are done.

Otherwise if $\varphi(\mathbb{R}_0^+) \cap C^{p\cap q}(x) = \emptyset$ for all $p, q \in \{1, 2, 3\}, p < q$, there exist $p, q \in \{1, 2, 3\}, p < q$, such that φ separates $C^{p\cap r}(x)$ and $C^{q\cap r}(x)$ with respect to $\{x\} + T_\leq^r(x), r \in \{1, 2, 3\} \setminus \{p, q\}$.

On one hand, $\mathcal{X}^*_{\text{w}-\text{Par}}(F^p, F^r)$ and $\mathcal{X}^*_{\text{w}-\text{Par}}(F^q, F^r)$ are connected sets both containing $\mathcal{X}^*(F^r)$.

On the other hand, by (28) the bicriteria chains $\mathcal{X}^*_{\text{w}-\text{Par}}(F^p, F^r)$ and $\mathcal{X}^*_{\text{w}-\text{Par}}(F^q, F^r)$ cannot cross $\{x\} - T_\leq^{p\cap r}(x)$ and $\{x\} - T_\leq^{q\cap r}(x)$, respectively.

Therefore we can distinguish the following three cases:

Case 1: $\mathcal{X}^*(F^r)$ is separated from $C^{p\cap r}(x)$ by φ with respect to $\{x\}+T^r_{\leq}(x), r \in \{1,2,3\}\backslash\{p,q\}$
$\Rightarrow\ \emptyset \neq \mathcal{X}^*_{\mathrm{w-Par}}(F^p, F^r) \cap \varphi(\mathrm{I\!R}^+_0) \subseteq \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^1, F^2, F^3) \cap \varphi(\mathrm{I\!R}^+_0)$

Case 2: $\mathcal{X}^*(F^r)$ is separated from $C^{q\cap r}(x)$ by φ with respect to $\{x\}+T^r_{\leq}(x), r \in \{1,2,3\}\backslash\{p,q\}$
$\Rightarrow\ \emptyset \neq \mathcal{X}^*_{\mathrm{w-Par}}(F^q, F^r) \cap \varphi(\mathrm{I\!R}^+_0) \subseteq \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^1, F^2, F^3) \cap \varphi(\mathrm{I\!R}^+_0)$

Case 3: $\emptyset \neq \mathcal{X}^*(F^r) \cap \varphi(\mathrm{I\!R}^+_0)\ \Rightarrow\ \emptyset \neq \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^1, F^2, F^3) \cap \varphi(\mathrm{I\!R}^+_0)$

That means all $\varphi \in C_\infty(\mathrm{I\!R}^+_0, \mathrm{I\!R}^2)$ with $\varphi(0) = x$ fulfill $\varphi(\mathrm{I\!R}^+_0) \cap \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^1, F^2, F^3) \neq \emptyset$. Hence $x \in \mathrm{encl}\,(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}(F^1, F^2, F^3))$. ∎

Finally we obtain the following theorem which provides a subset as well as a superset of

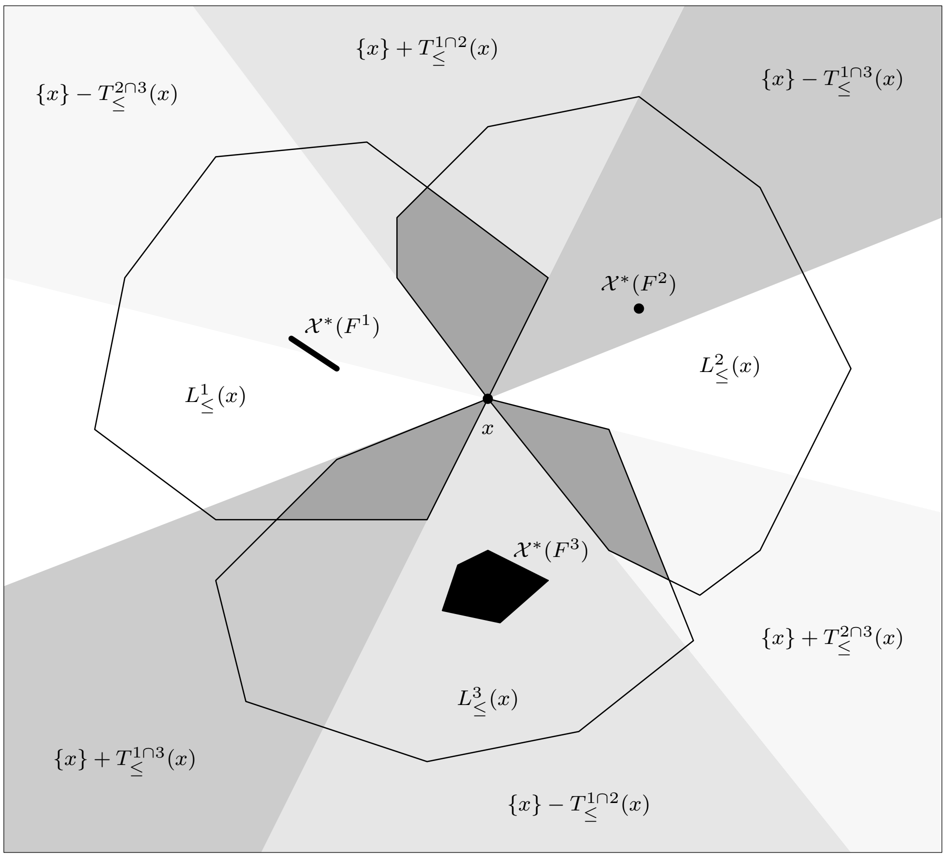

Figure 10: Illustration to the proof of Lemma 4.6

$\mathcal{X}^*_{\mathrm{Par}}(F^1, F^2, F^3)$.

Theorem 4.1

$$\mathrm{encl}\left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right)\right) \subseteq \mathcal{X}^*_{\mathrm{Par}}\left(F^1, F^2, F^3\right) \subseteq \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right) \cup \mathrm{encl}\left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^1, F^2, F^3\right)\right)$$

Proof:

$$
\begin{aligned}
\operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1,F^2,F^3\right)\right) &\subseteq \mathcal{X}^{*}_{\text{s}-\text{Par}}\left(F^1,F^2,F^3\right) && \text{by Lemma 4.3}\\
&\subseteq \mathcal{X}^{*}_{\text{Par}}\left(F^1,F^2,F^3\right)\\
&\subseteq \mathcal{X}^{*}_{\text{w}-\text{Par}}\left(F^1,F^2,F^3\right)\\
&\subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1,F^2,F^3\right)\cup \operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1,F^2,F^3\right)\right) && \text{by Lemma 4.6}
\end{aligned}
$$

■

Now it remains to consider the Pareto optimality of the *gap* $\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3)$ with respect to the three objective functions F^1,F^2,F^3. For a cell $C\in\mathcal{C}$ we define the collapsing and the remaining part of f with respect to Q-criteria optimality by

$$
\operatorname{col}_Q(C) := \left\{x\in C \,:\, x\notin \mathcal{X}^{*}_{\text{Par}}\left(F^1,\dots,F^Q\right)\right\} \tag{32}
$$

and

$$
\operatorname{rem}_Q(C) := \left\{x\in C \,:\, x\in \mathcal{X}^{*}_{\text{Par}}\left(F^1,\dots,F^Q\right)\right\} \quad . \tag{33}
$$

Using the differentiability of the objective functions in the interior of the cells we obtain the following lemma.

Lemma 4.7
For $C\in\mathcal{C}$ holds:

1. $\operatorname{col}_Q(C)\,\dot\cup\,\operatorname{rem}_Q(C) = C$.
2. *Either* $\operatorname{rem}_Q(C)=C$ *or* $\operatorname{rem}_Q(C)\subseteq\operatorname{bd}(C)$. *In the letter case* $\operatorname{rem}_Q(C)$ *is either empty or consists of complete faces and/or extreme points of C or*
3. *For $C\subseteq\mathcal{X}^*(F^p)$ with $p\in\{1,2,3\}$ and $x\in\operatorname{int}(C)$ holds:*

$$
x\in\operatorname{rem}_3(C) \;\Leftrightarrow\; \left\{\begin{array}{l} \exists\,\xi\in\mathbb{R} \textit{ with } \nabla F^q(x)=\xi\nabla F^r(x)\\ \textit{and } \xi<0 \textit{ for } q,r\in\{1,2,3\}\setminus\{p\},\, q<r \end{array}\right\} \tag{34}
$$

For $C\subseteq\mathcal{X}^{*}_{\text{Par}}(F^p,F^q)$ with $p,q\in\{1,2,3\},p<q$, and $x\in\operatorname{int}(C)$ holds:

$$
x\in\operatorname{rem}_3(C) \;\Leftrightarrow\; \left\{\begin{array}{l} \exists\,\xi^p,\xi^q\in\mathbb{R} \textit{ with } \nabla F^r(x)=\xi^p\nabla F^p(x)\,,\ \nabla F^r(x)=\xi^q\nabla F^q(x)\\ \textit{and } \xi^p\xi^q\le 0 \textit{ for } r\in\{1,2,3\}\setminus\{p,q\} \end{array}\right\} \tag{35}
$$

Proof:

1. Follows directly from the definition of $\operatorname{col}_Q(C)$ and $\operatorname{rem}_Q(C)$.
2. If $\operatorname{int}(C)\cap\mathcal{X}^{*}_{\text{Par}}(F^1,F^2,F^3)\neq\emptyset$ we have $C\subseteq\mathcal{X}^{*}_{\text{Par}}(F^1,F^2,F^3)$ and hence $\operatorname{rem}_Q(C)=C$. This follows analogously to the proof of Theorem 3.1

   If $\operatorname{int}(C)\cap\mathcal{X}^{*}_{\text{Par}}(F^1,F^2,F^3)=\emptyset$, then $\operatorname{rem}_Q(C)\subseteq\operatorname{bd}(C)$. The rest follows analogously to the proof of Theorem 3.1.

3. If $C \subseteq \mathcal{X}^*(F^p)$ for $p \in \{1,2,3\}$ and $x \in \operatorname{int}(C)$, we have $L^p_=(x) = L^p_\leq(x)$ and therefore

$$
\begin{aligned}
x \in \operatorname{rem}_3(f) \quad &\Leftrightarrow \quad L^q_=(x) \cap L^r_=(x) = L^q_\leq(x) \cap L^r_\leq(x) \text{ for } q,r \in \{1,2,3\} \setminus \{p\}, q<r \\
&\Leftrightarrow \quad \nabla F^q(x) = \xi \nabla F^r(x) \text{ with } \xi < 0 \text{ for } q,r \in \{1,2,3\} \setminus \{p\}, q<r \quad .
\end{aligned}
$$

If $C \subseteq \mathcal{X}^*_{\text{Par}}(F^p, F^q)$ for $p, q \in \{1,2,3\}$ and $x \in \operatorname{int}(C)$, there exists $\xi \in \mathbb{R}$ with

$$\nabla F^p(x) = \xi \nabla F^q(x) \text{ with } \xi < 0\,.$$

(Notice that the trivial case $\mathcal{X}^*(F^1) \cap \mathcal{X}^*(F^2) \neq \emptyset$, i. e. $\nabla F^p(x) = 0 = \nabla F^q(x)$ is included.)

The Pareto optimality condition $\bigcap_{q=1}^3 L^q_=(x) = \bigcap_{q=1}^3 L^q_\leq(x)$ for the 3 criteria is fulfilled if and only if the level curve $L^r_=(x),, r \in \{1,2,3\} \setminus \{p,q\}$, has the same slope than $L^p_=(x)$ and $L^q_=(x)$, i. e. if and only if $\xi^p, \xi^q \in \mathbb{R}$ exist with

$$\nabla F^r(x) = \xi^p \nabla F^p(x)\,,\ \nabla F^r(x) = \xi^q \nabla F^q(x) \text{ and } \xi^p \xi^q \leq 0\,.$$

■

Summing up the preceding results we get a complete geometric characterization of the set of Pareto solutions for three criteria.

Theorem 4.2

$$
\begin{aligned}
\mathcal{X}^*_{\text{Par}}\left(F^1,F^2,F^3\right) = \;& \left(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3) \cup \operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3)\right)\right) \\
& \setminus \{x \in \mathbb{R}^2 : \exists C \in \mathcal{C}, C \subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3), x \in \operatorname{col}_3(C)\}
\end{aligned}
$$

Proof:
Let $y \in \mathcal{X}^*_{\text{Par}}(F^1,F^2,F^3)$. Then we have by Theorem 4.1

$$y \in \mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1,F^2,F^3\right) \cup \operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^1,F^2,F^3\right)\right)\,.$$

Moreover for $C \in \mathcal{C}$ with $y \in C$ we have $y \in \operatorname{rem}_3(C)$, i. e. $y \notin \operatorname{col}_3(C)$. This implies

$$
\begin{aligned}
y \in \;& \left(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3) \cup \operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3)\right)\right) \\
& \setminus \{x \in \mathbb{R}^2 : \exists C \in \mathcal{C}, C \subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3), x \in \operatorname{col}_3(C)\} \quad .
\end{aligned}
$$

Now let

$$
\begin{aligned}
y \in \;& \left(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3) \cup \operatorname{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3)\right)\right) \\
& \setminus \{x \in \mathbb{R}^2 : \exists C \in \mathcal{C}, C \subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3), x \in \operatorname{col}_3(C)\} \quad .
\end{aligned}
$$

We distinguish the following cases:

Case 1: $y \in \operatorname{encl}(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3))$. Then $y \in \mathcal{X}^*_{\text{Par}}(F^1,F^2,F^3)$ by Theorem 4.1.

Case 2: $y \in \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3)$

Case 2.1: $\exists C \in \mathcal{C}, C \subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1,F^2,F^3)$ with $y \in C$

$$\Rightarrow \; y \notin \operatorname{col}_3(C) \;\Rightarrow\; y \in \operatorname{rem}_3(C) \;\Rightarrow\; y \in \mathcal{X}^*_{\text{Par}}(F^1,F^2,F^3).$$

Case 2.2 : $\nexists\, C \in \mathcal{C}, C \subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3)$ with $y \in C$

$\Rightarrow\; L^p_{\leq}(y) \cap L^q_{\leq}(y) = \{y\}$ for some $p, q \in \{1,2,3\}, p < q$

$\Rightarrow\; \bigcap_{q=1}^{3} L^q_{\leq}(y) = \{y\}$

$\Rightarrow\; y \in \mathcal{X}^*_{\text{s-Par}}(F^1, F^2, F^3) \subseteq \mathcal{X}^*_{\text{Par}}(F^1, F^2, F^3)\,.$

■

In the case of Ordered Weber functions the gradients $\nabla F^q(x), q \in \{1,2,3\}$, can be computed in $\mathbf{O}(M \log(MG_{\max}))$ time (analogous to the evaluation of the function). Therefore we can test with (34) and (35) in $\mathbf{O}(M \log(MG_{\max}))$ time if a cell $C \in \mathcal{C}, C \subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3)$ collapses. We obtain the following algorithm for $1/\mathbb{R}^2/\bullet/\gamma_m/3{-}(\sum_{ord})_{par}$ with time complexity $\mathbf{O}(M^5 G^2_{\max} \log(MG_{\max}))$.

ALGORITHM 4.1

(Solving $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/3 - (\sum_{ord})_{par}$ *.)*

Input:

1. *Demand points* $a_i \in \mathbb{R}^2$ *,* $i \in \mathcal{M}$ *.*
2. *Weights* λ^q_i *,* $i \in \mathcal{M}$ *,* $q = 1,2,3$ *satisfying* $0 \leq \lambda^q_1 \leq \ldots \leq \lambda^q_M$ *for* $q = 1,2,3$ *.*
3. *Weights* ω^q_i *,* $i \in \mathcal{M}$ *,* $q = 1,2,3$ *satisfying* $\omega^q_i \geq 0$ *for* $i \in \mathcal{M}$ *,* $q = 1,2,3$ *.*
4. *Polyhedral gauges* $\gamma_i : \mathbb{R}^2 \to \mathbb{R}$ *,* $i \in \mathcal{M}$ *.*

Output:

1. $\mathcal{X}^*_{\text{Par}}(F^1, F^2, F^3)$ *.*

Steps:

1. *Computation of the planar graph generated by the cells* $\mathcal{C}$*.*
2. *Computation of* $\mathcal{X}^*_{\text{w-Par}}(F^1, F^2)\,, \mathcal{X}^*_{\text{w-Par}}(F^1, F^3)\,, \mathcal{X}^*_{\text{w-Par}}(F^2, F^3)$ *using Algorithm 3.1.*
3. $\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3) := \mathcal{X}^*_{\text{w-Par}}(F^1, F^2) \cup \mathcal{X}^*_{\text{w-Par}}(F^1, F^3) \cup \mathcal{X}^*_{\text{w-Par}}(F^2, F^3)$*.*
4. $\mathcal{X}^*_{\text{Par}}(F^1, F^2, F^3) := \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3) \cup \text{encl}\,(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3))$*.*
5. **Forall** $C \in \mathcal{C}$ *with* $C \subseteq \mathcal{X}^{\text{gen}}_{\text{Par}}(F^1, F^2, F^3)$ **Do**
6. *Compute* $\text{col}_3(C)$ *using Lemma 4.7.*
7. $\mathcal{X}^*_{\text{Par}}(F^1, F^2, F^3) := \mathcal{X}^*_{\text{Par}}(F^1, F^2, F^3) \setminus \text{col}_3(C)$*.*
8. **Endfor**

■

Finally we present an example to illustrate the preceding results..

Example 4.1
We consider the four existing facilities $a_1 = (2, 6.5), a_2 = (5, 9.5), a_3 = (6.5, 2), a_4 = (11, 9.5)$ *(see Figure 3).* a_1 *and* a_4 *are associated with the* l_1*-norm, whereas* a_2 *and* a_3 *are associated with the* l_∞*-norm. We consider three Ordered Weber functions* F^q *defined by the following weights:*

q	ω_1^q	ω_2^q	ω_3^q	ω_4^q	λ_1^q	λ_2^q	λ_3^q	λ_4^q
1	1	1	0	1	1	1	1	1
2	1	1	1	0	1	1	1	1
3	1	1	1	1	0	0	0	1

We obtain the optimal solutions $\mathcal{X}^*(F^1) = \{a_2\}, \mathcal{X}^*(F^2) = \{a_1\}$ *and* $\mathcal{X}^*(F^3) = \overline{(6.5, 8), (8, 6.5)}$. *The sets* $\mathcal{X}_{\text{Par}}^{\text{gen}}(F^1, F^2, F^3)$ *and* $\mathcal{X}_{\text{Par}}^{*}(F^1, F^2, F^3)$ *are drawn in Figure 11 and Figure 12, respectively. Both figures show a part of the whole situation presented in Figure 3.*

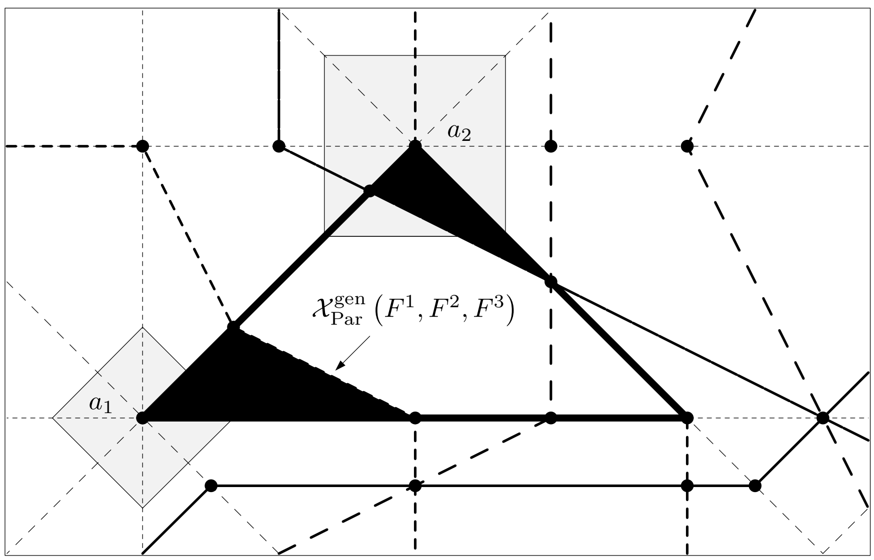

Figure 11: Illustration to Example 4.1

5 The Case $Q > 3$

Now we turn to the Q-Criteria case which is based on the 3-criteria case.

Theorem 5.1

1.

$$\bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \text{cl}\left(\text{encl}\left(\mathcal{X}_{\text{Par}}^{\text{gen}}(F^p, F^q, F^r)\right)\right) \subseteq \mathcal{X}_{\text{Par}}^{*}\left(F^1, \ldots, F^Q\right)$$

2.

$$\mathcal{X}_{\text{Par}}^{*}\left(F^1, \ldots, F^Q\right) \subseteq \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \mathcal{X}_{\text{Par}}^{\text{gen}}(F^p, F^q, F^r) \cup \bigcup_{\substack{p,q,r \in \mathcal{Q} \\ p<q<r}} \text{encl}\left(\mathcal{X}_{\text{Par}}^{\text{gen}}(F^p, F^q, F^r)\right)$$

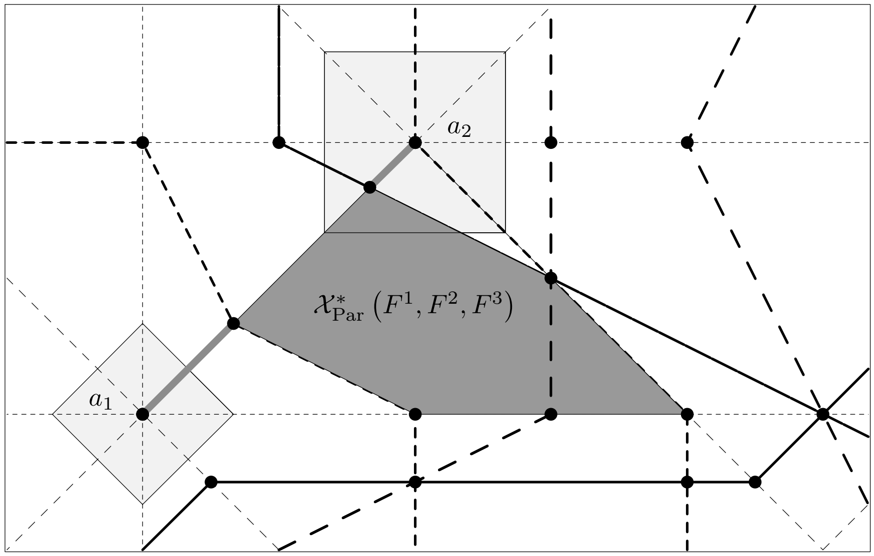

Figure 12: Illustration to Example 4.1

Proof:

1. $$x \in \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \mathrm{cl}\left(\mathrm{encl}\left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^p, F^q, F^r\right)\right)\right)$$

$\Leftrightarrow \quad x \in \mathrm{cl}\left(\mathrm{encl}\left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^p, F^q, F^r\right)\right)\right)$ for some $p, q, r \in \mathcal{Q}, p < q < r$

$\Rightarrow \quad x \in \mathcal{X}^*_{\mathrm{s-Par}}\left(F^p, F^q, F^r\right)$ for some $p, q, r \in \mathcal{Q}, p < q < r$, by Lemma 4.3

$\Leftrightarrow \quad L^p_{\leq}(x) \cap L^q_{\leq}(x) \cap L^r_{\leq}(x) = \{x\}$ for some $p, q, r \in \mathcal{Q}, p < q < r$, by (8)

$\Rightarrow \quad \bigcap_{q=1}^{Q} L^q_{\leq}(x) = \{x\}$, since $x \in L^q_{\leq}(x)$ for all $q \in \mathcal{Q}$

$\Leftrightarrow \quad x \in \mathcal{X}^*_{\mathrm{s-Par}}\left(F^1, \ldots, F^Q\right)$ by (8)

$\Rightarrow \quad x \in \mathcal{X}^*_{\mathrm{Par}}\left(F^1, \ldots, F^Q\right)$

2. $$x \in \mathcal{X}^*_{\mathrm{Par}}\left(F^1, \ldots, F^Q\right)$$

$\Rightarrow \quad x \in \mathcal{X}^*_{\mathrm{w-Par}}\left(F^1, \ldots, F^Q\right)$

$\Leftrightarrow \quad \bigcap_{q=1}^{Q} L^q_{<}(x) = \emptyset$ by (6)

$\Leftrightarrow \quad L^p_{<}(x) \cap L^q_{<}(x) \cap L^r_{<}(x) = \emptyset$ for some $p, q, r \in \mathcal{Q}, p < q < r$, by Helly's Theorem

$\Leftrightarrow \quad x \in \mathcal{X}^*_{\mathrm{w-Par}}\left(F^p, F^q, F^r\right)$ for some $p, q, r \in \mathcal{Q}, p < q < r$, by (6)

$\Rightarrow \quad x \in \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^p, F^q, F^r\right) \cup \mathrm{encl}\left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^p, F^q, F^r\right)\right)$ for some $p, q, r \in \mathcal{Q}, p < q < r$, by Lemma 4.6

$\Leftrightarrow \quad x \in \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^p, F^q, F^r\right) \;\cup \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \mathrm{encl}\left(\mathcal{X}^{\mathrm{gen}}_{\mathrm{Par}}\left(F^p, F^q, F^r\right)\right)$

$\blacksquare$

In the Q-criteria case the crucial region is now given by the cells $C \in \mathcal{C}$ with

$$C \subseteq \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^p,F^q,F^r\right) \ \setminus \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \text{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^p,F^q,F^r\right)\right)$$

$$= \bigcup_{\substack{p,q\in\mathcal{Q}\\ p<q}} \mathcal{X}^{*}_{\text{w}-\text{Par}}\left(F^p,F^q\right) \ \setminus \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \text{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^p,F^q,F^r\right)\right) \quad .$$

Similar to Lemma 4.7 one can test by comparing the gradients of the objective functions in $\text{int}(C)$, if the cell $C \in \mathcal{C}$ collapses with respect to $F^1,\dots,F^Q$ or not. Finally we obtain the following Theorem, which can be proven by the same technique as in the 3-criteria case (see proof of Theorem 4.2).

Theorem 5.2

$$\mathcal{X}^{*}_{\text{Par}}\left(F^1,\dots,F^Q\right) = \left(\bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^p,F^q,F^r\right) \ \cup \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \text{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^p,F^q,F^r\right)\right)\right)$$
$$\setminus \left\{x \in \mathbb{R}^2 \,:\, \exists C \in \mathcal{C}, C \subseteq \bigcup_{\substack{p,q\in\mathcal{Q}\\ p<q}} \mathcal{X}^{*}_{\text{w}-\text{Par}}\left(F^p,F^q\right) \ \setminus \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \text{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^p,F^q,F^r\right)\right), x \in \text{col}_Q(C)\right\}$$

For the Q-criteria Ordered Weber problems we obtain the following algorithm.

ALGORITHM 5.1
(Solving $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$, $Q > 3$ *.)*

Input:

1. *Demand points* $a_i \in \mathbb{R}^2$, $i \in \mathcal{M}$.
2. *Weights* λ^q_i , $i \in \mathcal{M}$, $q \in \mathcal{Q}$ *satisfying* $0 \le \lambda^q_1 \le \ldots \le \lambda^q_M$ *for* $q \in \mathcal{Q}$.
3. *Weights* ω^q_i , $i \in \mathcal{M}$, $q \in \mathcal{Q}$ *satisfying* $\omega^q_i \ge 0$ *for* $i \in \mathcal{M}$, $q \in \mathcal{Q}$.
4. *Polyhedral gauges* $\gamma_i : \mathbb{R}^2 \to \mathbb{R}$, $i \in \mathcal{M}$.

Output:

1. $\mathcal{X}^{*}_{\text{Par}}\left(F^1,\dots,F^Q\right)$.

Steps:

1. *Computation of the planar subdivision generated by the cells* $C \in \mathcal{C}$.
2. *Computation of* $\mathcal{X}^{*}_{\text{w}-\text{Par}}\left(F^p,F^q\right), p,q \in \mathcal{Q}, p<q,$ *using Algorithm 3.1.*
3. $\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^p,F^q,F^r\right) := \mathcal{X}^{*}_{\text{w}-\text{Par}}\left(F^p,F^q\right) \cup \mathcal{X}^{*}_{\text{w}-\text{Par}}\left(F^p,F^r\right) \cup \mathcal{X}^{*}_{\text{w}-\text{Par}}\left(F^q,F^r\right), p,q,r \in \mathcal{Q}, p<q<r.$
4. $\mathcal{X}^{*}_{\text{Par}}\left(F^1,\dots,F^Q\right) := \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^p,F^q,F^r\right) \ \cup \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \text{encl}\left(\mathcal{X}^{\text{gen}}_{\text{Par}}\left(F^p,F^q,F^r\right)\right).$

5. **Forall** $C \in \mathbf{C}$ *with* $C \subseteq \bigcup_{\substack{p,q\in\mathcal{Q}\\ p<q}} \mathcal{X}^*_{\text{w}-\text{Par}}(F^p, F^q) \setminus \bigcup_{\substack{p,q,r\in\mathcal{Q}\\ p<q<r}} \text{encl}\,(\mathcal{X}^{\text{gen}}_{\text{Par}}(F^p, F^q, F^r))$ **Do**

6. *Determine* $\text{col}_Q(C)$.

7. $\mathcal{X}^*_{\text{Par}}\left(F^1, \ldots, F^Q\right) := \mathcal{X}^*_{\text{Par}}\left(F^1, \ldots, F^Q\right) \setminus \text{col}_Q(C)$.

8. **Endfor**

6 Extensions

6.1 Multi-criteria Ordered Weber Problems with Attraction and Repulsion

If we allow the weights ω_i^q, $i \in \mathcal{M}$, $q \in \mathcal{Q}$, to be positive or negative, we cannot apply the procedures presented in the preceding sections. Especially, we do not have the following properties anymore:

- Convexity of the objective functions F^q, $q \in \mathcal{Q}$.
- Connectivity of the set of Pareto optimal points $\mathcal{X}^*_{\text{Par}}(F^1, \ldots, F^q)$.

As a consequence a solution algorithm for the multi-criteria ordered Weber problem with attraction and repulsion, classified as $1/\mathbb{R}^2/\omega_i^q \not\geqq 0/\gamma_i/Q - (\sum_{ord})_{par}$, has to have a completely different structure than the algorithm for the convex case $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$.

Note that for negative ω_i^q we cannot write $\omega_i^q \gamma_i(x) = \gamma_i(\omega_i^q x)$ anymore. Instead we have $\omega_i^q \gamma_i(x) = -\gamma_i(|\omega_i^q| x)$. Therefore, the increasing order of the weights $\lambda_1^q, \ldots, \lambda_M^q$ cannot be maintained and we drop the assumption $0 \leq \lambda_1^q \leq \ldots \leq \lambda_M^q$.

However the following properties are still fulfilled:

- The cell structure remains the same, since fundamental directions and bisector lines do not depend on λ_i^q and the sign of ω_i^q.
- Moreover we still have the linearity of the objective functions F^q inside each cell.

Consequently, we can compute the local Pareto solutions with respect to a single cell as described in the case of $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$. Of course we cannot be sure that the local Pareto solutions remain globally Pareto. Therefore to obtain the set of global Pareto solutions all local Pareto solutions have to be compared.

A schematic approach for solving $1/\mathbb{R}^2/\omega_i^q \not\geqq 0/\gamma_i/Q - (\sum_{ord})_{par}$ would be:

1. Compute the local Pareto solutions for each cell $C \in \mathcal{C}$.
2. Compare all solutions of step 1 and get $\mathcal{X}^*_{\text{Par}}\left(F^1, \ldots, F^Q\right)$.
3. Output: $\mathcal{X}^*_{\text{Par}}\left(F^1, \ldots, F^Q\right)$.

In general Step 2 might become very time consuming because we have to compare in the Q-criteria case $O(Q^2 M^4 G_{\max}^2)$ cells. However for more special cases efficient algorithms can be developed. In the single criteria case solution algorithms can be found in [PACFP98, ND97]. If we restrict ourselves to the bicriteria case, we can do a procedure similar to the one used in [HLN99] for network location problems:

After having finished step 1 we project all local Pareto sets into the 2-dimensional objective space. We get a set of $L \in O(M^4 G_{\max}^2)$ line segments. We apply the algorithm of [Her89], with the modification described in [HLN99], to get the global Pareto solution in the objective space in $O(L \log L)$ time. Afterwards we transform the solutions back to the decision space.

6.2 Higher Dimensions

The problem $1/\mathbb{R}^n/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$ has essentially the same structure as the problem in $\mathbb{R}^2$. Therefore, similar approaches to the ones given in Sections 4 and 5 could be applied. The only difference is that in $\mathbb{R}^n$ we should check for $n+1$ criteria at each time (see [EN96]), so that instead of considering bicriteria chains we must consider n-criteria chains in Step 2. Nevertheless, although theoretically possible this approach is computationally very time consuming, since many situations may occur when one intersects n level curves or level sets.

Alternatively, we can use a different approach based on checking for local Pareto optimality using convex analysis tools. This approach is just based on the null vector condition:

$$x^* \text{ is Pareto solution} \quad \Longleftrightarrow \quad 0 \in ri\left(conv \left(\bigcup_{q \in Q} \partial F^q(x^*) \right) \right)$$

where $\partial F^q(x^*)$ stands for the sub-differential set of F^q at x^* (see [FP95]).

In fact, it can easily be seen that the approach used for the planar case is nothing else than a geometric interpretation of this null vector condition exploiting the additional properties of the plane. It is straightforward to see that in the same way as it was done for the planar case, there exists a subdivision $\mathcal{C}$ of $\mathbb{R}^n$, such that in each element $C \in \mathcal{C}$ the objective functions $F^1, \ldots, F^Q$ are linear. Therefore, in the interior of each cell $C \in \mathcal{C}$, $\partial F^q(x^*)$ is constant and equals the vector defining the linear representation of F^q. This means that we can check the null vector condition in each cell in $O(Q \log Q)$ using [FGZ96]. Then using the connectivity of the set $\mathcal{X}^*_{\text{Par}}$ we can get this set by just following a backtracking search in the subdivision $\mathcal{C}$.

In order to obtain the set of Pareto solutions in each cell C, a scheme testing generalized elementary convex sets following a non-increasing order on its dimension can be performed.

A schematic procedure for solving $1/\mathbb{R}^n/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$ is given in what follows. The following notation is used:

1. $A(C)$ is the set of generalized elementary convex sets of any dimension adjacent to C.
2. $\mathcal{X}^*_{\text{Par}}(C)$ is the set of Pareto optimal solutions on C.
3. τ is an auxiliary set.

ALGORITHM 6.1
(Solving $1/\mathbb{R}^n/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$ *.)*

<u>*Input:*</u>

1. *Demand points* $a_i \in \mathbb{R}^n$, $i \in \mathcal{M}$.
2. *Weights* λ_i^q , $i \in \mathcal{M}$, $q = 1, 2, \ldots, Q$ *satisfying* $0 \leq \lambda_1^q \leq \ldots \leq \lambda_M^q$ *for* $q = 1, 2, \ldots, Q$.
3. *Weights* ω_i^q , $i \in \mathcal{M}$, $q = 1, 2$ *satisfying* $\omega_i^q \geq 0$ *for* $i \in \mathcal{M}$, $q = 1, 2, \ldots, Q$.
4. *Polyhedral gauges* $\gamma_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \mathcal{M}$.

<u>*Output:*</u>

1. $\mathcal{X}^*_{\text{Par}}\left(F^1, \ldots, F^Q\right)$.

<u>*Steps:*</u>

1. *Solve* $1/\mathbb{R}^n/\lambda_i^1, \omega_i^1/\gamma_i/\sum_{ord}$. *Let* D *be a generalized elementary convex set containing optimal solutions of* $1/\mathbb{R}^n/\lambda_1/\gamma_i/\sum_{ord}$.
   Initialize $\mathcal{X}^*_{\text{Par}} = \mathcal{X}^*_{\text{Par}}(D)$, $\tau = \emptyset$.
2. *WHILE* C *exists such that* $C \cap (\mathcal{X}^*_{\text{Par}} \setminus \tau) \neq \emptyset$ *DO*
3.   *BEGIN*
4.   *Compute* $A(C)$, *set* $C_0 = C$.
5.   *REPEAT*
6.   *Choose* $C_1 \in A(C_0)$.
7. $A(C_0) = A(C_0) \setminus C_1$
8.   *Compute* $\mathcal{X}^*_{\text{Par}}(C_1)$.
9.   *IF* $\mathcal{X}^*_{\text{Par}}(C_1) \not\subseteq \mathcal{X}^*_{\text{Par}}$
10.   *THEN*
11.   *BEGIN*
12. $\mathcal{X}^*_{\text{Par}} = \mathcal{X}^*_{\text{Par}} \cup \mathcal{X}^*_{\text{Par}}(C_1)$
13. $C = C_1$
14.   *EXITREPEAT*
15.   *END*
16.   *UNTIL* $A(C) = \emptyset$
17.   *IF* $A(C_0) = \emptyset$ *THEN* $\tau = \tau \cup C_0$
18.   *END*
19. *Output* $\mathcal{X}^*_{\text{Par}}$

$\blacksquare$

The reader can realize that the hardest part of this algorithmic scheme is the computation of the set $A(C)$ of adjacent generalized elementary convex sets to C. Once this step is efficiently done all the remaining steps can be performed with minor effort.

7 Conclusions

In this paper we showed the usefulness of ordered Weber problems for modeling multi-criteria locational decision problems. We developed efficient algorithms and proved structural results. Also a detailed complexity analysis of these algorithms is provided. Extensions to the multi-facility case as well as improvements for the complexity results for special cases are under research. Also a more detailed discussion of the problems mentioned in Section 6 is planned. Furthermore, we are working on an implementation of ordered Weber problems in **LOLA** (Library of Location Algorithms , [HKNS96]).

A Appendix

A.1 Computation of the bound on page 6

Theorem A.1 *An upper bound on the number of ordered regions is* $O(M^4G^2)$.

Proof:

Given two bisectors with $O(G)$ linear pieces, the maximum number of intersections is $O(G^2)$. The number of bisectors of M points is $\binom{M}{2}$, so, the maximum number of intersections between them is $O(G^2\binom{\binom{M}{2}}{2})$. By the Euler formula, the number of intersections has the same complexity as the number of regions. Hence, an upper bound for the number of ordered regions is $O(M^4G^2)$.

$\blacksquare$

A detailed analysis of this theorem shows that this bound is not too bad. Although, it is of order M^4G^2, it should be noted that the number of bisectors among the points in A is $\binom{M}{2}$ which is order M^2. Therefore, even in the most favorable case of straight lines, the number of regions in worst case analysis gives $O(\binom{M}{2}^2)$ which is, in fact $O(M^4)$. Since our bisectors are polygonal with G pieces, this bound is rather tight.

A.2 Case analysis for the proof of Lemma 3.1

$x^\perp := (-x_2, x_1)$ is perpendicular to $x = (x_1, x_2)$. For $q = 1, 2$ let s_q be the direction vector of the lines, which describe the level curves $L_=(F^q, .)$ in $int(C)$. Let H_C be the half-space with respect to the line

$$l_C := \left\{ y \in \mathbb{R}^2 \,:\, y = x + \eta(x_R - x) \,,\, \eta \in \mathbb{R} \right\}$$

which contains C (see Figure 13). Without loosing generality assume that s_q points into H_C, i.e. $\langle (x_R - x)^\perp, s_q \rangle \geq 0$ (If $\langle (x_R - x)^\perp, s_q \rangle < 0$, set $s_q := -s_q$.).

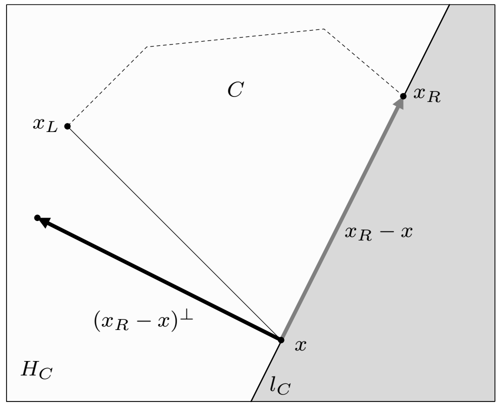

Figure 13: Halfspace H_C with $C \subseteq H_C$

The following case analysis shows how to determine which of the five situations (see Section 3.2) occurs.

Case 1 :

$$
\begin{aligned}
& x \in \operatorname{argmin}_{y\in C}\{f^1(y)\} \quad \wedge \quad x \in \operatorname{argmin}_{y\in C}\{f^2(y)\} \\
\Longleftrightarrow \quad & (\, f^1(x) \le f^1(x_L) \wedge f^1(x) \le f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) \le f^2(x_L) \wedge f^2(x) \le f^2(x_R)\,)
\end{aligned}
$$

Case 1.1 :

At least 3 of 4 inequalities are strict

$$
\begin{aligned}
\Longrightarrow \quad & \big((\, f^1(x) < f^1(x_L) \wedge f^1(x) < f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) < f^2(x_L) \wedge f^2(x) < f^2(x_R)\,)\big) \\
\vee \quad & \big((\, f^1(x) = f^1(x_L) \wedge f^1(x) < f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) < f^2(x_L) \wedge f^2(x) < f^2(x_R)\,)\big) \\
\vee \quad & \big((\, f^1(x) < f^1(x_L) \wedge f^1(x) = f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) < f^2(x_L) \wedge f^2(x) < f^2(x_R)\,)\big) \\
\vee \quad & \big((\, f^1(x) < f^1(x_L) \wedge f^1(x) < f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) = f^2(x_L) \wedge f^2(x) < f^2(x_R)\,)\big) \\
\vee \quad & \big((\, f^1(x) < f^1(x_L) \wedge f^1(x) < f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) < f^2(x_L) \wedge f^2(x) = f^2(x_R)\,)\big) \\
\Longrightarrow \quad & x \text{ dominates } x_L \text{ as well as } x_R \\
\Longrightarrow \quad & \text{Situation } \mathbf{E} \text{ occurs.}
\end{aligned}
$$

Case 1.2 :

Exactly 2 of 4 inequalities are strict.

Case 1.2.1 :

$$(\, f^1(x) < f^1(x_L) \,\wedge\, f^1(x) < f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) = f^2(x_L) \,\wedge\, f^2(x) = f^2(x_R)\,)$$

$\Longrightarrow$ x dominates x_L as well as x_R

$\Longrightarrow$ Situation **E** occurs.

stricter :

$\Longrightarrow$ $\{x\} = \text{argmin}_{y \in C}\{f^1(y)\}$ and $f^2/_C$ is constant

$\Longrightarrow$ $\{x\} = \mathcal{X}^*_{(2,1)}$, since f^2 convex and $int(C) \neq \emptyset$

$\Longrightarrow$ The end of the chain is achieved.

Case 1.2.2 :

$$\big(\,(\, f^1(x) = f^1(x_L) \,\wedge\, f^1(x) = f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) < f^2(x_L) \,\wedge\, f^2(x) < f^2(x_R)\,)\,\big)$$

$$\vee \quad \big(\,(\, f^1(x) < f^1(x_L) \,\wedge\, f^1(x) = f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) = f^2(x_L) \,\wedge\, f^2(x) < f^2(x_R)\,)\,\big)$$

$$\vee \quad \big(\,(\, f^1(x) = f^1(x_L) \,\wedge\, f^1(x) < f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) < f^2(x_L) \,\wedge\, f^2(x) = f^2(x_R)\,)\,\big)$$

$\Longrightarrow$ x dominates x_L as well as x_R

$\Longrightarrow$ Situation **E** occurs.

Case 1.2.3 :

$$\big(\,(\,f^1(x) < f^1(x_L)\,\wedge\,f^1(x) = f^1(x_R)\,)\quad\wedge\quad(\,f^2(x) < f^2(x_L)\,\wedge\,f^2(x) = f^2(x_R)\,)\,\big)$$

$\Longrightarrow$ x dominates x_L

$\Longrightarrow$ Situation **D** occurs.

Case 1.2.4 :

$$\big(\,(\,f^1(x) = f^1(x_L)\,\wedge\,f^1(x) < f^1(x_R)\,)\quad\wedge\quad(\,f^2(x) = f^2(x_L)\,\wedge\,f^2(x) < f^2(x_R)\,)\,\big)$$

$\Longrightarrow$ x dominates x_R

$\Longrightarrow$ Situation **C** occurs.

Case 1.3 :

Exactly 1 of 4 inequalities is strict.

Case 1.3.1 :

$$\big(\,(\,f^1(x) < f^1(x_L)\,\wedge\,f^1(x) = f^1(x_R)\,)\quad\wedge\quad(\,f^2(x) = f^2(x_L)\,\wedge\,f^2(x) = f^2(x_R)\,)\,\big)$$

$$\vee\quad\big(\,(\,f^1(x) = f^1(x_L)\,\wedge\,f^1(x) < f^1(x_R)\,)\quad\wedge\quad(\,f^2(x) = f^2(x_L)\,\wedge\,f^2(x) = f^2(x_R)\,)\,\big)$$

$\Longrightarrow$ $\{x\} \in \text{argmin}_{y\in C}\{f^1(y)\}$ and $f^2/_C$ is constant

$\Longrightarrow$ $\{x\} \in \mathcal{X}^*_{(2,1)}$, since f^2 convex and $int(C) \neq \emptyset$

$\Longrightarrow$ The end of the chain is achieved.

Case 1.3.2 :

$$(\,f^1(x) = f^1(x_L)\,\wedge\,f^1(x) = f^1(x_R)\,)\quad\wedge\quad(\,f^2(x) < f^2(x_L)\,\wedge\,f^2(x) = f^2(x_R)\,)$$

$\Longrightarrow$ $f^1/_C$ is constant and $\overline{xx_R} = \text{argmin}_{y\in C}\{f^2(y)\}$

$\Longrightarrow$ $\overline{xx_R} = \mathcal{X}^*_{(1,2)}$

$\Longrightarrow$ Situation **D** occurs.

Case 1.3.3 :

$$(\,f^1(x) = f^1(x_L)\,\wedge\,f^1(x) = f^1(x_R)\,)\quad\wedge\quad(\,f^2(x) = f^2(x_L)\,\wedge\,f^2(x) < f^2(x_R)\,)$$

$\Longrightarrow$ $f^1/_C$ is constant and $\overline{xx_L} = \text{argmin}_{y\in C}\{f^2(y)\}$

$\Longrightarrow$ $\overline{xx_L} = \mathcal{X}^*_{(1,2)}$

$\Longrightarrow$ Situation **C** occurs.

Case 1.4 :

None of the 4 inequalities is strict.

$\Longrightarrow$ $(\, f^1(x) = f^1(x_L) \,\wedge\, f^1(x) = f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) = f^2(x_L) \,\wedge\, f^2(x) = f^2(x_R)\,)$

$\Longrightarrow$ $f^1/_C$ is constant and $f^2/_C$ is constant

$\Longrightarrow$ $C \subseteq \mathcal{X}_1^*$ and $C \subseteq \mathcal{X}_2^*$, since f^1, f^2 convex and $int(C) \neq \emptyset$

$\Longrightarrow$ Contradiction to $\mathcal{X}_1^* \cap \mathcal{X}_2^* = \emptyset$.

Case 2 :

$$x \in \operatorname{argmin}_{y \in C}\{f^1(y)\} \qquad \wedge \qquad x \notin \operatorname{argmin}_{y \in C}\{f^2(y)\}$$

$\Longleftrightarrow$ $(\, f^1(x) \leq f^1(x_L) \,\wedge\, f^1(x) \leq f^1(x_R)\,) \quad \wedge \quad (\, f^2(x) > f^2(x_L) \,\vee\, f^2(x) > f^2(x_R)\,)$

Case 2.1 :

$\langle s_1^{\perp}, s_2 \rangle = 0$

$\Longrightarrow$ Situation **A** occurs (see Figure 14) .

Case 2.2 :

$\langle s_1^{\perp}, s_2 \rangle > 0$

$\Longrightarrow$ Situation **C** occurs (see Figure 15) .

Case 2.3 :

$\langle s_1^{\perp}, s_2 \rangle < 0$

Case 2.3.1 :

$\langle s_2^{\perp}, x_L - x \rangle \leq 0$

$\Longrightarrow$ Situation **D** occurs (see Figure 16) .

Case 2.3.2 :

$\langle s_2^{\perp}, x_L - x \rangle > 0$

Case 2.3.2.1 :

$f^2(x_L) = f^2(x_R)$

$\Longrightarrow$ $L_{=}(f^2, .)$ runs through x_L and x_R

$\Longrightarrow$ $\langle s_2^{\perp}, x_L - x \rangle \leq 0$

$\Longrightarrow$ Contradiction to **Case 2.3.2** .

Case 2.3.2.2 :

$f^2(x_L) > f^2(x_R)$

$\Longrightarrow$ Situation **D** occurs (see Figure 17) .

Case 2.3.2.3 :

$$f^2(x_L) < f^2(x_R)$$

$\Longrightarrow$ Situation **C** occurs (see Figure 18) .

Case 3 :

$$\begin{array}{rccc} & x \notin \operatorname{argmin}_{y \in C}\{f^1(y)\} & \wedge & x \in \operatorname{argmin}_{y \in C}\{f^2(y)\} \\ \Longleftrightarrow & (\, f^1(x) > f^1(x_L) \,\vee\, f^1(x) > f^1(x_R)\,) & \wedge & (\, f^2(x) \le f^2(x_L) \,\wedge\, f^2(x) \le f^2(x_R)\,) \end{array}$$

The case analysis corresponds to the case analysis of **Case 2** ,
with reversed roles of f^1 and f^2.

Case 4 :

$$\begin{array}{rccc} & x \notin \operatorname{argmin}_{y \in C}\{f^1(y)\} & \wedge & x \notin \operatorname{argmin}_{y \in C}\{f^2(y)\} \\ \Longleftrightarrow & (\, f^1(x) > f^1(x_L) \,\vee\, f^1(x) > f^1(x_R)\,) & \wedge & (\, f^2(x) > f^2(x_L) \,\vee\, f^2(x) > f^2(x_R)\,) \end{array}$$

Assumption :

$$f^1(x) > f^1(x_L) \,\wedge\, f^2(x) > f^2(x_L)$$

$\Longrightarrow$ x_L dominates x

$\Longrightarrow$ Contradiction to $x \in \mathcal{X}^*_{\text{Par}}$.

Assumption :

$$f^1(x) > f^1(x_R) \,\wedge\, f^2(x) > f^2(x_R)$$

$\Longrightarrow$ x_R dominates x

$\Longrightarrow$ Contradiction to $x \in \mathcal{X}^*_{\text{Par}}$.

Therefore holds :

$$\begin{array}{rccc} & \big(\, f^1(x) > f^1(x_L) \,\wedge\, f^1(x) \le f^1(x_R) & \wedge & f^2(x) \le f^2(x_L) \,\wedge\, f^2(x) > f^2(x_R)\,\big) \\ \vee & \big(\, f^1(x) \le f^1(x_L) \,\wedge\, f^1(x) > f^1(x_R) & \wedge & f^2(x) > f^2(x_L) \,\wedge\, f^2(x) \le f^2(x_R)\,\big) \\ \Longleftrightarrow & \big(\, f^1(x_L) < f^1(x) \le f^1(x_R) & \wedge & f^2(x_R) < f^2(x) \le f^2(x_L)\,\big) \\ \vee & \big(\, f^1(x_R) < f^1(x) \le f^1(x_L) & \wedge & f^2(x_L) < f^2(x) \le f^2(x_R)\,\big) \end{array}$$

Assumption :

$$f^1(x) = f^1(x_L) \,\vee\, f^1(x) = f^1(x_R) \quad \vee \quad f^2(x) = f^2(x_L) \,\vee\, f^2(x) = f^2(x_R)$$

$\Longrightarrow$ x is dominated by x_R or x_L

$\Longrightarrow$ Contradiction to $x \in \mathcal{X}^*_{\text{Par}}$.

Therefore holds :

$$\begin{array}{rccc} & \big(\, f^1(x_L) < f^1(x) < f^1(x_R) & \wedge & f^2(x_R) < f^2(x) < f^2(x_L)\,\big) \\ \vee & \big(\, f^1(x_R) < f^1(x) < f^1(x_L) & \wedge & f^2(x_L) < f^2(x) < f^2(x_R)\,\big) \end{array}$$

Case 4.1 :

$$\langle s_1^\perp, s_2 \rangle = 0$$

$\Longrightarrow$ Situation **A** occurs (see Figure 19) .

Case 4.2 :

$$\langle s_1^\perp, s_2 \rangle \neq 0$$

Case 4.2.1 :

$$\big(\, \langle s_1^\perp, s_2 \rangle > 0 \,\wedge\, (\, f^1(x_L) < f^1(x) < f^1(x_R) \,\wedge\, f^2(x_R) < f^2(x) < f^2(x_L)\,)\,\big)$$

$$\vee \quad \big(\, \langle s_1^\perp, s_2 \rangle < 0 \,\wedge\, (\, f^1(x_R) < f^1(x) < f^1(x_L) \,\wedge\, f^2(x_L) < f^2(x) < f^2(x_R)\,)\,\big)$$

$\Longrightarrow$ $\nexists\ y \in C$, which dominates x (see Figure 20)

$\Longrightarrow$ Contradiction to $x \in \mathcal{X}^*_{\text{Par}}$.

Case 4.2.2 :

$$\big(\, \langle s_1^\perp, s_2 \rangle > 0 \,\wedge\, (\, f^1(x_R) < f^1(x) < f^1(x_L) \,\wedge\, f^2(x_L) < f^2(x) < f^2(x_R)\,)\,\big)$$

$$\vee \quad \big(\, \langle s_1^\perp, s_2 \rangle < 0 \,\wedge\, (\, f^1(x_L) < f^1(x) < f^1(x_R) \,\wedge\, f^2(x_R) < f^2(x) < f^2(x_L)\,)\,\big)$$

$\Longrightarrow$ Situation **B** occurs (see Figure 21) .

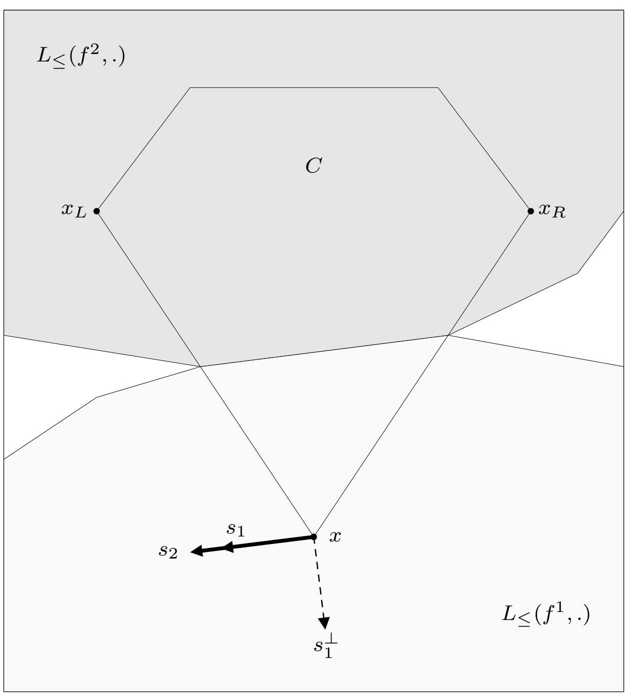

Figure 14: Illustration to Case 2.1

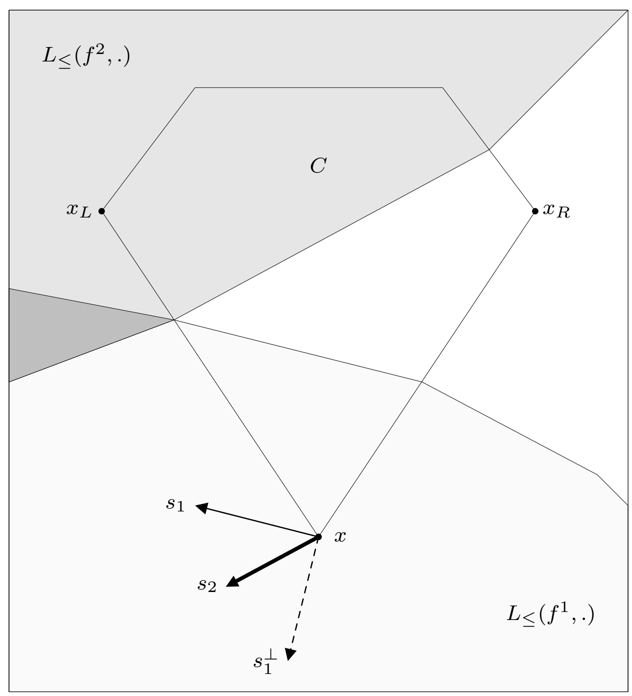

Figure 15: Illustration to Case 2.2

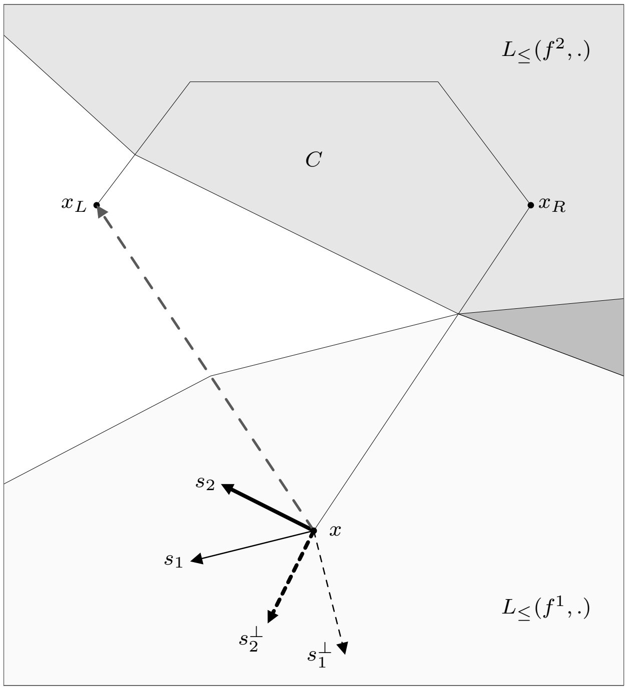

Figure 16: Illustration to Case 2.3.1

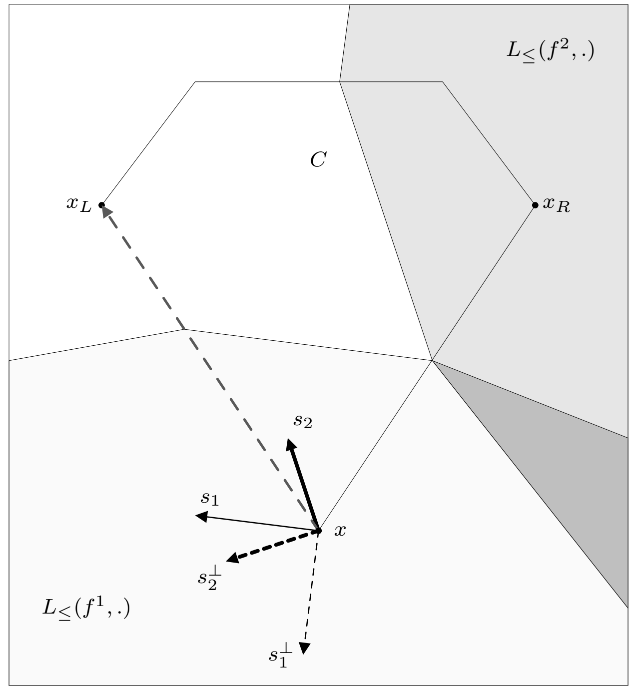

Figure 17: Illustration to Case 2.3.2.2

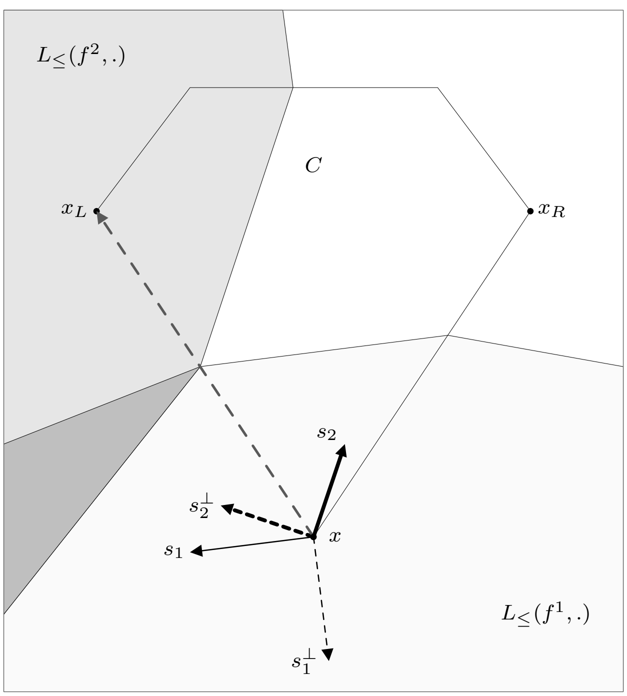

Figure 18: Illustration to Case 2.3.2.3

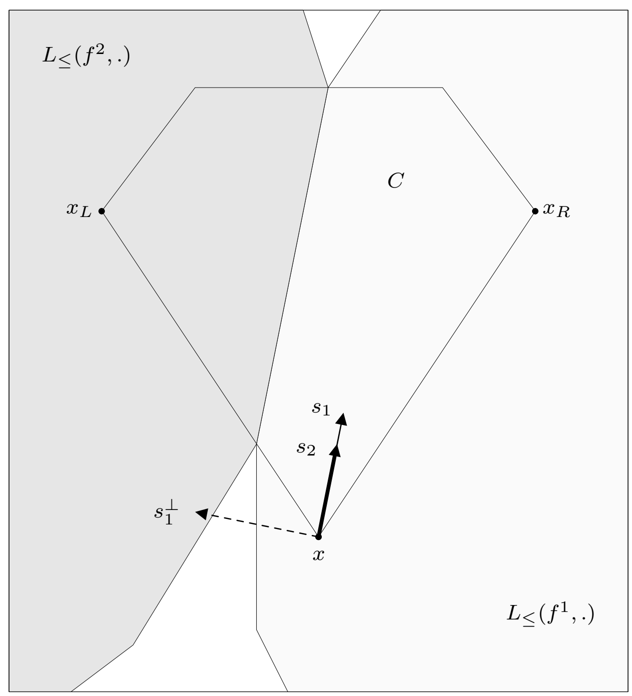

Figure 19: Illustration to Case 4.1

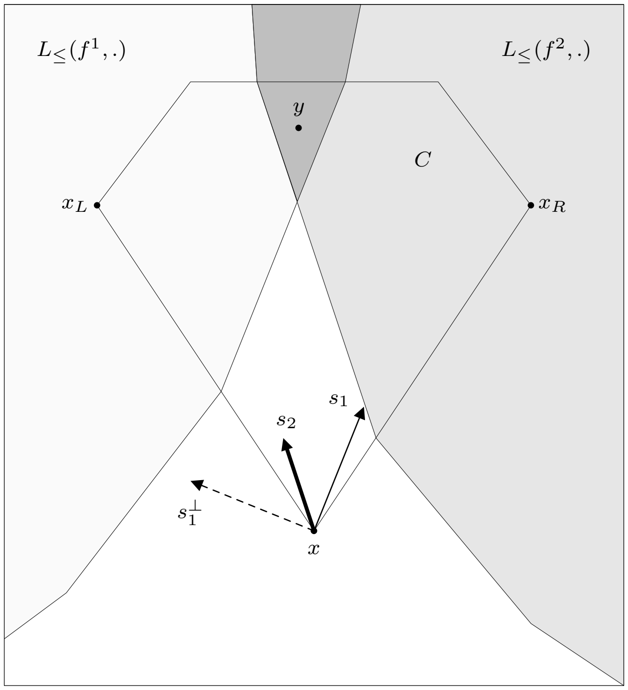

Figure 20: Illustration to Case 4.2.1

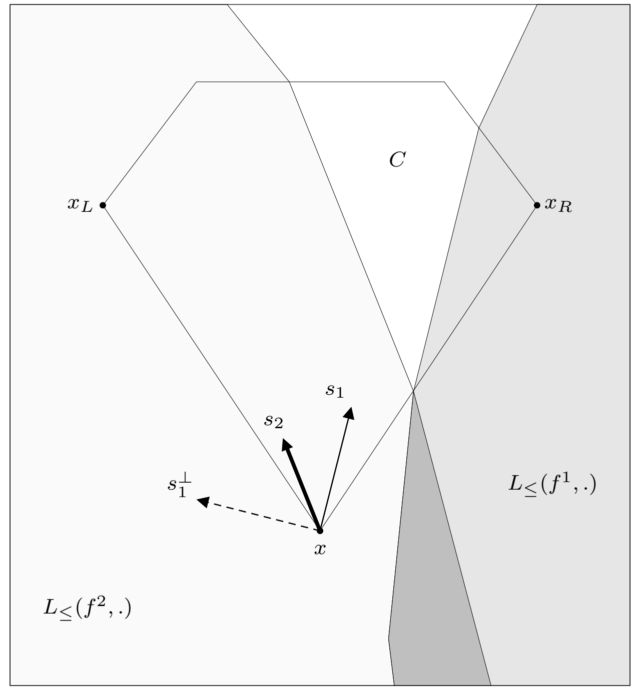

Figure 21: Illustration to Case 4.2.2

References

[BO79] J.L. Bentley and Th. Ottmann, *Algorithms for reporting and counting geometric intersections*, IEEE Trans. on Computers C 28 (1979), 643–647.

[DM85] Roland Durier and Christian Michelot, *Geometrical properties of the Fermat-Weber problem*, European Journal of Operational Research **20** (1985), 332–343.

[Dre95] Zvi Drezner (ed.), *Facility location - a survey of applications and methods*, Springer Series in Operations Research, 1995.

[EN96] M. Ehrgott and S. Nickel, *Reducing the number of criteria in quasi-convex multicriteria optimization*, Operations Research Proceedings, Springer, 1996.

[FGZ96] J.B.G. Frenk, J. Gromicho, and S. Zhang, *General models in min-max planar location: Checking optimality conditions*, JOTA **89** (1996), no. 1, 65–87.

[FP95] F.R. Fernandez and J. Puerto, *Multicriteria minisum facility location problem*, Tech. report, Universidad de Sevilla, 1995, Operations Research to appear.

[Her89] John Hershberger, *Finding the upper envelope of n line segments in o($n \log n$) time*, Information Processing Letters **33** (1989), no. 4, 169–174.

[HKNS96] H.W. Hamacher, K. Klamroth, S. Nickel, and A. Schoebel, *Library of location algorithms*, Tech. report, University of Kaiserslautern, 1996, http://www.mathematik.uni-kl.de/ lola.

[HLN99] H.W. Hamacher, M. Labbé, and S. Nickel, *Multicriteria network location problems with sum objectives*, Networks **33** (1999), 79–92.

[HN96] H.W. Hamacher and S. Nickel, *Multicriteria planar location problems*, European Journal of Operational Research **94** (1996), 66–86.

[HN98] H. W. Hamacher and S. Nickel, *Classification of location models*, Location Science **6** (1998), 229–242.

[HUL93] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms*, Springer-Verlag, 1993.

[ND97] Stefan Nickel and Eva-Maria Dudenhöffer, *Weber's Problem with Attraction and Repulsion under Polyhedral Gauges*, Journal of Global Optimization **11** (1997), 409–432.

[Nic97] Stefan Nickel, *Bicriteria and Restricted 2-Facility Weber Problems*, Mathematical Methods of Operations Research **45** (1997), 167–195.

[NP99] S. Nickel and J. Puerto, *A unified approach to network location problems*, Networks (1999), to appear.

[PACFP98] J Puerto, A.M.Rodriguez-Chia, and F.Fernandez-Palacin, *Ordered weber problems with attraction and repulsion*, Studies in Locational Analysis (1998), to appear.

[PF95] J. Puerto and F.R. Fernandez, *The symmetrical single facility location problem*, Tech. report, Prepublicación de la Faculdad de Matemáticas, Universidad de Sevilla, 1995, European Journal of Operations Research to appear.

[RCNPF96] Antonio M. Rodriguez-Chia, Stefan Nickel, Justo Puerto, and Francisco R. Fernandez, *A flexible approach to location problems*, Tech. report, University of Kaiserslautern, Department of Mathematics, 1996, Report in Wirtschaftsmathematik Nr.26/1997.

[Val64] Frederick A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

[War83] A.R. Warburton, *Quasiconcave vector maximization: Connectedness of the sets of pareto-optimal and weak pareto-optimal alternatives*, Journal of optimization theory and applications **40** (1983), no. 4, 537–557.

[Whi82] D.J. White, *Optimality and efficiency*, Wiley, Chicester, 1982.

[WW85] James E. Ward and Richard E. Wendell, *Using block norms for location modeling*, Operations Research **33** (1985), 1074–1090.