Nadir Values: Computation and Use in Compromise Programming

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Abstract

In this paper we investigate the problem of finding the Nadir point for multicriteria optimization problems (MOP). The Nadir point is characterized by the componentwise maximal values of efficient points for (MOP). It can be easily computed in the bicriteria case. However, in general this problem is very difficult. We review some existing methods and heuristics and propose some new ones. We propose a general method to compute Nadir values for the case of three objectives, based on theoretical results valid for any number of criteria. We also investigate the use of the Nadir point for compromise programming, when the goal is to be as far away as possible from the worst outcomes. We prove some results about (weak) Pareto optimality of the resulting solutions. The results are illustrated by examples.

1 Introduction

In this paper we consider optimization problems with multiple criteria, i.e.

$$\begin{array}{rl} \min & \big(f^1(x), \ldots, f^Q(x)\big) \\ \text{subject to} & x \in \mathcal{X}. \end{array} \qquad (MOP)$$

We investigate the determination of the ideal and anti-ideal, or Nadir point for (MOP). These points are characterized by the minimal (respectively maximal) objective values attained for Pareto optimal solutions of (MOP).

Definition 1 *A feasible solution x^* of (MOP) is called* Pareto optimal *if there does not exist another feasible solution x which is at least as good as x^* with respect to all criteria and strictly better for at least one objective i.e. if*

$$\nexists\, x \in \mathcal{X} \text{ s.t. } f^q(x) \le f^q(x^*) \quad \forall\, q = 1, \ldots, Q \text{ and } f(x) \ne f(x^*).$$

We write $f(x) < f(y)$ if $f^q(x) \le f^q(y) \quad \forall\, q = 1, \ldots, Q$ and $f(x) \ne f(y)$. Let $\mathcal{X}_{Par}$ be the set of all Pareto optimal solutions. If x^* is Pareto optimal, $f(x^*)$ is called efficient *or* nondominated*. The set of efficient points is denoted by $Y_{eff} := f(\mathcal{X}_{Par})$.*

Definition 2 *A feasible solution x^* of (MOP) is called* weakly Pareto optimal *if there does not exist another feasible solution x which is strictly better with respect to all criteria i.e. if*

$$\nexists\, x \in \mathcal{X} \text{ s.t. } f^q(x) < f^q(x^*) \quad \forall\, q = 1, \ldots, Q.$$

We write $f(x) \ll f(y)$ if $f^q(x) < f^q(y);\ \forall\, q = 1, \ldots, Q$.

Using Definition 1, we can formally define the Nadir and ideal point.

Definition 3 *Assume that at least one Pareto optimal solution exists, i.e. $\mathcal{X}_{Par} \ne \emptyset$. Then the* Nadir point *$y^N \in \mathbb{R}^Q$ is characterized by the componentwise supremum of all efficient points:*

$$y^N_q := \sup_{x \in \mathcal{X}_{Par}} f^q(x) \quad q = 1, \ldots, Q.$$

The ideal point *is defined to be the vector of the componentwise infima of all efficient solutions:*

$$y^I_q := \inf_{x \in \mathcal{X}_{Par}} f^q(x) \quad q = 1, \ldots, Q.$$

Throughout the paper we will assume that the efficient set Y_{eff} is externally stable in the sense of [23, p. 60], i.e. each nonefficient point $y \in f(X)$ is dominated by an efficient point $y = f(x)$, where $x \in X_{Par}$. In particular, the following investigations we assume that $\mathcal{X}_{Par}$ is a compact set such that the supremum and the infimum will be attained. We refer to [23] for results about existence, stability and other properties of the Pareto set and the efficient set. We also assume that the ideal point is not itself an efficient point. This would imply that the objectives are not conflicting, and induce a trivial case from the multicriteria perspective. From our assumption the Nadir point and ideal point can be computed as

$$\begin{aligned} y_q^N &= \max_{x \in \mathcal{X}_{Par}} f^q(x) \quad q = 1, \ldots, Q \\ y_q^I &= \min_{x \in \mathcal{X}_{Par}} f^q(x) \quad q = 1, \ldots, Q. \end{aligned}$$

The Nadir and ideal point provide important information about a multicriteria optimization problem (MOP). For a decision maker facing a multicriteria problem, they show the possible range of the objective values of all his criteria over the Pareto set: They are exact upper respectively lower bounds for the set of efficient points. But also in terms of methodology and solution algorithms knowing y^N and y^I is useful, as we shall explain now.

First note that determination of the ideal point for any number of criteria involves only the solution of Q single objective problems

$$\begin{aligned} \min \quad & f^q(x) \\ \text{subject to} \quad & x \in \mathcal{X}. \end{aligned} \tag{P_q}$$

over the whole feasible set $\mathcal{X}$. The following well known result implies that solving (P_q) yields y_q^I.

Lemma 1 *At least one optimal solution of problem (P_q) is Pareto optimal.*

Ideal points y^I and utopian points y^U defined by $y_i^U := y_i^I - \epsilon, \epsilon > 0$ are an essential component of compromise programming, see [29]. The idea is to find a feasible solution of (MOP) which is as close as possible to the ideal (or utopian) point.

$$\min_{x \in \mathcal{X}} ||f(x) - y^0||, \tag{1}$$

where $y^0 \in \{y^I, y^U\}$ and $||\cdot||$ is a norm on $\mathbb{R}^Q$. Whether an optimal solution of (1) is a Pareto optimal solution depends on properties of the norm.

Definition 4 *i) A norm $\|\cdot\| : \mathbb{R}^Q \to \mathbb{R}_+$ is called* **monotone**, *if for $a, b \in \mathbb{R}^Q$ $|a_i| \leq |b_i|,\ i = 1, \ldots, Q$ then $\|a\| \leq \|b\|$ holds, and $|a_i| < |b_i| \quad \forall\, i = 1, \ldots, Q$ then $\|a\| < \|b\|$ holds.*

ii) $\|\cdot\|$ is called **strictly monotone**, *if $|a_i| \leq |b_i|,\ i = 1, \ldots, Q$ and $\exists\ k$ s.t. $|a_k| \neq |b_k|$ then $\|a\| < \|b\|$ holds.*

We obtain the following results:

Theorem 1 *Let $\hat{x}$ be an optimal solution of (1). Then the following hold:*

i) If $\|\cdot\|$ is monotone then $\hat{x}$ is weakly Pareto optimal. If $\hat{x}$ is a unique optimal solution of (1) then $\hat{x} \in \mathcal{X}_{Par}$.

ii) If $\|\cdot\|$ is strictly monotone then $\hat{x}$ is Pareto optimal.

Proof:

i) Suppose $\hat{x}$ solves (1) and $\hat{x}$ is not weakly Pareto optimal. Then there is an $x \in \mathcal{X}$ such that

$$\begin{aligned} f^i(x) &< f^i(\hat{x}),\ i = 1, \ldots, Q \\ \Rightarrow\ 0 \le f^i(x) - y_i^0 &< f^i(\hat{x}) - y_i^0,\ i = 1, \ldots, Q \\ \Rightarrow\ \|f(x) - y^0\| &< \|f(\hat{x}) - y^0\|, \end{aligned}$$

contradicting optimality of $\hat{x}$.

Now suppose $\hat{x}$ is a unique optimal solution, but $\hat{x} \notin \mathcal{X}_{Par}$. Then there is an $x \in \mathcal{X}$ s.t. $f^i(x) \le f^i(\hat{x}),\ i = 1, \ldots, Q$ and there is some k s.t. $f^k(x) < f^k(\hat{x})$. Therefore $0 \le f^i(x) - y_i^0 \le f^i(\hat{x}) - y_i^0$ with strict inequality once. Thus $\|f(x) - y^0\| \le \|f(\hat{x}) - y^0\|$ and from optimality of $\hat{x}$ we obtain that equality holds, contradicting the uniqueness of $\hat{x}$.

ii) Suppose $\hat{x}$ solves (1) and $\hat{x} \notin \mathcal{X}_{Par}$. Then

$$\begin{aligned} \exists\, x \in \mathcal{X} \quad f^i(x) &\le f^i(\hat{x}), \quad i = 1, \ldots, Q \\ \exists\, k \text{ s.t. } f^k(x) &< f^k(\hat{x}) \\ \Longrightarrow\ 0 \le f^i(x) - y_i^0 &\le f^i(\hat{x}) - y_i, \quad i = 1, \ldots, Q \\ 0 \le f^k(x) - y_k^0 &< f^k(\hat{x}) - y_k^0 \\ \Longrightarrow\ \|f(x) - y^0\| &< \|f(\hat{x}) - y^0\|. \end{aligned}$$

□

Examples for strictly monotone norms are l_p norms

$$||y||_p = \left(\sum_{i=1}^{Q} |y_i|^p \right)^{\frac{1}{p}}$$

for $1 \le p < \infty$. The Chebychev norm is monotone, but not strictly monotone. Nevertheless compromise programming problems (1) are often formulated using a (weighted) Chebychev norm:

$$\min_{x \in \mathcal{X}} \max_{i=1,\ldots,Q} w_i |f^i(x) - y_i^I|, \tag{2}$$

where $w_i > 0;\ i = 1, \ldots, Q$.

From Theorem 1 it follows that an optimal solution of (2) is weakly Pareto optimal and Pareto optimal if it is unique. It is easily seen that (2) has at least one optimal solution which is Pareto optimal, under our assumptions.

In addition to the general result above, we obtain a stronger characterization of (weakly) Pareto optimal solutions from (2), if $y^0 = y^U$ is chosen.

Theorem 2 *A feasible point $\hat{x} \in \mathcal{X}$ is weakly Pareto optimal if and only if there exist $w_i > 0;\ i = 1, \ldots, Q$ such that $\hat{x}$ is an optimal solution of (2).*

Proof:
Sufficiency of the condition follows from i) in Theorem 1. For necessity consider $w_i = 1/(f^i(\hat{x}) - y_i^U) > 0;\ i = 1, \ldots, Q$.
Suppose $\hat{x}$ is not optimal for (2) with these weights. Then there is an $x \in \mathcal{X}$ such that

$$\begin{aligned} \max_i w_i(f^i(x) - y_i^U) &< \max_i \frac{1}{f^i(\hat{x}) - y_i^U}(f^i(\hat{x}) - y_i^U) = 1 \\ \Rightarrow\ f^i(x) - y_i^U &< f^i(\hat{x}) - y_i^U,\ i = 1, \ldots, Q \\ \Rightarrow\ f^i(x) &< f^i(\hat{x}),\ i = 1, \ldots, Q, \end{aligned}$$

contradicting the fact that $\hat{x}$ is weakly Pareto optimal. □

For a more detailed analysis of compromise programming we refer e.g. to [29, 23].
Besides the use of ideal and utopian points in compromise programming, important information is carried by y^I and y^N. In algorithms to find the Pareto set of an (MOP), or an approximation thereof, the search area in the objective space is restricted to a rectangular parallelepiped. This property is strongly exploited in one well known approach for the bicriteria ($Q = 2$) case, which we will explain in more detail in section 2.3.
Knowledge of y^N is also often assumed in interactive methods such as STEM [1].

2 Determining Nadir Values

In Lemma 1 we have seen that for computing y^I solving Q single objectives is enough. Finding the Nadir point, however, is much harder. We will discuss several existing exact as well as heuristic methods in this section. To illustrate these methods we will use the following example:

Example 1 *Consider the spanning tree problem with three objectives on the graph of Figure 1:*

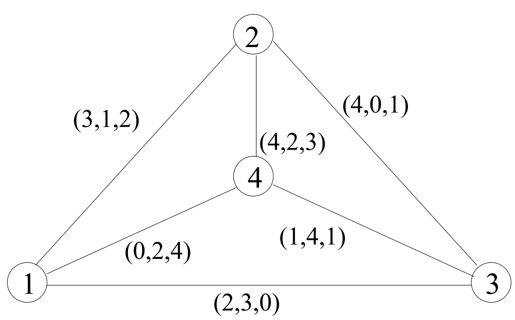

Figure 1: Graph $G = (V, E)$ with edge weights $w_{ij} \in \mathbb{Z}^3$

One can easily check that there are 8 Pareto optimal spanning trees. Their objective function vectors are

$$Y_{eff} = \left\{ \begin{pmatrix} 11 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \\ 5 \end{pmatrix} \right\}.$$

Hence the Nadir point is $y^N = (11, 8, 7)$.

2.1 Optimization over the Pareto Set

One could think of determining the components y_q^N of the Nadir point in a similar way to the computation of the components y_q^I of the ideal point:

$$y_q^N = \max_{x \in \mathcal{X}_{Par}} f^q(x) = \max_{y \in Y_{eff}} y_q. \tag{3}$$

Unfortunately it is not possible to replace $\mathcal{X}_{Par}$ by $\mathcal{X}$ in the maximization, as this would possibly lead to an overestimation of of y_q^N.

Lemma 2 *For all $q = 1, \dots, Q$:*

$$\max_{x \in \mathcal{X}} f^q(x) \geq y_q^N.$$

Example 2 *Denoting the estimate of Lemma 2 by $\tilde{y}^{max}$, i.e. $\tilde{y}_q^{max} = \max_{x \in \mathcal{X}} f^q(x)$, we check Example 1 and get $\tilde{y}^{max} = (11, 9, 8)^T$.*

Problem (3) is a problem of optimization over the Pareto set. As to the knowledge of the authors there are only a few papers published concerning this kind of problems, see [3, 4, 8, 2, 5, 10]. They are all restricted to the linear case, i.e. the optimization of a linear function over the Pareto optimal set of an (MOP). Let $\mathcal{X}_{Par}$ be the Pareto optimal set of the (MOP). Then the methods proposed in these articles solve the problem

$$\begin{array}{ll} \min & dx \\ \text{subject to} & x \in \mathcal{X}_{Par} \end{array}$$

If we assume that the objective functions of (MOP) are linear, $f^q(x) = c^q x$, then it is immediately clear how to apply these methods of the articles to our problem:
For all $q = 1, \dots, Q$ solve the problem

$$\begin{array}{ll} \max & c^q x \\ \text{subject to} & x \in \mathcal{X}_{Par}. \end{array}$$

This kind of problems is hard to solve since the efficient set of a multicriteria optimization problem is in general nonconvex, even if the (MOP) itself is linear. Hence in all of the papers mentioned above there are rather complex algorithms (or even just ideas of how to construct such algorithms) and most of them make use of global optimization techniques. Work related to this is done by a couple of authors, see e.g. [9, 25, 18] or [19], where the goal is to maximize not only linear but more general functions over the efficient set which does not lead to simpler algorithms.

2.2 Multiple Objective Linear Programming

A special case in multicriteria programming is the situation, where all objective functions and all constraints are linear, i.e.

$$\begin{array}{rl} \max & c^1 x \\ \max & c^2 x \\ & \vdots \\ \max & c^Q x \\ \text{s.t.} & x \in S = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} \end{array}$$

For this case Isermann and Steuer [20] proposed three deterministic approaches to compute y^N, where the first two are more or less theoretical investigations: The first approach consists of determining all efficient solutions and then using (3). The second idea is to solve a large primal-dual feasible program and the difficulty is on one hand the size of this program (roughly twice as many rows an twice as many columns as the original one) and on the other hand a set of highly nonlinear constraints. As a third approach the authors present a simplex-based procedure using the fact that the efficient extreme points are connected by efficient edges. Although better than the first and second approach this third idea is also not especially economical.

2.3 Lexicographic Optimization

A second special case in multicriteria programming is the situation, where only two objectives are to be considered. In this case the determination of the Nadir point is much easier. In order to explain it, we have to introduce lexicographic optimality.

Definition 5 *Let y and $z \in \mathbb{R}^Q$ be two vectors. Then $y <_{lex} z$ if there is a $q \in \{1, \dots, Q-1\}$ such that $y_k = z_k \ \forall k = 1, \dots, q$ and $y_{q+1} < z_{q+1}$ or $y_1 < z_1$. If $y <_{lex} z$ or $y = z$ then this will be denoted by $y \leq_{lex} z$.*
Let π be a permutation of $\{1, \dots, Q\}$. A feasible solution x^π of (MOP) is called lexicographically optimal with respect to π *if $f_\pi(x^\pi) \leq_{lex} f_\pi(x)$ for all feasible $x \in \mathcal{X}$, where $f_\pi(x) = (f_{\pi(1)}(x), \dots f_{\pi(Q)}(x))$.*
Finally, x^* is a global lexicographically optimal solution *if there exists a permutation π of $\{1, \dots, Q\}$ such that x^* is lexicographically optimal with respect to π. The set of all such solutions is denoted by $\mathcal{X}_{lex}$.*

A basic result is the following, see e.g. [11].

Lemma 3 *Let x^* be a global lexicographically optimal solution of (MOP). Then $x^* \in \mathcal{X}_{Par}$.*

In the bicriteria case, global lexicographically optimal solutions are all we need to determine y^I and y^N.

Lemma 4 *Consider an* (MOP) *with* $Q = 2$ *criteria and let* $x^{1,2}$ *and* $x^{2,1}$ *be two lexicographically optimal solutions with respect to permutations* $\pi_1 = (1,2)$ *and* $\pi_2 = (2,1)$*, respectively. Then*

1. $y^I = (f^1(x^{1,2}), f^2(x^{2,1}))$.
2. $y^N = (f^1(x^{2,1}), f^2(x^{1,2}))$.

Proof:
From Lemma 3 we know that $x^{1,2}$ and $x^{2,1}$ are Pareto optimal. Hence

$$\begin{aligned} y_i^N = \max_{x \in \mathcal{X}_{Par}} f^i(x) &\geq f^i(x^{1,2}) \\ y_i^N = \max_{x \in \mathcal{X}_{Par}} f^i(x) &\geq f^i(x^{2,1}) \\ y_i^I = \min_{x \in \mathcal{X}_{Par}} f^i(x) &\leq f^i(x^{1,2}) \\ y_i^I = \min_{x \in \mathcal{X}_{Par}} f^i(x) &\leq f^i(x^{2,1}) \end{aligned}$$

Assume now there is some $x^* \in \mathcal{X}_{Par}$ for which there holds $f^1(x^*) = y_1^N > f^1(x^{2,1})$. Since $x^* \in \mathcal{X}_{Par}$, $f^2(x^*) \leq f^2(x^{2,1})$. On the other hand $x^{2,1} \in \mathcal{X}_{lex}$ implies $f^2(x^*) = f^2(x^{2,1})$. Hence x^* is dominated by $x^{2,1}$. The rest is analogous. □

Now the image of each Pareto optimal solution of (MOP) is contained in the rectangle defined by the four vectors $f(x^{1,2}), f(x^{2,1}), y^N$, and y^I. Starting from one of the two lexicographically optimal solutions one can now proceed to explore the efficient set. To do so, the ϵ-constraint method [6] can be used in general. Other possibilities are parametric programming methods for linear problems [24], or ranking methods for combinatorial problems [7].
We also note that the determination of $x^{1,2}$ (or $x^{2,1}$) basically involves solving two single objective problems: First minimize f^1 over $\mathcal{X}$, and second, minimize f^2 over $\mathcal{X}$ under the additional constraint, that the optimal value of f^1 computed before is retained. If (MOP) is linear, these are two LP's. In combinatorial optimization the same algorithms that solve single objective problems can often be easily adapted to solve lexicographic problems, too. Given that the single objective problem is solvable in polynomial time, the same is then true for computation of $\mathcal{X}_{lex}$. Thus considerable gain over solving restricted problems, which are often NP-hard [16] is achieved. Recall that combinatorial (MOP) are usually NP-hard even in the bicriteria case [13].
Now we look at a generalization of Lemma 4. Can we determine Nadir objective values, using global lexicographic optimality? Then we could compute the Nadir point from $\tilde{y}^{lex}$ defined as follows:

$$\tilde{y}_q^{lex} := \max_{x \in \mathcal{X}_{lex}} f^q(x).$$

The answer is no, as can be seen from the example:

Example 3 *We continue the Example 1: The image of all global lexicographically optimal spanning trees is*

$$f(\mathcal{X}_{lex}) = \left\{ \begin{pmatrix} 11 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix} \right\}.$$

Hence the vector of all maximal entries found using these solutions only is $\tilde{y}^{lex} = (11, 7, 7)$, *and* $\tilde{y}_2^{lex} < y_2^N$.

However, Lemma 3 implies that using lexicographic optimization we can never overestimate Nadir values.

Lemma 5 *For all* $q = 1, \ldots, Q$:

$$\max_{x \in \mathcal{X}_{lex}} f^q(x) \leq y_q^N.$$

Furthermore, global lexicographically optimal solutions determine the ideal point:

Lemma 6 *The ideal point* y^I *is given as follows:*

$$y_q^I = \min_{x \in \mathcal{X}_{lex}} f^q(x).$$

To complete this subsection, we address the question of computing $\mathcal{X}_{lex}$. It is interesting to note that – even though an exponential number of permutations has to be considered – $\mathcal{X}_{lex}$ can often be computed efficiently. This is true for problems with a finite set of alternatives, such as in multiattribute decision making, see [12], or when the set of Pareto optimal solutions can be restricted to a finite set of candidates, e.g. in network location problems, see [17].

2.4 Pay-Off Tables and Other Heuristics

Due to the difficulty of computing y^N, some authors propose heuristics to compute the Nadir point. The most popular are dealing with the so-called pay-off table, e.g. [22] and references therein. We will present this approach, which provides the decision maker only with estimates (see e.g. [28]), briefly here.
The tables are computed by solving a single criterion optimization problem (P_q) for each objective to find the minimal value. The optimal solutions x_q are then evaluated for all criteria and the pay-off table is a matrix given by $P = (p_{qi}) := (f^i(x_q))$. The Nadir value is estimated by

$$\tilde{y}_i^{PT} := \max_{q=1,\ldots,Q} p_{qi} = \max\{f^i(x_q) : q = 1, \ldots, Q\}.$$

Note that the entries on the diagonal of P determine y^I.

Example 4 *In Example 1 the pay-off table looks like:*

	$f^1(x_q)$	$f^2(x_q)$	$f^3(x_q)$
$x_1 \in argmin\{f^1(x) : x \in \mathcal{X}\}$	4	7	7
$x_2 \in argmin\{f^2(x) : x \in \mathcal{X}\}$	11	3	6
	7	3	7
$x_3 \in argmin\{f^2(x) : x \in \mathcal{X}\}$	7	7	2

Hence the Nadir point $y^N = (11, 8, 6)$ *would be estimated by* $\tilde{y}^{PT} = (11, 7, 7)$ *or by* $\tilde{y}^{PT} = (7, 7, 7)$ *(depending on the solution* x_2 *chosen in the minimization of* f^2*), underestimating the exact values. Using payoff-tables an overestimation is also possible, see e.g. [21].*

Lemma 7 *If the solution of* (P_q) *is unique for all* $q = 1, \ldots, Q$ *then* $\tilde{y}^{PT}$ *from the payoff-table will never overestimate the Nadir point* y^N.

Proof:
If the solution of (P_q) is unique for all $q = 1, \ldots, Q$ then these solutions x_q are never just weakly Pareto optimal. Hence all x_q are global lexicographically optimal solutions, thus Pareto optimal. Using $y_q^N = \max_{x \in \mathcal{X}_{Par}} f^q(x)$ the assertion follows. □

Note that this pay-off table approach is kind of comparable to the global lexicographic optimization approach. The main difference is that by determining the pay-off table we are not sure to find all global lexicographically optimal solutions.
Instead of solving problem (P_q) one could think of solving

$$\begin{array}{ll} \min & f^q(x) + \sum_{i \neq q}^{Q} \epsilon_i f^i(x) \\ \text{subject to} & x \in \mathcal{X} \end{array} \qquad (P_q(\epsilon))$$

for each q, where $\epsilon_i \geq 0,\ i \neq q$ are small numbers, e.g. $\epsilon_i = 1/Q^2$. Note that for $\epsilon_i = 0,\ i \neq q$ we again get the ordinary payoff-table. Solving these kind of problems for $\epsilon_i > 0$ we are sure to obtain a Pareto optimal solution, hence never an overestimation of y^N.
Eskandari et. al. [15] suggested to solve a second single criterion optimization problem for each objective as a maximization problem, as in Lemma 2. As pointed out in Example 2, in our Example 1 this would yield the vector $\tilde{y}^{max} = (11, 9, 8)$, which now overestimates the correct value.
Yet another approach one could expect to work is to determine the Nadir point in all $(Q-1)$-criteria subproblems $(MOP(i))$ (see Definition 6 below). Then letting Y^N be the set of all Nadir points of these subproblems, choose

$$\tilde{y}_q^{Q-1} := \max\{y_q^N : y^N \in Y^N\}.$$

Example 5 *In Example 1 again,* $\tilde{y}^{Q-1}$ *does not give the correct result.*

	Nadir point for			
objective	(f^1, f^2)	(f^1, f^3)	(f^2, f^3)	max
f^1	7	7		7
f^2	7		7	7
f^3		7	6	7

Again there is a big difference between $y^N = (11, 8, 7)$ *and* $\tilde{y}^{Q-1} = (7, 7, 7)$.

Concluding this short presentation of simple heuristics we can ascertain that none of them produced the right result, some of them overestimating, some of them underestimating the Nadir point.
With these kinds of heuristics the over- or underestimation of the Nadir point can be even arbitrarily large, see [21] for an example. In the latter article another heuristic is given based on the use of reference directions.

3 A New Algorithm for the Three Objectives Case

In the last section we have seen that for bicriteria optimization problems the determination of the Nadir point is easy using lexicographic optimization. For Q strictly greater than 2 we will encounter big difficulties since the knowledge of one objective function value does not give us the possibility of controlling the others. Hence the lexicographic approach is no longer useful. In this paper we present an algorithm for $Q = 3$ which yields on one hand the exact values for the Nadir point and on the other hand is easy to implement, just using algorithms for determining the Pareto optimal solutions of a bicriteria optimization problem. Before we pass over to the algorithm (Section 3.2) we will present some theory which gives the background and the motivation for our method (see Section 3.1) and is valid for any number of criteria Q. In Section 4 we will present some results concerning the use of the Nadir point in compromise programming. We will end our investigations with some concluding remarks in Section 5.

3.1 Theoretical Results

In this section we give some theoretical results which hold not only for three criteria but also for the general case of an (MOP) with Q criteria. For the rest of the paper subproblems of (MOP) with $Q-1$ criteria will be essential. We define them as follows:

Definition 6 *Given a Q-criteria (MOP), we consider Q related (MOP) with $Q-1$ objectives*

$$\begin{array}{rl} \min & \left(f^1(x), \dots, f^{i-1}(x), f^{i+1}(x), \dots f^Q(x)\right) \\ \textit{subject to} & x \in \mathcal{X}. \end{array} \qquad (MOP(i))$$

Beside Pareto optimal solutions for the Q-criteria (MOP) we are dealing with Pareto optimal solutions for the $(Q-1)$-criteria problems $(MOP(i))$.

Definition 7 *Let $x^* \in \mathbb{R}^n$, let $f : \mathbb{R}^n \to \mathbb{R}^Q$. x^* is called* Q-Pareto*, if x^* is Pareto optimal for (MOP). x^* is called* $(Q-1)$-Pareto*, if there exists an index $i \in \{1, \dots, Q\}$ such that x^* is Pareto optimal for $(MOP(i))$.*

These $(Q-1)$-Pareto solutions are also very interesting for the original problem:

Proposition 1 *Given an (MOP) with Q criteria. Then there holds: If x is $(Q-1)$-Pareto, then*

- *either x is Q-Pareto*
- *or $f(x)$ is dominated by $f(y)$, where y is Q-Pareto such that $\exists$ a unique index j with $f^j(y) < f^j(x)$ and $f^i(x) = f^i(y)$ $\forall i \neq j$.*

Proof:
If x is $(Q-1)$-Pareto then either x is Q-Pareto or x is not Q-Pareto. If x is not Q-Pareto then there exists a Q-Pareto solution y such that $f^i(y) \leq f^i(x)$ for all $i = 1, \ldots, Q$ and $f^j(y) < f^j(x)$ for at least one index $j \in \{1, \ldots, Q\}$.
Assume $\|\{k : f^k(y) < f^k(x)\}\| \geq 2$. Then for each subset of $\{1, \ldots, Q\}$ with $Q-1$ elements there exists at least one index l such that $f^l(y) < f^l(x)$, while for all other indices $f^i(y) \leq f^i(x)$ holds. Hence there does not exists an index q such that x is Pareto optimal for $(MOP(q))$ and x cannot be $(Q-1)$-Pareto which contradicts our assumption.

□

Before we state the result which is fundamental for our algorithm, we introduce a notation.

Notation 1 *We denote the set of all $(Q-1)$-Pareto solutions, where the dominated solutions are removed, by Opt^{Q-1}, hence:*

$$Opt^{Q-1} = \{x : x \text{ is } (Q-1)\text{-Pareto and } \nexists\, \bar{x} \in Opt^{Q-1} \text{ with } f(\bar{x}) < f(x)\}$$

Theorem 3 *Assume a multicriteria optimization problem as given in (MOP). Then the set Opt^{Q-1} of all $(Q-1)$-Pareto solutions (except dominated solutions) contains*

1. *a set of solutions such that in every component the maximal entry of any Q-Pareto solution is found, i.e. the Nadir point is*

$$y_q^N = \max\{f^q(x) : x \in Opt^{Q-1}\},$$

2. *the set $\mathcal{X}_{lex}$ of all global lexicographically optimal solutions of (MOP), i.e. the ideal point is*

$$y_q^I = \min\{f^q(x) : x \in Opt^{Q-1}\}.$$

Proof:

1. Assume there exists a Q-Pareto solution $\bar{x}$ which is not $(Q-1)$-Pareto but for which exists an index $m \in \{1, \ldots, Q\}$ such that $f^m(\bar{x}) > f^m(x)$ for all $x \in Opt^{Q-1}$. Consider now the problem $MOP(m)$:
   - $\bar{x}$ is not Pareto optimal for $(MOP(m))$. Therefore there exists $x^* \in Opt^{Q-1}$ such that $f^i(x^*) \leq f^i(\bar{x})$ for all $i \in \{1, \ldots, Q\} \setminus \{m\}$ and $f^j(x^*) < f^j(\bar{x})$ for some $j \in \{1, \ldots, Q\} \setminus \{m\}$.
   - $f^m(\bar{x}) > f^m(x^*)$. Thus $f^1(x^*) \leq f^1(\bar{x}), \ldots, f^m(x^*) < f^m(\bar{x}), \ldots, f^Q(x^*) \leq f^Q(\bar{x})$ and $\bar{x}$ is not Q-Pareto, a contradiction.

   Thus $y_q^N \leq \max\{f^q(x) : x \in Opt^{Q-1}\}$, but from Proposition 1 we know that $Opt^{Q-1} \subset Opt^Q$ hence $y_q^N \geq \max\{f^q(x) : x \in Opt^{Q-1}\}$ and the first part of the Theorem is proven.

2. From Lemma 3 we know that for (MOP) $\mathcal{X}_{lex} \subset \mathcal{X}_{Par}$. Hence if we find all $(Q-1)$-Pareto solutions x then all global lexicographically optimal solutions for all $Q-1$-criteria problems $(MOP(i))$, $i = 1, \ldots, Q$ are found.

We will show that if $x^* \in \mathcal{X}_{lex}$ is a global lexicographically optimal solution of (MOP) then there exists a subproblem $(MOP(i))$ such that x^* is a global lexicographically optimal solution of $(MOP(i))$.

Since x^* is global lexicographically optimal for (MOP) there exists a permutation π of $\{1,\ldots,Q\}$ such that $f_\pi(x^*) \leq_{lex} f_\pi(x)$ for all $x \in \mathcal{X}$. For all x for which $f_\pi(x^*) = f_\pi(x)$ there holds also $f_{\pi(q)}(x^*) = f_{\pi(q)}(x)$ $q \in \{1,\ldots,Q\} \setminus \{i\}$ for all $(Q-1)$-criteria problems $(MOP(i))$

Thus we can restrict our attention to those x feasible for which $f_\pi(x^*) <_{lex} f_\pi(x)$. We define

$$K := \max_{x\in\mathcal{X}}\{k : f_{\pi(q)}(x^*) = f_{\pi(q)}(x)\ \forall\ q = 1,\ldots,k \wedge f_{\pi(k+1)}(x^*) < f_{\pi(k+1)}(x)\}$$

Then K is the largest index for which a feasible x exists such that $f_\pi(x^*)$ and $f_\pi(x)$ are identical in the first K positions.

If $K = Q-1$ (there is at least one x for which the first difference between $f_\pi(x^*)$ and $f_\pi(x)$ occurs in the Qth component) then consider the problem $(MOP(Q))$. Either $f_{\pi(Q)}(x^*) = f_{\pi(Q)}(x)$ or $f_{\pi(Q)}(x^*) <_{lex} f_{\pi(Q)}(x)$.

If $K < Q-1$ then consider $MOP(K+2)$. For this problem it must hold that

$$f_{\pi(K+1)}(x^*) <_{lex} f_{\pi(K+1)}(x)$$

for all x under consideration.

Hence in either case we found a problem $(MOP(i))$ for which x^* is lexicographically optimal with respect to the permutation π restricted to $\{1,\ldots,Q\} \setminus \pi^{-1}(i)$. This implies that $\mathcal{X}_{lex} \subset Opt^{Q-1}$.

□

Note that in the bicriteria case $Q = 2$, Theorem 3 yields Lemma 4 as a special case. Hence, it is a proper generalization of that well known result.

3.2 The Algorithm for the Three Objective Case

We now present a procedure based on the first statement 1 of Theorem 3 to find the Nadir point of an (MOP) with three objective functions. In this procedure we assume that an algorithm to compute the Pareto optimal solutions for a bicriteria optimization problem is given. We call this algorithm $\mathbf{PARETO}^2$. Then we can state the algorithm for computing the Nadir point. The algorithm also provides us with the Ideal point without further efforts:

Nadir and Ideal Point in Three Dimensions

Input: Instance P of a multicriteria optimization problem (MOP) with 3 criteria
Output: The corresponding Nadir point y^N
The corresponding ideal point y^I

Step 1: **for** $i = 1, \ldots, 3$ do
Apply $\mathbf{PARETO}^2$ to compute the 2-Pareto-solutions of $(MOP(i))$
Complete the solutions by their adequate ith component
Step 2: Remove all dominated solutions
and let Opt^2 be defined as in Theorem 3
Step 3: **for** $i = 1, \ldots, 3$ do
$y_i^N = \max\{f^i(x) : x \in Opt^2\}$
$y_i^I = \min\{f^i(x) : x \in Opt^2\}$

We also remark, that the algorithm can in principle be used for (MOP) with any number of criteria. However, an algorithm to solve the resulting $Q-1$ objective problems is needed, and only few algorithms are known for computing $\mathcal{X}_{Par}$ for $Q > 2$. A recursive procedure to achieve this goal is currently under investigation.

Let us discuss the algorithm $\mathbf{PARETO}^2$ here. Solving bicriteria problems is usually much easier than solving general multicriteria problems. For linear problems for example, it is known that the set of Pareto optimal solutions is equal to the set of optimal solution of a parametric LP with parameter $\lambda \in (0, 1)$.

$$\min_{x \in X} \lambda c^1 x + (1 - \lambda) c^2 x.$$

This kind of problem can easily be solved by parametric linear programming, and no specific multicriteria methodology is needed, see e.g. [24].
The difference between two and three objectives is even wider in combinatorial optimization. With the exception of only a few problems such as shortest path, the existing algorithms can only find all Pareto optimal solutions when two criteria are involved. This is in particular true for a very successful approach called the two phases method (see e.g. [26, 27]), which has no known generalization to three objectives. For more details we refer to a recent survey on the subject, see [14].
We illustrate our algorithm with two examples, one combinatorial and one linear problem.

3.3 Examples

Example 6 *We will again consider a spanning tree problem with three objectives. The edge weights corresponding to objectives one are the same as in Example 1, objectives two and three change as depicted in Figure 2.*
There are 9 Pareto optimal solutions and the efficient set is

$$Y_{eff} = \left\{ \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 10 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 11 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 9 \\ 5 \end{pmatrix} \right\}.$$

Hence the Nadir point is $y^N = (7, 11, 8)$ *and the ideal point is* $y^I = (4, 6, 2)$.
In this example the algorithm behaves as follows:

Step 1: $i = 1$*: Searching the 2-Pareto optimal solutions for* $(MOP(1))$ *yields the efficient vectors*

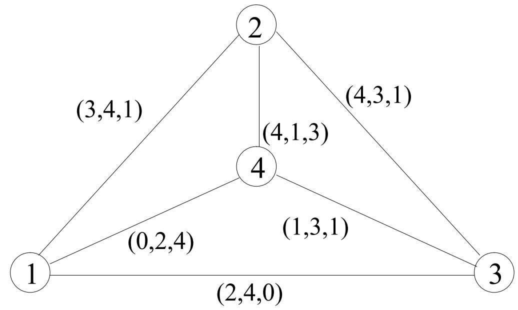

Figure 2: Graph $G = (V, E)$ with edge weights $w_{ij} \in \mathbb{Z}^3$

$$\begin{pmatrix} 10 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

Adequate completion:

$$\begin{pmatrix} 7 \\ 10 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 10 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 4 \end{pmatrix}$$

$i = 2$: *Searching the 2-Pareto optimal solutions for* $(MOP(2))$ *yields the efficient vectors*

$$\begin{pmatrix} 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Adequate completion:

$$\begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 11 \\ 2 \end{pmatrix}$$

$i = 3$: *Searching the 2-Pareto optimal solutions for* $(MOP(3))$ *yields the efficient vectors*

$$\begin{pmatrix} 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

Adequate completion:

$$\begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 6 \end{pmatrix}$$

Step 2: After removing all dominated solutions we get the following set of 2-efficient solutions:

$$f(Opt^2) = \left\{ \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 10 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 11 \\ 2 \end{pmatrix} \right\}$$

Step 3: Calculating the maximum respectively minimum of the set $\{f^q(x) : x \in Opt^2\}$ *yields*

$$y^N = \begin{pmatrix} 7 \\ 11 \\ 8 \end{pmatrix}, \; y^I = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$$

Example 7 *In this example we consider a linear programming problem with three criteria* *and three constraints.*

$$\begin{array}{rrrrcl} \min & -x_1 & -2x_2 & & & \\ \min & -x_1 & & +2x_3 & & \\ \min & x_1 & & -x_3 & & \\ s.t. & x_1 & +x_2 & & \leq & 1 \\ & & x_2 & & \leq & 2 \\ & x_1 & -x_2 & +x_3 & \leq & 4 \\ & & & x_i & \geq & 0,\ i = 1,2,3 \end{array}$$

The feasible set $\mathcal{X}$ of this problem (with the Pareto set indicated by bold lines) is shown in Figure 3.

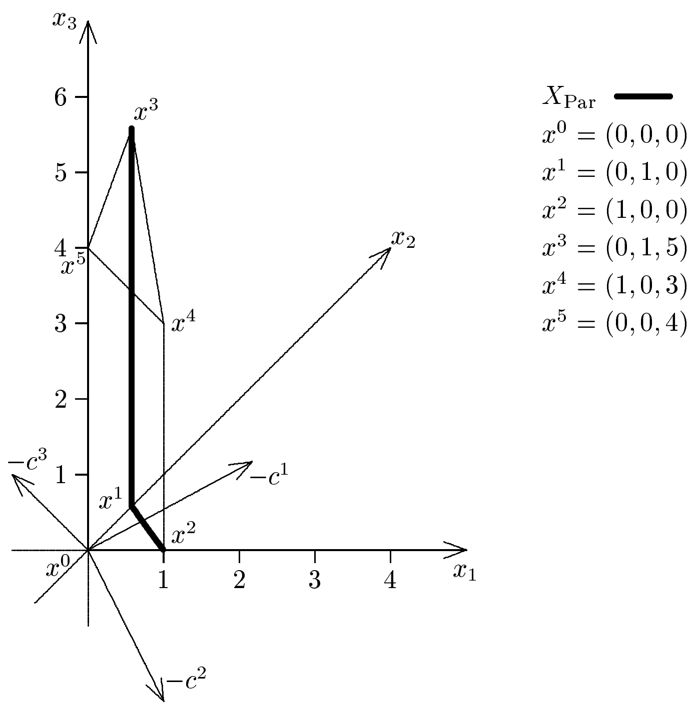

Figure 3: Feasible Set $\mathcal{X}$ and Pareto Set $\mathcal{X}_{Par}$ for Example 7

It turns out that

$$\mathcal{X}_{Par} = conv((0,1,0),(1,0,0)) \cup conv((0,1,0),(0,1,5))$$

and

$$Y_{eff} = conv((-2,0,0),(-1,-1,1)) \cup conv((-2,0,0),(-2,10,-5)).$$

Thus the Nadir point is $y^N = (-1, 10, 1)$.
Solving the three possible subproblems $MOP(i)$ we get the following results.

$$\begin{aligned} MOP(1):\ & \mathcal{X}_{Par} = conv((0,0,0),(0,1,0)) \cup conv((0,1,0)(0,1,5)) \\ & Y_{eff} = conv((0,0),(10,-5)) \end{aligned}$$

$$MOP(2): \mathcal{X}_{Par} = (0,1,5)$$
$$Y_{eff} = (-2,-5)$$

$$MOP(3): \mathcal{X}_{Par} = conv((1,0,0),(0,1,0))$$
$$Y_{eff} = conv((-1,-1)(-2,0))$$

After removing all dominated solutions the efficient set of the original (MOP) *coincides with the image of* Opt^2*, thus the Nadir point is found. We also note that computing* $\mathcal{X}_{lex}$ *would have been sufficient here.* $\mathcal{X}_{lex} = \{(0,1,0),(0,1,5),(1,0,0)\}$ *contains all points needed to compute* y^N*. However, as we have seen in the previous example, this is not true in general.*
The advantage of our method here, is that we just have to solve three parametric LP's (with one parameter in the objective function) corresponding to the three $MOP(i)$ *problems. To compute* $\mathcal{X}_{Par}$ *for the original LP, one would has to apply a multicriteria simplex algorithm (see e.g. [24]), which is computationally much more expensive, especially when* $\mathcal{X}$ *has many extreme points and facets.*

4 Using the Nadir Point as Reference Point in Compromise Programming

Similarly to using the ideal/utopian point in compromise programming one could think of finding a solution as far as possible from the Nadir point. This is more a conservative point of view, as one tries to avoid the worst instead of striving to achieve the best. This idea yields the problem

$$\begin{aligned} &\max_{x \in \mathcal{X}} ||f(x) - y^N|| \\ &\text{subject to } f^i(x) \quad \leq \quad y_i^N \quad i = 1, \dots Q \end{aligned} \tag{4}$$

The additional constraints are needed in order to guarantee to consider only solution which yield objective values below the Nadir value.

Proposition 2 *An optimal solution of (4) is weakly Pareto optimal if the norm* $||\cdot||$ *is monotone, and Pareto optimal, if the norm is strictly monotone.*

Proof:
Suppose x^* solves (4) and is not weakly Pareto optimal. Then there is some $x \in \mathcal{X}$ such that $f^i(x) < f^i(x^*) \leq y_i^N$ for all $i = 1, \dots, Q$. Therefore $0 < y_i^N - f^i(x) < y_i^N - f^i(x^*)$, $i = 1, \dots, Q$ and $||f(x) - y^N|| < ||f(x^*) - y^N||$, due to monotonicity of $||\cdot||$.
Now suppose $||\cdot||$ is strictly monotone and x^* is an optimal solution of (4). If x^* is not Pareto optimal there is some $x \in \mathcal{X}$ such that $f^i(x) \leq f^i(x^*)$, $i = 1, \dots, Q$ with strict inequality for at least one $i \in \{1, \dots, Q\}$. As above, strict monotonicity now implies $||f(x) - y^N|| < ||f(x^*) - y^N||$, contradicting the choice of x^*.

□

Despite this result, the fact that $y_i^I \leq f^i(x) \leq y_i^N$, the optimal solution of (4) will likely be such that $f^i(x^*) = y_i^I$ for some i, i.e. on the extremity of the efficient set.

Example 8 *Consider $Q = 2$ and the Chebychev norm. Then choosing $x^{1,2}$ and $x^{2,1}$ from Lemma 4 (the lexicographically optimal solutions), we have that $x^{1,2}$ or $x^{2,1}$ solves (4). We know that $x^{1,2}$ and $x^{2,1}$ are Pareto optimal and $y^I = (f^1(x^{1,2}), f^2(x^{2,1}))$ and $y^N = (f^2(x^{1,2}), f^1(x^{2,1}))$. Therefore if $|y_1^I - y_1^N| \leq |y_2^I - y_2^N|$ we have*

$$|f^1(x) - y_1^N| \leq |y_1^I - y_1^N| \leq |y_2^I - y_2^N| = |f^2(x^2) - y_2^N|.$$

Since also $|f^2(x) - y_2^N| \leq |y_2^I - y_2^N|$ for all x feasible for (4), $x^{2,1}$ solves the problem. The case for $x^{1,2}$ is analogous.

Therefore, to avoid these extreme cases, which are not likely candidates for a compromise among the conflicting objectives, an alternative option for (4) with the Chebychev norm is

$$\begin{aligned} \max_{x \in \mathcal{X}} \min_{i=1,\ldots,Q} w_i |f^i(x) - y_i^N| & \\ \text{subject to } f^i(x) \leq y_i^N \quad i = 1, \ldots Q, & \end{aligned} \tag{5}$$

where $w_i > 0, i = 1, \ldots, Q$. Note that this is not a special case of (4) with the Chebychev norm, but note also the analogy to (2). For (5) we have the following result.

Proposition 3 *An optimal solution of (5) is weakly Pareto optimal.*

Proof:
Let x^* be an optimal solution of (5). Suppose x^* is not weakly Pareto optimal. Then there is some $x \in \mathcal{X}$ such that $f^i(x) < f^i(x^*) \leq y_i^N, i = 1, \ldots, Q$. Therefore x is feasible for (5) and

$$\begin{aligned} w_i(y_i^N - f^i(x^*)) &< w_i(y_i^N - f^i(x)),\ i = 1, \ldots, Q \\ \Rightarrow \min_{i=1,\ldots,Q} w_i(y_i^N - f^i(x^*)) &< \min_{i=1,\ldots,Q} w_i(y_i^N - f^i(x)), \end{aligned}$$

contradicting the choice of x^*.

□

However, an optimal solution of (5) is not necessarily Pareto optimal.

Example 9 *Consider a bicriteria problem where $Y = f(\mathcal{X})$ is as shown in Figure 4. The efficient set consists of two curve segments. For all efficient points left of $\bar{y}$ the minimal distance to y^N is vertical and less than 1. For all efficient points below y^* the minimal distance is horizontal and less than or equal to 2. Thus y^*, for which $y_1^N - y_1^* = y_2^N - y_2^* = 2$ is optimal.*

Of course, the reason for the behaviour shown in the example is the nonconvexity of the efficient set. And indeed, for $Q = 2$ objectives and under convexity assumptions we can prove the stronger Theorem 4.

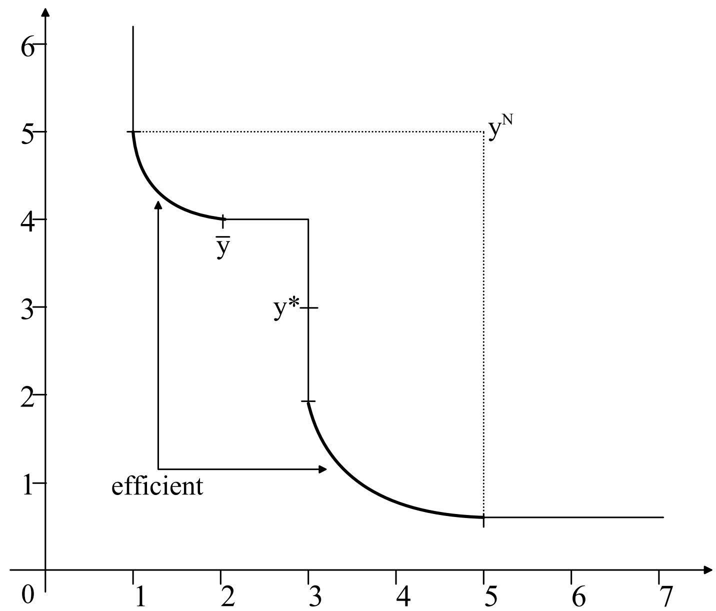

Figure 4: Bicriteria Problem

Theorem 4 *Consider the bicriteria (MOP) and assume that the objective functions f^1 and f^2 of (MOP) are convex and the image of $\mathcal{X}$ under (f^1, f^2) is a convex set. Then any optimal solution of*

$$\max_{x \in \mathcal{X}} \min_{q=1,2} \{w_q |f^q(x) - y_q^N|, f^q(x) \leq y_q^N, \ q = 1, 2\} \tag{6}$$

for $w_1, w_2 > 0$ is a Pareto optimal solution of (MOP).

Proof:
The proof is based on the fact that the optimal solution values of

$$\begin{array}{rl} \max & \lambda \\ \text{subject to} & \left(f^1(x), f^2(x)\right)^T \leq y^N - \lambda \mathbf{w} \\ & x \in \mathcal{X} \\ & \lambda \geq 0, \end{array} \tag{7}$$

where $\mathbf{w} = (1/w_1, 1/w_2) > 0$ is the vector of the inverse of the weights, and of (6) are the same. Let (λ^*, x^*) be an optimal solution of (7). Hence we have to show that x^* is a Pareto solution of (MOP).
Assume now the opposite, i.e. assume that x^* is dominated. We have to investigate two cases: either x^* is weakly Pareto optimal or x^* is not even weakly Pareto optimal.

Case 1: x^* is not weakly Pareto optimal. Then there exists a Pareto solution $\hat{x}$ such that $f^q(\hat{x}) < f^q(x^*), \ q = 1, 2$. But this contradicts the fact that λ^* is maximal.

Case 2: x^* is weakly Pareto optimal. Then there exists a Pareto optimal solution $\hat{x}$ dominating x^* such that the solution vectors of x^* and $\hat{x}$ coincide in at least one component. Wlog assume

$$f^1(\hat{x}) = \ f^1(x^*) \ \leq y_1^N \tag{8}$$
$$f^2(\hat{x}) < \ f^2(x^*) \ \leq y_2^N \tag{9}$$

If $f^2(x^*) = y_2^N$ then $\max_x \min_q |f^q(x) - y_q^N| = \min_q |f^q(x^*) - y_q^N| = \lambda^* = 0$. Due to the convexity of $\mathcal{X}$ this implies $|\mathcal{X}_{Par}| = 1$ and $\{y^N\} = \mathcal{X}_{Par} = \{\hat{x}\}$, thus $f^2(\hat{x}) = f^2(x^*)$ which is a contradiction to (9).

Hence let $f^2(x^*) < y_2^N$. Due to Lemmas 3 and 4 there is a Pareto optimal solution $\bar{x}$ such that

$$f^2(\bar{x}) = y_2^N > f^2(x^*) > f^2(\hat{x}) \tag{10}$$

We need to find a point $\tilde{x}$ such that $f^q(\tilde{x}) < f^q(x^*) \ \forall \ q = 1, 2$, but inequalities (10) imply

$$f^1(\bar{x}) < f^1(\hat{x}) = f^1(x^*)$$

Consider now the line between $f(\bar{x})$ and $f(\hat{x}), l := \{y \in \mathbb{R}^2 : y = (1-\mu)f(\bar{x}) + \mu f(\hat{x}), \ \mu \in (0,1)\}$. Then $l \subseteq \mathcal{X}$ (again due to convexity) and $f^1(z) < f^1(x^*) \forall \ z \in l$. Because of (9) there exists $\tilde{x} \in l : f^2(\tilde{x}) < f^2(x^*)$, hence λ^* is not maximal.

□

The natural question is now: Is it possible to generalize Theorem 4 to three and more criteria or is the restriction to the bicriteria necessary?
We will give an example that shows that in fact the generalization is wrong. The only conclusion we can draw for more than two criteria is that any solution of (6) is weakly Pareto (Proposition 3).

Example 10 *Let*

$$\begin{aligned} P_1 &= (0,0,0) \\ P_2 &= (1,-1,1) \\ P_3 &= (0,2,-1) \\ C &= conv(P_1, P_2, P_3) \end{aligned}$$

and assume that $\mathcal{X} := C \cup \{x \in \mathbb{R}^3 : x$ *is dominated by* $y, y \in C\}$. *Furtheron let the objective functions be defined as* $f^1(x) := x_1, f^2(x) := x_2$ *and* $f^3(x) := x_3$. *Then the Nadir point is given by* $y^N = (1,2,1)$. *We will show that an optimal solution of (6) is* $x^* = (0,1,0)$, *a weakly Pareto solution dominated by* P_1. *First we will give the dominated region in terms of halfspaces. Consider first Figure 5. In this figure the region dominated by the points on the line between* P_1 *and* P_2 *is indicated. Including the line it is fully described by*

$$x_1 \geq 0 \tag{11}$$
$$x_2 \geq -1 \tag{12}$$
$$x_3 \geq 0 \tag{13}$$
$$x_1 + x_2 \geq 0 \tag{14}$$
$$x_2 + x_3 \geq 0\,. \tag{15}$$

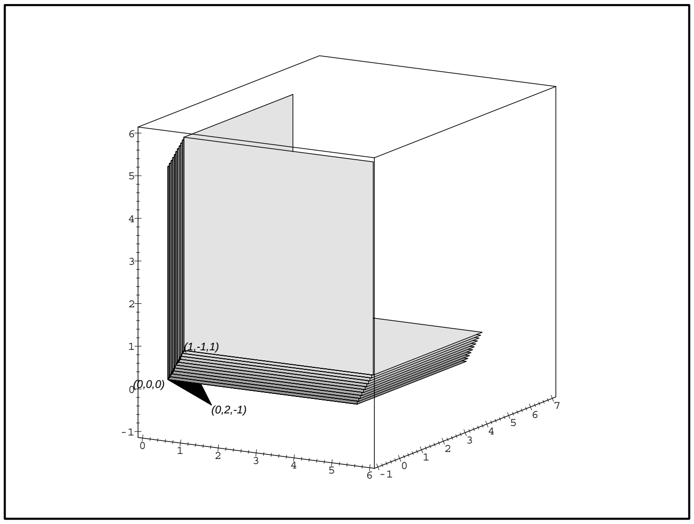

Figure 5: Region dominated by points on the line between P_1 and P_2

The region dominated by points on the line between P_1 and P_3 (as depicted in Figure 6) can be described by the inequalities

$$
\begin{aligned}
x_1 &\geq 0 && (16)\\
x_2 &\geq 0 && (17)\\
x_3 &\geq -1 && (18)\\
x_2 + 2x_3 &\geq 0 && (19)
\end{aligned}
$$

(again including the line itself). Note that the points on these two lines are the only efficient points. All other points in the convex hull of P_1, P_2, P_3 are dominated. We will now show that the complete feasible set (both in decision and objective space) is characterized by

$$
\begin{aligned}
x_1 &\geq 0 && (20)\\
x_2 &\geq -1 && (21)\\
x_3 &\geq -1 && (22)\\
x_1 + x_2 &\geq 0 && (23)\\
x_2 + x_3 &\geq 0 && (24)\\
x_2 + 2x_3 &\geq 0 && (25)
\end{aligned}
$$

Therefore let

$$
\begin{aligned}
S_1 &:= \{x \in \mathbb{R}^n : (11) - -(15) \textit{ hold}\} && (26)\\
S_2 &:= \{x \in \mathbb{R}^n : (16) - -(19) \textit{ hold}\} && (27)\\
S_3 &:= \{x \in \mathbb{R}^n : (20) - -(25) \textit{ hold}\} && (28)
\end{aligned}
$$

We have to show that $S_1 \bigcup S_2 = S_3$.

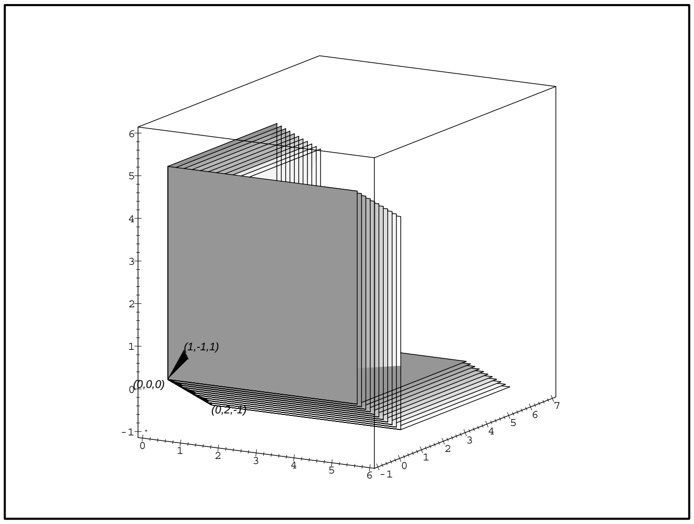

Figure 6: Region dominated by points on the line between P_1 and P_3

Let $x \in S_1 \bigcup S_2$. *Then* $x \in S_1$ *or* $x \in S_2$. *Assume first* $x \in S_1$, *hence (20), (21), (22), (23), (24) are already fulfilled. But* $x_2 + 2x_3 = \underbrace{x_2 + x_3}_{\geq 0\ (15)} + \underbrace{x_3}_{\geq 0\ (13)} \geq 0$ *hence* $x \in S_3$. *Assume now* $x \in S_2$, *hence (20), (21), (22) and (25) hold. Because of (16) and (17) the inequality (23) also holds. The proof for (24) is done by case differentiation. In the first case* $x_3 \in [-1, 0)$. *Then* $x_2 + x_3 = \underbrace{x_2 + 2x_3}_{\geq 0\ (19)} \underbrace{-x_3}_{\geq 0} \geq 0$. *In the second case* $x_3 \geq 0$. *Then due to (17)* $x_2 + x_3 \geq 0$ *hence (24).*

Up to now we have shown $S_1 \bigcup S_2 \subseteq S_3$. *To show the other inclusion let* $x \in S_3$. *Assume first* $x \notin S_1$. *We have to show that (17) holds. The only possibility for* $x \in S_3$ *and* $x \notin S_1$ *is* $x_3 \in [-1, 0)$. *But due to (25)* $x_2 \geq \underbrace{-2x_3}_{\in (0,2]} \geq 0$. *If we assume* $x \in S_3, x \notin S_2$ *we have to show (13). As before we can conclude* $x_2 \in [-1, 0)$ *and due to (24)* $x_3 \geq \underbrace{-x_2}_{\in (0,1]} \geq 0$ *holds. Thus* $S_3 \subseteq S_1 \bigcup S_2$ *and hence* $S_3 = S_1 \bigcup S_2$.

We will now show that in fact S_3 *is a complete description of the feasible set which is defined as the union of the convex hull of* P_1, P_2 *and* P_3 *and the points dominated by this convex hull, i.e.*

$$D := \left\{ x \in \mathbb{R}^n : x \geq \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right.$$
$$\left. \textit{for some } \lambda_i \in [0, 1]; \lambda_1 + \lambda_2 + \lambda_3 = 1 \right\}$$

Assume that $x \in D$. *Hence* $\exists \lambda_1, \lambda_2, \lambda_3$:

$$\begin{aligned} \lambda_1, \lambda_2, \lambda_3 &\in [0,1] && (29)\\ \lambda_1 + \lambda_2 + \lambda_3 &= 1 && (30)\\ x_1 &\geq \lambda_2 && (31)\\ x_2 &\geq -\lambda_2 + 2\lambda_3 && (32)\\ x_3 &\geq \lambda_2 - \lambda_3 && (33) \end{aligned}$$

Then there holds:

$$\begin{aligned} (31) \Rightarrow \quad & x_1 \geq \underbrace{\lambda_2}_{\geq 0\ (29)} && \geq 0 && \Rightarrow (20)\\ (32) \Rightarrow \quad & x_2 \geq \underbrace{-\lambda_2}_{\geq -1\ (29)} + 2\underbrace{\lambda_3}_{\geq 0\ (29)} && \geq -1 && \Rightarrow (21)\\ (33) \Rightarrow \quad & x_3 \geq \underbrace{\lambda_2}_{\geq 0\ (29)} \underbrace{-\lambda_3}_{\geq -1\ (29)} && \geq -1 && \Rightarrow (22)\\ (31),(32) \Rightarrow \quad & x_1 + x_2 \geq 2\underbrace{\lambda_3}_{\geq 0\ (29)} && \geq 0 && \Rightarrow (23)\\ (32),(33) \Rightarrow \quad & x_2 + x_3 \geq \underbrace{\lambda_3}_{\geq 0\ (29)} && \geq 0 && \Rightarrow (24)\\ (32),(33) \Rightarrow \quad & x_2 + 2x_3 \geq \underbrace{\lambda_2}_{\geq 0\ (29)} && \geq 0 && \Rightarrow (25) \end{aligned}$$

Hence $D \subseteq S_3$.
Now assume that $x \in S_3 = S_1 \bigcup S_2$. *It will be shown that there exist* $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$: $\lambda_1 + \lambda_2 + \lambda_3 = 1$ *and* x *fulfills (31), (32), (33). If* $x \in S_1$ *then define* $\lambda_3 := 0$. *For those* x *for which* $x_2 \in [-1,0]$ *define* $\lambda_2 := -x_2$ *and* $\lambda_1 := 1 - \lambda_2$, *hence (29) and (30) hold. Since* $x_1 \geq \max(0, -x_2) = -x_2 = \lambda_2$ *(31) also holds and analogously (33) can be shown. (32) is obviously true. If* $x_2 > 0$ *then let* $\lambda_2 := \lambda_3 = 0, \lambda_1 := 1$ *and (29) - (33) follow immediately, hence* $S_1 \subseteq D$.
If $x \in S_2$ *then define* $\lambda_2 := 0$. *If* $x_3 \in [-1,0]$ *define* $\lambda_3 := -x_3$ *and* $\lambda_1 := 1 - \lambda_3$, *if* $x_3 > 0$ *define* $\lambda_3 := \lambda_2 = 0, \lambda_1 := 1$. *Using the same arguments as for* $x \in S_1$ *we can conclude* $S_2 \subseteq D$ *and hence* $S_3 = S_1 \bigcup S_2 = D$.
Of course S_3 *is a convex set. If we now solve (6), i.e.* $\max\{\lambda : y^N - (\lambda, \lambda, \lambda)^T \in S_3\}$ *especially (20) has to be met, hence* $1 - \lambda \geq 0$ *or* $\lambda \leq 1$. *It is easy to show that all other inequalities yield the same or weaker restrictions on* λ *hence* $x^* = (0,1,0)^T$, *which is dominated by* P_1.

5 Conclusions

In this paper we discussed the computation of Nadir and ideal objective values in multicriteria optimization problems. We first reviewed some literature concerning the determination of Nadir points by exact and heuristic methods. We illustrated that the heuristics either over- or underestimated correct Nadir values. We then gave some theoretical background to justify our approach, before we presented our algorithm to solve the problem of finding the Nadir point in the three criteria case. In contrast to most of the algorithms given so far this new one also works for general continuous as well as combinatorial multicriteria problems and not just in the linear case. After illustrating the algorithm by two examples we pointed out how the Nadir point can be used in compromise programming to achieve acceptable solutions. We proved some results on (weak) Pareto optimality of the resulting solutions. A topic of future research is to extend the ideas presented here in order to develop recursive algorithms to solve multicriteria problems. These could then alo be used to compute Nadir values in the general case of Q criteria.

References

[1] R. Benayoun, J. de Montgolfier, J. Tergny, and O. Laritchev. Linear programming with multiple objective functions: Step method (stem). *Mathematical Programming*, 1(3):366–375, 1971.

[2] H. P. Benson. A finite, nonadjacent extreme-point search algorithm for optimization over the efficient set. *Jorunal of Optimization Theory and Applications*, 73(1):47–64, 1992.

[3] Harold P. Benson. Optimization over the efficient set. *Journal of Mathematical Analysis and Applications*, 98:562–580, 1984.

[4] Harold P. Benson. An all-linear programming relaxation algorithm for optimizing over the efficient set. *Journal of Global Optimization*, 1(1):83–104, 1991.

[5] Harold P. Benson. A bisection-extreme point search algorithm for optimizing over the efficient set in the linear dependence case. *Journal of Global Optimization*, 3:95–111, 1993.

[6] V. Chankong and Y.Y. Haimes. *Multiobjective Decision Making: theory and Methodology*. Elsevier Science Pulishing Co., Inc, New York, NY, 1983.

[7] J.C.M. Climaco and E.Q.V. Martins. A bicriterion shortest path algorithm. *European Journal of Operational Research*, 11:399–404, 1982.

[8] J. P. Dauer. Optimization over the efficient set using an active constraint approach. *Zeitschrift für Operations Research*, 35:185–195, 1991.

[9] Jerald P. Dauer and Timothy A. Fosnaugh. Optimization oveer the efficient set. *Journal of Global Optimization*, 7(3):261–277, 1995.

[10] J. G. Ecker and J. H. Song. Optimizing a linear function over an efficient set. *Journal of Optimization Theory and Applications*, 83(3):541–563, 1994.

[11] M. Ehrgott. *Multiple Criteria Optimization – Classification and Methodology*. Shaker Verlag, Aachen, 1997.

[12] M. Ehrgott. Discrete decision problems, multiple criteria optimization classes and lexicographic max-ordering. In T.J. Stewart and R.C. van den Honert, editors, *Trends in Multicriteria Decision Making*, volume 465 of *Lecture Notes in Economics and Mathematical Systems*, pages 31–44. Springer Verlag, Berlin, 1998.

[13] M. Ehrgott. Approximation algorithms for combinatorial multicriteria optimization problems. *International Transcations in Operational Research*, 7:5–31, 2000.

[14] M. Ehrgott and X. Gandibleux. An annotated bibliography of multiobjective combinatorial optimization. Technical report, University of Kaiserslautern, Department of Mathematics, 2000. Report in Wirtschaftsmathematik Nr. 62/2000, submitted to OR Spektrum.

[15] A. Eskandari, Ffolliott P., and F. Szidarovszky. Uncertainty and method choice in descrete multiobjective programming problems. *Applied Mathematics and Computation*, 69:335–351, 1995.

[16] M.R. Garey and D.S. Johnson. *Computers and Intractability – A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, 1979.

[17] H.W. Hamacher, M. Labbé, and S. Nickel. Multicriteria network location problems with sum objectives. *Networks*, 33:79–92, 1999.

[18] R. Horst and N. V. Thoai. Utility function programs and optimization over the efficient set in multiple-objective decision making. *Journal of Optimization Theory and Applications*, 92(3):605–631, 1997.

[19] R. Horst and N.V. Thoai. Maximizing a concave function over the efficient or weakly-efficient set. *European Journal of Operational Research*, 117(2):239–252, 1999.

[20] Heinz Isermann and Ralph E. Steuer. Computational experience cnocerning payoff tables and minimum criterion values over the efficient set. *European Journal of Operational Research*, 33:91–97, 1987.

[21] Pekka Korhonen, Seppo Salo, and Ralph E. Steuer. A heuristic for estimating nadir criterion values in multiple objective linear programming. *Operations Research*, 45(5):751–757, 1997.

[22] Gary R. Reeves and Randall C. Reid. Minimum values over the efficient set in multiple decision making. *European Journal of Operational Research*, 36(3):334–338, 1988.

[23] Y. Sawaragi, H. Nakayama, and T. Tanino. *Theory of Multiobjective Optimization.* Academic Press, Orlando, 1985.

[24] R.E. Steuer. *Multiple Criteria Optimization: Theory, Computation and Application.* John Wiley & Sons, New York, 1985.

[25] P. T. Thach, H. Konno, and D. Yokota. Dual approach to minimization of the set of pareto-optimal solutions. *Journal of Optimization Theory and Applications*, 88(3):689–707, 1996.

[26] E.L. Ulungu. *Optimisation Combinatoire multicritere: Determination de l'ensemble des solutions efficaces et methodes interactives.* PhD thesis, Faculté des Sciences, Université de Mons-Hainaut, Mons, Belgium, 1993.

[27] E.L. Ulungu and J. Teghem. The two-phases method: An efficient procedure to solve bi-objective combinatorial optimization problems. *Foundations of Computing and Decision Sciences*, 20(2):149–165, 1994.

[28] H. R. Weistroffer. Careful usage of pessimistic values is needed in multiple objectives optimization. *Operations Research Letters*, 4(1):23–25, 1985.

[29] P.L. Yu. *Multiple Criteria Decision Making: Concepts, Techniques and Extensions.* Plenum Press, New York, NY, 1985.