Linear Facility Location in Three Dimensions - Models and Solution Methods

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January 28, 2000

Abstract

We consider the problem of locating a line or a line segment in three-dimensional space, such that the sum of distances from the linear facility to a given set of points is minimized. An example is planning the drilling of a mine shaft, with access to ore deposits through horizontal tunnels connecting the deposits and the shaft. Various models of the problem are developed and analyzed, and efficient solution methods are given.

1 Introduction

The problem of locating a line in two-dimensional space was considered early by Wesolowsky [15] and further developed by Morris and Norback [10, 11, 12]. Schöbel's recent dissertation [14] describes what has been done in the area of locating lines in the plane and hyperplanes in $\mathbb{R}^n$ up till now. In computational geometry line and hyperplane location problems are also of interest [7]. For the location of line segments only a few special cases have been discussed [6, 1, 13]. Here we consider a new problem: the location of a line (or a line segment) in three-dimensional space. A practical setting for this problem is found in mining. An area contains deposits of some mineral in various locations underground. Instead of digging down separately to each deposit, it may be cheaper to construct a main shaft and reach the deposits by tunnels. One setting may prescribe that the shaft be vertical and the tunnels horizontal, for construction purposes. In another setting the shaft may be angled and the tunnels perpendicular to the shaft. In both cases we want to locate the shaft so as to minimize the annual transportation costs of moving the mineral through the tunnels (and up the shaft). A third case may take into account the heavy cost of digging the shaft by including the length of the shaft as a decision variable; here the deep-lying deposits are reached by straight line tunnels from the bottom of the shaft.

2 Notation

We first introduce some necessary notation from location theory. The classical location problem is the so-called Weber or Fermat-Toricelli-Problem in which a set of existing facilities $\mathcal{A} = \{A_1, \ldots, A_M\}$ in the plane is given. The objective is to locate a point X such that the sum of distances from the existing facilities to the point X is minimized, i.e.

$$\min \sum_{m=1,\ldots,M} w_m d(A_m, X).$$

The parameters $w_m \geq 0$ are weights assigned to the existing facilities. The function $d(A_m, X)$ calculates the distance between any two points A_m and X in $\mathbb{R}^2$. For an overview about location theory we refer to the textbooks by Love, Morris, and Wesolowsky [8] or Francis, McGinnis, and White [4].
In the classification scheme of [5] the Weber problem with Euclidean distance is classified as $1/\mathbb{R}^2/\cdot/l_2/\sum$ meaning that we want to locate one point (1) in the plane $\mathbb{R}^2$ with no special assumptions $(\cdot)$, using the Euclidean norm l_2 to measure the distance from the existing facilities to the new point and minimizing the sum $(\sum)$ of distances as objective function. This problem has a lot of generalizations. One of them is to locate not a point, but a line l. Then the objective function can be written as

$$\min \sum_{m=1,\ldots,M} w_m d(A_m, l),$$

where the distance between a point A and a line is given by

$$d(A, l) = \min_{X \in l} d(A, X). \tag{1}$$

The classification of this problem is given by $1l/\mathbb{R}^2/\cdot/d/\sum$, where $1l$ indicates that we want to locate one line instead of one point. Analogously, one can formulate the problem of locating a line segment s with fixed length. While line location problems can be solved efficiently for l_p norm distances, very little is known about the location of line segments. For a recent overview of line and line segment location problems, see [14, 9].
In this paper we extend line location problems in the plane to $\mathbb{R}^3$. Given a set of existing facilities in $\mathbb{R}^3$,

$$\mathcal{A} = \{A_1, A_2, \ldots, A_M\}, \quad \mathcal{M} = \{1, 2, \ldots, M\}$$

with $A_m = (a_{m1}, a_{m2}, a_{m3}) \in \mathbb{R}^3$, we look for a straight line $l \subset \mathbb{R}^3$. As distance measure we mainly deal with the p-norms, $1 \leq p \leq \infty$.
Consequently the three-dimensional line location problem $1l/\mathbb{R}^3/\cdot/\sum/l_p$ is given as follows. Find a line l such that we minimize

$$f(l) = \sum_{m \in \mathcal{M}} w_m l_p(A_m, l).$$

In the mining example mentioned above the existing facilities represent the deposits and the line l models the mining shaft. The objective is to minimize the costs of the tunnel system which we assume to be related to the length of the tunnels. The length of a tunnel from a deposit A to the shaft l is given by $d(A, l)$ where d is mainly dependent on the properties of the tunnel system.
Apart from defining the distance between a point and a line as in (1), the mining example motivates also the following model. We assume that the paths connecting the line to an existing facility A (the tunnel from the deposit A to the shaft in the mining example) have to be horizontal. Therefore, the three-dimensional distance l_p simplifies to the two-dimensional distance l_p in the horizontal plane through A.
The remainder of the paper is organized as follows. We start with locating a vertical line in the next section and discuss the case of a vertical line segment in Section 4. In Section 5 we deal with arbitrary lines, but assuming horizontal paths. In Section 6 we forget about both restrictions and present results for locating an arbitrary line in $\mathbb{R}^3$.

3 Locating a vertical line

In a mining application a natural restriction is that the main shaft must be dug vertically to lower the digging costs and the costs of operating the elevator in the shaft. Thus it is of interest to consider the special case of locating a vertical line. A vertical line l is completely described by only one point β on it. Without restriction let $\beta = (\beta_1, \beta_2, 0) \in \mathbb{R}^3$, i.e.

$$l_\beta = \{X = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = \beta_1, x_2 = \beta_2\}.$$

To calculate the distance from a point A to l_β we use the following lemma.

Lemma 1 *Suppose l is a vertical line and let $A \in \mathbb{R}^3$. Then all shortest paths (with respect to $l_p, p \geq 1$) from A to l lie completely in the horizontal plane through A.*

Proof: Let $X = (\beta_1, \beta_2, \lambda)$ be any point on the vertical line l_β. Then

$$l_p(A, X) = (|a_1 - \beta_1|^p + |a_2 - \beta_2|^p + |a_3 - \lambda|^p)^{\frac{1}{p}}$$

is minimized for $\lambda = a_3$.

QED

Using Lemma 1 we can specify the distance d between a point $A = (a_1, a_2, a_3)$ and a vertical line l as

$$d(A, l) = l_p((a_1, a_2), (\beta_1, \beta_2)). \tag{2}$$

and our problem can be restated as

$$\min_{\beta^2 \in \mathbb{R}^2} \sum_{m \in \mathcal{M}} w_m l_p(A_m^2, \beta^2)$$

where A_m^2 denotes the projection of A_m onto the horizontal plane and $\beta^2 = (\beta_1, \beta_2)$. This means, the three-dimensional line location problem with variables β_1 and β_2 reduces to the location of a point $\beta^2 = (\beta_1, \beta_2)$ in the plane. This is the classical Weber problem which can be solved efficiently for all l_p distances, see e.g. [8]. We summarize the result of this section in the following lemma.

Lemma 2 *Locating a vertical line in $\mathbb{R}^3$ with distance measure l_p is equivalent to a Weber problem with distance measure l_p in the plane.*

Note that using l_p norms here is essential; Lemma 1 and Lemma 2 cannot be generalized to all distances d derived from norms.

4 Locating a vertical line segment

In most applications the costs for building the new linear facility may not be neglected, such that the line cannot be assumed to be infinite as in the previous section. In our approach we do not fix the length of the line segment, but we introduce additional costs for establishing the facility. Assuming that these costs are proportional to the length of the line segment s we derive the following objective function,

$$f(s) = \sum_{m \in \mathcal{M}} w_m d(A_m, s) + v \text{ length}(s)$$

where $v \geq 0$ is a weight or cost per unit length.
We define a vertical line segment $s_{(\beta_1, \beta_2, h_1, h_2)}$ by its starting point (β_1, β_2, h_1) and its endpoint (β_1, β_2, h_2). Without loss of generality let us assume that $h_2 \geq h_1$, such that $\text{length}(s_{(\beta_1, \beta_2, h_1, h_2)}) = h_2 - h_1$. In the mining example, the special case $h_1 = 0$ corresponds to the shaft extending from the ground surface down to the depth h_2. Using Lemma 1, the p-norm distance from an existing facility $A = (a_1, a_2, a_3)$ to the line segment $s = s_{(\beta_1, \beta_2, h_1, h_2)}$ is then given by

$$d(A,s) = \begin{cases} l_p\left((a_1,a_2,a_3),(\beta_1,\beta_2,h_1)\right) & \text{if } a_3 < h_1 \\ l_p\left((a_1,a_2),(\beta_1,\beta_2)\right) & \text{if } h_1 \le a_3 \le h_2 \\ l_p\left((a_1,a_2,a_3),(\beta_1,\beta_2,h_2)\right) & \text{if } a_3 > h_2. \end{cases}$$

Using the definition of l_p we rewrite $d(A,s)$ as

$$\begin{aligned} d(A,s) &= \left(|a_1-\beta_1|^p + |a_2-\beta_2|^p + (\max\{h_1-a_3, 0, a_3-h_2\})^p\right)^{\frac{1}{p}} \\ &= \|(a_1-\beta_1, a_2-\beta_2, \max\{h_1-a_3,0,a_3-h_2\})\|_p. \end{aligned} \tag{3}$$

Note that for $h_2 < h_1$ we have $d(A, s_{(\beta_1,\beta_2,h_1,h_2)}) \ge d(A, s_{(\beta_1,\beta_2,h_1,h_1)})$. This is needed to get rid of the restriction $h_2 \ge h_1$ later on.

Lemma 3 *$d(A,s_X)$ is a convex function of $X = (\beta_1,\beta_2,h_1,h_2) \in \mathbb{R}^4$.*

Proof: Let $X, Y \in \mathbb{R}^4$, and $Z = \lambda X + (1-\lambda)Y$, $\lambda \in [0,1]$. For $1 \le p < \infty$ we then get

$$\begin{aligned} d(A,s_Z) &= (|a_1-(\lambda x_1+(1-\lambda)y_1)|^p + |a_2-(\lambda x_2+(1-\lambda)y_2)|^p \\ &\quad + (\max\{(\lambda x_3+(1-\lambda)y_3)-a_3, 0, a_3-(\lambda x_4+(1-\lambda)y_4)\})^p)^{\frac{1}{p}} \\ &= (|\lambda(a_1-x_1)+(1-\lambda)(a_1-y_1)|^p + |\lambda(a_2-x_2)+(1-\lambda)(a_2-y_2)|^p \\ &\quad + (\max\{\lambda(x_3-a_3)+(1-\lambda)(y_3-a_3), 0, \\ &\qquad\qquad \lambda(a_3-x_4)+(1-\lambda)(a_3-y_4)\})^p)^{\frac{1}{p}} \\ &\le (|\lambda(a_1-x_1)+(1-\lambda)(a_1-y_1)|^p + |\lambda(a_2-x_2)+(1-\lambda)(a_2-y_2)|^p \\ &\quad + (\max\{\lambda(x_3-a_3), 0, \lambda(a_3-x_4)\} \\ &\qquad\qquad + \max\{(1-\lambda)(y_3-a_3), 0, (1-\lambda)(a_3-y_4)\})^p)^{\frac{1}{p}} \\ &\le (|\lambda(a_1-x_1)|^p + |\lambda(a_2-x_2)|^p + (\max\{\lambda(x_3-a_3),0,\lambda(a_3-x_4)\})^p)^{\frac{1}{p}} \\ &\quad + (|(1-\lambda)(a_1-y_1)|^p + |(1-\lambda)(a_2-y_2)|^p \\ &\qquad\qquad + (\max\{(1-\lambda)(y_3-a_3),0,(1-\lambda)(a_3-y_4)\})^p)^{\frac{1}{p}} \\ &\quad \text{by the triangle inequality of norms} \\ &= \lambda d(A,s_X) + (1-\lambda) d(A,s_Y). \end{aligned}$$

Thus, we conclude that $d(A,s_Z)$ is a convex function in $Z \in \mathbb{R}^4$, if $p \ge 1$. For $p = \infty$ the proof can be done analogously.

QED

The extension of Lemma 3 to arbitrary norms is not straightforward, since $\|(a,b,z_1)\| \le \|(a,b,z_2)\|$ if $|z_1| \le |z_2|$ is not necessarily true for arbitrary norms.

Lemma 4 *The objective function,*

$$f(\beta_1, \beta_2, h_1, h_2) = \sum_{m=1}^{M} w_m d(A_m, s_{\beta_1,\beta_2,h_1,h_2}) + v|h_2 - h_1|$$

is a convex function of $(\beta_1, \beta_2, h_1, h_2)$.

Proof: Using Lemma 3 and keeping in mind that the weights w_m are nonnegative, it follows that f is the sum of $M+1$ convex functions and hence, f is itself convex.

QED

Lemma 4 implies that it is easy to solve the unconstrained problem to obtain the minisum solution $(\beta_1^*, \beta_2^*, h_1^*, h_2^*)$, e.g., by a gradient descent approach. For $h_2 < h_1$ the solution can be improved by setting $h_1' := h_1$ and $h_2' := h_1$. Thus, the constraint $h_2 \geq h_1$ will be satisfied in any optimal solution, and therefore it does not need to be included explicitly.
Once a local minimum is obtained, the convexity of f guarantees that it is a global solution. In the following we give a more efficient solution approach which utilizes a well-known technique for locating a point facility in the plane. For the mine-shaft example, the origin of the segment s coincides with ground level, and may arbitrarily be set to $h_1 = 0$.

Algorithm 1 (for locating a vertical line segment with p-norm distances)

Step 1. Choose initial solution $(\beta_1^0, \beta_2^0, h_1^0, h_2^0)$ and set counter $g = 0$.

Step 2. Holding $h_1 = h_1^g$ and $h_2 = h_2^g$ fixed, perform Weiszfeld iterations (see [8]) until a stopping criterion is reached. Denote the current solution by $X^g := (\beta_1^{g+1}, \beta_2^{g+1}, h_1^g, h_2^g)$.

Step 3. Holding $\beta_1 = \beta_1^{g+1}$ and $\beta_2 = \beta_2^{g+1}$ fixed, optimize for h_1 and h_2 until a stopping criterion is reached. Denote the current solution by $X^{g+1} := (\beta_1^{g+1}, \beta_2^{g+1}, h_1^{g+1}, h_2^{g+1})$.

Step 4. If $f(s_{X^g}) - f(s_{X^{g+1}}) < \delta$, STOP;
else set g=g+1 and return to Step 2.

In steps 2 and 3, the algorithm iteratively examines subspaces (β_1, β_2) and (h_1, h_2). The stopping criterion in each subspace may take the form of a δ-accuracy as in step 4. Alternatively, the number of iterations (descent moves) in each subspace may be fixed in a manner to improve the overall computational efficiency of the algorithm.
In step 3, if h_1 is fixed at 0, a simple one-dimensional search will find the optimal value of h_2. Otherwise, the objective function f in step 3 is given by

$$f(h_1, h_2) = \sum_{m \in \mathcal{M}} w_m \left(c_m + \max\{h_1 - a_{m3}, 0, a_{m3} - h_2\}^p\right)^{\frac{1}{p}} + v|h_2 - h_1|,$$

where c_m is only dependent on β_1^{g+1} and β_2^{g+1} and therefore constant in this context. Defining

$$\begin{aligned}\mathcal{M}_1(h_1) &= \{m \in \mathcal{M} : a_{m3} < h_1\}\\ \mathcal{M}_2(h_2) &= \{m \in \mathcal{M} : a_{m3} > h_2\}\end{aligned}$$

we can reformulate f as

$$\begin{aligned}f(h_1, h_2) &= \sum_{m \in \mathcal{M}_1(h_1)} w_m \left(c_m + (h_1 - a_{m3})^p\right)^{\frac{1}{p}} - v h_1\\ &+ \sum_{m \in \mathcal{M}_2(h_2)} w_m \left(c_m + (a_{m3} - h_2)^p\right)^{\frac{1}{p}} + v h_2\\ &= f_1(h_1) + f_2(h_2),\end{aligned}$$

where we require $h_2 \geq h_1$. To minimize f_1 (or f_2, respectively), one can determine the partial derivative in each layer where $\mathcal{M}_1$ ($\mathcal{M}_2$) changes and solve numerically to zero. Thus, f_1 and f_2 can be minimized separately leading to minimizers h_1^* and h_2^*. We have to distinguish two cases:

Case 1: $h_1^* \leq h_2^*$. Then the solution is feasible and therefore minimizes f.

Case 2: $h_1^* < h_2^*$. Then any minimizer of f satisfies $h_1 = h_2$, i.e. the line segment degenerates to a point $(\beta_1^{g+1}, \beta_2^{g+1}, h_1)$ which can be found by minimizing the one-dimensional function

$$\sum_{m \in \mathcal{M}} w_m \left(c_m + |h_1 - a_{m3}|^p\right)^{\frac{1}{p}}.$$

Lemma 5 *Let $p \in [1, 2]$. Algorithm 1 converges uniformly to the optimal solution as $\delta \to 0$.*

Proof: Referring to [2], we may show that each Weiszfeld iteration in step 2 results in an improvement of the objective function. Each completion of step 3 is a descent move in the corresponding subspace. Thus, we may conclude in a similar fashion as in [2] that the series converges to a unique attraction point, and due to the convexity of the objective function, this coincides with the global optimum.

QED

In the unlikely event that an iterate coincides with a singular point of the iteration functions, a hyperbolic approximation of the distance may be used (see [8]). If $p > 2$, the descent property of the iterates in step 2 is no longer guaranteed ([2]). However, computational results in [3] indicate that a step-size adjustment factor will remedy this problem when it occurs.

For the special case $p = 1$ we show that an exact optimal solution can be found in linear time. Replacing p by 1 in (3), the objective function f can be separated into the following two functions f_1 and f_2.

$$
\begin{aligned}
f(\beta_1, \beta_2, h_1, h_2) &= \sum_{m \in M} w_m(|a_{m1} - \beta_1| + |a_{m2} - \beta_2| \\
&\qquad + \max\{h_1 - a_{m3}, 0, a_{m3} - h_2\}) + v|h_2 - h_1| \\
&= f_1(\beta_1, \beta_2) + f_2(h_1, h_2).
\end{aligned}
$$

Both functions f_1 and f_2 can be minimized separately. The problem of minimizing f_1 is a Weber problem in the plane with rectangular metric, where the set of existing facilities is given by $\mathcal{A}^2 = \{A_m^2 : m \in \mathcal{M}\}$. As a consequence, the optimal parameters β_1^*, β_2^* are independent of the cost v for establishing the line segment in this case, and f_1 can be minimized in linear time.
To minimize f_2 we can proceed along the lines of step 3 of Algorithm 1. We assume that the existing facilities are sorted according to their third coordinates, i.e. $a_{13} \leq a_{23} \leq \ldots \leq a_{M3}$. Defining

$$
x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases},
$$

we can rewrite f_2 as

$$
\begin{aligned}
f_2(h_1, h_2) &= \sum_{m \in \mathcal{M}} w_m \left((h_1 - a_{m3})^+ + (a_{m3} - h_2)^+\right) + v|h_2 - h_1| \\
&= \sum_{m \in \mathcal{M}} w_m (h_1 - a_{m3})^+ - v h_1 + \sum_{m \in \mathcal{M}} w_m (a_{m3} - h_2)^+ + v h_2 \\
&= f_{21}(h_1) + f_{22}(h_2),
\end{aligned}
$$

where we require that $h_2 \geq h_1$ holds. Looking at the slopes of the piecewise linear functions f_{21} and f_{22}, it turns out that the respective optimal solutions h_1^* and h_2^* are given by the following expressions.
Let i_1 be such that $\sum_{m=1}^{i_1} w_m \geq v$ and $\sum_{m=1}^{i_1 - 1} w_m < v$. If the first inequality holds strictly, then $h_1^* = a_{i_1 3}$ is the unique solution for h_1. Otherwise all values in the interval $[a_{i_1 3},\ a_{i_1+1,3}]$ are optimal.
Analogously, for finding the best value h_2^*, let i_2 be such that $\sum_{m=i_2}^{M} w_m \geq v$ and $\sum_{m=i_2+1}^{M} w_m < v$; now either $h_2^* = a_{i_2 3}$ is the unique optimum or the interval $[a_{i_2-1,3},\ a_{i_2 3}]$ is the set of optimizers.
We distinguish two cases:

Case 1: $h_1^* \leq h_2^*$. Then the solution is feasible and therefore minimizes f_2.

Case 2: $h_1^* < h_2^*$. Then any minimizer of f_2 satisfies $h_1 = h_2$ yielding the median problem $\min \sum_{m \in M} w_m |h_1 - a_{m3}|$, which can be solved in linear time.

The above results are summarized as follows.

Lemma 6 *The location of a vertical line segment with respect to the l_1 norm can be solved in linear time, if the existing facilities are sorted (according to their third coordinates).*

5 Arbitrary line with horizontal paths

Given two parameters $\alpha, \beta \in \mathbb{R}^3$, we define an arbitrary line $l_{\alpha,\beta}$ by

$$l_{\alpha,\beta} = \{X \in \mathbb{R}^3 : X = \lambda\alpha + \beta, \lambda \in \mathbb{R}\}. \tag{4}$$

Throughout this section, we assume that the paths connecting an existing facility with the line have to be horizontal, i.e. we can calculate the distance from l to $A = (a_1, a_2, a_3)$ as the (two-dimensional) distance between A and the closest point $P = (p_1, p_2, p_3)$ on the line l with $p_3 = a_3$. The classification of problems of these kind is given by $1l/\mathbb{R}^3/\cdot/l_{p,\text{horizontal}}/\sum$.
If all existing facilities lie in the same horizontal plane, and we assume horizontal paths, the three-dimensional line location problem reduces to a two-dimensional line location problem in the plane and can therefore be solved efficiently for all distances derived from norms (see, e.g., [14]). In the following we therefore exclude this trivial case and assume that not all existing facilities lie in the same horizontal plane. Then, due to the assumption of horizontal paths, no horizontal line can be optimal. Therefore we let

$$\alpha = (\alpha_1, \alpha_2, 1) \quad \text{and} \quad \beta = (\beta_1, \beta_2, 0).$$

Then the point P on l with $p_3 = a_3$ is given by

$$P = (\alpha_1 a_3 + \beta_1, \alpha_2 a_3 + \beta_2, a_3).$$

For the distance from $A = (a_1, a_2, a_3)$ to the line we consequently get:

$$\begin{aligned} d(A, l_{\alpha,\beta}) &= l_p((a_1, a_2, a_3), (p_1, p_2, p_3)) \\ &= l_p((a_1, a_2), (\alpha_1 a_3 + \beta_1, \alpha_2 a_3 + \beta_2)). \end{aligned} \tag{5}$$

Lemma 7 *$d(A, l_{\alpha,\beta})$ is a convex function of $(\alpha_1, \beta_1, \alpha_2, \beta_2)$.*

Proof: Consider any $X = (x_1, \ldots, x_4), Y = (y_1, \ldots, y_4) \in \mathbb{R}^4$, and let
$Z = \lambda X + (1-\lambda)Y$, $\lambda \in [0, 1]$. Let $\|\cdot\|$ denote any l_p norm. Then

$$\begin{aligned} d(A, l_Z) &= d^2\left((a_1, a_2), (z_1 a_3 + z_2, z_3 a_3 + z_4)\right) \\ &= \|\left(a_1 - (z_1 a_3 + z_2), a_2 - (z_3 a_3 + z_4)\right)\| \\ &= \|\left(a_1 - ((\lambda x_1 + (1-\lambda) y_1) a_3 + \lambda x_2 + (1-\lambda) y_2\right), \\ &\quad\ a_2 - ((\lambda x_3 + (1-\lambda) y_3) a_3 + \lambda x_4 + (1-\lambda) y_4))\| \end{aligned}$$

$$
\begin{aligned}
&= \| (\lambda(a_1 - x_1 a_3 - x_2) + (1-\lambda)(a_1 - y_1 a_3 - y_2), \\
&\quad \lambda(a_2 - x_3 a_3 - x_4) + (1-\lambda)(a_2 - y_3 a_3 - y_4)) \| \\
&\leq \| (\lambda(a_1 - x_1 a_3 - x_2), \lambda(a_2 - x_3 a_3 - x_4)) \| + \\
&\quad \| ((1-\lambda)(a_1 - y_1 a_3 - y_2), (1-\lambda)(a_2 - y_3 a_3 - y_4)) \| \\
&\quad \text{ by the triangle inequality for norms} \\
&= \lambda \| (a_1 - x_1 a_3 - x_2, a_2 - x_3 a_3 - x_4) \| + \\
&\quad (1-\lambda) \| (a_1 - y_1 a_3 - y_2, a_2 - y_3 a_3 - y_4) \| \\
&= \lambda d^2 ((a_1, a_2), (x_1 a_3 + x_2, x_3 a_3 + x_4)) + \\
&\quad (1-\lambda) d^2 ((a_1, a_2), (y_1 a_3 + y_2, y_3 a_3 + y_4))
\end{aligned}
$$

We conclude that $d(A, l_{\alpha,\beta})$ is a convex function of $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ as required.

QED

Lemma 8 *The objective function*

$$f(\alpha, \beta) = \sum_{m=1}^{M} w_m d(A_m, l_{\alpha,\beta})$$

is a convex function of $\alpha_1, \beta_1, \alpha_2, \beta_2$.

Proof: Since $w_m \geq 0$ we can use Lemma 7 to conclude that f is the sum of M convex functions and hence f itself is convex.

QED

Note that Lemma 7 and Lemma 8 do not only hold for l_p distances, $1 \leq p \leq \infty$ but for all distances d derived from norms.
As mentioned before, the minimization of a convex function is relatively simple, since a local minimum is also global. Although the problem may therefore be solved by a standard approach, some special cases will be studied in more detail in the following. First, we consider the case where the line is required to pass through a specified point. Then we discuss two special distance measures for this problem, namely l_1 and l_2.

5.1 Fixed starting point of the line

In this section we suppose that we are looking for a line passing through one specified point $\beta = (\beta_1, \beta_2, 0)$, i.e. the parameter β in formula (4) can be fixed. With the general assumptions of Section 5 our problem reduces to the two-dimensional problem of calculating α_1 and α_2. Using Equation 5 we reformulate the distance between an existing facility A and a line $l_{\alpha,\beta}$ as follows.

$$\begin{aligned}
d(A, l_{\alpha,\beta}) &= l_p((a_1, a_2), (\alpha_1 a_3 + \beta_1, \alpha_2 a_3 + \beta_2)) \\
&= \|(\alpha_1 a_3 + \beta_1 - a_1, \alpha_2 a_3 + \beta_2 - a_2)\| \\
&= |a_3| \|(\alpha_1 + \frac{\beta_1 - a_1}{a_3}, \alpha_2 + \frac{\beta_2 - a_2}{a_3}\| \\
&= |a_3| \|(\alpha_1 - a_1', \alpha_2 - a_2')\|,
\end{aligned}$$

where

$$\begin{aligned}
a_1' &= \frac{a_1 - \beta_1}{a_3} \quad \text{and} \\
a_2' &= \frac{a_2 - \beta_2}{a_3}
\end{aligned}$$

Defining $\alpha^2 = (\alpha_1, \alpha_2)$, $A_m' = (a_{m1}', a_{m2}')$ and weights $w_m' = |a_{m3}| w_m$ the objective function can be rewritten as

$$f(l_{\alpha,\beta}) = \sum_{m \in \mathcal{M}} w_m' l_p(\alpha^2, A_m').$$

Since this is a classical Weber problem in the plane we have proven the next lemma.

Lemma 9 *Locating a line in* $\mathbb{R}^3$ *with fixed origin* $(\beta_1, \beta_2, 0)$, *horizontal paths and distance measure* l_p *is equivalent to a Weber problem with distance measure* l_p *in the plane.*

Note that this approach works not only for $d = l_p$ but also for all distances d derived from norms.

5.2 Horizontal paths with rectangular distance

Now let us assume that the distance from a point $A \in \mathbb{R}^3$ to the line l is measured by the two-dimensional rectangular distance l_1 in the horizontal plane passing through A. Using Equation 5 we obtain the following minimization problem.

$$\min_{\alpha_1, \beta_1, \alpha_2, \beta_2} \sum_{m \in \mathcal{M}} w_m l_1((a_{m1}, a_{m2}), (\alpha_1 a_{m3} + \beta_1, \alpha_2 a_{m3} + \beta_2))$$

Using the definition of the l_1-distance, the problem can be separated into the following two subproblems.

$$(P_k) \qquad \min_{\alpha_k, \beta_k} \sum_{m \in \mathcal{M}} w_m(|a_{mk} - \beta_k - a_{m3}\alpha_k|) \quad k = 1, 2$$

Both problems can be solved in linear time by linear programming. For an exact formulation of the linear programs describing (P_k) see [16]. In the following we mention a geometric interpretation of the subproblems (P_k):
Since the vertical distance between a point $Z = (z_1, z_2)$ in the plane and a non-vertical line with slope s and intercept b is given by

$$d_{ver}(l_{s,b}, Z) = |z_2 - b - sz_1|,$$

both problems (P_k) can be interpreted as line location problems in the plane with vertical distance d_{ver} where the existing facilities for subproblem (P_k) are determined by

$$A_m^k = (a_{m3}, a_{mk}) \text{ for all } m \in \mathcal{M},\ k = 1, 2.$$

The result of both problems (P_k) is a non-vertical line l_k^* in the plane with intercept β_k^* and slope α_k^*, yielding the optimal solution for the parameters $\alpha_1, \alpha_2, \beta_1$, and β_2 for the three-dimensional line $l^* = l_{\alpha^*,\beta^*}$. We remark that l_1^* is the projection of l^* into the xz-plane while l_2^* is the projection of l^* into the yz-plane.

Lemma 10 *Locating a line in* $\mathbb{R}^3$ *with horizontal paths with respect to the* l_1 *norm is equivalent to two planar line location problems with vertical distance and can therefore be solved in linear time.*

5.3 Horizontal paths with Euclidean distance

Using the Euclidean norm to calculate the distance from a point $A \in \mathbb{R}^3$ to the line l within the horizontal plane through A, the objective function can be rewritten as

$$f(l_{\alpha,\beta}) = \sum_{m\in\mathcal{M}} w_m \sqrt{(\beta_1 + \alpha_1 a_{m3} - a_{m1})^2 + (\beta_2 + \alpha_2 a_{m3} - a_{m2})^2}$$

The derivatives are given by

$$\begin{aligned}
\frac{\partial f}{\partial \beta_i} &= \sum_{m\in\mathcal{M}} \frac{w_m(\beta_i + \alpha_i a_{m3} - a_{mi})}{\sqrt{(\beta_1 + \alpha_1 a_{m3} - a_{m1})^2 + (\beta_2 + \alpha_2 a_{m3} - a_{m2})^2}},\ i = 1, 2,\\
\frac{\partial f}{\partial \alpha_i} &= \sum_{m\in\mathcal{M}} \frac{w_m a_{m3}(\beta_i + \alpha_i a_{m3} - a_{mi})}{\sqrt{(\beta_1 + \alpha_1 a_{m3} - a_{m1})^2 + (\beta_2 + \alpha_2 a_{m3} - a_{m2})^2}},\ i = 1, 2.
\end{aligned}$$

Setting the derivatives equal to zero and rearranging the terms leads to the modified Weiszfeld algorithm given below. The main idea is to iterate separately on β and α, using updated values each time. Convergence of the sequence of iterates to the optimal solution is readily shown. As in Algorithm 1 a hyperbolic approximation may be used, if necessary.

Algorithm 2 (for locating a line with horizontal Euclidean distance)

Step 1. Choose initial solution $(\beta_1^0, \beta_2^0, \alpha_1^0, \alpha_2^0)$, and set counter $g = 0$.

Step 2a. Compute $c_m = \sqrt{(\beta_1^g + \alpha_1^g a_{m3} - a_{m1})^2 + (\beta_2^g + \alpha_2^g a_{m3} - a_{m2})^2}$, $m \in \mathcal{M}$.

Step 2b. Iterate on β as follows:

$$\beta_i^{g+1} = \frac{\sum_{m\in\mathcal{M}} w_m(a_{mi} - \alpha_i^g a_{m3})/c_m}{\sum_{m\in\mathcal{M}} w_m/c_m}, i = 1, 2,$$

Step 3a. Compute $c'_m = \sqrt{(\beta_1^{g+1} + \alpha_1^g a_{m3} - a_{m1})^2 + (\beta_2^{g+1} + \alpha_2^g a_{m3} - a_{m2})^2}$, $m \in \mathcal{M}$.

Step 3b. Iterate on α as follows:

$$\alpha_i^{g+1} = \frac{\sum_{m\in\mathcal{M}} w_m a_{m3}(a_{mi} - \beta_i^{g+1})/c'_m}{\sum_{m\in\mathcal{M}} w_m a_{m3}^2/c'_m}, i = 1, 2.$$

Step 4. If $f(l_{\alpha,\beta}^g) - f(l_{\alpha,\beta}^{g+1}) < \delta$, STOP;
else set $g = g + 1$ and return to step 2a.

6 Locating an arbitrary line with shortest distances

In this section we relax the condition that the paths between the facilities and the line must be horizontal, and instead use formula (1) to determine the distance from a point to a line.

6.1 Euclidean distance

For the Euclidean distance l_2 the classification of the problem is given by $1l/\mathbb{R}^3/\cdot/l_2/\sum$. With decision variables $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ (assuming without loss of generality that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$) and $\beta = (\beta_1, \beta_2, 0)$ the line is given by $l_{\alpha,\beta} = \{x \in \mathbb{R}^3 : x = \lambda\alpha + \beta, \lambda \in \mathbb{R}\}$. For any given point $A_m = (a_{m1}, a_{m2}, a_{m3}) \in \mathbb{R}^3$ the closest point on the line is found as the one with λ being the inner product $\lambda_m^* = \langle \alpha, A_m - \beta \rangle$, i.e., we get the following formula for calculating the distance between $A_m \in \mathbb{R}^3$ and $l = l_{\alpha,\beta}$, if α is normed to 1.

$$d(A_m, l) = \sqrt{(a_{m1} - \alpha_1\lambda_m^* - \beta_1)^2 + (a_{m2} - \alpha_2\lambda_m^* - \beta_2)^2 + (a_{m3} - \alpha_3\lambda_m^*)^2}$$

$$= \sqrt{\langle A_m - \beta, A_m - \beta\rangle - \langle A_m - \beta, \alpha\rangle\langle A_m - \beta, \alpha\rangle}$$

The objective function is given by

$$f(l_{\alpha,\beta}) = \sum_{m\in\mathcal{M}} w_m \sqrt{\langle A_m - \beta, A_m - \beta\rangle - \langle A_m - \beta, \alpha\rangle\langle A_m - \beta, \alpha\rangle}.$$

For the problem in the plane it has been shown by several authors (the earliest proof is in [15]) that with Euclidean distance there always exists an optimal line passing through two of the existing facilities. In [7] this statement was sharpened: For the Euclidean distance, *all* optimal lines pass through two of the existing facilities. Generalizations of this incidence property to other distances than the Euclidean can be found in [14]. With this background one might suspect that such an incidence property is also true for locating a line in three-dimensional space. But in the following counterexample *no* optimal line passes through two existing facilities, so the two-dimensional incidence property cannot be generalized.
Assume $M = 8$ existing facilities as the vertices of a cuboid, given by the following coordinates.

$$A_1 = (0,0,0), A_2 = (1,0,0), A_3 = (1,1,0), A_4 = (0,1,0),$$

$$A_5 = (0,0,e), A_6 = (1,0,e), A_7 = (1,1,e), A_8 = (0,1,e),$$

where $e > 0$.
Consider the line l_1 passing through the points $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2}, e)$. We get that $d(A_m, l_1) = \frac{1}{2}\sqrt{2}$ for all $m = 1, \ldots, 8$, such that

$$f(l_1) = 4\sqrt{2},$$

independent of e, when all weights are one.
We want to show that for large e the line l_1 is better than any line passing through two of the existing facilities. For the line $l_2 = l_{\alpha,\beta}$ with $\alpha = (1,1,e)$ and $\beta = (0,0,0)$, passing through A_1 and A_7 we get

$$\begin{aligned}
d(A_1, l_2) &= d(A_7, l_2) = 0, \\
d(A_2, l_2) &= d(A_4, l_2) = d(A_6, l_2) = d(A_8, l_2) = \sqrt{\frac{1+e^2}{2+e^2}}, \\
d(A_3, l_2) &= d(A_5, l_2) = \sqrt{\frac{2e^2}{2+e^2}}, \\
\Longrightarrow f(l_2) &= \frac{1}{\sqrt{2+e^2}}(4\sqrt{1+e^2} + 2e\sqrt{2}).
\end{aligned}$$

For $e \to \infty$ we get $f(l_2) \to 4 + 2\sqrt{2} > 4\sqrt{2} = f(l_1)$. The vertical and horizontal lines passing through two of the facilities are even worse, and the lines which are diagonals in one of the faces (as the line through A_2 and A_7) are also worse than l_2. This means that, for large enough e, the line l_1 is better than all lines passing through two of the existing facilities, so no such line is optimal.

Unfortunately, the objective function of $1l/\mathrm{IR}^3/\cdot/l_2/\sum$ is neither convex nor concave, so without extensive search we can only expect a local minumum. The following property for the Euclidean distance is helpful for developing an algorithm.

Lemma 11 *Let $l = l_{\alpha,\beta} \subset \mathrm{IR}^3$ be a line and $A \in \mathrm{IR}^3$ be a point. Then the shortest Euclidean path from A to l is a line segment orthogonal to l, i.e. it lies in a plane with normal vector α.*

This means, if the slope of the line $l_{\alpha,\beta}$ is already fixed (i.e. the vector α is given) then the problem reduces to a classical Weber problem in the plane orthogonal to $l_{\alpha,\beta}$. To use the results of Section 3 this problem can further be reduced to the location of a vertical line with respect to the Euclidean distance (by applying a rotation, such that l becomes a vertical line). The following heuristic method makes use of this property.

Algorithm 3 (for locating a line with shortest Euclidean distance)

Step 1. Choose an initial solution l^0, $g = 0$.

Step 2. Find a rotation r which maps l^g to a vertical line. Determine $\mathcal{A}^r = \{r(A) : A \in \mathcal{A}\}$.

Step 3. Determine l_r by solving the problem with respect to $\mathcal{A}^r$ using the horizontal Euclidean distance by Algorithm 2.

Calculate $l^{g+1} = r^{-1}(l_r)$ by retransforming l_r.

Step 4. If $f(l^g) - f(l^{g+1}) < \delta$, STOP;
else set $g = g + 1$ and return to step 2.

For a quicker solution, step 3 in Algorithm 3 may be replaced by

Step 3a. Let $l^g = l_{\alpha^g,\beta^g}$. Fix α^g and find the best starting point $\beta^{g+1} = (\beta_1^{g+1}, \beta_2^{g+1}, 0)$ for the vertical line l^g by using Lemma 2.

Step 3b. Fix β^{g+1} and optimize for α^{g+1} with respect to the horizontal Euclidean distance by using Lemma 9. Let $l^{g+1} = l_{\alpha^{g+1},\beta^{g+1}}$.

6.2 l_p distance

If we use a p-norm distance instead of the Euclidean distance, the property of Lemma 11 is in general not true.
To determine the distance between a point A_m and a line $l = l_{\alpha,\beta}$ we have to find λ_m^* such that $P_m = \lambda_m^* \alpha + \beta$ is the closest point on the line (by solving a one-dimensional minimization problem). We get

$$l_p(A_m, l) = \min_{P \in l} l_p(A_m, P) = l_p(A_m, \lambda_m^* \alpha + \beta).$$

The objective function

$$f(l_{\alpha,\beta}) = \sum_{m \in \mathcal{M}} w_m \left(\sum_{j=1}^{3} |a_{mj} - \alpha_j \lambda_m^* - \beta_j|^p\right)^{\frac{1}{p}}$$

is neither convex nor concave, but a local minimum may be found by the following scheme.

Algorithm 4 (for locating a line with shortest l_p distance)

Step 1. Choose an initial solution (α^0, β^0), compute the λ_m^* values and the objective function value $f(l^0_{\alpha^0,\beta^0})$, and set counter $g = 0$.

Step 2a. Holding α^g and the λ_m^* values fixed find the best starting point $\beta^{g+1} = (\beta_1^{g+1}, \beta_2^{g+1}, 0)$ for the line by the classical Weiszfeld algorithm for $1/\mathbb{R}^2/\cdot/l_p/\sum$.

Step 2b. Holding β^{g+1} and the λ_m^* values fixed perform Weiszfeld-type iterations on α until a stopping criterion is reached.

Denote the current solution by $(\alpha^{g+1}, \beta^{g+1})$.

Step 3. Compute $\lambda_m^*, m \in \mathcal{M}$ for the current solution.
If $f(l^g_{\alpha,\beta}) - f(l^{g+1}_{\alpha,\beta}) < \delta$, STOP;
else set $g = g + 1$ and return to step 2a.

In step 2a it turns out that the problem to find β_1^{g+1} and β_2^{g+1} reduces to a classical one facility problem in the plane with l_p distance where the existing facilities are given by

$$A'_m = (a_{m1} - \lambda_m^* \alpha_1, a_{m2} - \lambda_m^* \alpha_2) \in \mathbb{R}^2 \text{ for all } m \in \mathcal{M}.$$

Note however that a third fixed dimension (with $\beta_3 = 0$) must be included in the distance formula. It leads to a constant term within the formulation for $l_p(A_m, l)$ which otherwise has no effect on the minimization procedure.

In step 2b, on the other hand, we optimize for α_1, α_2, and α_3, and get the objective function

$$\sum_{m \in \mathcal{M}} \frac{w_m}{\lambda_m^*} (\sum_{j=1}^{3} |(a_{mj} - \beta_j)\lambda_m^* - \alpha_j|^p)^{\frac{1}{p}},$$

which is a Weber problem of type $1/\mathbb{R}^3/\cdot/l_p/\sum$ in $\mathbb{R}^3$.
The Weiszfeld iterations in both parts of step 2 result in a sequence of descent moves for the fixed values of $\lambda_m^*, m \in \mathcal{M}$. By updating the λ_m^* values in step 3 for the new line $l_{\alpha,\beta}^{g+1}$, we are replacing distances to the line by shortest distances, thereby providing a further improvement of the objective function. The iteration scheme thus converges to a stationary point. A multi-start version of Algorithm 4 with random initial solutions may be used to improve the likelihood of finding the global optimum.

6.3 Rectangular distance

In the special case of the rectangular distance l_1 (the classification of the problem is given by $1l/\mathbb{R}^3/\cdot/l_1/\sum$) we present the following formula for determining the distance between a point and a line in $\mathbb{R}^3$.

Lemma 12 *Let $A = (a_1, a_2, a_3) \in \mathbb{R}^3$ and let $l_{\alpha,\beta} \subset \mathbb{R}^3$ be a line defined by the parameters $\alpha, \beta \in \mathbb{R}^3$. Then*

$$l_1(A, l_{\alpha,\beta}) = \min \left\{ \sum_{j=1,2,3} \left| a_j - \frac{a_i - \beta_i}{\alpha_i}\alpha_j - \beta_j \right|, i = 1, 2, 3 \right\}$$

Proof:

$$\begin{aligned} l_1(A, l_{\alpha,\beta}) &= \min_{X \in l} l_1(A, X) \\ &= \min_{\lambda \in \mathbb{R}} l_1(A, \lambda\alpha + \beta) \\ &= \min_{\lambda \in \mathbb{R}} (|a_1 - \lambda\alpha_1 - \beta_1| + |a_2 - \lambda\alpha_2 - \beta_2| + |a_3 - \lambda\alpha_3 - \beta_3|) \\ &= \min_{\lambda \in \mathbb{R}} \sum_{j=1,2,3} |\alpha_j| \left| \frac{a_j - \beta_j}{\alpha_j} - \lambda \right| \end{aligned}$$

Since this is a weighted median problem there exists $i \in \{1, 2, 3\}$ such that

$$\lambda = \frac{a_i - \beta_i}{\alpha_i}$$

is optimal. Defining

$$P_i = \frac{a_i - \beta_i}{\alpha_i}\alpha + \beta \in \mathbb{R}^3, \ i = 1, 2, 3,$$

the distance between A and $l_{\alpha,\beta}$ is given by

$$l_1(A, l_{\alpha,\beta}) = \min\{l_1(A, P_1), l_1(A, P_2), l_1(A, P_3)\},$$

which proves the result. QED

Note that one shortest rectangular path from the point A to the line l in the three-dimensional space always keeps within one plane (since P_i and A share the same coordinate i). In particular, if the index i for the optimal λ in the proof of Lemma 12 is given by $i = 3$ then the path from A to l stays completely in the horizontal plane passing through A. Analogously, if $i = 1, 2$ the path lies completely in a plane parallel to the yz-plane or parallel to the xz-plane, respectively. Unfortunately, the choice of the index i for λ is not only dependent on the parameters of the line (as in the two-dimensional case), but also on the position of the point A, so the property of Lemma 11 does not hold. To solve problems of type $1l/\mathbb{R}^3/\cdot/l_1/\sum$ one may use a local search to find a local minimum as for the p-norm case, but steps 2a and 2b of Algorithm 4 can be combined to run in linear time, as the following approach shows.

Algorithm 5 (for locating a line with shortest rectangular distance)

Step 1. Choose an initial solution (α^0, β^0), compute the λ_m^* values and the objective function value $f(l^0_{\alpha^0,\beta^0})$, and set counter $g = 0$.

Step 2. Holding the λ_m^* values fixed optimize for α and β. Denote the solution by $l^{g+1} = l_{\alpha^{g+1},\beta^{g+1}}$.

Step 3. Compute $\lambda_m^*, m \in \mathcal{M}$ for l^{g+1}
If $f(l^g_{\alpha,\beta}) - f(l^{g+1}_{\alpha,\beta}) < \delta$, STOP;
else set $g = g + 1$ and return to step 2.

The minimization problem of step 2 is given by

$$\min_{\alpha,\beta} \sum_{m \in \mathcal{M}} w_m \left(|a_{m1} - \lambda_m^* \alpha_1 - \beta_1| + |a_{m2} - \lambda_m^* \alpha_2 - \beta_2| + |a_{m3} - \lambda_m^* \alpha_3 - \beta_3| \right).$$

It can be separated into three independent subproblems $P_k, k = 1, 2, 3$, each being a line location problem in the plane of type $1l/\mathbb{R}^2/\cdot/d_{ver}/\sum$, where the existing facilities in subproblem P_k are given by

$$A'_m = (\lambda_m^*, a_{mk}) \text{ for all } m \in \mathcal{M},$$

the weights are given by the original weights w_m, and the optimal solution yields a line with slope α_k^* and intercept β_k^*. All three subproblems can be solved in linear time by linear programming [16].

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