Polyhedral Properties of the Uncapacitated Multiple Allocation Hub Location Problem

Horst W. Hamacher *† Martine Labbé ‡ Stefan Nickel * Tim Sonneborn *§

June 21, 2000

Abstract

We examine the feasibility polyhedron of the uncapacitated hub location problem (UHL) with multiple allocation, which has applications in the fields of air passenger and cargo transportation, telecommunication and postal delivery services. In particular we determine the dimension and derive some classes of facets of this polyhedron. We develop some general rules about lifting facets from the uncapacitated facility location (UFL) for UHL and projecting facets from UHL to UFL. By applying these rules we get a new class of facets for UHL which dominates the inequalities in the original formulation. Thus we get a new formulation of UHL whose constraints are all facet–defining. We show its superior computational performance by benchmarking it on a well known data set.

Keywords:

integer programming, hub location, facility location, valid inequalities, facets, branch–and–cut.

*Institut für Techno– und Wirtschaftsmathematik (ITWM), Kaiserslautern, Germany
†Fachbereich Mathematik, Universität Kaiserslautern, Germany
‡Service de Mathématiques de la Gestion (SLG), Université Libre de Bruxelles, Belgium
§correspondig author, email: sonneborn@itwm.uni-kl.de

1 Introduction

The Uncapacitated Hub Location Problem (UHL) with multiple allocation involves so called transshipment or *hub* nodes, which have the function to collect commodities from their origin, transfer them to other hubs and distribute them to their final destination. The problem is to locate the hub nodes and to route the commodities through the hubs. As we allow multiple allocation, commodities having the same origin (or destination) may be allocated to different hubs (see Figure 1). The objective is to minimize the total costs, which consist of transportation costs per unit and fixed charge costs for establishing hubs at nodes, under the constraint that all commodities have to be routed via one or two hub nodes .

During the last years, different kinds of *hub location problems* have been discussed in the literature (for an overview of some basic problems see [Campbell, 1994]). Main applications of hub location problems concern air passenger and cargo transportation, telecommunication and postal delivery services. The main types of problems which are dealt with are p–hub location, where the number of hubs to be located is fixed to p (see e.g. [Skorin-Kapov et al., 1996]), and *fixed charge* hub location problems, where this number is unlimited, but a certain fixed cost has to be paid for establishing a hub facility (see e.g. [Ebery et al., 1998], [Campbell, 1994]). Furthermore one distinguishes between single allocation (see e.g. [Ernst and Krishnamoorthy, 1996], [O'Kelly, 1987]) and multiple allocation (see e.g. [Ernst and Krishnamoorthy, 1998]) problems. In the single allocation case all commodities having the same origin (or destination, respectively) must be allocated to the same first (or second, respectively) hub, while in multiple allocation they can be allocated to different hubs.

Very little is known about the polyhedral aspect of hub location problems. For the single allocation problem with two fixed hub locations, the allocation part can be written as a linear program and therefore solved in polynomial time [Sohn and Park, 1997], while in case of three fixed hub locations, the allocation part is NP–hard and some facets of the feasibility polytope were computed [Sohn and Park, 1996].

UHL is NP–hard because it generalizes the One– and Two–Level Uncapacitated Facility Location Problems (UFL and TUFL) which are known to be NP–hard (see e.g. [Cornuéjols and Thizy, 1982], [Aardal et al., 1996]).

The remainder of this paper is organized as follows: in Section 2 we will present the mixed

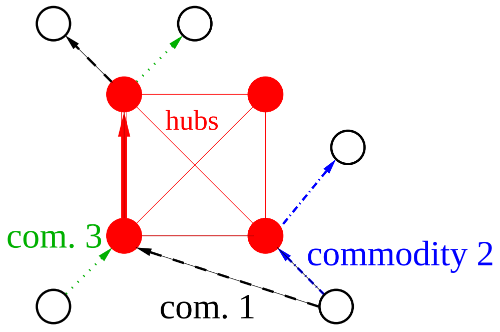

Figure 1: Example for a hub network (multiple allocation)

integer formulation of UHL.

In the following Section 3 we compare UHL with the uncapacitated facility location problem. We determine the dimension of the feasibility polytope of UHL and develop a general rule how facets from UFL can be lifted to obtain new facets of UHL. These new facets can be used for a tighter and more compact formulation of UHL, which is also presented in Section 3.

In addition we consider the other direction and show how facets from UHL can be projected to UFL in Section 4.

We show the efficiency of a new facet–based UHL formulation by benchmarking it on a well–known data set and comparing its performance with different other formulations in Section 5. Finally we give some conclusions in Section 6.

2 Mixed Integer Formulation of UHL

Let $\mathcal{K}$ be a set of commodities and $\mathcal{H}$ be a set of potential hub nodes. For every commodity $k \in \mathcal{K}$ and every ordered pair of hubs $(i, j) \in \mathcal{H} \times \mathcal{H}$ let C_{ijk} denote the transportation costs for routing commodity k via hubs i and j (in this direction). Moreover, F_j represents the fixed costs for establishing node j ($j \in \mathcal{H}$) as a hub node.

Let Y_j ($j \in \mathcal{H}$) be equal to 1, if node j is established as a hub node and 0 otherwise; and let $X_{ijk} \geq 0$ ($i, j \in \mathcal{H}, k \in \mathcal{K}$) determine the fraction of commodity k which is routed via first hub node i and second hub node j.

We want to determine which hub nodes should be opened and to which hubs each commodity should be assigned such that the total costs are minimized under the constraint that all commodities have to be routed via one or two hubs.

The Uncapacitated Hub Location Problem with multiple allocation can be modeled as the following mixed–integer linear program [Skorin-Kapov et al., 1996]:
(UHL)

$$
\begin{aligned}
\min \quad & \sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} C_{ijk}\, X_{ijk} + \sum_{j\in\mathcal{H}} F_j\, Y_j && \\
\text{s.t.} \quad & \sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}} X_{ijk} = 1 \quad && \text{for all } k\in\mathcal{K}, \qquad (1)\\
& \sum_{j\in\mathcal{H}} X_{ijk} \le Y_i && \text{for all } i\in\mathcal{H}, k\in\mathcal{K}, \qquad (2)\\
& \sum_{i\in\mathcal{H}} X_{ijk} \le Y_j && \text{for all } j\in\mathcal{H}, k\in\mathcal{K}, \qquad (3)\\
& X_{ijk} \ge 0 && \text{for all } i,j\in\mathcal{H}, k\in\mathcal{K}, \qquad (4)\\
& 0 \le Y_j \le 1 && \text{for all } j\in\mathcal{H}, \qquad (5)\\
& Y_j \in \mathbb{Z} && \text{for all } j\in\mathcal{H}. \qquad (6)
\end{aligned}
$$

In the objective function we minimize the total (variable plus fixed) costs. All flow of every commodity k has to be routed via one or two nodes i and j (1), but only if i and j are hub nodes ((2) and (3)).

We note that there always exists an optimal solution of UHL in which all X_{ijk} variables are integer–valued because there are no capacity constraints on the hubs.

Let $q := |\mathcal{K}|$ and $n := |\mathcal{H}|$. UHL involves $n^2 q + n$ variables, n of them are binary. There are $(2n+1)q$ linear constraints to be satisfied.

For sake of simplicity let $X := (X_{ijk})_{i,j\in\mathcal{H},k\in\mathcal{K}}$ and $Y := (Y_j)_{j\in\mathcal{H}}$.

Furthermore let $\mathcal{X}_{UHL}$ be the set of feasible solutions of UHL, that is $\mathcal{X}_{UHL} := \{(X,Y) \in \mathbb{R}^{n^2q+n} : (X,Y)\,\text{satisfies (1) — (6)}\}$, $\mathcal{Z}_{UHL}$ be the set of feasible integral points of UHL, that is $\mathcal{Z}_{UHL} := \{(X,Y) \in \mathcal{X}_{UHL} : X_{ijk} \in \{0,1\} \text{ for all } i,j\in\mathcal{H}, k\in\mathcal{K}\}$, and let $\mathcal{P}_{UHL}$ be the polyhedron obtained by the convex hull of $\mathcal{Z}_{UHL}$, that is $\mathcal{P}_{UHL} := \operatorname{conv}(\mathcal{Z}_{UHL})$.

3 Lifting facets from UFL to UHL

The (One–Level) Uncapacitated Facility Location Problem (UFL) (see e.g. [Cho et al., 1983a], [Cho et al., 1983b], [Cornuéjols et al., 1990], [Guignard, 1980]) can be modeled as the following linear mixed integer program:
(UFL)

$$\begin{aligned}
\min \quad & \sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} c_{jk}\, x_{jk} + \sum_{j\in\mathcal{H}} f_j\, y_j \\
\text{s.t.} \quad & \sum_{j\in\mathcal{H}} x_{jk} = 1 \quad \text{for all } k\in\mathcal{K}, & (7)\\
& x_{jk} \le y_j \quad \text{for all } j\in\mathcal{H}, k\in\mathcal{K}, & (8)\\
& x_{jk} \ge 0 \quad \text{for all } j\in\mathcal{H}, k\in\mathcal{K}, & (9)\\
& 0 \le y_j \le 1 \quad \text{for all } j\in\mathcal{H}, & (10)\\
& y_j \in \mathbb{Z} \quad \text{for all } j\in\mathcal{H}, & (11)
\end{aligned}$$

where c_{jk} are the transportation costs for facility j to serve customer k, f_j are the fixed costs for establishing a facility at node j, x_{jk} is the fraction of client k's demand served by facility $j \in \mathcal{H}$; and where $y_j = 1$ if facility j is open, and $y_j = 0$ otherwise. As before, we assume that n and q are both greater or equal to 2.

Let $x := (x_{jk})_{j\in\mathcal{H},k\in\mathcal{K}}$, $y := (y_j)_{j\in\mathcal{H}}$, $\mathcal{X}_{UFL} := \{(x,y) \in \mathbb{R}^{nq+n} : (x,y) \text{ satisfies } (7) — (11)\}$, $\mathcal{Z}_{UFL} := \{(x,y) \in \mathcal{X}_{UFL} : x_{jk} \in \{0,1\} \text{ for all } j\in\mathcal{H}, k\in\mathcal{K}\}$ and $\mathcal{P}_{UFL} := \operatorname{conv}(\mathcal{Z}_{UFL})$.

The dimension of $\mathcal{P}_{UFL}$ can be derived straightforwardly by showing that the k equality constraints in (7) are linearly independent and every other equality satisfied by all points in $\mathcal{P}_{UFL}$ is a linear combination of equalities of (7) (see [Cornuéjols et al., 1990]).

Proposition 3.1 *The dimension of $\mathcal{P}_{UFL}$ is $dim \mathcal{P}_{UFL} = nq + n - q$.*

We define a function $\sigma : \mathcal{P}_{UFL} \to \mathcal{P}_{UHL}$ by

$$\begin{aligned}
Y_j &:= y_j \text{ for all } j\in\mathcal{H},\\
X_{jjk} &:= x_{jk} \text{ for all } j\in\mathcal{H}, k\in\mathcal{K},\\
X_{ijk} &:= 0 \text{ for all } i\in\mathcal{H}, j\in\mathcal{H} : i\neq j, k\in\mathcal{K}.
\end{aligned}$$

for all $(x, y) \in \mathcal{P}_{UFL}$ and denote by σ-UFL the following mixed integer program:

$$\begin{aligned} \min \quad & \sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} C_{ijk}\,X_{ijk} + \sum_{j\in\mathcal{H}} F_j\,Y_j \\ \text{s.t.}\ \ & (X, Y) \in \mathcal{X}_{UHL} \text{ and} \\ & X_{ijk} = 0 \text{ for all } i \in \mathcal{H}, j \in \mathcal{H} : i \neq j, k \in \mathcal{K}. \end{aligned} \tag{12}$$

We choose $c_{jk} := C_{jjk}$ and $f_j := F_j$ for all $j \in \mathcal{H}, k \in \mathcal{K}$ as data in UFL. Then σ-UFL is equivalent to UFL in the sense that (x, y) is a feasible (or optimal, respectively) solution of UFL if and only if $\sigma(x, y)$ is a feasible (or optimal, respectively) solution of σ-UFL.

We define $\mathcal{X}_{\sigma-UFL}$, $\mathcal{Z}_{\sigma-UFL}$ and $\mathcal{P}_{\sigma-UFL}$ analogously as for UFL. Clearly $\sigma(\mathcal{P}_{UFL}) = \mathcal{P}_{\sigma-UFL}$ and $\dim \mathcal{P}_{\sigma-UFL} = \dim \mathcal{P}_{UFL}$. Furthermore $\mathcal{F}_{UFL}$ is a p–dimensional face of $\mathcal{P}_{UFL}$ if and only if $\mathcal{F}_{\sigma-UFL}$ is a p–dimensional face of $\mathcal{P}_{\sigma-UFL}$. (for $0 \leq p \leq nq + n - q$).

We will use the σ-UFL formulation in the remainder of the paper whenever it is helpful.

As $\mathcal{P}_{\sigma-UFL} \subseteq \mathcal{P}_{UHL}$ we have the following result (see Figure 2).

Proposition 3.2 *UHL is a relaxation of σ-UFL. In particular every valid inequality for $\mathcal{P}_{UHL}$ is also valid for $\mathcal{P}_{\sigma-UFL}$.* □

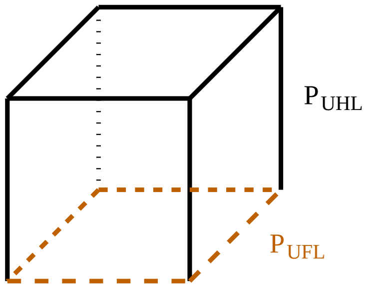

Figure 2: The polyhedra $\mathcal{P}_{UHL}$ and $\mathcal{P}_{UFL}$.

By means of $\mathcal{P}_{\sigma-UFL}$ we can derive the dimension of the polytope $\mathcal{P}_{UHL}$.

Theorem 3.3 *The dimension of the polytope $\mathcal{P}_{UHL}$ is $\dim \mathcal{P}_{UHL} = n^2q + n - q$.*

Proof: We have to show that there are $n^2q+n-q+1$ affinely independent points lying on $\mathcal{P}_{UHL}$. First, by Proposition 3.1 we have $nq+n-q+1$ affinely independent points on the polytope $\mathcal{P}_{\sigma-UFL}$. In every of these vectors of $\mathcal{P}_{\sigma-UFL}$ all entries of the form X_{ijk} $(k \in \mathcal{K}, i \in \mathcal{H}, j \in \mathcal{H} : i \neq j)$ are zero.

Then, for every $k' \in \mathcal{K}$ and every $i' \in \mathcal{H}, j' \in \mathcal{H}$ with $i' \neq j'$ we define a point (X, Y) on $\mathcal{P}_{UHL}$ with $Y_{i'} = Y_{j'} = 1$, $X_{i'j'k'} = 1$ and $X_{i'i'k} = 1$ for all $k \neq k'$, all other values equal to zero. In every of those vectors there is exactly one of the entries of the form X_{ijk} $(k \in \mathcal{K}, i \in \mathcal{H}, j \in \mathcal{H} : i \neq j)$ not equal to zero, so these $(n^2-n)q$ points are affinely independent.

As all X_{ijk} entries are zero for $i \neq j$ in the $nq+n-q+1$ points of $\mathcal{P}_{\sigma-UFL}$ defined first, in total we have $nq+n-q+1+(n^2-n)q = n^2q+n-q+1$ affinely independent points on $\mathcal{P}_{UHL}$. $\square$

Now we develop a general rule for lifting facets from UFL to UHL because for UFL many classes of facets are known (see e.g. [Cho et al., 1983a], [Cho et al., 1983b],[Cornuéjols and Thizy, 1982]).

Theorem 3.4 *Let $a_{jk}, b_j, d \in \mathbb{R}$ for all $j \in \mathcal{H}, k \in \mathcal{K}$ such that for all $\{i,j\}(i \in \mathcal{H}, j \in \mathcal{H})$ there exists a $k' \in \mathcal{K}$ with $b_j \geq \min\{a_{ik'} - a_{jk'}, 0\}$ (*), and let*

$$\sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{K}} a_{jk} x_{jk} + \sum_{j \in \mathcal{H}} b_j y_j \leq d \tag{13}$$

represent a facet of $\mathcal{P}_{UFL}$ that is not a non–negativity constraint for some x_{jk}. Then

$$\sum_{i \in \mathcal{H}} \sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{K}} \max\{a_{ik}, a_{jk}\} X_{ijk} + \sum_{j \in \mathcal{H}} b_j Y_j \leq d \tag{14}$$

represents a facet of $\mathcal{P}_{UHL}$.

Proof: First we verify the validity of (14). Assume (14) is not valid. This means there exists an $(\overline{X}, \overline{Y}) \in \mathcal{Z}_{UHL}$ with

$$\sum_{i \in \mathcal{H}} \sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{K}} \max\{a_{ik}, a_{jk}\} \overline{X}_{ijk} + \sum_{j \in \mathcal{H}} b_j \overline{Y_j} > d.$$

For every $k \in \mathcal{K}$ there is exactly one $(i_k, j_k) \in \mathcal{H}^2$ with $\overline{X}_{i_k j_k k} = 1$. By (2) and (3) we have $Y_{i_k} = Y_{j_k} = 1$, such that the following solution $(\overline{x}, \overline{y})$ is feasible for UFL:

$$\begin{aligned}
\overline{y}_i &:= \overline{Y}_i && \text{for all } i \in \mathcal{H}, \\
\overline{x}_{i_k k} &:= 1, \text{ if } a_{i_k k} \geq a_{j_k k}, && \text{for all } k \in \mathcal{K}, \\
\overline{x}_{j_k k} &:= 1, \text{ if } a_{i_k k} < a_{j_k k}, && \text{for all } k \in \mathcal{K}, \\
\text{and } \overline{x}_{ik} &:= 0 && \text{for all other } i \in \mathcal{H}, k \in \mathcal{K}.
\end{aligned}$$

If we evaluate (13) in $(\overline{x}, \overline{y})$, we get

$$\sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{K}} a_{jk} \overline{x}_{jk} + \sum_{j \in \mathcal{H}} b_j \overline{y}_j = \sum_{i \in \mathcal{H}} \sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{K}} \max\{a_{ik}, a_{jk}\} \overline{X}_{ijk} + \sum_{j \in \mathcal{H}} b_j \overline{Y}_j > d$$

so (13) is not valid for $\mathcal{P}_{UFL}$, which is a contradiction.

We note that the assumption (*) is not required for the validity part.

To show that (14) is facet–defining we have to show that there are $\dim \mathcal{P}_{UHL} = n^2 q + n - q$ affinely independent points of $\mathcal{P}_{UHL}$ lying on the face

$$\mathcal{F}_{UHL} := \{(X, Y) \in \mathcal{P}_{UHL} : \sum_{i \in \mathcal{H}} \sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{K}} \max\{a_{ik}, a_{jk}\} X_{ijk} + \sum_{j \in \mathcal{H}} b_j Y_j = d\}$$

We get $\dim \mathcal{P}_{UFL} = nq + n - q$ affinely independent points (X, Y) on $\mathcal{F}_{UHL}$ by taking $nq + n - q$ affinely independent points on

$$\mathcal{F}_{\sigma-UFL} := \{(X, Y) \in \mathcal{P}_{\sigma-UFL} : \sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{K}} a_{jk} X_{jjk} + \sum_{j \in \mathcal{H}} b_j Y_j = d\}$$

Since (14) is not a non–negativity constraint and $\mathcal{F}_{\sigma-UFL}$ is a facet of $\mathcal{P}_{\sigma-UFL}$, for every $i \in \mathcal{H}, k \in \mathcal{K}$ there is a point on $\mathcal{F}_{\sigma-UFL}$ with $X_{iik} = 1$.

Now we will define another $(n^2 - n)q$ affinely independent points on $\mathcal{F}_{UHL}$. For every $i', j' \in \mathcal{H} : i' \neq j'$ and $k' \in \mathcal{K}$ we will define a point P_1 on $\mathcal{F}_{UHL}$ with $X_{i'j'k'} = 1$ and $X_{ijk'} = 0$ for all $(i, j) \neq (i', j')$.

According to the assumptions of this theorem w.l.o.g. there exists a $k'' \in \mathcal{K}$ such that $b_{j'} \geq \min\{a_{i'k''} - a_{j'k''}, 0\}$. Let P_0 describe a point $(\overline{X}, \overline{Y}) \in \mathcal{F}_{UHL}$ with $\overline{Y}_{i'} = 1$ and $\overline{X}_{i'i'k'} = 1$.

case (a): $b_{j'} \geq \min\{a_{i'k'} - a_{j'k'}, 0\}$:

If $a_{i'k'} \le a_{j'k'}$, then $b_{j'} \ge a_{i'k'} - a_{j'k'}$.
If $a_{i'k'} \ge a_{j'k'}$, then $b_{j'} \ge 0$.

In both cases the following point $P_1 = (X, Y) \in \mathcal{P}_{UHL}$ with

$$\begin{aligned} Y_{i'} = Y_{j'} &:= 1, \\ X_{i'j'k'} &:= 1, \\ X_{i'i'k'} &:= 0, \\ \text{all other values as in } P_0& \end{aligned}$$

satisfies

$$\begin{aligned} &\sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} \max\{a_{ik}, a_{jk}\} X_{ijk} + \sum_{j\in\mathcal{H}} b_j Y_j \\ = &\sum_{\substack{i\in\mathcal{H},k\in\mathcal{K}: \\ (i,k)\neq(i',k')}} a_{ik} X_{iik} + \max\{a_{i'k'}, a_{j'k'}\} X_{i'j'k'} + \sum_{\substack{j\in\mathcal{H}: \\ j\neq j'}} b_j Y_j + b_{j'} Y_{j'} \\ \ge &\sum_{\substack{i\in\mathcal{H},k\in\mathcal{K}: \\ (i,k)\neq(i',k')}} a_{ik} \overline{X}_{iik} + a_{i'k'} \overline{X}_{i'i'k'} + \sum_{\substack{j\in\mathcal{H}: \\ j\neq j'}} b_j \overline{Y}_j + b_{j'} \overline{Y}_{j'} = d \end{aligned}$$

and, since (14) is a valid inequality, $P_1 \in \mathcal{F}_{UHL}$.

case (b): There is a $k'' \neq k' \in \mathcal{K}$ with $\min\{a_{i'k'} - a_{j'k'}, 0\} \ge b_{j'} \ge \min\{a_{i'k''} - a_{j'k''}, 0\}$.

Then we can define a point $P_1 = (X, Y) \in \mathcal{F}_{UHL}$ as in (a) with

$$\begin{aligned} Y_{i'} = Y_{j'} &:= 1, \\ X_{i'j'k''} &:= 1, \\ X_{i'i'k''} &:= 0, \\ \text{all other values as in } P_0.& \end{aligned}$$

From this point we define a point $P_2 = (\widetilde{X}, \widetilde{Y}) \in \mathcal{P}_{UHL}$ with

$$\begin{aligned} \widetilde{Y}_{i'} = \widetilde{Y}_{j'} &:= 1, \\ \widetilde{X}_{i'j'k'} = \widetilde{X}_{i'j'k''} &:= 1, \\ \widetilde{X}_{i'i'k'} = \widetilde{X}_{i'i'k''} &:= 0, \\ \text{all other values as in } P_1.& \end{aligned}$$

P_2 satisfies again

$$\sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}}\max\{a_{ik},a_{jk}\}\widetilde{X}_{ijk}+\sum_{j\in\mathcal{H}}b_j\widetilde{Y}_j\geq\sum_{i\in\mathcal{H}}\sum_{k\in\mathcal{K}}a_{iik}\overline{X}_{iik}+\sum_{j\in\mathcal{H}}b_j\overline{Y}_j=d,$$

and, since (14) is a valid inequality, $P_2\in\mathcal{F}_{UHL}$.

All points defined so far are affinely independent because they can be written as rows of a $(n^2q+n-q)\times(n^2q+n)$ matrix in the following way: The first $(nq+n-q)$ rows are those affinely independent points of $\mathcal{F}_{UHL}$ with $X_{ijk}=0$ for all $i\neq j$. The second part of rows of the matrix corresponds to the points defined in case (a), while the last part of rows describes the points defined in case (b). Then the columns of the matrix corresponding to X_{ijk} for $i\neq j$ form a lower triangle matrix. $\square$

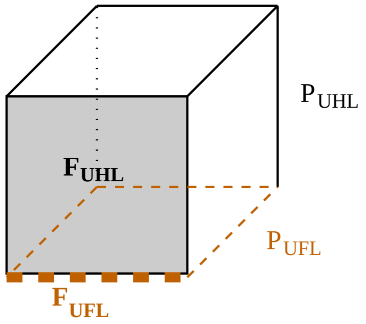

Figure 3: Facet lifting from $\mathcal{P}_{UFL}$ to $\mathcal{P}_{UHL}$

Corollary 3.5 *The following inequalities are valid for $\mathcal{X}_{UHL}$ and define facets of $\mathcal{P}_{UHL}$:*

$$X_{ijk}\geq 0 \qquad \text{for all } i,j\in\mathcal{H}:i\neq j,k\in\mathcal{K}, \tag{15}$$

$$X_{iik}\geq 0 \qquad \text{for all } i\in\mathcal{H},k\in\mathcal{K},\ \text{if } n\geq 3, \tag{16}$$

$$Y_j\leq 1 \qquad \text{for all } j\in\mathcal{H}, \tag{17}$$

$$\sum_{i\in\mathcal{H}}X_{ijk}+\sum_{i\in\mathcal{H}\setminus\{j\}}X_{jik}\leq Y_j \qquad \text{for all } j\in\mathcal{H},k\in\mathcal{K}. \tag{18}$$

Proof: The proofs of (15) and (16) are straightforward and therefore omitted. (17) and (18) are applications of Theorem 3.4 to the inequalities $y_j\leq 1$ and $x_{jk}\leq y_j$ of UFL, which are facet–defining for $\mathcal{P}_{UFL}$ (see [Cornuéjols et al., 1990]). $\square$

Lemma 3.6 *Let $q \geq n$ and let $\mathcal{S} = \{k(1), k(2), \ldots, k(n)\} \subseteq \mathcal{K}$ be a set of n different commodities. Then*

$$\sum_{j \in \mathcal{H}} (Y_j + X_{jjk(j)}) \geq 2 \tag{19}$$

is valid for $\mathcal{X}_{UHL}$.

Proof: If $Y_{j_1} = Y_{j_2} = 1$ for some $j_1 \neq j_2$, (19) follows immediately.
If $Y_{j_1} = 1$ for one $j_1 \in \mathcal{H}$ and all other $Y_j = 0$, it follows that $X_{j_1 j_1 k} = 1$ for all $k \in \mathcal{K}$ and especially $X_{j_1 j_1 k(j_1)} = 1$, so that (19) is implied. □

Corollary 3.7 *If $q \geq n \geq 3$, then (19) defines a facet of $\mathcal{P}_{UHL}$.*

Proof: This is an application of Theorem 3.4 to the inequalities $\sum_{j \in \mathcal{H}} (y_j + x_{jk(j)}) \geq 2$, which are facet–defining for $\mathcal{P}_{UFL}$ (see [Aardal et al., 1996]). □

At the end of this section we note that the inequalities of type (2) and (3) do not define facets of $\mathcal{P}_{UHL}$ because they are dominated by the inequalities of (18). Thus a replacement of (2) and (3) by (18) provides a better formulation of UHL.

A more compact formulation which makes use of (18) is obtained by using a single index e for every subset of $\mathcal{H}$ containing one or two hubs. To this purpose, let $\mathcal{E} := \{\mathcal{S} \subseteq \mathcal{H} : 1 \leq |\mathcal{S}| \leq 2\}$. We define some modified transportation costs $\widetilde{C}_{ek}$ as $\widetilde{C}_{ek} := \min\{C_{ijk}, C_{jik}\}$ if $e = \{i, j\}$, and $\widetilde{C}_{ek} := C_{iik}$ if $e = \{i\}$, for all $e \in \mathcal{E}$ and $k \in \mathcal{K}$. Then we can formulate UHL as the following equivalent mixed integer program, which is stronger than UHL and the similar formulations given in [Klincewicz, 1996] and [Krispin and Wagner, 1998].

(FACET-UHL)

$$\begin{aligned}
\min \quad & \sum_{e \in \mathcal{E}} \sum_{k \in \mathcal{K}} \widetilde{C}_{ek} X_{ek} + \sum_{j \in \mathcal{H}} F_j Y_j, && \\
\text{s.t.} \quad & \sum_{e \in \mathcal{E}} X_{ek} = 1 \quad && \text{for all } k \in \mathcal{K}, \quad (20)\\
& \sum_{e \in \mathcal{E} : e \ni j} X_{ek} \leq Y_j \quad && \text{for all } j \in \mathcal{H}, k \in \mathcal{K}, \quad (21)\\
& X_{ek} \geq 0 \quad && \text{for all } e \in \mathcal{E}, k \in \mathcal{K}, \quad (22)\\
& Y_j \leq 1 \quad && \text{for all } j \in \mathcal{H}, \quad (23)\\
& Y_j \in \mathbb{Z} \quad && \text{for all } j \in \mathcal{H}. \quad (24)
\end{aligned}$$

Here (21) corresponds to (18). We show that in case that there are only two potential hub nodes, it is sufficient to solve the LP relaxation of FACET-UHL.

Theorem 3.8 *If $n = 2$, the polyhedron of the LP relaxation of FACET-UHL, described by* (20) — (23)*, has only integer vertices.*

Proof: Let $\mathcal{H} := \{1, 2\}$. We define $Z_{1k} := X_{\{1\}k} + X_{\{1,2\}k}$ and $Z_{2k} := X_{\{2\}k} + X_{\{1,2\}k}$ for all $k \in \mathcal{K}$. Then by (20) we have $X_{\{1\}k} = 1 - Z_{2k}$, $X_{\{2\}k} = 1 - Z_{1k}$ and $X_{\{1,2\}k} = Z_{1k} + Z_{2k} - 1$. In this sense the polyhedron of the LP relaxation described by (20) — (23) is equivalent to the polyhedron obtained by following inequalities:

$$Z_{1k} + Z_{2k} \geq 1 \quad \text{for all } k \in \mathcal{K}, \tag{25}$$

$$Z_{jk} \leq Y_j \quad \text{for all } j \in \{1, 2\}, k \in \mathcal{K}, \tag{26}$$

$$Z_{jk} \geq 0 \quad \text{for all } j \in \{1, 2\}, \tag{27}$$

$$Y_j \leq 1 \quad \text{for all } j \in \{1, 2\}. \tag{28}$$

(25) — (28) describe an instance of a full–dimensional UFL with two facilities.
It has been shown e.g. by [Cho et al., 1983a] that in this case the corresponding constraint matrix is totally unimodular, so all the extreme points of the LP relaxation polyhedron are integral. □

Another advantage of FACET-UHL is its smaller number of X_{ijk} variables. We compare the computational performance of FACET-UHL and UHL in Section 5.

4 Projecting facets from UHL to UFL

In this section we show that some facets of the polytope of UHL also induce facets for UFL. In our first statement we show that a certain facet–defining inequality for $\mathcal{P}_{UHL}$ can be projected to a facet–defining inequality for $\mathcal{P}_{UFL}$ by means of the formulation σ-UFL.

Theorem 4.1 *Let $a_{ijk}, b_j, d \in \mathbb{R}$ for all $i, j \in \mathcal{H}, k \in \mathcal{K}$ such that*
$a_{ijk} \leq \max\{a_{iik}, a_{jjk}\}$ for all $i, j \in \mathcal{H}, k \in \mathcal{K}$, (**)
and let

$$\sum_{i \in \mathcal{H}} \sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{K}} a_{ijk} X_{ijk} + \sum_{j \in \mathcal{H}} b_j Y_j \leq d, \tag{29}$$

represent a facet of $\mathcal{P}_{UHL}$ that is not a non–negativity constraint for some X_{ijk}. Then

$$\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{jjk}x_{jk} + \sum_{j\in\mathcal{H}} b_j y_j \leq d \tag{30}$$

represents a facet for $\mathcal{P}_{UFL}$.

Proof: First we prove the validity of (30). Assume that (30) is not valid. Then there exists an $(\overline{x},\overline{y}) \in \mathcal{Z}_{UFL}$ with $\sum\limits_{j\in\mathcal{H}}\sum\limits_{k\in\mathcal{H}} a_{jjk}\overline{x}_{jk} + \sum\limits_{j\in\mathcal{H}} b_j\overline{y}_j > d$.

For every $k \in \mathcal{K}$ there is exactly one $j_k \in \mathcal{H}$ with $\overline{x}_{j_k k} = 1$.

Now we can define a feasible solution $(\overline{X},\overline{Y})$ of UHL with

$$\begin{aligned}
\overline{X}_{j_k j_k k} &= 1 \quad \text{for all } k \in \mathcal{K},\\
\overline{X}_{ijk} &= 0 \quad \text{for all } (i,j) \neq (j_k, j_k), k \in \mathcal{K},\\
\overline{Y}_j &= \overline{y}_j \text{ for all } j \in \mathcal{H}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{H}} a_{ijk}\overline{X}_{ijk} + \sum_{j\in\mathcal{H}} b_j\overline{Y}_j\\
=&\sum_{k\in\mathcal{K}} a_{j_k j_k k}\overline{X}_{j_k j_k k} + \sum_{j\in\mathcal{H}} b_j\overline{Y}_j\\
=&\sum_{k\in\mathcal{K}} a_{j_k j_k k}\overline{x}_{j_k k} + \sum_{j\in\mathcal{H}} b_j\overline{y}_j\\
=&\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{jjk}\overline{x}_{jk} + \sum_{j\in\mathcal{H}} b_j\overline{y}_j > d,
\end{aligned}$$

so (29) is not valid for $\mathcal{P}_{UHL}$, which is a contradiction.

We note that the assumption (**) is not required for the validity part.

To show that (30) is facet-defining we have to show that there are $dim\mathcal{P}_{UFL} = nq + n - q$ affinely independent points on

$$\mathcal{F}_{\sigma-UFL} := \{(X,Y) \in \mathcal{P}_{\sigma-UFL} : \sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{jjk}X_{jjk} + \sum_{j\in\mathcal{H}} b_j Y_j = d\}.$$

We have $dim\mathcal{P}_{UHL} = n^2q + n - q$ affinely independent points on

$$\mathcal{F}_{UHL} := \{(X,Y) \in \mathcal{P}_{UHL} : \sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{ijk}X_{ijk} + \sum_{j\in\mathcal{H}} b_j Y_j = d\}$$

If we write these points as rows of an $(n^2q+n-q)\times(n^2q+n)$ matrix A, the definition of affine independence implies that the $(n^2q+n-q)\times(n^2q+n+1)$ matrix $(A|1)$ has $rank(A|1)=n^2q+n-q$.

Let $(A'|1)$ be the matrix which is obtained from $(A|1)$ by the following operations:

- Perform some elementary column operations: replace the entries in the columns X_{jjk} by $\sum\limits_{\substack{i\in\mathcal{H}:\\ a_{ijk}\le a_{jjk}}} X_{ijk} + \sum\limits_{\substack{i\in\mathcal{H}:\\ a_{jik}>a_{iik}}} X_{jik}$ for all $j\in\mathcal{H}, k\in\mathcal{K}$.
- Delete the columns X_{ijk} for all $i,j\in\mathcal{H}: i\neq j, k\in\mathcal{K}$.

Then

$$rank(A'|1)\ge n^2q+n-q-(n^2-n)q = nq+n-q = dim\mathcal{P}_{UFL}.$$

Every row in $(A'|1)$ corresponds to a point $P=(\overline{X},\overline{Y})\in\sigma(\mathcal{P}_{UFL})$, for which the following inequality holds by (**):

$$\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{H}} a_{jjk}\overline{X}_{jjk}+\sum_{j\in\mathcal{H}} b_j\overline{Y}_j \ge \sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{ijk}X_{ijk}+\sum_{j\in\mathcal{H}} b_jY_j = d, \tag{31}$$

and, since (30) is valid, P is on $\mathcal{F}_{\sigma-UFL}$.

Since (29) is not a non–negativity constraint for some X_{ijk}, $\mathcal{F}_{\sigma-UFL}$ is not an improper $(\dim\mathcal{P}_{UFL}+1)$–dimensional face (i.e. no linear combination of $\sum\limits_{j\in\mathcal{H}} x_{jk}=1$ for some $k\in\mathcal{K}$), so $\mathcal{F}_{\sigma-UFL}$ is a facet of $\mathcal{P}_{UFL}$. □

For our next theorem we use another class of functions from $\mathcal{P}_{UFL}$ to $\mathcal{P}_{UHL}$. Instead of letting $X_{jjk}:=x_{jk}$ as in σ, for every $k\in\mathcal{K}$ we define an $i_k\in\mathcal{H}$ with $X_{i_kjk}:=x_{jk}$.

Theorem 4.2 *Let $a_{ijk}, b_j, d\in\mathbb{R}$ for all $i,j\in\mathcal{H}, k\in\mathcal{K}$ such that for every $k\in\mathcal{K}$ there exists an $i_k\in\mathcal{H}$ with $a_{i_kjk}\ge a_{ijk}$ for all $j\in\mathcal{H}$ and $b_{i_k}\ge 0$, (***) and let*

$$\sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{ijk}X_{ijk}+\sum_{j\in\mathcal{H}} b_jY_j\le d, \tag{32}$$

represent a facet of $\mathcal{P}_{UHL}$ that is not a non–negativity constraint for some X_{ijk}. Then

$$\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{i_kjk}x_{jk}+\sum_{j\in\mathcal{H}} b_jy_j\le d \tag{33}$$

represents a facet for $\mathcal{P}_{UFL}$.

Proof: First we prove the validity of (33). Assume that (33) is not valid. Then there exists an $(\overline{x},\overline{y}) \in \mathcal{Z}_{UHL}$ with $\sum\limits_{j\in\mathcal{H}}\sum\limits_{k\in\mathcal{H}} a_{i_k jk}\overline{x}_{jk} + \sum\limits_{j\in\mathcal{H}} b_j\overline{y}_j > d$.

For every $k \in \mathcal{K}$ there is exactly one $j_k \in \mathcal{H}$ with $\overline{x}_{j_k k} = 1$.

Then we can define a feasible solution $(\overline{X},\overline{Y})$ of UHL with

$$\begin{aligned}
\overline{X}_{i_k j_k k} &= 1 \quad \text{for all } k \in \mathcal{K},\\
\overline{X}_{ijk} &= 0 \quad \text{for all } (i,j) \neq (i_k, j_k), k \in \mathcal{K},\\
\overline{Y}_{i_k} &= 1 \quad \text{for all } i_k \in \mathcal{H},\\
\overline{Y}_j &= \overline{y}_j \text{ for all other } j \in \mathcal{H}.
\end{aligned}$$

Then by (***)

$$\begin{aligned}
&\sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{H}} a_{ijk}\overline{X}_{ijk} + \sum_{j\in\mathcal{H}} b_j\overline{Y}_j\\
=&\sum_{k\in\mathcal{K}} a_{i_k j_k k}\overline{X}_{i_k j_k k} + \sum_{j\in\mathcal{H}} b_j\overline{Y}_j\\
\geq&\sum_{k\in\mathcal{K}} a_{i_k j_k k}\overline{x}_{j_k k} + \sum_{j\in\mathcal{H}} b_j\overline{y}_j\\
=&\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{i_k jk}\overline{x}_{jk} + \sum_{j\in\mathcal{H}} b_j\overline{y}_j > d,
\end{aligned}$$

so (32) is not valid for $\mathcal{P}_{UHL}$, which is a contradiction.

To show that (33) is facet-defining we have to show that there are $dim\mathcal{P}_{UFL} = nq + n - q$ affinely independent points on

$$\mathcal{F}_{UFL} := \{(x,y) \in \mathcal{P}_{UFL} : \sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{i_k jk}x_{jk} + \sum_{j\in\mathcal{H}} b_j y_j = d\}.$$

We have $dim\mathcal{P}_{UHL} = n^2q + n - q$ affinely independent points on

$$\mathcal{F}_{UHL} := \{(x,y) \in \mathcal{P}_{UHL} : \sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}}\sum_{k\in\mathcal{K}} a_{ijk}X_{ijk} + \sum_{j\in\mathcal{H}} b_j Y_j = d\}$$

If we write these points as rows of an $(n^2q + n - q) \times (n^2q + n)$ matrix A, the definition of affine independence implies that the $(n^2q + n - q) \times (n^2q + n + 1)$ matrix $(A|1)$ has $rank(A|1) = n^2q + n - q$.

Let $(A'|1)$ be the matrix which is obtained from $(A|1)$ by the following operations:

- Perform some elementary column operations: replace the entries in the columns $X_{i_k jk}$ by $\sum_{i \in \mathcal{H}} X_{ijk}$.
- Delete the columns X_{ijk} for all $i, j \in \mathcal{H} : i \neq i_k, k \in \mathcal{K}$.

Then

$$rank(A'|1) \geq n^2 q + n - q - (n^2 - n)q = nq + n - q = dim \mathcal{P}_{UFL}.$$

If we now define

$$\begin{aligned} x_{jk} &:= X_{i_k jk} \text{ for all } j \in \mathcal{H}, k \in \mathcal{K}, \\ y_j &:= Y_j \quad\quad \text{for all } j \in \mathcal{H}, \end{aligned}$$

then every row in $(A'|1)$ corresponds to a point $P = (x, y) \in \mathcal{P}_{UFL}$, for which the following inequality holds:

$$\sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{H}} a_{i_k jk} x_{jk} + \sum_{j \in \mathcal{H}} b_j y_j \geq \sum_{i \in \mathcal{H}} \sum_{j \in \mathcal{H}} \sum_{k \in \mathcal{K}} a_{ijk} X_{ijk} + \sum_{j \in \mathcal{H}} b_j Y_j = d, \tag{34}$$

and, since (33) is valid, P is on $\mathcal{F}_{UFL}$.

Since (32) is not a non–negativity constraint for some X_{ijk}, $\mathcal{F}_{UFL}$ is not an improper $(\dim \mathcal{P}_{UFL} + 1)$–dimensional face (i.e. no linear combination of $\sum_{j \in \mathcal{H}} x_{jk} = 1$ for some $k \in \mathcal{K}$), so $\mathcal{F}_{UFL}$ is a facet of $\mathcal{P}_{UFL}$. □

5 Computational Results

In this section we test the computational behavior of the new formulation (FACET-UHL) of the Uncapacitated Hub Location Problem with multiple allocation, which is given in Section 3.

We compare the performance of FACET-UHL with the original formulation UHL (see Section 2) and a reformulation called EK-UHL. The latter is based on a formulation for the p–hub location problem given in [Ernst and Krishnamoorthy, 1998], which has the advantage of fewer ($O(qn)$) variables and constraints because every commodity is represented only by its origin.

We implemented all three formulations in AMPLPlus [Fourer et al., 1993] and used the dual simplex algorithm and the built–in branch–and–bound routine of CPLEX 6.5.2 [ILOG, 1996] to solve the integer programs on a Pentium II PC with 266 Mhz and 64 megabyte RAM.

We use the CAB data set described in [O'Kelly et al., 1995] to benchmark our algorithms. These data contain the passenger flows and distances between 25 major cities in the U.S. Every origin–destination pair of these cities represents a different commodity. Every city is a potential hub node. The transportation costs for an origin–destination pair $k = (k_1, k_2)$ routed via first hub i and second hub j are defined by

$$C_{ijk} := W_{k_1k_2}\,(d_{k_1i} + \alpha d_{ij} + d_{jk_2}),$$

where $W_{k_1k_2}$ is the given passenger requirement between k_1 and k_2, d_{vw} is the Euclidean distance between two cities v and w, and $\alpha \in [0,1]$ is a given discount factor for transportation between two hub nodes.

We get different instances by choosing different subsets of 10, 15, 20 or 25 cities and $\alpha \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$. As there are no fixed costs given in these data, we define $F_k := 100 \cdot \sum_{l=1}^{25} \sum_{m=1}^{25} d_{lm}$ for all $k \in \mathcal{H}$. The results are shown in Table 1.

From Table 1 it can be seen that the LP relaxation of (FACET-UHL) produces optimal solutions of the integer program in almost all cases. (Even in the last case $n = 25$, $\alpha = 1.0$, the gap of the value of the LP relaxation and the optimal integer value is only 0.22%.) The new formulation FACET-UHL performs better both in computation time and branch&bound nodes than the original UHL. Although in some cases of $\alpha = 1$ FACET-UHL needs more computation time than EK-UHL, the number of branch&bound nodes in FACET-UHL is usually much less than in EK-UHL.

We note that in the optimal solution in case $\alpha = 1$ every commodity k is allocated to exactly one hub because there is no discount given to use the interhub connections. This special case can be solved by a UFL formulation with the advantage of less variables and less constraints.

		UHL			EK-UHL			FACET-UHL			
cities	α	CPU sec.	gap %	B&B nod.	CPU sec.	gap %	B&B nod.	CPU sec.	gap %	B&B nod.	opt. sol. # hubs
10	0.2	1.70	3.01	2	0.58	0.09	3	0.27	-	1	6
	0.4	1.70	3.35	8	0.55	0.45	4	0.20	-	1	4
	0.6	0.95	0.00	2	0.22	-	1	0.20	-	1	3
	0.8	0.31	-	1	0.21	-	1	0.21	-	1	3
	1.0	0.35	-	1	0.19	-	1	0.15	-	1	3
15	0.2	1.50	-	1	1.70	0.13	3	0.81	-	1	10
	0.4	3.50	0.00	3	2.00	0.05	8	0.85	-	1	9
	0.6	1.60	-	1	2.10	0.34	6	0.84	-	1	8
	0.8	5.60	0.00	3	2.50	0.23	4	0.87	-	1	6
	1.0	1.40	-	1	0.65	-	1	0.87	-	1	5
20	0.2	12.00	0.27	5	5.20	0.27	5	2.60	-	1	15
	0.4	20.00	0.53	12	6.30	0.51	8	2.50	-	1	14
	0.6	50.00	0.60	17	16.00	0.54	66	2.70	-	1	12
	0.8	50.00	0.01	7	19.00	0.24	14	2.80	-	1	8
	1.0	6.60	-	1	1.80	-	1	0.74	-	1	6
25	0.2	52.00	0.39	6	10.00	0.12	3	6.10	-	1	20
	0.4	43.00	0.52	13	12.00	0.23	8	6.20	-	1	20
	0.6	92.00	0.68	28	21.00	0.37	23	6.20	-	1	16
	0.8	180.00	0.29	13	49.00	0.19	23	6.60	-	1	12
	1.0	100.00	0.21	10	41.00	0.21	15	32.00	0.22	12	8

Table 1: Comparison between UHL, EK-UHL and FACET-UHL using CPLEX 6.5.2

6 Conclusions

In this paper we determined the dimension and some classes of facets for the Uncapacitated Hub Location (UHL) polyhedron. We developed some rules how to lift facets from the Uncapacitated Facility Location (UFL) Polyhedron to UHL and vice versa. By applying these rules to the inequalities in the UFL formulation we got new classes of facets for

UHL, which provide a closer UHL polyhedron and a better solving time.

Theorem 3.4 can be applied to other classes of facets for UFL, which have been found e.g. in [Cornuéjols and Thizy, 1982], [Cho et al., 1983a], [Cho et al., 1983b], [Guignard, 1980] to obtain new facets of $\mathcal{P}_{UHL}$. In all these classes the additional requirement (*) of Theorem 3.4 is satisfied. Thus the goal can be either to prove that this requirement is necessary in a facet for $\mathcal{P}_{UFL}$ or to find an example for which this theorem cannot be applied. However, incorporating additional new UHL facets into a branch&cut algorithm will lead to even better computational results.

Recently new hub location models based on network design formulations have been developed in [Nickel et al., 2000] for applications in urban public transportation. Polyhedral examinations of these new models would be of interest in order to obtain fast solution algorithms for different kinds of real world problems.

References

[Aardal et al., 1996] Aardal, K., Labbé, M., Leung, J., and Queyranne, M. (1996). On the two-level uncapacitated facility location problem. *INFORMS Journal on Computing*, 8(3):289–301.

[Campbell, 1994] Campbell, J. F. (1994). Integer programming formulations of discrete hub location problems. *European Journal of Operational Research*, 72:387–405.

[Cho et al., 1983a] Cho, D., Johnson, E., Padberg, M., and Rao, M. (1983a). On the uncapacitated plant location problem i: Valid inequalities and facets. *Mathematics of Operations Research 8*, pages 579–589.

[Cho et al., 1983b] Cho, D., Johnson, E., Padberg, M., and Rao, M. (1983b). On the uncapacitated plant location problem ii: Facets and lifting theorems. *Mathematics of Operations Research 8*, pages 590–612.

[Cornuéjols et al., 1990] Cornuéjols, G., Padberg, M., and Wolsey, L. (1990). The uncapacitated facility location problem. In Mirchandani, P. and Francis, R., editors, *Discrete Location Theory*, pages 119–172. Wiley, New York.

[Cornuéjols and Thizy, 1982] Cornuéjols, G. and Thizy, J.-M. (1982). Some facets of the simple plant location polytope. *Mathematical Programming*, 23:50–74.

[Ebery et al., 1998] Ebery, J., Krishnamoorthy, M., Ernst, A., and Boland, N. (1998). The capacitated multiple allocation hub location problem: Formulations and algorithms. *European Journal of Operational Research*, forthcoming.

[Ernst and Krishnamoorthy, 1996] Ernst, A. T. and Krishnamoorthy, M. (1996). Efficient algorithms for the uncapacitated single allocation p-hub median problem. *Location Science*, 4(3):139–154.

[Ernst and Krishnamoorthy, 1998] Ernst, A. T. and Krishnamoorthy, M. (1998). Exact and heuristic algorithms for the uncapacitated multiple allocation p-hub median problem. *European Journal of Operational Research*, 104.1:100–112.

[Fourer et al., 1993] Fourer, R., Gay, D. M., and Kerningham, B. W. (1993). *AMPL: A Modeling Language for Mathematical Programming.* The Scientific Press, San Francisco.

[Guignard, 1980] Guignard, M. (1980). Fractional vertices, cuts and facets of the simple plant location problem. *Mathematical Programming 12*, pages 150–162.

[ILOG, 1996] ILOG (1996). Using CPLEX for AMPL. Manual Pages.

[Klincewicz, 1996] Klincewicz, J. G. (1996). A dual algorithm for the uncapacitated hub location problem. *Location Science*, 4:173–184.

[Krispin and Wagner, 1998] Krispin, G. and Wagner, B. (1998). HubLocater: An exact solution method for the multiple allocation hub location problem. Technical report, Institute of Operations Research, Darmstadt University of Technology.

[Nickel et al., 2000] Nickel, S., Schöbel, A., and Sonneborn, T. (2000). Hub location problems in urban traffic networks. In Niittymaki and Pursula, editors, *Mathematical Methods and Optimization in Transportation Systems*. KLUWER academic publishers. to appear.

[O'Kelly et al., 1995] O'Kelly, M., Skorin-Kapov, D., and Skorin-Kapov, J. (1995). Lower bounds for the hub location problem. *Management Science*, 4:713–721.

[O'Kelly, 1987] O'Kelly, M. E. (1987). A quadratic integer program for the location of interacting hub facilities. *European Journal of Operational Research*, 32:393–404.

[Skorin-Kapov et al., 1996] Skorin-Kapov, D., Skorin-Kapov, J., and O'Kelly, M. (1996). Tight linear programming relaxations of uncapacitated p-hub median problems. *European Journal of Operational Research*, 94:582–593.

[Sohn and Park, 1996] Sohn, J. and Park, S. (1996). The single allocation problem in the interacting three–hub network. Technical report, Department of Industrial Engineering, KAIST, Taejon, Korea.

[Sohn and Park, 1997] Sohn, J. and Park, S. (1997). A linear program for the two-hub location problem. *European Journal of Operational Research*, 100:617–622.