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On value preserving and growth optimal portfolios.

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Abstract.
In a discrete–time financial market setting, the paper relates various concepts introduced for dynamic portfolios (both in discrete and in continuous time). These concepts are: value preserving portfolios, numeraire portfolios, interest oriented portfolios, and growth optimal portfolios. It will turn out that these concepts are all associated with a unique martingale measure which agrees with the minimal martingale measure only for complete markets.

Key words: value preserving portfolios, numeraire portfolios, interest oriented portfolios, growth optimal portfolios, logarithmic utility, martingale measure, minimal martingale measure, incomplete financial markets.

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§1 Introduction and Summary.
The concepts of value preserving portfolios, numeraire portfolios, growth optimal portfolios, or that of the minimal martingale measure have all been developed independently and with totally different intentions. Some were introduced in a discrete–time setting and some in a continuous–time setting whereas we here restrict attention to the discrete–time framework. While the minimal martingale measure has been introduced by Föllmer and Schweizer (1991) in the context of option hedging and pricing in incomplete financial markets, the growth optimal portfolio is defined as the dynamic portfolio maximizing the expected logarithm of the associated value process at every future time instant. In contrast to that, the concepts of a value preserving portfolio [cf. Hellwig (1989)] and of the numeraire portfolio [cf. Long (1990)] can be seen as lying somewhere in between the valuation and the portfolio optimization problem. However, the main aim of this paper is to demonstrate that all these concepts have close relations to each other. In fact, they are in many cases equivalent in a sense that will be made more precise later on.
The paper is organized as follows. Section 2 contains the description of the discrete–time financial market setting. The above mentioned concepts are introduced in section 3 while section 4 will be specially devoted to the study of the characteristics of growth optimal portfolios. Finally, in section 5 we present various results relating the different market concepts.

§2 The financial market.

On the market an investor can observe the prices of 1+d securities at the dates $t = 0,1,\ldots,T$ where T is the time horizon. Uncertainty is modelled by a probability space $(\Omega,\mathfrak{F},P)$. One of the securities is a bond (or savings account) with **interest rate** r_t, $1\le t\le T$. The **bond price process** is defined by

(2.1) $$B_t := (1+r_1)\cdot\ldots\cdot(1+r_t)\,,\ 0\le t\le T, \text{ where } B_0 = 1.$$

The other d securities are called stocks. The evolution of the prices will be modelled by a d–dimensional stochastic process $\{S_t, t=0,1,\ldots,T\}$ where S_0 is deterministic. Then B_t and the components S_t^k of S_t, $1\le k\le d$, are assumed to be positive. The information structure will be represented by random variables H_t with values in some space Ω_t, $0\le t\le T$, (which is endowed with some σ–algebra $\mathfrak{F}_t$ if Ω_t is uncountable). There H_t describes the **history** of the market at time t where H_0 is a given constant, $H_0 := 0$ say. Previous histories are never forgotten; therefore we assume that H_{t-1} is a function of H_t (which is measurable if Ω_t is uncountable). We say that a stochastic process $\{Z_t\}$ is **adapted** (to the information structure) if $Z_t = \zeta_t(H_t)$ for some measurable function ζ_t. This implies that Z_0 is deterministic. It is natural to assume that $\{r_t\}$ and $\{S_t\}$ are adapted, since the investor can observe r_t and S_t. In many cases r_t will be deterministic or predictable (i.e. a function of H_{t-1}), but we don't need such an assumption for the theory of this paper.

Remark 2.2. If Ω_t is finite or countable then one obtains a partition $\mathfrak{H}_t = \{\,\{H_t=h\},\ h\in\Omega_t\}$ of Ω. The assumption that H_{t-1} is a function of H_t is expressed by the property that $\mathfrak{H}_{t-1} \subset \mathfrak{H}_t$. The assumed information structure is as general as assuming that the information structure is given by a filtration, i.e. by an increasing family of sub–σ–algebras $\{\mathfrak{F}_t\}$ of $\mathfrak{F}$. In that situation namely, one can choose $(\Omega_t,\mathfrak{F}_t) := (\Omega,\mathfrak{F}_t)$ and H_t as the identity on Ω, now considered as measurable function from $(\Omega,\mathfrak{F})$ to $(\Omega,\mathfrak{F}_t)$. Then any $\mathfrak{F}_t$–measurable mapping is a measurable function of H_t. []

For any vector–valued process $\{Z_t\}$, let us define the backward increment by $\Delta Z_t := Z_t - Z_{t-1}$. Further, we write $x^\top$ for the transposed vector and $x^\top\cdot y$ for the inner product of $x,y\in\mathbb{R}^d$.

As was shown by Harrison and Kreps (1979), one can use a reduction to the case where the interest rates of the bond are zero upon defining the **discounted stock price process** $\breve{S}_t = (\breve{S}_t^1,\ldots,\breve{S}_t^d)^\top$ by

(2.3) $$\breve{S}_t^k := S_t^k / B_t\ ,\ k= 1,\ldots,d,\ t = 0,\ldots,T.$$

The **relative risk process** $\{R_t = (R_t^1,\ldots,R_t^d)^\top, 1\le t\le T\}$ [cf. Karatzas & Kou (1996)] is defined by

(2.4) $$1 + R_t^k := 1 + \Delta\breve{S}_t^k / \breve{S}_{t-1}^k = (1 + \Delta S_t^k / S_{t-1}^k) / (1 + r_t)$$

where $\{\ \Delta S_t^k/S_{t-1}^k,\ 1\le t\le T\ \}$ is the **return process** corresponding to $\{S_t^k, 0\le t\le T\}$ [cf. Pliska (1997) § 3.2]. Then we get

(2.5) $$\check{S}^k_t = \check{S}^k_{t-1}\cdot(1+R^k_t) = S^k_0\cdot(1+R^k_1)\cdot\ldots\cdot(1+R^k_t).$$

The reader may think of the history as $H_t := (r_1,R_1,\ldots,r_t,R_t)$, but more general situations are also allowed where nontraded assets can be included in the history. It is convenient to choose $\{(r_t,R_t)\}$ as underlying basic process because assumptions like independence are easier to state in terms of this process. Then the other processes are defined through (2.1), (2.5), and (2.3).

A **portfolio plan** is given by an $\mathbb{R}^d$–valued adapted stochastic process $\xi = \{\xi_t,\ 0\le t<T\}$. During $(t-1,t]$ the investor holds a portfolio $\xi_{t-1} = (\xi^1_{t-1},\ldots,\xi^d_{t-1})^\top$ where ξ^k_{t-1} denotes the number of shares of the k–th stock. A **consumption plan** is a real–valued adapted stochastic process $\{c_t,\ 1\le t\le T\}$. One should note that negative consumption is not forbidden. Sometimes c_t is called dividend and $\{-c_t,\ 1\le t\le T\}$ is called the cost process . A **portfolio and consumption plan** is described by (ξ,c) and just called a **plan** for short. Given the **initial wealth** x, the number η_t of shares of the bond in $[t,t+1)$ is then specified by (ξ,c) according to the **budget equation**

(2.6) $$\eta_0 + \xi_0^\top\cdot S_0 = x,\quad \eta_t B_t + \xi_t^\top\cdot S_t = \eta_{t-1}B_t + \xi_{t-1}^\top\cdot S_t - c_t,\ 1\le t<T.$$

The **value process** $\{V^{\xi,c}_t(x)\}$ and the **discounted value process** $\{\check{V}^{\xi,c}_t(x)\}$ are given through

(2.7) $$V^{\xi,c}_t(x) := \eta_t B_t + \xi_t^\top\cdot S_t =: B_t\cdot\check{V}^{\xi,c}_t(x),\ 0\le t<T,$$
$$V^{\xi,c}_T(x) := \eta_{T-1}B_T + \xi_{T-1}^\top\cdot S_T - c_T =: B_T\cdot\check{V}^{\xi,c}_T(x).$$

It is convenient to introduce the **pre–consumption value process** $\{V^{\xi,c}_{t-}(x),\ 1\le t\le T\}$ by

(2.8) $$V^{\xi,c}_{t-}(x) = \eta_{t-1}\cdot B_t + \xi_{t-1}^\top\cdot S_t = V^{\xi,c}_t(x) + c_t.$$

It is not difficult to see that $\{V^{\xi,c}_t(x)\}$, $\{\check{V}^{\xi,c}_t(x)\}$, and $\{V^{\xi,c}_{t-}(x)\}$ are adapted stochastic processes and to show

(2.9) $$\check{V}^{\xi,c}_t(x) + \check{c}_t = \check{V}^{\xi,c}_{t-1}(x) + \xi_{t-1}^\top\cdot\Delta\check{S}_t\ ,\text{ where } c_t =: B_t\cdot\check{c}_t,$$
$$V^{\xi,c}_t(x) + c_t = V^{\xi,c}_{t-1}(x) + \eta_{t-1}\cdot\Delta B_t + \xi_{t-1}^\top\cdot\Delta S_t = B_t\cdot[\check{V}^{\xi,c}_{t-1}(x) + \xi_{t-1}^\top\cdot\Delta\check{S}_t],$$
$$\Delta V^{\xi,c}_t(x) + c_t = \eta_{t-1}\Delta B_t + \xi_{t-1}^\top\cdot\Delta S_t.$$

A portfolio plan ξ describes a **self–financing portfolio plan** with value process $\{V^\xi_t(x),\ 0\le t\le T\}$ if the equations (2.6) and (2.7) hold for $c_t = 0,\ 1\le t\le T$. We assume the well–known **no–arbitrage condition**, i.e. that one of the following two equivalent conditions holds:

(NA) For any self–financing portfolio plan ξ one has:

(1) $V^\xi_T(0)\ge 0$ a.s. implies $V^\xi_T(0)=0$ a.s. ;

(2) for $1\le t\le T$: $\xi_{t-1}^\top\cdot\Delta\check{S}_t \ge 0$ a.s. implies $\xi_{t-1}^\top\cdot\Delta\check{S}_t = 0$ a.s.

[cf. Schachermayer (1992)]. The (NA) condition immediately implies that $V_t^{\xi}(0) \geq 0$ a.s. $\forall$ $1 \leq t \leq T$ if only $V_T^{\xi}(0) \geq 0$.

A plan (ξ,c) is called **admissible** if $V_t^{\xi,c}(x) > 0$ a.s., $0 \leq t < T$, and $V_T^{\xi,c}(x) \geq 0$ a.s.. An admissible self–financing portfolio plan can also be described by a **portfolio process** $\pi = (\pi_t, 0 \leq t < T)$ defined through

(2.10) $\pi_t = (\pi_t^1,\dots,\pi_t^d)^\top, \quad \pi_t^k := \xi_t^k \cdot S_t^k / V_t^{\xi}(x)$.

As a portfolio plan, a portfolio process π is an $\mathbb{R}^d$–valued adapted stochastic process. Then $\pi_t^k = \xi_t^k \cdot \breve{S}_t^k / \breve{V}_t^{\xi}(x)$ and $\breve{V}_t^{\xi}(x) = \breve{V}_{t-1}^{\xi}(x) + \xi_{t-1}^\top \cdot \Delta \breve{S}_t = \breve{V}_{t-1}^{\xi}(x) \cdot (1 + \pi_{t-1}^\top \cdot R_t)$. Thus we obtain

(2.11) $V_t^{\xi}(x) =: V_t^{\pi}(x) = x \cdot V_t^{\pi}(1)$ where

$$V_t^{\pi}(1) = \Pi_{m=1}^{t} (1 + r_m) \cdot (1 + \pi_{m-1}^\top \cdot R_m) = B_t \, \Pi_{m=1}^{t} (1 + \pi_{m-1}^\top \cdot R_m).$$

Thus, by use of π, one is able to write V_t as an exponential. This representation is also used in continuous time [cf. Karatzas & Kou (1996), Korn (1997a)]. A portfolio process π is **admissible** if

(2.12) $1 + \pi_{t-1}^\top \cdot R_t > 0$ a.s. for $1 \leq t < T$, $1 + \pi_{T-1}^\top \cdot R_T \geq 0$ a.s..

Here admissibility is independent of the initial wealth x and thus easier to handle. Conversely if any admissible process π is given and we define $V_t^{\pi}(x)$ as in (2.11) and ξ through $\pi_t^k \cdot V_t^{\pi}(x)/S_t^k =: \xi_t^k$, then we obtain $V_t^{\pi}(x) = V_t^{\xi}(x)$ by induction.

Thus, one can describe each admissible self–financing portfolio plan by an admissible portfolio process and vice versa. Let us write Π for the set of admissible portfolio processes π. In the present paper, we will use the concept of a portfolio process only if it is self–financing. In order to use an L^1–L^∞–framework, some boundedness of the risk process will be assumed, which is also used in continuous time [cf. Karatzas & Kou (1996), Korn (1997a)].

2.13 **Boundedness Assumption.** r_t and the components of R_t are bounded and bounded away from -1, $1 \leq t \leq T$.

A plan (ξ,c) will be called **bounded** if there exists some $M_t < \infty$ such that $\|\xi_t\| \leq M_t$ a.s. and $|c_t| \leq M_t$ a.s. for all t. Similarly, a portfolio process π is called bounded if $\|\pi_t\| \leq M_t$ a.s. for some $M_t < \infty$, $0 \leq t < T$. Obviously, if the plan (ξ,c) [or $\pi \in \Pi$] is bounded then $V_t^{\xi,c}(x)$ [or $V_t^{\pi}(x)$] is bounded and we have no difficulties with the integrability of $V_t(x)$ or similar expressions.

In order to use the important concept of martingales w.r.t. to the given information structure, we have to use conditional expectations $E[Z|H_t=\cdot]$ defined on Ω_t and $E[Z|H_t]$ defined on Ω for random variables Z bounded from below or from above. There we use the convention that $E[Z|H_t] := E[Z|\sigma(H_t)]$, i.e. $E[Z|H_t](\omega) := \zeta(H_t(\omega))$, $\omega \in \Omega$, where $\zeta(h) := E[Z|H_t=h]$, $h \in \Omega_t$.

Then a stochastic process $\{Z_t, 0 \le t \le T\}$ is a **martingale** if it is adapted, if Z_t is integrable and if

$$E[\Delta Z_t | H_{t-1}] = 0, \text{ i.e. } E[Z_t | H_{t-1}] = Z_{t-1} \quad \text{a. s.}, 1 \le t \le T.$$

To formulate some market concepts later on, it is necessary to consider further probability measures Q on $(\Omega, \mathfrak{F})$ which are equivalent (to the given physical probability measure P), i.e.

$$E_Q[X] = E[L \cdot X] \text{ for some positive density } L =: \frac{dQ}{dP},$$

where we write $E_Q[X] = \int X\, dQ$ for the expectation of the random variable X under Q whereas $E[X]$ is the expectation under P as usual. If Ω is finite or countable and $P(\{\omega\})>0$ for all $\omega \in \Omega$, then Q is equivalent (to P) if and only if $Q(\{\omega\})>0$ for all $\omega \in \Omega$. Of course, we have $E_Q[\ldots | H_0 = h] = E_Q[\ldots]$ as H_0 is constant.

Then $\{\check{S}_t\}$ is called a **Q–martingale** and Q a **martingale measure** iff $\check{S} = \{\check{S}_0, \ldots, \check{S}_T\}$ forms a martingale under Q, i.e.

(2.14a) $\qquad E_Q[\Delta \check{S}_t^k | H_{t-1}] = 0 \quad ,1 \le k \le d, \ \forall\ 1 \le t \le T.$

Obviously (2.14a) is equivalent to

(2.14b) $\qquad E_Q[R_t^k | H_{t-1}] = 0 \quad ,1 \le k \le d, \ \forall\ 1 \le t \le T.$

Then we set

(2.15) $\qquad \mathcal{Q} := \{Q; Q \text{ is an equivalent martingale measure }\}.$

From the 'Fundamental Theorem of Asset Pricing' [cf. Dalang et al (1990), Schachermayer (1992), Rogers (1994), Jacod & Shiryaev (1998)] we know that $\mathcal{Q}$ is not empty if and only if the no–arbitrage condition (NA) holds. Then there even exists some $Q' \in \mathcal{Q}$ such that dQ'/dP is bounded. Then for all $Q \in \mathcal{Q}$, $P[B] = 0 \Leftrightarrow Q[B] = 0$. Thus, when writing a.s. we do not need to specify the underlying measure.

It is known [cf. Harrison & Pliska (1981), Jacod & Shiryaev (1998)] that in discrete time the assumption of completeness (i.e. $\mathcal{Q}$ is a singleton) is a severe restriction and in general incomplete market situations one has several choices of equivalent martingale measures (from the convex set $\mathcal{Q}$). Further, it is well–known [cf. Harrison & Kreps (1979)] that each martingale measure corresponds to a consistent price system. Thus in incomplete markets, no preference independent pricing of contingent claims is possible.

2.16 Bayes' Formula. For $Q \in \mathcal{Q}$ and $dQ/dP =: L$ we have for any real random variable X bounded from below: $\qquad E_Q[X | H_t] = E[L \cdot X | H_t] \,/\, E[L | H_t]$.

For a proof see Karatzas & Shreve (1988, p. 193).

§ 3 Market concepts.

Before describing the different market concepts, we give the following lemma which completely characterizes the form of an equivalent martingale measure in our market.

3.1 Factorization Lemma. $Q \in \mathcal{Q}$ if and only if there is some adapted stochastic process $\{L_t(Q), 1 \le t \le T\}$ such that $L_t(Q) > 0$, $\frac{dQ}{dP} = \Pi_{t=1}^{T} L_t(Q)$ and :

(3.2a) $$E[L_t(Q) \mid H_{t-1}] = 1 \quad ,$$

(3.2b) $$E[L_t(Q) \cdot R_t^k \mid H_{t-1}] = 0 \, , \; , 1 \le k \le d.$$

Proof. If $Q \in \mathcal{Q}$ then $Q[A] = {}_A\int L \, dP$, $A \in \mathfrak{F}$, for some positive random variable $L = dQ/dP$. Set $L_1(Q) := E[L \mid H_1]$. Suppose we have defined $L_1(Q),\ldots, L_{t-1}(Q)$, then define $L_t(Q)$ by

(3.3) $$L_1(Q) \cdot \ldots \cdot L_t(Q) := E[L \mid H_t] \, , 1 \le t \le T.$$

Now let $Z_t = \zeta_t(H_t)$ be some bounded random variable depending on H_t, then:

$E[L \cdot Z_t \mid H_t] = Z_t \cdot E[L \mid H_t]$ and

$E[L \cdot Z_t \mid H_{t-1}] = E[Z_t \cdot E[L \mid H_t] \mid H_{t-1}] = E[L_1(Q) \cdot \ldots \cdot L_t(Q) \cdot Z_t \mid H_{t-1}]$

$= L_1(Q) \cdot \ldots \cdot L_{t-1}(Q) \cdot E[L_t(Q) \cdot Z_t \mid H_{t-1}] = E[L \mid H_{t-1}] \cdot E[L_t(Q) \cdot Z_t \mid H_{t-1}]$.

From the Bayes' formula we now obtain for $Z_t = \zeta_t(H_t)$:

(3.4) $$E[L_t(Q) \cdot Z_t \mid H_{t-1}] = E[L \cdot Z_t \mid H_{t-1}] / E[L \mid H_{t-1}] = E_Q[Z_t \mid H_{t-1}] \, .$$

Now choosing $Z_t \equiv 1$ and $Z_t \equiv R_t^k$ we obtain (3.2a,b).

By the same sort of argument one can prove that $\frac{dQ}{dP} = \Pi_{t=1}^{T} L_t(Q)$ with (3.2a,b) implies that Q is a martingale measure. However, we will not need this part of the lemma. []

3.5 Corollary. Let $Q \in \mathcal{Q}$ and $L_t(Q)$ be given as in 3.1, $1 \le t \le T$. Then one has for any bounded random variable $Z_t = \zeta_t(H_t)$ depending on H_t and $0 \le m < t \le T$:

(3.5a) $$E_Q[Z_t/B_t] = E[L_1(Q) \cdot \ldots \cdot L_t(Q) \cdot Z_t/B_t] \; ,$$

(3.5b) $$E_Q[\frac{1}{B_t} \cdot Z_t \mid H_m] = \frac{1}{B_m} \cdot E[L_{m+1}(Q) \cdot \ldots \cdot L_t(Q) \cdot \frac{1}{1+r_{m+1}} \cdot \ldots \cdot \frac{1}{1+r_t} \cdot Z_t \mid H_m] \, .$$

Proof. For m=t–1 the formula (3.5b) follows from (3.4) with Z_t/B_t in place of Z_t. Now one can use backward induction on m. The formula (3.5a) considers the case m=0. []

The corollary above gives rise to define the **risk–adjusted interest rates (marginal returns)**

(3.6) $$\frac{1}{1+r_t(Q)} := \frac{1}{1+r_t} \cdot L_t(Q) \; , 1 \le t \le T,$$

and the **state–price deflator**

(3.7) $\qquad D_t(Q) := \frac{1}{1+r_1(Q)} \cdot \ldots \cdot \frac{1}{1+r_t(Q)}$ where $D_0(Q) := 1$

In view of (3.5b), we have for any bounded random variable $Z_t = \zeta_t(H_t)$

(3.8a) $\qquad E_Q[\frac{1}{1+r_{m+1}} \cdot \ldots \cdot \frac{1}{1+r_t} \cdot Z_t | H_m] = \frac{1}{D_m(Q)} E[\, D_t(Q) \cdot Z_t | H_m] \,.$

Furthermore, we obtain from (3.1a,b):

(3.8b) $\qquad \frac{1}{D_{t-1}(Q)} E[\, D_t(Q) \cdot B_t | H_{t-1}] = B_{t-1} \,,$

(3.8c) $\qquad \frac{1}{D_{t-1}(Q)} E[\, D_t(Q) \cdot S^k_t | H_{t-1}] = S^k_{t-1} \,, \quad 1 \le t \le T.$

The relations (3.8) justify the name state–price deflator [cf. Duffie (1992, p. 23)] . By noting that the left–hand side of (3.8a) is a possible price of the contingent claim Z_t (w.r.t. the pricing system given by Q) and by writing the right–hand side for m=0 as

$$\int_\Omega D_t(Q,\omega)\, Z_t(\omega)\, P[d\omega] \,,$$

one can interpret $D_t(Q,\omega)$ as the current price of one unit of money paid at the future time t when the economy is in state ω. This interpretation also justifies the use of the name path dependent portfolio value for (3.9) where

(3.9) $\qquad V^{\xi,c}_{m,t}(x|Q) := \frac{1}{D_m(Q)} \cdot \left\{ \Sigma^t_{n=m+1} D_n(Q) \cdot c_n + D_t(Q) \cdot V^{\xi,c}_t(x) \right\},$

$\qquad V^{\xi,c}_{t,t}(x|Q) = V^{\xi,c}_t(x).$

3.10 Lemma. For any $Q \in \mathcal{Q}$ and any bounded plan (ξ,c) one has:

the process $\{V^{\xi,c}_{0,t}(x|Q), 0 \le t \le T\}$ is a martingale.

We remind the reader that we can interpret a martingale as a stochastically constant process.

Proof. From (2.9) we conclude

$$\begin{aligned} E_Q[\check{V}^{\xi,c}_t(x)|H_{t-1}] &= E_Q[\check{V}^{\xi,c}_{t-1}(x) - \check{c}_t|H_{t-1}] + E_Q[\xi^T_{t-1} \cdot \Delta\check{S}_t|H_{t-1}] \\ &= E_Q[\check{V}^{\xi,c}_{t-1}(x) - \check{c}_t|H_{t-1}] + \xi^T_{t-1} \cdot E_Q[\Delta\check{S}_t|H_{t-1}] \\ &= \check{V}^{\xi,c}_{t-1}(x) - E_Q[\check{c}_t|H_{t-1}] \end{aligned}$$, which implies by (3.5):

$$E[\frac{1}{1+r_t(Q)} \cdot V^{\xi,c}_t(x)|H_{t-1}] = V^{\xi,c}_{t-1}(x) - E[\frac{1}{1+r_t(Q)} \cdot c_t|H_{t-1}] \quad \text{or}$$

$$E[D_t(Q) \cdot \{c_t + V^{\xi,c}_t(x)\}|H_{t-1}] = D_{t-1}(Q) \cdot V^{\xi,c}_{t-1}(x)$$

and the assertion follows. []

In accordance with Hellwig (1996a,b), $V^{\xi,c}_{t,T}(x|Q)$, $0 \le t \le T$, is called the present economic value of (ξ,c) at time t associated with $Q \in \mathcal{Q}$, where one has:

(3.11a) $V^{\xi,c}_{t-1,T}(x|Q) = \frac{1}{1 + r_t(Q)} [c_t + V^{\xi,c}_{t,T}(x|Q)]$;

(3.11b) $V^{\xi,c}_t(x) = E[\, V^{\xi,c}_{t,T}(x|Q) \,|H_t]$, $0 \le t \le T$.

The proof follows from Lemma 3.10 which implies: $V^{\xi,c}_{0,t}(x|Q) = E[V^{\xi,c}_{0,T}(x|Q)|H_t]$, i.e. $\Sigma^t_{n=1} D_n(Q)\cdot c_n + D_t(Q)\cdot V^{\xi,c}_t(x) = E[\Sigma^T_{n=1} D_n(Q)\cdot c_n + D_T(Q)\cdot V^{\xi,c}_T(x)|H_t] =$ $\Sigma^t_{n=1} D_n(Q)\cdot c_n + E[\Sigma^T_{n=t+1} D_n(Q)\cdot c_n + D_T(Q)\cdot V^{\xi,c}_T(x)|H_t]$ which yields (3.11).

Now we can define the concept of a value conserving plan introduced by Hellwig (1989) and the generalization presented by Hellwig (1996b,c).

3.12 Definition. (a) A bounded plan (ξ,c) is called **value preserving** for the initial wealth x if one of the following equivalent conditions holds for some $Q \in \Omega$:

(3.13a) $V^{\xi,c}_{t,T}(x|Q) = x$ for $1 \le t \le T$;

(3.13b) $c_t = r_t(Q)\cdot x$ for $1 \le t \le T$ and $V^{\xi,c}_T(x) = x$.

(b) Suppose that for each $Q \in \Omega$ there is given an adapted bounded stochastic process $\{X_t(Q), 0 \le t \le T\}$ where $X_0(Q) = x$. Then a bounded plan (ξ,c) is called **value oriented** for the initial wealth x if one of the following equivalent conditions holds for some $Q \in \Omega$:

(3.14a) $V^{\xi,c}_{t,T}(x|Q) = X_t(Q)$ for $0 \le t \le T$;

(3.14b) $c_t = [1 + r_t(Q)]\cdot X_{t-1}(Q) - X_t(Q)$ for $1 \le t \le T$ and $V^{\xi,c}_T(x) = X_T(Q)$.

Relation (3.13b) means that consumption coincides exactly with the marginal return due to the risk–adjusted interest rate. Note especially that this could lead to a negative consumption ! The sequence $\{X_t(Q), 0 \le t \le T\}$ is the desired value sequence. Examples will be given below where $X_t(Q)$ indeed depends on Q. In the situation of 3.12, we say that (ξ,c) is value preserving (oriented) with associated $Q \in \Omega$ if we want to specify Q. However, it will turn out that $Q = Q^*$ is uniquely determined by the properties in 3.12. The following lemma is known from Wiesemann (1995a),(1995b).

3.15 Lemma. The conditions (3.13a) and (3.13b) as well as (3.14a) and (3.14b) are indeed equivalent.

Proof. It is sufficient to prove the equivalence in (3.14). If (3.14a) holds then we get from (3.11a): $X_{t-1}(Q) = \frac{1}{1 + r_t(Q)} [c_t + X_t(Q)]$ and thus (3.14b). On the other hand, if (3.14b) holds we obtain from (3.11a):

$$V^{\xi,c}_{t-1}(x|Q) = \frac{1}{1 + r_t(Q)} \{[1 + r_t(Q)] \cdot X_{t-1}(Q) - X_t(Q) + V^{\xi,c}_t(x|Q)\}$$

$$= X_{t-1}(Q) + \frac{1}{1 + r_t(Q)} [V^{\xi,c}_t(x) - X_t(Q)] .$$

Starting from $V^{\xi,c}_{T,T}(x|Q) = V^{\xi,c}_T(x) = X_T(Q)$ we get the relation (3.14a) by backward induction. []

3.16 Definition. (a) A bounded plan (ξ,c) is called **weakly value preserving for** x if one of the following equivalent conditions holds:

(a1) $V^{\xi,c}_t(x) = x$ for $1 \leq t \leq T$;

(a2) $c_t = r_t \cdot x + \xi^T_{t-1} \cdot [S_t - (1+r_t) \cdot S_{t-1}]$ for $1 \leq t \leq T$.

(b) A bounded plan (ξ,c) is called **weakly value oriented for** x w.r.t. $\{X_t\}$ where $X_0 = x$ if one of the following equivalent conditions holds:

(b1) $V^{\xi,c}_t(x) = X_t$ for $1 \leq t \leq T$;

(b2) $c_t = (1+r_t) \cdot X_{t-1} - X_t + \xi^T_{t-1} \cdot [S_t - (1+r_t) \cdot S_{t-1}]$ for $1 \leq t \leq T$.

By use of (2.9) and forward induction, it is easy to prove the equivalence of the conditions in 3.16 (a) and (b). From (3.11b) one immediately obtains:

3.17 Lemma. (a) A value preserving plan is weakly value preserving.

(b) A value oriented plan with associated Q is weakly value oriented w.r.t. $\{X_t(Q)\}$.

In order to show that the generalization in 3.12b of the concept of value preserving is useful we consider two examples.

3.18 Example (Self–financing portfolios). [cf. Hellwig (1996c), Schäl (1998)].
Let us consider the case

(3.18a) $X_t(Q) := \Pi^t_{m=1} [1 + r_m(Q)]$, $0 \leq t \leq T$, in particular $x = 1$.

With the interpretation of the risk adjusted rates of return as the return rate of the market, one can think of $X_t(Q)$ as the value process of one unit of money invested in the market at time 0 and left there until time t. In this special case, (3.14b) is obviously equivalent to

(3.18b) $c_t = 0$, $1 \leq t \leq T$, ξ is a self–financing portfolio plan with $V^{\xi}_T(1) = \Pi^T_{m=1} [1 + r_m(Q)]$.
However, there is still the question of existence of such a self–financing portfolio plan. This example will become important in §4. []

3.19 Example (Option pricing). Let X be a bounded random variable which is a measurable function of H_T. One may think of X as a contingent claim corresponding to some option. In the case of a European option, one has $X = (S_T^1 - K)^+ \geq 0$ where K is the strike price. We here set

(3.19a) $X_t(Q) := B_t \cdot E_Q[\, X/B_T \mid H_t], 0 \leq t \leq T$, in particular $X_T(Q) = X$, $X_0(Q) = E_Q[X/B_T]$.

Then $\{X_t(Q)\}$ is a price process of the option. Now consider the case that there exists some $Q \in \Omega$ and some plan (ξ,c) which is value oriented with associated Q and w.r.t. $\{X_t(Q)\}$ in sense of definition 3.12b. Then starting from an initial wealth

(3.19b) $x = E_Q[X/B_T]$

the contingent claim can be hedged by the value oriented plan (ξ,c), i.e. we have $V_T^{\xi,c}(x) = X$. Note that this strategy will in general not be a self–financing one ! We will however demonstrate in section 5 that it has some attractive features. In particular we have:

(3.19c) $V_t^{\xi,c}(x) = B_t \cdot E_Q[\, X/B_T \mid H_t] := \frac{1}{D_t(Q)} \cdot \left\{ \Sigma_{n=t+1}^T D_n(Q) \cdot c_n + D_T(Q) \cdot X \right\}$.

Below, it will be shown that there exists at most one such Q and sufficient conditions for the existence will be given. Thus $x = E_Q[X/B_T]$ can be considered as a candidate for a price of the contingent claim. []

The following property was introduced by Korn (1997b) for a continuous–time model.

3.20 Definition [cf. Korn (1997b)]. A plan (ξ,c) is called **interest oriented** if the following condition holds for some $Q \in \Omega$:

(3.20a) $V_{t-}^{\xi,c}(x) - V_{t-1}^{\xi,c}(x) = \Delta V_t^{\xi,c}(x) + c_t = r_t(Q) \cdot V_{t-1}^{\xi,c}(x)$, $1 \leq t \leq T$.

For an admissible plan one can define the **portfolio return** of (ξ,c) in t by

$r_t^{\xi,c}(x) := [V_{t-}^{\xi,c}(x) - V_{t-1}^{\xi,c}(x)] \,/\, V_{t-1}^{\xi,c}(x) = [\Delta V_t^{\xi,c}(x) + c_t] \,/\, V_{t-1}^{\xi,c}(x)$.

Then (ξ,c) is interest oriented if $r_t^{\xi,c}(x) = r_t(Q)$ for some $Q \in \Omega$.

The first identity of (3.20a) just recalls the definition of V_{t-}. Example 3.18 gives rise to the following definition:

3.21 Definition [cf. Long (1990)]. An admissible portfolio process $\pi \in \Pi$ is called a **numeraire portfolio** if one of the following equivalent conditions holds:

(3.22a) $\{1/V_t^{\pi}(1)\ ,\ 1 \leq t \leq T\}$ defines a state–price deflator $\{D_t\}$, i.e. the relations (3.8b,c) hold for $D_t := 1/V_t^{\pi}(1)$;

(3.22b) $L_t := 1/(1 + \pi_{t-1}^T \cdot R_t)$, $1 \leq t \leq T$, defines some $Q \in \Omega$ by $\frac{dQ}{dP} = \Pi_{t=1}^T L_t$.

In view of (2.11), (3.6), and (3.7) the two conditions (3.22a) and (3.22b) are indeed equivalent.
In the situation of 3.22 we can write (3.8b,c) as:

(3.23a) $\quad E[B_t/V_t^{\pi}(1)\,|H_{t-1}] = B_{t-1}/V_{t-1}^{\pi}(1),$

(3.23b) $\quad E[S_t^k/V_t^{\pi}(1)\,|H_{t-1}] = S_{t-1}^k/V_{t-1}^{\pi}(1),\ 1 \le t \le T.$

Thus, if one replaces the discount factor $1/B_t$ by $1/V_t^{\pi}(1)$ then the discounted price processes are martingales under the given physical probability measure. Recall that B_t is the value at time t of one unit of money put on the bank account at time 0 whereas $V_t^{\pi}(1)$ is the value of one unit of money due to profit of investment according to π. Thus, one can replace the **change of measure** from P to Q (defined by (3.8a)) by a **change of numeraire**. The value process defined by some numeraire portfolio (if it exists at all) is known to be unique [cf. Theorem 5.4 below].

3.24 Definition. A portfolio process π^* is called a **growth–optimal portfolio** if

$$E[\ln(V_t^{\pi^*}(1)] = \max_{\pi\in\Pi} E[\ln(V_t^{\pi}(1)]\ ,\ 1 \le t \le T.$$

A growth–optimal portfolio π maximizes the expected value of the logarithm of the terminal value, or equivalently, π maximizes the expected growth rate of wealth invested in the market.

§4 Admissible growth–optimal portfolios.

Let $q_1(B) := P[R_1 \in B]$, $B \subset \mathbb{R}^d$, be the distribution of R_1. We need the support Σ_1 of q_1 defined as the smallest closed subset B of $\mathbb{R}^d$ such that $q_1(B) = 1$. Furthermore, let $\mathscr{L}_1$ be the smallest linear space in $\mathbb{R}^d$ containing Σ_1, i.e. the smallest linear space $\mathscr{L}$ in $\mathbb{R}^d$ such that $P[R_1 \in \mathscr{L}] = 1$. In most cases one will have $\mathscr{L}_1 = \mathbb{R}^d$. If Ω is finite, then $P[R_1 \in \Sigma_1] = 1$ for some finite subset Σ_1 of $\mathbb{R}^d$. In that case, Σ_1 is the support of $q_1(\cdot)$ if w.l.o.g. $q_1(\{\sigma\}) > 0\ \ \forall\, \sigma \in \Sigma_1$.
For $t > 1$, we similarly define $q_t(B\,|\,h) := P[R_t \in B\,|\,H_{t-1}{=}h]$ as the conditional distribution of R_t given the past $H_{t-1}{=}h$. Then for fixed h, $\Sigma_t(h)$ is the support of $q_t(\cdot\,|\,h)$ and $\mathscr{L}_t(h)$ is the smallest linear space in $\mathbb{R}^d$ containing $\Sigma_t(h)$. We set $q_1(\cdot\,|\,h) := q_1(\cdot)$, $\Sigma_1(h) := \Sigma_1$, $\mathscr{L}_1(h) := \mathscr{L}_1$ and

(4.1) $\quad \Theta_t(h) := \{\, \vartheta \in \mathbb{R}^d;\ \ 1 + \vartheta^T\cdot\sigma \ge 0\ \ \forall\, \sigma \in \Sigma_t(h)\,\}\,,\ \ \Theta_1(h) =: \Theta_1$

$$\partial\Theta_t(h) := \{\, \vartheta \in \Theta_t(h);\ \ \exists\, \sigma \in \Sigma_t(h) \text{ with } 1 + \vartheta^T\cdot\sigma = 0\},\ \partial\Theta_1(h) =: \partial\Theta_1.$$

In Lemma 4.3a below, we will provide another characterization of $\Theta_t(h)$. In order to get admissible portfolio processes we look for portfolios ϑ in $\Theta_t(h)\backslash\partial\Theta_t(h)$. But for reasons of compactness we first start with $\Theta_t(h)$. It is known that the no–arbitrage condition also holds locally a.s. [cf. Dalang et al. (1990. Lemma 2.3) , Pliska (1997, (3.22)), Jacod & Shiryaev (1998), Schäl (1999,§2)]. By an appropriate choice of the conditional distributions one obtains that the no–arbitrage condition locally holds everywhere. Therefore we assume

(NA)* $P[\vartheta^T \cdot R_t \geq 0 | H_{t-1}=h] = 1$ implies $P[\vartheta^T \cdot R_t = 0 | H_{t-1}=h] = 1$, i.e.

$$\vartheta^T \cdot \sigma \geq 0 \ \ \forall\, \sigma \in \Sigma_t(h) \ \text{ implies } \ \vartheta^T \cdot \sigma = 0 \ \ \forall\, \sigma \in \Sigma_t(h)\ ,\ h \in \Omega_{t-1} \ \text{ for all } \vartheta \in \mathbb{R}^d.$$

The equivalence of the two characterizations of (NA)* can be proved as Lemma 4.3a below. The following geometric characterization is given by Jacod & Shiryaev (1998).

4.2 Remark. Let $h \in \Omega_{t-1}$ be fixed and let $\mathscr{H}_t(h)$ be the smallest affine hyperplane containing $\Sigma_t(h)$ and thus the convex hull conv($\Sigma_t(h)$) of $\Sigma_t(h)$. $\mathscr{H}_t(h)$ reduces to one point if and only if $\Sigma_t(h)$ is a singleton set. Similarly, $\mathscr{L}_t(h)$ reduces to one point if and only if $\Sigma_t(h) = \{0\}$. Otherwise $\mathscr{H}_t(h)$ and $\mathscr{L}_t(h)$ have a dimension between 1 and d. Then the interior of conv($\Sigma_t(h)$) relative to $\mathscr{H}_t(h)$ [resp. $\mathscr{L}_t(h)$] is constructed as follows. A point σ is an interior point of conv($\Sigma_t(h)$) relative to $\mathscr{H}_t(h)$ [$\mathscr{L}_t(h)$] if σ is contained in the interior of conv($\Sigma_t(h)$) for the relative topology within the hyperplane $\mathscr{H}_t(h)$ [$\mathscr{L}_t(h)$]. For example, if $\mathscr{H}_t(h)$ has dimension 1 then conv($\Sigma_t(h)$) is a closed line segment and the open segment is the relative interior.

The following conditions are equivalent:

(1) condition (NA)* holds for the fixed h;

(2) 0 is an interior point of conv($\Sigma_t(h)$) relative to $\mathscr{H}_t(h)$;

(3) 0 is an interior point of conv($\Sigma_t(h)$) relative to $\mathscr{L}_t(h)$.

By definition, the conditions are always satisfied in the case $\Sigma_t(h) = \{0\}$. The equivalence of (1) and (2) was proved by Jacod & Shiryaev (1998, Theorem 3).
"(2) ⇒ (3)" Assume that (2) holds. Then $0 \in \mathscr{H}_t(h)$ and thus $\mathscr{H}_t(h) = \mathscr{L}_t(h)$.
"(3) ⇒ (2)" If (3) holds, then $0 \in \text{conv}(\Sigma_t(h)) \subset \mathscr{H}_t(h)$ and we again have $\mathscr{H}_t(h) = \mathscr{L}_t(h)$. []

We recall that the random variable R_t is assumed to be bounded; thus for any t one can find some $\rho_t < \infty$ such that $\Sigma_t(h) \subset B(0,\rho_t)$, $h \in \Omega_{t-1}$, where $B(0,\rho)$ is the ball in $\mathbb{R}^d$ around 0 with radius ρ.

4.3 Lemma. (a) $\Theta_t(h) = \{\ \vartheta \in \mathbb{R}^d;\ P[1 + \vartheta^T \cdot R_t \geq 0 | H_{t-1}=h] = 1\ \}$.

(b) If $\Sigma_t(h) \subset B(0,\rho)$ for some $0<\rho<\infty$, then $B(0,1/\rho) \subset \Theta_t(h)$.

(c) If $B(0,\varepsilon) \cap \mathscr{L}_t(h)$ is contained in the convex hull conv($\Sigma_t(h)$) of $\Sigma_t(h)$, then $\Theta_t(h) \cap \mathscr{L}_t(h) \subset B(0,1/\varepsilon)$.

Proof. Let t and h be fixed. a) Let Θ^* denote the right–hand of (a). Then one obviously has "$\Theta^* \supset \Theta_t(h)$". In order to prove "$\Theta^* \subset \Theta_t(h)$" choose some $\vartheta \in \Theta^*$ and some $\sigma \in \Sigma_t(h)$. Then for each ball $B_n := B(\sigma,\frac{1}{n})$ around σ we know that

$$0 < P[R_t \in B_n | H_{t-1}=h] = P[R_t \in B_n, 1 + \vartheta^T \cdot R_t \geq 0 | H_{t-1}=h].$$

Thus there exists some $\sigma_n \in B_n$ with $1 + \vartheta^T \cdot \sigma_n \geq 0$ which implies that $1 + \vartheta^T \cdot \sigma \geq 0$.

b) The proof of part (b) is easy. c) Obviously $1 + \vartheta^T \cdot \sigma \geq 0 \quad \forall\, \sigma \in \Sigma_t(h)$ implies that $1 + \vartheta^T \cdot \sigma \geq 0 \quad \forall\, \sigma \in \text{conv}(\Sigma_t(h))$ and thus $1 + \vartheta^T \cdot \sigma \geq 0 \quad \forall\, \sigma \in B(0,\varepsilon) \cap \mathscr{L}_t(h)$.

If the latter relation holds for $\vartheta \in \mathscr{L}_t(h)$, we can choose $\sigma = -\frac{\varepsilon}{\|\vartheta\|}\vartheta$ and obtain $\varepsilon \cdot \|\vartheta\| \leq 1$. []

4.4 Lemma. $\Theta_t(h) \cap \mathscr{L}_t(h)$ is compact.

Proof. Since $\Theta_t(h)$ and $\mathscr{L}_t(h)$ are closed it is sufficient to prove that $\Theta_t(h) \cap \mathscr{L}_t(h)$ is bounded. One can give a proof using Remark 4.2 and Lemma 4.3c. But a direct proof is also available.
Assume that there is some sequence $\{\vartheta_n\}$ such that $\vartheta_n \in \Theta_t(h_n) \cap \mathscr{L}_t(h_n)$ such that $0 < \|\vartheta_n\| \to \infty$ and hence $\varepsilon_n = 1/\|\vartheta_n\| \to 0$. Then $e_n := \varepsilon_n \cdot \vartheta_n \in \mathscr{L}_t(h_n) \cap S^{d-1}$ where $S^{d-1} = \{\vartheta \in \mathbb{R}^d;\ \|\vartheta\|=1\}$ denotes the sphere. By assumption we have $1 + \vartheta_n^T \cdot \sigma \geq 0$ and thus $\varepsilon_n + e_n^T \cdot \sigma \geq 0\ \forall\, \sigma \in \Sigma_t(h)$. Since $\mathscr{L}_t(h_n) \cap S^{d-1}$ is compact there are some subsequence $(n') \subset \mathbb{N}$ and some $e \in \mathscr{L}_t(h_n) \cap S^{d-1}$ such that $e_{n'} \to e$. This implies that $e^T \cdot \sigma \geq 0 \quad \forall\, \sigma \in \Sigma_t(h)$. By (NA)* this implies $e^T \cdot \sigma = 0$ $\forall\, \sigma \in \Sigma_t(h)$. But then $e^T \sigma = 0 \quad \forall\, \sigma \in \mathscr{L}_t(h)$ which contradicts $e \in \mathscr{L}_t(h) \setminus \{0\}$. []

Now we use the logarithmic utility function and define the conditional expected utility as

(4.5) $\quad I_t(h,\vartheta) := E[\ln(1 + \vartheta^T \cdot R_t) | H_{t-1}=h] = \int \ln(1 + \vartheta^T \cdot \sigma)\, q_t(h;d\sigma)$, $\vartheta \in \Theta_t(h)$, $h \in \Omega_{t-1}$.

4.6 Lemma. Let $\mathbb{P}$ denote the compact metric space of all d×d orthogonal projection matrices and let $\Gamma_t : \Omega_{t-1} \mapsto \mathbb{P}$ be defined such that $\Gamma_t(h)$ is the orthogonal projection on $\mathscr{L}_t(h)$. Then the mapping $(h,\vartheta) \mapsto \Gamma_t(h)\vartheta$ is measurable and

(4.6a) $\quad P[\,\vartheta^T \cdot R_t = (\Gamma_t(h)\vartheta)^T \cdot R_t | H_{t-1}=h] = 1,\ \ h \in \Omega_{t-1}$.

Proof. Rogers (1994, Proposition 2.4) proved that Γ_t is measurable on a set B_{t-1} where $P[H_{t-1} \in B_{t-1}] = 1$. This fact is sufficient for the application below. But an analysis of the proof shows that one can here choose $B_{t-1} = \Omega_{t-1}$. Then $\Gamma_t(h)\vartheta$ is measurable in h und continuous in ϑ, hence measurable in (h,ϑ). Finally (4.6a) follows from $P[R_t \in \mathscr{L}_t(h) | H_{t-1}=h] = 1$. []

4.7 Lemma. a) $\Theta_t(h) \ni \vartheta \mapsto I_t(h,\vartheta)$ is upper semi–continuous;

b) $\{(h,\vartheta) \in \Omega_{t-1} \times \mathbb{R}^d;\ \vartheta \in \Theta_t(h) \cap \mathscr{L}_t(h)\}$ is a measurable subset of $\Omega_{t-1} \times \mathbb{R}^d$;

c) there exists a unique measurable mapping $\varphi_t : \Omega_t \mapsto \mathbb{R}^d$ such that $\varphi_t(h) \in \Theta_t(h) \cap \mathscr{L}_t(h)$ and $I_t(h,\varphi_t(h)) = \sup_{\vartheta \in \Theta_t(h)} I_t(h,\vartheta)$, $h \in \Omega_{t-1}$.

Proof. a) Suppose $\vartheta_n \to \vartheta_0$, then we may assume that $\|\vartheta_n\| \le \rho_0$ for some $\rho_0<\infty$. Further, choose $\rho_t<\infty$ such that $R_t \le \rho_t$ a.s.; then $\vartheta_n^\top \cdot R_t \le \rho_0 \cdot \rho_t$ a.s. . Now, we conclude from Fatou's lemma:

$$\limsup_{n\to\infty} E[\ln(1+\vartheta_n^\top \cdot R_t) \mid H_t=h] \le E[\ln(1+\vartheta_0^\top \cdot R_t) \mid H_t=h].$$

b) We have in view of Lemma 4.3a and Lemma 4.6:

$$\{(h,\vartheta) \in \Omega_{t-1}\times\mathbb{R}^d;\ \vartheta \in \Theta_t(h)\cap\mathcal{L}_t(h)\}$$
$$= \{(h,\vartheta) \in \Omega_{t-1}\times\mathbb{R}^d;\ \int 1_{[0,\infty)}(1+\vartheta^\top\cdot\sigma)\, q_t(h;d\sigma) = 1\ ,\ \Gamma_t(h)\vartheta = \vartheta\ \}.$$

where also $(h,\vartheta) \mapsto \int 1_{[0,\infty)}(1+\vartheta^\top\cdot\sigma)\, q_t(h;d\sigma)$ is measurable.

c) From Lemma 4.4 and part a) we know that $I_t(h,\cdot)$ attains the maximum on $\Theta_t(h)\cap\mathcal{L}_t(h)$ which is the maximum on $\Theta_t(h)$. Since $I_t(h,\cdot)$ is strictly concave, this maximum point is unique. Now we can apply a selection theorem: If Ω_{t-1} is a Borel subset of some polish space one can refer to Brown & Purves (1973, Corollary 1) or Bertsekas & Shreve (1978, Proposition 7.33) and in the general case to Schäl (1974,Theorem 2). []

As usual, $v^- := \max(0,-v)$ and $E[\ln(V_t^\pi(x))] := -\infty$ if $E[\{\ln(V_t^\pi(x))\}^-] = \infty$.

4.8 Theorem. There exists a growth–optimal portfolio π^*, i.e.

$$E[\ln(V_t^{\pi^*}(x))] = \sup_{\pi\in\Pi} E[\ln(V_t^\pi(x))] \quad \forall\ x>0,\ 1\le t\le T.$$

Proof. Choose $\varphi_t : \Omega_t \mapsto \mathbb{R}^d$ as in Lemma 4.7 and define the stepwise optimal ("myopic") portfolio process by

(4.8a) $\qquad \pi_t^* := \varphi_t(H_t),\ 0 \le t<T.$

Then $I_t(h,\varphi_t(h)) \ge I_t(h,0) = 0$ if 0 denotes here a vanishing portfolio. In particular, we have

(4.8b) $\qquad P[1+\varphi_{t-1}(h)^\top\cdot R_t > 0 \mid H_t=h] = 1,$

otherwise $I_t(h,\varphi_t(h)) = -\infty$. Therefore, π^* is admissible. Now we have by (2.11)

$$E[\ln(V_t^{\pi^*}(x))] = E[\ln\{x\cdot\Pi_{m=1}^t (1+r_m)\cdot(1+\pi_{m-1}^{*\top}\cdot R_m)\}]$$
$$= \ln(x) + \Sigma_{m=1}^t E[\ln(1+r_m)] + \Sigma_{m=1}^t E[\ln(1+\pi_{m-1}^{*\top}\cdot R_m)]$$

where

$$E[\ln(1+\pi_{m-1}^{*\top}\cdot R_m)] = E[\,E[\ln(1+\pi_{m-1}^{*\top}\cdot R_m) \mid H_{m-1}]\,]$$
$$= E[\,E[\ln(1+\varphi_{m-1}^\top(H_{m-1})\cdot R_m) \mid H_{m-1}]\,] = E[\,\sup_\vartheta E[\ln(1+\vartheta^\top\cdot R_m) \mid H_{m-1}]\,]$$
$$\ge E[\,E[\ln(1+\pi_{m-1}^\top\cdot R_m) \mid H_{m-1}]\,] \quad \text{for } \pi\in\Pi$$

and $E[\,E[\ln(1+\pi_{m-1}^{*\top}\cdot R_m)^- \mid H_{m-1}]\,] = 0$ as was shown above. []

Theorem 4.8 does not guarantee that $\sup_{\pi\in\Pi} E[\ln(V_t^\pi(x))]$ is finite. For this and other purposes we make an additional assumption which is assumed to hold now throughout the remainder of the paper and which can be looked upon as a uniform (NA)–condition in view of Remark 4.2.

4.9 Uniform (NA)–Assumption. For all $1 \le t \le T$ there exists some $\varepsilon_t > 0$ such that

$$B(0,\varepsilon_t)\cap\mathscr{L}_t(h) \subset \mathrm{conv}(\Sigma_t(h)) \quad \forall\, h \in \Omega_{t-1}.$$

4.10 Remark. Assumption 4.9 holds under the condition (NA)* in each of the following cases:

(1) Ω is finite;

(2) $\Sigma_t(h)$ is independent of $h \in \Omega_{t-1}$ for $1 < t \le T$.

Proof. From Remark 4.2 we know that for each $h \in \Omega_{t-1}$ there exists some $\varepsilon_t(h)>0$ such that $B(0,\varepsilon_t(h))\cap\mathscr{L}_t(h) \subset \mathrm{conv}(\Sigma_t(h))$. In the case (1) we may assume w.l.o.g. that Ω_{t-1} is finite. Then define $\varepsilon_t := \min\{\varepsilon_t(h), h \in \Omega_{t-1}\}$. In the case (2), $\Sigma_t(h)$ and hence $\mathscr{L}_t(h)$ are independent of h. Then just define $\varepsilon_t := \varepsilon_t(h_o)$ for some $h_o \in \Omega_{t-1}$. []

In many papers it is assumed that the random variables $(r_1,R_1),\dots,(r_T,R_T)$ are independent. Then one chooses $H_t = (r_1,R_1,\dots,r_t,R_t)$ and R_t and H_{t-1} are independent. Then $q_t(\cdot\,|h)$ and thus $\Sigma_t(h)$ are independent of h, i.e. 4.10 (2) holds.

4.11 Lemma. (a) $\Theta_t(h)\cap\mathscr{L}_t(h) \subset B(0,1/\varepsilon_t) \quad \forall\, h \in \Omega_{t-1}$.

(b) If $\pi_t \in \Theta_t(H_t)\cap\mathscr{L}_t(H_t)$, $0 \le t < T$, then π is bounded.

(c) For any $\pi \in \Pi$, define $\pi' \in \Pi$ by $\pi'_t = \Gamma_t(H_t)\pi_t$, then $\pi'_t \in \Theta_t(H_t)\cap\mathscr{L}_t(H_t)$, π' is bounded, and $V_t^\pi(x) = V_t^{\pi'}(x)$ a.s., $1 \le t \le T$.

Proof. Part (a) is an immediate consequence of Lemma 4.3c and Assumption 4.9.

b) From part (a) we know that $\|\pi_t\| \le \max\{1/\varepsilon_t, 1 \le t \le T\}$. Finally, part (c) follows from Lemma 4.6. []

4.12 Theorem. Set $V_t^* := V_t^{\pi^*}(1)$ where π^* is defined as in 4.8.

(a) There is some $0 < M < \infty$ such that $|V_t^\pi(1)| \le M$ a.s., $0 \le t \le T$, for any $\pi \in \Pi$.

(b) $\sup_{\pi\in\Pi} E[\ln(V_t^\pi(x))] < \infty$.

(c) For any growth–optimal $\pi \in \Pi$ one has: $V_t^\pi(1) = V_t^*(1)$ a.s..

Proof. a) Choose π' as in 4.11(c) and $\varepsilon_0 := \min\{\varepsilon_t, 1 \le t \le T\}$. Let M' be an upper bound for B_t and $\|R_t\|$, then by (2.11): $0 \le V_t^\pi(1) = V_t^{\pi'}(1) \le \{M'(1 + M'/\varepsilon_0)\}^T =: M$ a.s. .

b) We have $E[\ln(V_t^\pi(x))] \le \ln x + \ln M$.

c) Let $\pi \in \Pi$ be growth–optimal. Then $\pi_t = \psi_t(H_t)$ for some function ψ_t ; we set $\psi'_t(h) := \Gamma_t(h)\psi_t(h)$. In view of Lemma 4.6, we have as in the proof of Theorem 4.8:

$$E[\ln(1+\psi_{m-1}^{T}(H_{m-1})\cdot R_m)] = E[\ln(1+\psi'^{T}_{m-1}(H_{m-1})\cdot R_m)]$$
$$= E[\,E[\ln(1+\psi'^{T}_{m-1}(H_{m-1})\cdot R_m)\,|\,H_{m-1}]\,] = E[\,E[\ln(1+\varphi_{m-1}^{T}(H_{m-1})\cdot R_m)\,|\,H_{m-1}]\,]$$
$$= E[\,\sup_{\vartheta} E[\ln(1+\vartheta^{T}\cdot R_m)\,|\,H_{m-1}]\,] \geq E[\,E[\ln(1+\pi_{m-1}^{T}\cdot R_m)\,|\,H_{m-1}]\,].$$

Thus equality holds throughout where now $E[\ln(1+\psi_{m-1}^{T}(H_{m-1})\cdot R_m)] < \infty$. Therefore

$E[\ln(1+\psi'^{T}_{m-1}(H_{m-1})\cdot R_m)\,|\,H_{m-1}] = \sup_{\vartheta} E[\ln(1+\vartheta^{T}\cdot R_m)\,|\,H_{m-1}]$ a.s. , i.e.

$$I_m(H_{m-1}, \psi'_{m-1}(H_{m-1})) = \sup_{\vartheta} I_m(H_{m-1}, \vartheta).$$

Actually, we have just used Bellman's optimality principle. From the uniqueness result in Lemma 4.7b we conclude that $\psi'_{m-1}(H_{m-1}) = \varphi_{m-1}(H_{m-1})$ and thus $V_t^{\pi^*}(1) = V_t^{\pi}(1)$ a.s.. []

Theorem 4.12c was proved by Becherer (1999) for a semi–martingale framework and gives rise to the following:

4.13 Definition. Define V_t^* as in Theorem 4.12. Then $\{V_t^*,\ 0 \le t \le T\}$ is called the **growth–optimal value process.**

In order to obtain relations to the concept of the numeraire portfolio we need the following result proved in Schäl (1999a,b):

4.14 Theorem. Let φ_t be defined as in Lemma 4.7. Then, the condition

(4.15) $\quad E[\,\vartheta^{T}\cdot R_t/(1+\vartheta^{T}\cdot R_t)\,|\,H_t{=}h] < 0$ for all $\vartheta \in \partial\Theta_t(h) \cap \mathscr{L}_t(h)$, $h \in \Omega_{t-1}$, $1 \le t \le T$,

implies the **first order condition:** $E[R_t^k/(1+\varphi_t(h)^{T}\cdot R_t)\,|\,H_t{=}h] = 0$, $h \in \Omega_{t-1}$ $1 \le t \le T$.

4.16 Corollary. Let $\pi^* = (\pi_t^*)$ be defined by (4.8a) and 4.7c. Then under condition (4.15),

(a) $L_t(Q^*) := 1/(1+\pi_t^{*T}\cdot R_t)$ defines a martingale measure Q^* according to Lemma 3.1;

(b) π^* is a numeraire portfolio.

Proof. We obtain from Theorem 4.14:

$1 = 1 - \Sigma_{k=1}^{d}\ \varphi_t^k(h)\, E[R_t^k/(1+\varphi_t(h)^{T}\cdot R_t)\,|\,H_t{=}h] = 1 - E[\varphi_t(h)^{T}\cdot R_t/(1+\varphi_t(h)^{T}\cdot R_t)\,|\,H_t{=}h]$, thus

(4.17) $\quad E[\,1/(1+\varphi_t(h)^{T}\cdot R_t)\,|\,H_t{=}h] = 1$, $\ h \in \Omega_{t-1}$, $1 \le t \le T$.

In view of (4.8a) we now have:

(4.18a) $\quad E[R_t^k/(1+\pi_t^{*T}\cdot R_t)\,|\,H_t{=}h] = E[R_t^k\cdot L_t(Q)\,|\,H_t{=}h] = 0;$

(4.18b) $\quad E[1/(1+\pi_t^{*T}\cdot R_t)\,|\,H_t{=}h] = E[L_t(Q)\,|\,H_t{=}h] = 1.$

Therefore, the equations (3.2) are satisfied. Thus π^* is a numeraire portfolio. []

4.19 Example. The one–dimensional case.

Consider the case d=1, then $\Sigma_t(h)$ is a compact subset of $\mathbb{R}$. Set

$$-\alpha_t(h) = \min \Sigma_t(h),\ \beta_t(h) = \max \Sigma_t(h).$$

Then $\mathrm{conv}(\Sigma_t(h)) = [-\alpha_t(h),\beta_t(h)]$. Because of Remark 4.2 we know that

$$\alpha_t(h) > 0,\ \beta_t(h) > 0.$$

Then condition (4.15) is satisfied if and only if

(4.20) $$E[\,R_t/(1 + \frac{1}{\alpha_t(h)} R_t) \,|\, H_t{=}h] < 0 < E[\,R_t/(1 - \frac{1}{\beta_t(h)} R_t) \,|\, H_t{=}h]\ ,\ h \in \Omega_{t-1},\ 1{\le}t{\le}T.$$

For a proof, we consider the case t= 1 and we omit the indices 1 and h.

Then $\min_{\sigma\in\Sigma} 1 + \vartheta\cdot\sigma = \min_{-\alpha\le\sigma\le\beta} 1 + \vartheta\cdot\sigma = 1 - \vartheta\cdot\alpha$ for $\vartheta{>}0$ and $= 1 + \beta\cdot\vartheta$ for $\vartheta{<}0$.

Hence, $$\min_{-\alpha\le\sigma\le\beta} 1 + \vartheta\cdot\sigma \ge 0 \Leftrightarrow \vartheta \in \Theta = [-\tfrac{1}{\beta}, \tfrac{1}{\alpha}],$$
$$\min_{-\alpha\le\sigma\le\beta} 1 + \vartheta\cdot\sigma = 0 \Leftrightarrow \vartheta \in \partial\Theta = \{-\tfrac{1}{\beta}, \tfrac{1}{\alpha}\}.$$

Then $E[\frac{\vartheta\cdot R}{1 + \vartheta\cdot R}] = \vartheta\cdot E[\frac{R}{1 + \vartheta\cdot R}] < 0$ for $\vartheta \in \{-\frac{1}{\beta}, \frac{1}{\alpha}\}$ if and only if

(4.20)' $$E[\frac{R}{1 - R/\beta}] > 0 > E[\frac{R}{1 + R/\alpha}]\,.$$

In fact, the latter condition (4.20)' is weak. It can be looked upon as a kind of no–arbitage condition. The martingale case $E[R] = 0$ is not interesting as we can choose $\vartheta = 0$ then. Let us suppose that $E[R] > 0$. Then $E[R/(1 - R/\beta)] \ge E[R] > 0$ and the condition $E[R/(1 + R/\alpha)] < 0$ requires that negative values of R should not have too less probability in the following sense: If the values of R are multiplied by a weight $1/(1 + R/\alpha)$ where $1/(1 + R/\alpha) \le 1$ for $R{\ge}0$ and $1/(1 + R/\alpha) \ge 1$ for $R{\le}0$ and even $1/(1 + R/\alpha) \approx \infty$ for $R \approx -\alpha$. The condition (4.20) can easily be proved to be also necessary for the first order condition [cf. Schäl (1995) or example 5.7 below]. []

We will give a sufficient condition for (4.15) which is far from being necessary, however. Consider the situation where $\Sigma_t(h)$ is a polytope in $\mathcal{L}_t(h)$, i.e. a bounded polyhedral set as in the important case where $\Sigma_t(h)$ is a rectangle

(4.21) $$\Sigma_t(h) = \mathop{\times}_{1}^{d} [-\alpha_{it}(h),\beta_{it}(h)]\ ,\ \alpha_{it}(h),\ \beta_{it}(h) > 0,\ 1{\le}i{\le}d.$$

We assume that the vertices have positive probability or more generally that each ball $B(\sigma,\varepsilon)$ around a vertex σ with radius $\varepsilon{>}0$ has enough probability.

4.22 Theorem. (a) A sufficient condition for (4.15) is the following:

(4.23) there exists some finite (or compact) subset $\Sigma_{ot}(h)$ of $\Sigma_t(h)$ such that $\Sigma_t(h)$ is contained in the convex hull $\mathrm{conv}(\Sigma_{ot}(h))$ of $\Sigma_{ot}(h)$ and
$$\limsup_{\varepsilon\downarrow 0} \tfrac{1}{\varepsilon} P[R_t \in B(\sigma,\varepsilon) \,|\, H_{t-1}{=}h] > \tfrac{1}{\varepsilon_t} \quad \text{for } \sigma \in \Sigma_{ot}(h),\ h \in \Omega_{t-1},\ 1 \le t \le T.$$

(b) If Ω is finite, the condition (4.15) is always satisfied and thus the statements of Corollary 4.16 hold true.

Proof. a) W.l.o.g. consider the case t=1, then Σ_1, Θ_1, $\mathscr{L}_1$, and Σ_{o1} are independent of h. Choose $\vartheta \in \partial\Theta_1 \cap \mathscr{L}_1$. Then one obtains the following relation from the assumption :

$$0 = \min_{\sigma \in \Sigma_1} (1 + \vartheta^T \cdot \sigma) = \min_{\sigma \in \Sigma_{o1}} (1 + \vartheta^T \cdot \sigma) = 1 + \vartheta^T \cdot \sigma_o \quad \text{for some } \sigma_o \in \Sigma_{o1}.$$

Further, by Lemma 4.11, we have then for $\sigma \in B(\sigma_o, \varepsilon)$:

$$1 + \vartheta^T \cdot \sigma = \vartheta^T \cdot (\sigma - \sigma_o) \le \|\vartheta\| \cdot \varepsilon \le \varepsilon/\varepsilon_1 < 1 \quad \text{for sufficiently small } \varepsilon$$

and hence $\{R_1 \in B(\sigma_o,\varepsilon)\} \subset \{1 + \vartheta^T \cdot R_1 \le \varepsilon/\varepsilon_1\} \subset \{\vartheta^T \cdot R_1 < 0\}$. Now

$$E[\vartheta^T \cdot R_1/(1 + \vartheta^T \cdot R_1)] \le E[1_{\{R_1 \in B(\sigma_o,\varepsilon)\}} \cdot \vartheta^T \cdot R_1/(1 + \vartheta^T \cdot R_1)]$$

$$+ E[1_{\{\vartheta^T R_1 > 0\}} \cdot \vartheta^T \cdot R_1/(1 + \vartheta^T \cdot R_1)] \le -\frac{1 - \varepsilon/\varepsilon_1}{\varepsilon/\varepsilon_1} \cdot P[R_1 \in B(\sigma_o,\varepsilon)] + 1.$$

The last expression is negative if

$$\text{(4.24)} \qquad P[R_1 \in B(\sigma_o,\varepsilon)] \cdot \{\tfrac{1}{\varepsilon} - \tfrac{1}{\varepsilon_1}\} > \tfrac{1}{\varepsilon_1}.$$

Now, if $\lim_{\varepsilon \downarrow 0} P[R_1 \in B(\sigma_o,\varepsilon)] = P[R_1 = \sigma_o] > 0$ then $\frac{1}{\varepsilon} P[R_1 \in B(\sigma_o,\varepsilon)]$ is arbitrarily large and hence (4.22) is fulfilled for ε small enough. Otherwise,

$$\limsup_{\varepsilon \downarrow 0} P[R_1 \in B(\sigma_o,\varepsilon)] \cdot \{\tfrac{1}{\varepsilon} - \tfrac{1}{\varepsilon_1}\} = \limsup_{\varepsilon \downarrow 0} P[R_1 \in B(\sigma_o,\varepsilon)] \cdot \tfrac{1}{\varepsilon} > \tfrac{1}{\varepsilon_1}$$

by assumption and again (4.22) is fulfilled for ε small enough.

b) If Ω is finite, then Σ_1 is finite and we may choose $\Sigma_{o1} = \Sigma_1$. Then

$P[R_1 \in B(\sigma_o,\varepsilon)] \ge P[R_1 = \sigma_o] > 0$ and $\frac{1}{\varepsilon} P[R_1 \in B(\sigma_o,\varepsilon)] \to \infty$ for $\varepsilon \downarrow 0$. []

For Theorem 4.22(b) one can also use a result by Hakansson (1971) that the optimal portfolio can be chosen as an interior point in case of the log–utility.

§5 Relations between the market concepts.

Let us define the measure Q* by

$$\text{(5.1)} \qquad \frac{dQ^*}{dP} := B_T / V_T^*, \text{ where } \{V_t^*, 0 \le t \le T\} \text{ is the growth–optimal value process.}$$

It is well known that there are strong relations between the growth–optimal portfolio and the numeraire portfolio [cf. Long (1990)] or the value preserving portfolio [cf. Hellwig (1993), Wiesemann (1995a)]. We start with the following relation [cf. Conze & Viswanathan (1991)]:

5.2 Theorem. Let $\pi \in \Pi$ be a numeraire portfolio, then π is a growth–optimal portfolio.

Proof. We know from (3.22b) that $L_t := 1/(1+\pi_{t-1}^{T}\cdot R_t)$, $1 \le t \le T$, defines some $Q \in \mathcal{Q}$ by $\frac{dQ}{dP} = L = \Pi_{t=1}^{T} L_t = 1/\check{V}_t^{\pi}(1)$. Therefore we can conclude from (3.2a,b) that

(5.3) $\quad E[L\cdot\check{V}_t^{\pi'}(1)] = 1, \quad 1 \le t \le T$, for any $\pi' \in \Pi$.

Now we obtain from Jensen's inequality

$$E[\ln(\check{V}_t^{\pi'}(1))] - E[\ln(\tfrac{1}{L})] = E[\ln(L\cdot\check{V}_t^{\pi'}(1))] \le \ln(1) = 0. \text{ i.e.}$$

$$E[\ln(\check{V}_t^{\pi'}(1))] \le E[\ln(\tfrac{1}{L})] = E[\ln(\check{V}_t^{\pi}(1))].$$

In view of (2.11), π is optimal for any initial value $x>0$. []

On the other hand, it is known that the existence of a growth–optimal portfolio will not imply the existence of a numeraire portfolio [cf. Becherer (1999)]. We will give an example.

5.4 **Example.** We may restrict attention to the case $d=1$ and $T=1$ [cf. Example 4.19]. Let the distribution of $R := R_1$ on $\Sigma := \Sigma_1 := [-1,1]$ be given by

$$E[g(R)] := \lambda\cdot{}_{-1}\!\int^{0} \tfrac{3}{2}(1-\sigma^2)\, g(\sigma)\, d\sigma + (1-\lambda)\cdot{}_{0}\!\int^{1} \tfrac{3}{2}(1-\sigma^2)\, g(\sigma)\, d\sigma$$

where we choose $\lambda>0$ sufficient small, e.g. $\lambda = 1/12$. Then

$$E[R] := \lambda\cdot{}_{-1}\!\int^{0} \tfrac{3}{2}(1-\sigma^2)\,\sigma\, d\sigma + (1-\lambda)\cdot{}_{0}\!\int^{1} \tfrac{3}{2}(1-\sigma^2)\,\sigma\, d\sigma$$

$$= (1-2\lambda)\,{}_{0}\!\int^{1} \tfrac{3}{2}(1-\sigma^2)\,\sigma\, d\sigma = \tfrac{3}{8}(1-2\lambda) > 0.$$

Obviously, by the choice of $\lambda^* = \frac{1}{2}$ one obtains an equivalent martingale measure. Now set

$$f(\vartheta) := E[\frac{R}{1+\vartheta\cdot R}],$$

then f is strictly decreasing on $\Theta := [-1,1]$ where $f(-1) \ge f(\vartheta) \ge f(1)$ for $\vartheta \in \Theta$. Now

$$f(1) = E[\frac{R}{1+R}] = \lambda\cdot{}_{-1}\!\int^{0} \tfrac{3}{2}(1-\sigma)\,\sigma\, d\sigma + (1-\lambda)\cdot{}_{0}\!\int^{1} \tfrac{3}{2}(1-\sigma)\,\sigma\, d\sigma = \tfrac{1}{4} - \tfrac{3}{2}\lambda > 0,$$

$$f(-1) = E[\frac{R}{1-R}] = \lambda\cdot{}_{-1}\!\int^{0} \tfrac{3}{2}(1+\sigma)\,\sigma\, d\sigma + (1-\lambda)\cdot{}_{0}\!\int^{1} \tfrac{3}{2}(1+\sigma)\,\sigma\, d\sigma = \tfrac{5}{4} - \lambda > 0.$$

Hence there is no $\vartheta \in \Theta$ such that $f(\vartheta) = 0$, i.e. such that $dQ/dP = 1/(1+\vartheta\cdot R)$ defines some $Q \in \mathcal{Q}$ and ϑ is hence a numeraire portfolio. On the other hand, we have

$$\infty > f(-1) \ge f(\vartheta) = \frac{d}{d\vartheta} E[\ln(1+\vartheta\cdot R)] \ge f(1) > 0 \quad \text{for } -1 < \vartheta < 1.$$

Thus, we finally obtain that $\max_{\vartheta\in\Theta} E[\ln(1+\vartheta\cdot R)] = E[\ln(1+R)]$ $(= {}_{0}\!\int^{1} f(\vartheta)\, d\vartheta < \infty)$ and $\vartheta^* = 1$ is the growth–optimum portfolio. []

Becherer (1999) gives a more general definition of a numeraire portfolio such that each growth–optimal portfolio is a numeraire portfolio.

5.5 **Theorem.** If $Q \in \mathcal{Q}$ is a martingale measure defined by a numeraire portfolio $\pi \in \Pi$ according to (3.22b) then $Q = Q^*$. Especially this implies that $Q^* \in \mathcal{Q}$ if a numeraire portfolio exists.

Proof. (i) Let π be a numeraire portfolio and $\frac{dQ}{dP} = B_T/V_T^{\pi}(1)$. Then π is growth–optimal. From Theorem 4.12 we conclude $dQ/dP = dQ^*/dP$ a.s. and thus $Q^* = Q \in \Omega$.
(ii) One can also give a direct proof of the uniqueness of Q [cf. Conze & Viswanathan (1991)]. Suppose that $\pi, \tilde{\pi} \in \Pi$ are numeraire portfolios. Because of Proposition 4.12 we may assume that π and $\tilde{\pi}$ are bounded. Therefore, the following expectations are well–defined. Define $L_t = (1 + \pi_{t-1}^{\top} \cdot R_t)^{-1}$ and similarly $\tilde{L}_t$. Then we have
$E[\tilde{L}_t/L_t] = 1$ since $E[\tilde{L}_t/L_t | H_{t-1}] = E[\tilde{L} | H_{t-1}] + (1/\tilde{S}_{t-1}) \cdot \pi_{t-1}^{\top} \cdot E[\tilde{L} \cdot \Delta\tilde{S}_t | H_{t-1}] = 1$
in view of (3.2). By the same argument we obtain $E[L_t/\tilde{L}_t] = 1$. Since $x \mapsto 1/x$ is strictly convex, we would have by the strict form of Jensen's inequality: $1 = E[L_t/\tilde{L}_t] > 1/E[\tilde{L}_t/L_t] = 1$ if $P[\tilde{L}_t/L_t = 1] < 1$. Thus $P[\tilde{L}_t = L_t] = 1$ and $\Pi_{t=1}^{T} L_t = \Pi_{t=1}^{T} \tilde{L}_t$ a.s.. []

5.6 Theorem. The conditions in (a) and those in (b) are mutually equivalent for a plan (ξ,c) where we assume the situation of definition 3.12b for part (b).

(a)
- (1) (ξ,c) is value preserving (w.r.t. Q);
- (2) (ξ,c) is weakly value preserving and interest oriented (w.r.t.Q);
- (3) $c_t = r_t(Q) \cdot V_{t-1,T}^{\xi,c}(x | Q)$, $1 \le t \le T$, and $V_T^{\xi,c}(x) = x$.

(b)
- (1) (ξ,c) is value oriented w.r.t. Q ;
- (2) (ξ,c) is weakly value oriented w.r.t. $\{X_t(Q)\}$ and interest oriented w.r.t. Q;
- (3) $c_t = [r_t(Q) - r_t(X | Q)] \cdot V_{t-1,T}^{\xi,c}(x | Q)$, $1 \le t \le T$, and $V_T^{\xi,c}(x) = X_T(Q)$ where $r_t(X | Q) := [X_t(Q) - X_{t-1}(Q)] / X_{t-1}(Q)$.

The quantity $r_t(X | Q)$ is called the **desired growth rate** in Hellwig (1996c) and is only defined if $X_{t-1}(Q) > 0$. However this property is necessary if the plan (ξ,c) should be admissible.

Proof. We only consider (b) and write $X_t := X_t(Q)$, $V_t := V_t^{\xi,c}(x)$ and $V_{t,T} := V_{t,T}^{\xi,c}(x | Q)$.
"(1) $\Rightarrow$ (2)" Suppose (3.14b) holds. By Lemma 3.17, (ξ,c) is weakly value oriented. Thus, we can then replace X_t by V_t and obtain (3.20a).
"(2) $\Rightarrow$ (1) Suppose $V_t = X_t$ for $1 \le t \le T$. From (3.20a) we get
$X_t + c_t = X_{t-1} \cdot [1 + r_t(Q)]$, $1 \le t \le T$, i.e. (3.14b) holds.
"(1) $\Rightarrow$ (3)" We can apply both (3.14a) and (3.14b).
"(3) $\Rightarrow$ (1)" We want to show (3.14a), i.e. $V_{t,T} = X_t$, which holds for t=T by assumption. Now assume (3.14a) for t. Then we obtain from (iii): $[1 + r_t(Q)] \cdot V_{t-1,T} - c_t = V_{t-1,T} \cdot X_t/X_{t-1} = V_{t-1,T} \cdot V_{t,T})/X_{t-1}$. On the other hand we know from (3.11a):
$[1 + r_t(Q)] \cdot V_{t-1,T} - c_t = V_{t,T}$; hence $X_{t-1} = V_{t-1,T}$. []

In the next theorems, we will use the transformation (2.10) from ξ to π also for a not necessarily self–financing plan (ξ,c).

5.7 Theorem. Suppose that (ξ,c) is an admissible plan and define $\pi \in \Pi$ by

$$\pi_t^k := \xi_t^k \cdot S_t^k / V_t^{\xi,c}(x),\ 1 \le k \le d,\ 0 \le t < T.$$

Then the following statements are equivalent for $Q \in \mathcal{Q}$:

(1) (ξ,c) is interest oriented w.r.t. Q ;

(2) (ξ,c) is interest oriented w.r.t. $Q = Q^*$;

(3) π forms a numeraire portfolio and $\frac{dQ}{dP} = \Pi_{t=1}^T\ 1/(1 + \pi_{t-1}^\top \cdot R_t)$;

(4) π forms a numeraire portfolio and $Q = Q^*$.

Proof. By definition of π we have:

$$1 + (1/\breve{V}_{t-1}^{\xi,c}(x))\ \xi_{t-1}^\top \cdot \Delta\breve{S}_t = 1 + \Sigma_{k=1}^d\ (1/\breve{V}_{t-1}^{\xi,c}(x))\xi_{t-1}^k\ R_t^k \cdot \breve{S}_{t-1}^k$$
$$= 1 + (\xi_{t-1}^k \cdot S_{t-1}^k / V_{t-1}^{\xi,c}(x))\ R_t^k = 1 + \pi_{t-1}^\top \cdot R_t$$

Now let be $Q \in \mathcal{Q}$ with $L_t(Q) = [1 + r_t]/[1 + r_t(Q)]$, then for $1 \le t \le T$:

(3.20a) $\Leftrightarrow$ $V_t^{\xi,c}(x) + c_t = V_{t-1}^{\xi,c}(x) \cdot [1 + r_t(Q)]$ $\Leftrightarrow$ $\breve{V}_t^{\xi,c}(x) + \breve{c}_t = \breve{V}_{t-1}^{\xi,c}(x) \cdot [1 + r_t(Q)]/[1 + r_t]$,

and hence by (2.9): (3.20a) $\Leftrightarrow$ $\breve{V}_{t-1}^{\xi,c}(x) + \xi_{t-1}^\top \cdot \Delta\breve{S}_t = \breve{V}_{t-1}^{\xi,c}(x)/L_t(Q)$

$\Leftrightarrow$ $1/L_t(Q) = 1 + (1/\breve{V}_{t-1}^{\xi,c}(x))\ \xi_{t-1}^\top \cdot \Delta\breve{S}_t$.

Thus we conclude from the first relation: (1) $\Leftrightarrow$ $1/L_t(Q) = 1 + \pi_{t-1}^\top \cdot R_t$, $1 \le t \le T$, $\Leftrightarrow$ (3).

We know from Theorem 5.5 that "(3) $\Leftrightarrow$ (4)" ; finally we have (1) $\Leftrightarrow$ (2). []

From Theorems 5.6 and 5.7 we obtain:

5.8 Corollary. Let be $x > 0$, $Q \in \mathcal{Q}$, and (ξ,c) some plan.

(a) If the plan (ξ,c) is value preserving in x with associated Q then $Q = Q^*$.

(b) If in the situation of definition 3.12b where $X_t(Q) > 0$ for $t<T$ (ξ,c) is value oriented with associated Q, then $Q = Q^*$.

The corollary and Theorem 5.6 (a) in particular imply the following suggestive representation of the consumption of a value preserving plan (ξ,c):

(5.8a) $$c_t = r_t(Q^*) \cdot V_{t-1}^{\xi,c}(x).$$

We can now explain the construction of a value preserving plan.

5.9 Theorem. Suppose π is a numeraire portfolio and $\{X_t, 0\le t\le T\}$ is an adapted stochastic process. Define $\xi_t^k = (X_t/S_t^k)\,\pi_t$, $0\le t<T$, $c_t = (1+r_t)\cdot(1+\pi_{t-1}^{\top}\cdot R_t)\cdot X_{t-1} - X_t$. Then (ξ,c) is value oriented w.r.t. $\{X_t, 0\le t\le T\}$ in $x := X_0$ [necessarily with associated $Q = Q^* \in \Omega$ and $\{X_t(Q^*), 0\le t\le T\} := \{X_t, 0\le t\le T\}$].

The construction of (ξ,c) in Theorem 5.9 was also given by Wiesemann (1995).

Proof. Set $\check{X}_t := X_t/B_t$ and $L_t(Q^*) := 1/(1+\pi_{t-1}^{\top}\cdot R_t) =: [1+r_t]/[1+r_t(Q^*)]$. Then we have by assumption $\check{c}_t = (1+\pi_{t-1}^{\top}\cdot R_t)\cdot\check{X}_{t-1} - \check{X}_t$.
We want to consider the case where we start with an initial wealth $x = X_0$ and we first will show: $V_t^{\xi,c}(x) = X_t$. The property holds for t=0. Now suppose it is true for t−1. Then we have by (2.9):

$$\check{V}_t^{\xi,c}(x) = \check{V}_{t-1}^{\xi,c}(x) + \xi_{t-1}^{\top}\cdot\Delta\check{S}_t - \check{c}_t = \check{X}_{t-1} + \Sigma_{k=1}^{d} X_{t-1}\cdot\pi_{t-1}^{k}\cdot\Delta\check{S}_t^k/S_{k-1}^k - \check{c}_t$$
$$= \check{X}_{t-1} + \Sigma_{k=1}^{d} \check{X}_{t-1}\cdot\pi_{t-1}^{k}\cdot\Delta\check{S}_t^k/\check{S}_{k-1}^k - \check{c}_t = \check{X}_{t-1}\cdot(1+\pi_{t-1}^{\top}\cdot R_t) - \check{c}_t = \check{X}_t.$$

Now we know that $V_T^{\xi,c}(x) = X_T$. From the definition of c_t we further have:
$c_t = [1+r_t(Q^*)]\cdot X_{t-1} - X_t$. Thus, relation (3.14b) is satisfied. []

The theorem provides the following construction of the value oriented plan:
We start with an initial wealth $V_0 = x$. Suppose we have constructed the value V_{t-1} of the plan at t−1 such that $V_{t-1} = X_{t-1}$. Then we choose $\pi_{t-1} := \pi_{t-1}^*$ according to (4.8a) which is growth optimal in the sense of Lemma 4.7c. [We know by Theorem 5.2 that the numeraire portfolio π is necessarily growth optimal]. Then we get according to (2.9)

(5.9a)
$$V_{t-} = B_t\cdot[\check{V}_{t-1} + \xi_{t-1}^{\top}\cdot\Delta\check{S}_t]$$
$$= \check{X}_{t-1}\cdot B_t\cdot[1+\pi_{t-1}^{\top}\cdot R_t] = X_{t-1}\cdot(1+r_t)\cdot[1+\pi_{t-1}^{\top}\cdot R_t].$$

The second identity can be shown as (2.11). Now we choose c_t such that $V_t = V_{t-} - c_t = X_t$.

5.10 Corollary. If Ω is finite or if condition (4.23) holds or more generally if condition (4.15) holds, then there exists a numeraire portfolio, an interest oriented plan, a value preserving plan in x and a value oriented plan in x w.r.t. any adapted stochastic process $\{X_t, 0\le t\le T\}$ such that $X_t>0$ for $t<T$.

Proof. From Theorem 4.22 and Corollary 4.16 we know the existence of a numeraire portfolio. From Theorem 5.9 we then obtain a value oriented plan which is value preserving for $X_t = x$, $0\le t\le T$, and interest oriented by Theorem 5.6. []

5.11 Remark on option pricing. Under the conditions of Corollary 5.10, a possible candidate for the price process of a contingent claim X is given by

$$X_t^* := X_t(Q^*) = B_t \cdot E_{Q^*}[\frac{X}{B(T)} | H_t]$$

where Q* is the equivalent martingale measure corresponding to the numeraire portfolio π (according to Theorem 5.5). As in general incomplete markets there is no possibility to replicate the claim X, there remains the question of a suitable hedging strategy. By Theorem 5.9, a value oriented hedging plan corresponding to $\{X_t^*\}$ is given by

$$\xi_t^k = (X_t^*/S_t^k) \cdot \pi_t^k, \quad c_t = (1+r_t) \cdot (1+\pi_{t-1}^T \cdot R_t) \cdot X_{t-1}^* - X_t^*.$$

Using the notation $\check{X}_t^* := X_t^*/B_t$ we obtain

$$E_{Q^*}[c_t/B_t | H_{t-1}] = \check{X}_{t-1}^* \cdot E_{Q^*}[\pi_{t-1}^T \cdot R_t | H_{t-1}] - E_{Q^*}[\Delta \check{X}_t^* | H_{t-1}] = \check{X}_{t-1}^* \cdot 0 - 0 \text{, i.e.}$$

(5.12) $$E_{Q^*}[c_t/B_t | H_{t-1}] = 0 \ , \ 1 \le t \le T,$$

which can be interpreted as a condition for (ξ,c) being mean–self–financing with respect to Q*, i.e. the conditional expected discounted signed consumption has a zero value. In other words, the aggregated discounted consumption is a Q*–martigale. As the consumption payments are real physical payments it should be interesting for the investor how the (aggregated) consumption process evolves with respect to the physical measure P. Let us again use the notation :

$$L_t := 1/(1 + \pi_{t-1}^T \cdot R_t) \text{, } \frac{dQ^*}{dP} = \Pi_{t=1}^T L_t \text{, } \Pi_{m=1}^t L_m = B_t/V_t^*.$$

Then we get $(c_t/B_t) \cdot \Pi_{m=1}^t L_m = c_t/V_t^*$ and by (3.5b) :

$$\Pi_{m=1}^{t-1} L_m \cdot E_{Q^*}[c_t/B_t | H_{t-1}] = \Pi_{m=1}^{t-1} L_m \cdot \frac{1}{B_{t-1}} \cdot E[L_t \cdot \frac{1}{1+r_t} \cdot c_t | H_{t-1}] = E[c_t/V_t^* | H_{t-1}] \text{, i.e.}$$

(5.13) $$E[c_t/V_t^* | H_{t-1}] = 0 \text{, } \ 1 \le t \le T.$$

in view of (5.12). The investor is free to use the value process of the numeraire portfolio as numeraire instead of the bond price process (cf. Artzner (1997)). With the new numeraire. however, it turns out by (5.13) that (ξ,c) is now mean–self–financing with respect to P. Thus our plan (ξ,c) enjoys the two properties of being a mean–self–financing hedging plan and of being locally growth optimal in the sense of (5.9a). []

5.14 Example. [**minimal martingale measure, Girsanov transformation**].
Again we consider the case d = 1, then we have as in Example 4.19:

(5.15) $\text{conv}(\Sigma_t(h)) = [-\alpha_t(h), \beta_t(h)]$ for some $\alpha_t(h) > 0$, $\beta_t(h) > 0$ with $-\alpha_t(h) \in \Sigma_t(h)$, $\beta_t(h) \in \Sigma_t(h)$.

By the discrete–time Girsanov transformation one obtains the minimal martingale Q^o according to $L_t(Q^o) = b_{t-1}(H_{t-1}) + a_{t-1}(H_{t-1}) \cdot R_t$ [cf. Schweizer (1995)]. From (3.2) one can compute:

(5.16) $b_{t-1}(H_{t-1}) = 1 + \{\mu_{t-1}/\sigma_{t-1}\}^2$, $a_{t-1}(H_{t-1}) = -\mu_{t-1}/\sigma^2_{t-1}$ where
$\mu_t := E[R_{t+1}|H_t]$ and $\sigma^2_t := Var[R_{t+1}|H_t] := E[R^2_{t+1}|H_t] - \mu^2_t$.

One difficulty with the Girsanav transformation in discrete time is that it may lead to a density with positive and negative values. The resulting martingale measure is then called a **signed martingale maesure.**

For example, take T=1, $\Sigma_1 = \{-1, \frac{1}{2}, 1\}$, $P[R = \pm 1] = 1/12$, hence $P[R = \frac{1}{2}] = 5/6$.

Then $\mu_0 = 5/12$, $\sigma^2_0 = 29/(12)^2$, hence $a_0 = -60/29$, $b_0 = 54/29$ and $L_1(Q^o) = b_0 + a_0 \cdot R_1$.
But $L_1(Q^o) < 0$ on $\{R_1 = 1\}$.

On the other hand we know from Theorem 4.22b and Corollary 4.16a that $L_1 = \{1 + \pi^*_0 R_1\}^{-1} > 0$ always defines a martingale measure if Ω is finite. Now we will prove that the martingale measure from the Girsanov transformation coincides with the martingale measure from the numeraire portfolio only in a binomial model that means only for a complete market according to Harrison & Pliska (1981) and Jacod & Shiryaev (1998).. A (non–Markovian) binomial model is characterized by the fact

(5.17) $R_t \in \{-\alpha_t(H_{t-1}), \beta_t(H_{t-1})\}$ a.s. $1 \le t \le T$.

5.18 Theorem. Let Q* be the measure defined by (5.1) and let Q^o be the (possibly signed) minimal martingale measure. Then $Q^* = Q^o$ if and only if (5.17) holds.

Proof. If (5.17) holds then we conclude from Theorem 4.22 and Corollary 4.16 that $Q^* \in \mathcal{Q}$. Moreover it is easy to derive from (5.17) that there is exactly one martingale measure (even in the larger class of signed martingale measures) which necessarily agrees with Q* and Q^o [cf. Jacod & Shiryaev (1998)]. Now assume that $Q^o = Q^*$. Then we have $dQ^o/dP = dQ^*/dP \ge 0$ a.s. and thus Q^o is not only a signed martingale measure but even $Q^o \in \mathcal{Q}$, and hence $Q^* \in \mathcal{Q}$. Thus

$$L_t(Q^o) = b_{t-1}(H_{t-1}) + a_{t-1}(H_{t-1}) \cdot R_t = \{1 + \pi^*_{t-1} \cdot R_t\}^{-1} =: L^*_t \quad \text{a.s.}.$$

Assume that $P[-\alpha_t(H_{t-1}) < R_t < \beta_t(H_{t-1})\}] > 0$. We have by (4.8a): $\pi^*_{t-1} = \varphi_{t-1}(H_{t-1})$.
Then there is some $h \in \Omega_{t-1}$ such that:

(1) $P[-\alpha_t(h) < R_t < \beta_t(h)\} | H_{t-1}=h] > 0$,

(2) $P[b_{t-1}(h) + a_{t-1}(h) \cdot R_t = \{1 + \varphi_{t-1}(h) \cdot R_t\}^{-1} | H_{t-1}=h] = 1$,

(3) $E_{Q^*}[R_t | H_{t-1}=h] = 0$.

Now we let $h \in \Omega_{t-1}$ be fixed and drop the indices t and h and we define:

$$f(\sigma) := \{1 + \varphi \cdot \sigma\}^{-1} \text{ on } [-\alpha, \beta].$$

As in the proof of Lemma 4.3a one can show that (2) is equivalent to

(2)' $b + a \cdot \sigma = f(\sigma) \quad \forall \sigma \in \Sigma_t(h)$.

In particular, $b + a \cdot \sigma = f(\sigma)$ for $\sigma \in \{-\alpha, \beta\}$.

By solving this system of two equations one obtains

$$b = \frac{1}{\alpha+\beta}\left[f(\beta)\cdot\alpha + f(-\alpha)\cdot\beta\right].$$

(The solution is known from the binomial model.) From the strict convexity of f we conclude that:

$$f(\sigma) < \frac{\sigma+\alpha}{\alpha+\beta}\cdot f(\beta) + \frac{\beta-\sigma}{\alpha+\beta}\cdot f(\alpha) = b + \sigma \cdot \frac{f(\beta) - f(-\alpha)}{\alpha + \beta} \quad \text{for } -\alpha < \sigma < \beta .$$

Thus we obtain from (1) and (3): $E_{Q^*}[f(R_t) \mid H_{t-1}=h] < b$.

On the other hand we know from (2) and the Bayes formula (3.4):

$$E_{Q^*}[f(R_t) \mid H_{t-1}=h] = E[L_t^* \cdot f(R_t) \mid H_{t-1}=h] = E[L_t^* \cdot (b + a \cdot R_t) \mid H_{t-1}=h]$$
$$= E_{Q^*}[b + a \cdot R_t \mid H_{t-1}=h] = b.$$

A similar kind of argument was used by Schäl (1999) to get an upper bound for the price of a contingent claim in terms of the price of a binomial model. In continuous–time models with continuous asset prices the situation is completely different. There Q* agrees with the minimal martingale measure although the market may be incomplete [cf. Korn (1998)]. []

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