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Hyperplane transversals of homothetical, centrally symmetric polytopes

By

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Abstract

Let $P \subset \mathbb{R}^n$, $n \geq 2$, be a centrally symmetric, convex n-polytope with $2r$ vertices, and $\mathcal{P}$ be a family of $m \geq n+1$ homothetical copies of P. We show that a hyperplane transversal of all members of $\mathcal{P}$ (it it exists) can be found in $O(rm)$ time.

Keywords: center hyperplane, centrally symmetric polytope, common transversal, hyperplane transversal, Minkowski space, polyhedral norm, scaled translates, stabbing problem

1. INTRODUCTION

Let $\mathcal{C} := \{C_1, \cdots, C_m\}$ be a family of convex sets in n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$. Then a hyperplane $H \subset \mathbb{R}^n$ is said to be a *hyperplane transversal* (or a *stabbing hyperplane*) with respect to $\mathcal{C}$ if $H \cap C_i \neq \emptyset$ for each $i \in \{1, \cdots, m\}$. The *hyperplane transversal problem* is to find out whether there exist hyperplane transversals with respect to such a family $\mathcal{C}$. (Note that the replacement of the hyperplane above by a k-dimensional affine flat, $1 \leq k \leq n-1$, yields the more general *k-flat transversal problem.*)

In this paper we solve a special case of the hyperplane transversal problem, namely the case when $\mathcal{C}$ is restricted to a family $\mathcal{P}$ of $m \geq n+1$ scaled translates of a centrally symmetric, convex n-polytope P with $2r$ vertices (i.e., $\mathcal{P}$ consists of m homothetical copies of P). To exclude trivial subcases, we will always assume that the affine hull of the centers of all polytopes from $\mathcal{P}$ is n-dimensional. Using results from location theory (referring to an extension of the *point set width problem*) we will show that this problem can be solved in $O(rm)$ time for any fixed dimension $n \geq 2$.

With respect to the k-flat transversal problem, the following results are known. In the planar case a line transversal of a family $\mathcal{C}$ of n convex sets can be found in $O(n \log n)$ time (cf. [12]), and this time complexity has been shown to be optimal by [3], even in the case when all members of $\mathcal{C}$ are translates of each other. If, in addition, the members of $\mathcal{C}$ are pairwise disjoint translates of a convex set, then linear time is enough, see [10]. Finding a line stabber for m translates of a convex polygon with s vertices can be done in $O(sm)$ time, cf. [6].

Whereas there are mumerous approaches to line stabbing problems in the plane, only a few algorithms for analogous problems in higher dimensions are known. For example, [1] succeeded in stabbing m line segments in $\mathbb{R}^n$ by a hyperplane with $O(m^n)$ time, and for $n = 3$ a plane stabber for a set of m convex polyhedra with a total of sm vertices can be found in $O(s^2m^3)$ time. For further results on stabbing convex polyhedra in $\mathbb{R}^3$ see [8]. In higher dimensions, e.g. the following results are known: If $\mathcal{P}$ is a family of convex n-polytopes having a total of s *vertices*, then the space of hyperplane transversals of $\mathcal{P}$ can be constructed in $O(s^n)$ time, cf. [9] and [7]. If $\mathcal{P}$ consists of m convex n-polytopes with a total of a *edges*, then a hyperplane transversal of $\mathcal{P}$ can be found in $O(m \cdot a^{n-1})$ time (see [2]), and from [5] and [6] it follows that the same time complexity is sufficient if a denotes the total number of *directions determined by polytope edges* of all members of $\mathcal{P}$.

More algorithmical approaches to stabbing problems are discussed in section 5 of [11], and mainly theoretical results about k-flat transversals (e.g., related to Helly-type theorems) can be found in [21], [4], [26], [27], and [23] (the above mentioned survey [11] contains also a lot of theoretical results). For example, [20] investigates the problem of stabbing boxes in higher dimensions, and related results for general convex polytopes were obtained by [6], [1] and others. However, we could not find our result presented here in the known literature.

2. BASIC NOTIONS AND A RELATED LOCATION PROBLEM

Since our result on hyperplane transversals is shown to be strongly related to a result from locaton science (namely, that of finding *center hyperplanes* of finite point sets in Minkowski spaces), we have to introduce some notions related to distance measures and finite-dimensional normed spaces. According to [19] (see also [25] for a modern representation) we define norms geometrically, with the help of the respective unit balls. For $x \in \mathbb{R}^n, n \geq 2$, and $B \subset \mathbb{R}^n$ a compact, convex set with nonempty interior and centred at the origin, the *norm* $\gamma : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by

$$\gamma(x) := \min\{\lambda > 0 : x \in \lambda B\},$$

and B is said to be the *unit ball* of the n-dimensional Minkowski space M^n equipped with γ. In the usual way, this yields the *distance*

$$d(x, y) = \gamma(y - x)$$

between two points $x, y \in \mathbb{R}^n$, and the distance between a point $x \in \mathbb{R}^n$ and a hyperplane $H \subset \mathbb{R}^n$ is given by

$$d(x, H) = \inf_{y \in H} d(x, y).$$

If, in particular, B is a centrally symmetric, convex polytope with vertex set

$$\text{vert } B = \{b_1, \cdots, b_r, -b_1, \cdots, -b_r\},$$

then γ is called a *polyhedral norm*, and the vectors $b_i, -b_i$, $i = 1, \cdots, r$, are also said to be the *fundamental directions* of γ. On the other hand, l_2 denotes the *Euclidean distance* with unit ball

$$B_e = \{x = (a_1, \cdots, a_n) \in \mathbb{R}^n : \sqrt{a_1^2 + \cdots + a_n^2} \leq 1\}.$$

Another distance measure (which is neither a norm nor a metric, since the respective interpoint distance may be infinite) is the so-called *t-distance* which, for a given direction $t \in \mathbb{R}^n$, is denoted by

$$d_t(x, y) := \gamma_t(y - x),$$

where

$$\gamma_t(x) := \begin{cases} \mid \alpha \mid & \text{if } x = \alpha t, \\ \infty & \text{otherwise}. \end{cases}$$

It is clear that (instead of using d_t as an interpoint distance) the consideration of t-distances *between points and hyperplanes*, given by

$$d_t(x, H) := \min\{\mid \lambda \mid : x + \lambda t \in H\}$$

with $\min \emptyset := \infty$, makes sense.

In particular, if t equals the n-th unit vector e_n of $\mathbb{R}^n$, we get the *vertical distance* between $x = (a_1, \cdots, a_n)$ and $y = (b_1, \cdots, b_n)$:

$$d_{\text{ver}}(x, y) = \begin{cases} \mid b_n - a_n \mid & \text{if } a_i = b_i \text{ for all } i = 1, \cdots, n-1, \\ \infty & \text{otherwise}. \end{cases}$$

(The reason for introducing the distances d_t and d_{ver} is that hyperplane location problems with resect to these distances can be solved in a convenient manner, and that these restricted location problems form the basic building block for solving the general case referring to $d(x, H)$.)

Now we are ready to present the announced location problem: Given a set of m points

$$X = \{x_1, \cdots, x_m\} \subset \mathbb{R}^n$$

with corresponding weights $w_i > 0$, $i = 1, \cdots, m$, find a hyperplane H such that

$$g(H) := \max_{i \in \{1, \cdots, m\}} w_i \cdot d(x_i, H)$$

is minimized. (To exclude trivial subcases, also for the location problem we always assume that the affine hull of X is the whole space $\mathbb{R}^n$.) A hyperplane minimizing g is called a (weighted) *center hyperplane* with respect to the given point set, and it is obvious that for equal weights ($w_1 = \cdots = w_m = 1$, say) the search of such an optimal hyperplane (i.e., the *center hyperplane problem*) is equivalent to the *point set width problem.* For surveys on the center hyperplane problem with Euclidean distance we refer to [13], [14], and [15], and its extension to finite-dimensional normed spaces is studied by [17] and [22], see also [18] for strongly related results.

3. THE RELATION BETWEEN BOTH THE PROBLEMS

It is mentioned already in [13] and [15] that the hyperplane $H \subset \mathbb{R}^n$ is a hyperplane transversal of the family of balls

$$A = \{\{x : w_i \cdot l_2(x_i, x) \leq 1\} : i = 1, \cdots, m\}$$

if the objective value of the corresponding center hyperplane problem (with respect to the point set $X = \{x_1, \cdots, x_m\}$ with corresponding positive weights w_i) is less than or equal to 1. In the following we prove a more general statement which will be needed to transform algorithms from center problems to transversal problems.

For any real number $\lambda \geq 0$ and any distance measure d let $A_i(\lambda)$ denote the convex hull of all points whose weighted distance to $x_i \in X$ is less than or equal to λ, i.e.,

$$A_i(\lambda) := \text{ conv } \{x : w_i \cdot d(x_i, x) \leq \lambda\}$$

for all $i \in \{1, \cdots, m\}$, and $\mathcal{A}(\lambda) := \{A_i(\lambda) : i = 1, \cdots, m\}$. Recall that for a hyperplane H the objective value of the respective center problem is

$$g(H) = \max_{i \in \{1, \cdots, m\}} w_i \cdot d(x_i, H) .$$

Lemma 1. *For $X = \{x_1, \cdots, x_m\} \subset \mathbb{R}^n$ and $w_1, \cdots, w_m > 0$, let d be a distance measure of a norm such that all sets in $\mathcal{A}(\lambda_0)$ are connected for $\lambda_0 \geq 0$. Then for any hyperplane H*

$$g(H) \leq \lambda_0 \Leftrightarrow H \textit{ is a hyperplane transversal of } \mathcal{A}(\lambda_0) .$$

Proof.

$$\begin{aligned} g := g(H) \leq \lambda_0 &\Leftrightarrow w_i \cdot d(x_i, H) \leq \lambda_0 \text{ for all } i \in \{1, \cdots, m\} \\ &\Leftrightarrow \text{for each } i \in \{1, \cdots, m\} \text{ there exists an } \overline{x}_i \in H \text{ with} \\ & w_i \cdot d(x_i, \overline{x}_i) \leq \lambda_0 \\ &\Leftrightarrow H \cap A_i(g) \neq \emptyset \text{ for all } i \in \{1, \cdots, m\} \\ &\Leftrightarrow H \text{ is a hyperplane transversal of } \mathcal{A}(g) . \quad \blacksquare \end{aligned}$$

In a direct way, this lemma implies

Theorem 1. *Let $X = \{x_1, \cdots, x_m\} \subset \mathbb{R}^n, n \geq 2$, and $w_1, \cdots, w_m > 0$ be given as above. Then g^* is the objective value of the center hyperplane problem with respect to this weighted point set and the distance measure d if and only if g^* is the smallest real number $\lambda \geq 0$ for which $\mathcal{A}(\lambda)$ has a hyperplane transversal. Moreover, a hyperplane $H \subset \mathbb{R}^n$ is a center hyperplane with objective value g^* if and only if H is a hyperplane transversal of $\mathcal{A}(g^*)$, and for all $\lambda < g^*$ no hyperplane transversal of $\mathcal{A}(\lambda)$ exists.*

We remark that this equivalence is similar to the equivalence of *center problems* in the sense of *point location* (i.e., the center hyperplane is replaced by a center point) and *piercing problems.* For a definition of piercing problems, a proof of the corresponding equivalence and various approaches we refer to [24].

4. FINDING CENTER HYPERPLANES

As already mentioned, the following results are the basic building block for getting hyperplane transversals of centrally symmetric, homothetical convex polytopes. So it is necessary to present them here in a compressed form; for proofs of all results in this section we refer to [17] and [22].

Theorem 2. *For all distances derived from norms there exists a center hyperplane $H \subset \mathbb{R}^n, n \geq 2$, which is at maximum distance from $n+1$ affinely independent points from the given set X.*

Thus, by enumerating all hyperplanes at maximum distance from $n+1$ affinely independent points from X, one can find a center hyperplane in polynomially bounded time. For the special case that the distance measure $d = d_B$ is derived from a polyhedral norm (i.e., the unit ball B is a polytope in $\mathbb{R}^n$) we have

Lemma 2. *Let $H \subset \mathbb{R}^n, n \geq 2$, be a hyperplane and* vert $B = \{b_1, \cdots, b_r, -b_1, \cdots, -b_r\}$ *the set of fundamental directions of a block norm γ_B. Then there exists an index $j \in \{1, \cdots, r\}$ such that for all $x \in \mathbb{R}^n$*

$$d_B(x, H) = d_{b_j}(x, H) .$$

Thus one can decompose the center hyperplane problem for block norms into r independent subproblems, by solving the location problem separately for each $j \in \{1, \cdots, r\}$ with respect to d_{b_j} and then by chosing a hyperplane with the smallest objective value among the corresponding r values.

Further on, each of the r subproblems can be simplified by

Lemma 3. *Let $p, q \in \mathbb{R}^n, n \geq 2$, and D be a linear transformation with $D(p) = q$ and with* det $(D) \neq 0$. *Then*

$$d_q(D(x), D(H)) = d_p(x, H) ,$$

where $D(H) := \{D(y) : y \in H\}$.

Obviously, each subproblem with respect to $d_t = d_{b_j}$ can be transformed to a center hyperplane problem with respect to vertical distance d_{ver}, and it is well-known that such a location problem can be solved efficiently by linear programming methods, see [16]. Thus we can present the following decomposition algorithm.

Algorithm 1

<u>**Input:**</u> Block norm distance d_B with fundamental directions $b_1, -b_1, \cdots, b_r, -b_r$; $x_i \in \mathbb{R}^n, n \geq 2$; $w_i > 0$ for all $i \in \{1, \cdots, m\}$

<u>**Output:**</u> Hyperplane $H^* \subset \mathbb{R}^n$ with objective value z^* which solves the center hyperplane problem with respect to d_B

1. $z^* := \infty$

2. For $j = 1$ to r do

1. Determine a transformation D such that $D(b_j) = e_n$ and det $(D) \neq 0$.

2. For $i \in \{1, \cdots, m\}$ do: $D(x_i) = Dx_i$.

3. Find a hyperplane H_j^* minimizing

$$g(H) = \max_{i \in \{1,\cdots,m\}} w_i \cdot d_{\text{ver}}(Dx_i, H)\,.$$

(Use the algorithm due to [16].)

4. If $g(H_j^*) < z^*$, then set $z^* := g(H_j^*)$ and $H^* := D^{-1}(H_j^*)$.

3. Output: H^* with objective value z^*.

This algorithm runs in $O(rm)$ time, since the corresponding center hyperplane problems with respect to d_{ver} can be solved in linear time for any fixed dimension, cf. [16].

5. TRANSVERSAL ALGORITHM

With the help of Algorithm 1 we are able to solve the hyperplane stabbing problem efficiently. Let $\mathcal{A}$ be a given family of scaled translates of a centrally symmetric, convex n-polytope $B \subset \mathbb{R}^n, n \geq 2$. Using Theorem 1 we can establish the following algorithm.

Algorithm 2 (for finding a hyperplane transversal of a family of homothetical, centrally symmetric and convex n-polytopes in $\mathbb{R}^n, n \geq 2$)

Input: A convex n-polytope B centred at the origin, defined by its vertex set vert $\{b_1, \cdots, b_r, -b_1, \cdots, -b_r\}$, and a family $\mathcal{A} = \{A_1, \cdots, A_m\}$ of scaled translates of B, i.e.,

$$A_i = x_i + \lambda_i B, \;\; x_i \in \mathbb{R}^n, \lambda_i > 0 \;\text{ for all }\; i \in \{1, \cdots, m\}\,.$$

Output: A hyperplane transversal of $\mathcal{A}$, if it exists.

1. Define for all $i \in \{1, \cdots, m\}$

$$\overline{x}_i := x_i \;\text{ and }\; w_i := \frac{1}{\lambda_i}$$

as a set of points $\overline{x}_i \in \mathbb{R}^n$ with positive weights w_i.

2. Use Algorithm 1 to obtain a hyperplane H^* and the objective value z^* for the set of given points $\{\overline{x}_1, \cdots, \overline{x}_m\}$ with corresponding weights $w_1, \cdots, w_m > 0$.

3. If $z^* \leq 1$, then H^* is a hyperplane transversal of $\mathcal{A}$. If $z^* > 1$, then there is no hyperplane transversal of $\mathcal{A}$.

It is obvious that (based on Algorithm 1 above, and on [16]) also this algorithm is running in $O(rm)$ time. And it is also worth mentioning that the assumption that the polytopes are *n-dimensional* is not explicitly used; i.e., this time complexity also refers to the case of degenerate convex polytopes with a center of symmetry, e.g. also to line segments. To our best knowledge, these results are new.

A final view on closely related algorithms might be interesting. For example, stabbing m *parallel line segments* in $\mathbb{R}^n$ yields $r = 1$ and therefore $O(m)$ time; for *arbitrary line segments* the best known approach needs $O(m^n)$ time, cf. [1]. Further on, to find a line transversal of m *translates* of a convex polygon $P \subset \mathbb{R}^2$ also runs in $O(rm)$ time, where r is the number of different slopes of all edges of P (comparable with the number of fundamental directions in our considerations), see [6]. Indeed, this algorithm does not require central symmetry of P, but on the other hand our Algorithm 2 refers to *scaled* translates. And for $n = 3$ a plane stabber of a family of m convex polyhedra with a total of rm vertices can be found in $O(r^2m^3)$, see [1]. Again, this is comparable with our result: if these polyhedra are scaled translates of a polyhedron centred at the origin, $O(rm)$ time is enough.

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