Orthogonal Non–Bandlimited Wavelets on the Sphere

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Abstract

This paper introduces orthogonal non–bandlimited wavelets on the sphere with respect to a certain Sobolev space topology. The construction of those kernels is based on a clustering of the index set $\mathcal{N} = \{(n,k) \in \mathbb{N}_0 \times \mathbb{Z} | -n \leq k \leq n\}$ associated to the system of spherical harmonics $\{Y_{n,k}\}_{(n,k)\in\mathcal{N}}$. The wavelets presented here form reproducing kernels of the spans of the clustered harmonics. More explicitly, the horizontal partition $\mathcal{M}_n = \{(n,k) \in \mathcal{N}\}$, $n \in \mathbb{N}_0$ yields the usual Shannon wavelets, which are bandlimited, whereas non-bandlimited kernels can be obtained from a vertical clustering $\mathcal{B}_k = \{(n,k),(n,-k) \in \mathcal{N}\}$, $k \in \mathbb{N}_0$. For this case a particular kernel is investigated in detail, and a wavelet representation is derived explicitly.

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1 Introduction

For a long time the only remarkable tool in signal processing was the Fourier transform. In this context a signal needs not only be a sound or an image. More generally, it can be an observable of a physical or other system. However, as it is well–known, the Fourier transform merely offers very restricted information of such a signal, since it contains the total amplitude for the frequency under consideration rather than the distribution of the harmonic modes in each individual location. In fact, the uncertainty principle tells us that a perfect localization in frequency domain coincides with the impossibility of a perfect localization in space domain. In consequence, local anomalies of a signal cannot be isolated sufficiently by using the Fourier transform. Moreover, in orthogonal (Fourier) expansions, a local change of a signal requires the modification of all orthogonal (Fourier) coefficients.
In constructive approximation in Euclidean spaces, the situation completely changed when wavelets came into play in the 1980s (see, for example, S. Mallat (1998) for a review). Wavelets allow a compromise of frequency and space localization. In fact, a multitude of improvements of the original concept helped to let the wavelet approach become the most important tool in today's signal processing. Two essential milestones of this process are the theory of orthogonal wavelets by Y. Meyer (1992) and the concept of orthogonal wavelets with compact support by I. Daubechies (1992).
However, relevant signals are not only functions in Euclidean spaces. Functions describing geophysical quantities, such as the Earth's gravitational or magnetic potential, the air pressure and wind field, the elastic field of the Earth's crust etc, are significant sources of information in the geosciences. For more than two centuries such quantities have been analyzed in spherical approximation by orthogonal (Fourier) expansions in terms of spherical harmonics, i.e. homogeneous harmonic polynomials restricted to a sphere. But this approach is not efficiently and economically applicable to data sets of today's geosciences. For example, the extreme local variation of the density of data points and the huge amount of satellite observations (as e.g. offered by the German satellite CHAMP, launched in 2000) cannot be managed by approximation techniques involving trial functions with global support such as the spherical harmonics. Furthermore, local changes and undulations of geodata, as e.g. caused by tectonic movements, seismic activities, etc, unavoidably require a completely new calculation within a Fourier model by means of globally supported trial functions, which is a feature that should be avoided in future approximation.
Fortunately, wavelet variants and advancements have become more and more

important in all geosciences during the last years, too. For an overview of relevant trends the reader is referred e.g. to R. Klees, R. Haagmans (2000) and the references therein. In particular, spherical wavelets, which are adequately applicable to problems concerned with geomagnetics or Earth's gravitation have been developed by W. Freeden, U. Windheuser (1996) and W. Freeden, M. Schreiner (1998). A detailed discussion of spherical wavelets and their applications in potential theory can be found e.g. in W. Freeden et al. (1998) and W. Freeden (1999), respectively. More general approaches to wavelets on geoscientifically relevant domains like ellipsoid, geoid, (regular) Earth's surface etc, are due to W. Freeden, F. Schneider (1998). Wavelets on a ball and its outer space, respectively, which are of significance in geophysics, geodesy and Earth's seismology, and their applications to a class of inverse problems are introduced and discussed by V. Michel (1998, 1999, 2000).
Yet, the construction of spherical orthogonal wavelets has been restricted to the bandlimited case. However, as motivated by the uncertainty principle (see W. Freeden, V. Michel (1999)) orthogonal non–bandlimited wavelets have several advantages in constructive approximation, for example, fast convergence, strong space localization properties etc. This paper fills the gap of realizing orthogonal non-bandlimited wavelets on the sphere. The idea of constructing orthogonal wavelets on the sphere, as proposed here, is to have a closer look at the index set

$$\mathcal{N} = \{(n,k) : n = 0,1,\ldots; k = -n,\ldots,+n\}$$

characterizing degree n and order k of the system $\{Y_{n,k}\}_{(n,k)\in\mathcal{N}} \subset \mathcal{L}^2(\Omega)$ of the spherical harmonics $Y_{n,k}$ (with Ω being the unit sphere in $\mathbb{R}^3$). We essentially distinguish two cases:
(i) The Shannon wavelets have an orthogonal bandlimited kernel (cf. W. Freeden, M. Schreiner (1998)), where the finite–dimensional detail spaces

$$\mathcal{W}_n = \operatorname{span}\{Y_{n,k} : k = -n,\ldots,n\};\ n = 0,1,\ldots;$$

are constructed in accordance with a 'horizontal' partition (cf. Fig. 1)

$$\mathcal{M}_n = \{(n,k) \in \mathcal{N} : k = -n,\ldots,n\}\,;\ n = 0,1,\ldots;$$

i.e. in the standard way a function is being approximated by a truncated Fourier series in terms of spherical harmonics

$$\lim_{N\to\infty} \int\limits_{\Omega} \left(F(\xi) - \sum_{n=0}^{N}\sum_{k=-n}^{n} (F,Y_{n,k})_{\mathcal{L}^2(\Omega)} Y_{n,k}(\xi) \right)^2 d\omega(\xi) = 0,$$

where $d\omega$ denotes the surface element on Ω.
(ii) Non-bandlimited Shannon wavelets can be based on 'vertical' partitions

$$\mathcal{B}_k = \{(n,k),(n,-k) \in \mathcal{N} : n = k, k+1, \ldots\}; \; k = 0,1,\ldots$$

(cf. Fig. 2). They even have better convergence properties than the standard horizontal systems. Moreover, as already mentioned, the uncertainty principle implies that non–bandlimited wavelet kernels (such as defined by a vertical subdivision) show much stronger space localization than their bandlimited counterparts. Since the wavelets and scaling functions are in the vertical case reproducing kernels of infinite–dimensional detail and scale spaces, respectively, the price to be paid is to use the topologies of a certain class of Sobolev spaces (in order to guarantee the existence of the occuring kernels).
Altogether, forming orthogonal bases the sequence of non–bandlimited wavelets allows representations of signals (functions) in non–redundant form and automatically adapted amount of localization in space and frequency. On the other hand, orthogonal wavelets as proposed in this paper, cannot be expected to be obtained by discretizing a 'generic' continuous frame. In fact, they are constructed in discrete way corresponding to an a priori chosen partition, from which the non–bandlimited (vertical) variant can be regarded as being a fast realization of a coverage of $\mathcal{N}$. This advantage is compensated by the fact that much more efforts must be made to convert the infinite series expansions into expressions in terms of elementary functions. A 'horizontal' partition generates radially symmetric kernels (i.e. radial basis functions on the sphere) due to the addition theorem of spherical harmonics, i.e., the wavelets admit as simplest realization the representation

$$\Psi_n(\xi,\eta) = \sum_{k=-n-1}^{n+1} Y_{n+1,k}(\xi) Y_{n+1,k}(\eta) = \frac{2n+3}{4\pi} \, P_{n+1}(\xi \cdot \eta),$$

$(\xi,\eta) \in \Omega^2$, where P_{n+1} is the Legendre polynomial of degree $n+1$. Consequently, such kernels can be interpreted as univariate functions and are easy to evaluate in numerical calculations. By contrast, a 'vertical' partition yields wavelets of the form

$$\Psi_k(\xi,\eta) = \sum_{n=k+1}^{\infty} Y_{n,k+1}(\xi) Y_{n,k+1}(\eta),$$

$(\xi,\eta) \in \Omega^2$. These product kernels are by far more difficult to evaluate. Usually, a truncation of the series is the only possible way to determine

the values of the kernel function at least approximately. One of the main objectives of this paper, however, is to show that explicit series–free representations of non–bandlimited wavelets can be given on the sphere, if a particular example of such a wavelet is taken into account.
The layout of the paper is as follows: Section 2 is concerned with the discussion of spherical harmonics, the introduction of Sobolev spaces, and the definition of product kernels. Section 3 presents the theory of orthogonal wavelets in terms of bandlimited as well as non–bandlimited product kernels. It is shown that a particular example of a bandlimited orthogonal wavelet kernel can be generated by the (axisymmetric) Green function corresponding to the iterated Beltrami–operator. Finally, in Section 4, explicit representations of non–bandlimited orthogonal wavelets are developed for the series expansions of the Green function corresponding to the iterated Legendre–operator on the interval $[-1,+1]$.

2 Preliminaries

For all $x \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)^T$, different from the origin 0, we let $x = r\xi$, $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, where $\xi = (\xi_1, \xi_2, \xi_3)^T$ is the uniquely determined directional unit vector of $x \in \mathbb{R}^3$. The unit sphere in $\mathbb{R}^3$ is denoted by Ω. If the vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$ form the canonical orthonormal basis in $\mathbb{R}^3$, we may represent the points $\xi \in \Omega$ by

$$
\begin{aligned}
\xi &= t_\xi \varepsilon^3 + \sqrt{1 - t_\xi^2}\left(\cos\varphi_\xi \varepsilon^1 + \sin\varphi_\xi \varepsilon^2\right), \\
t_\xi &= \cos\vartheta_\xi, \vartheta_\xi \in [0,\pi], \varphi_\xi \in [0, 2\pi]
\end{aligned}
$$

(ϑ_ξ: latitude, φ_ξ : longitude, t_ξ: polar distance).
By convention, a sum $\sum_{n=k}^{l}$ with $l < k$ is always assumed to be zero.

2.1 Spherical Harmonics

The *spherical harmonics* Y_n of degree n are defined as the everywhere on the unit sphere Ω twice continuously differentiable eigenfunctions of the Beltrami operator Δ^* corresponding to the eigenvalues $(\Delta^*)^\wedge(n) = -n(n+1)$, $n = 0, 1, \ldots$. The *Legendre polynomials* P_n are the only everywhere on the interval $[-1,+1]$ infinitely differentiable eigenfunctions of the *Legendre operator* $L_t = (1-t^2)(d/dt)^2 - 2t(d/dt)$ which in $t = 1$ satisfy $P_n(1) = 1$. Apart from a multiplicative constant, the 'ε^3–Legendre function' $P_n(\varepsilon^3 \cdot)$: $\xi \mapsto P_n(\varepsilon^3 \cdot \xi)$, $\xi \in \Omega$, is the only spherical harmonic of degree n which is invariant under orthogonal transformations leaving ε^3 fixed. The linear space $Harm_n$ of all spherical harmonics of order n has the dimension $\dim(Harm_n) = 2n+1$. Thus, there exist $2n+1$ linearly independent spherical harmonics $Y_{n,-n}, \ldots, Y_{n,n}$. Throughout the remainder of this paper we assume this system to be orthonormalized in the sense of the $\mathcal{L}^2(\Omega)$–inner product. $Harm_{0,\ldots,m} = \oplus_{n=0}^{m} Harm_n$, $m \geq 0$, denotes the space of all spherical harmonics of degree $\leq m$. Clearly, $\dim(Harm_{0,\ldots,m}) = \sum_{n=0}^{m}(2n+1) = (m+1)^2$.

An outstanding result of the theory of spherical harmonics is the *addition theorem*

$$\sum_{k=-n}^{n} Y_{n,k}(\xi) Y_{n,k}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta),\ (\xi, \eta) \in \Omega \times \Omega \ .$$

The addition theorem, therefore, relates the spherical harmonics on Ω to a univariate function, viz. the Legendre polynomial defined on the interval $[-1, 1]$.

As is well–known, an upper bound of the $C(\Omega)$-norm of the functions $Y_{n,k}$; $n = 0, 1, ...$; $k = -n, ..., n$; can be given by

$$||Y_{n,k}||_{C(\Omega)} \leq \sqrt{\frac{2n+1}{4\pi}}.$$

A function $G(L; \cdot, \cdot)$ is called Green's function with respect to the Legendre operator L, if it satisfies the following properties (cf. W. Freeden (1980)):

(i) *(differential equation)* for every $s \in [-1, 1]$, $G(L; s, \cdot)$ is a continuous function on $[-1, 1]$ that fulfills the boundedness conditions

$$|G(L; s, \pm 1)| < +\infty.$$

For every $s \in [-1, 1]$, $t \mapsto G(L; s, t)$ is twice continuously differentiable on the set $\{t \in [-1, 1] : t \neq s\}$, and

$$L_t\, G(L; s, t) = -\frac{2n+1}{2}\, P_0(s) P_0(t) = -\,\frac{2n+1}{2} \quad .$$

(ii) *(characteristic singularity)* for every $s \in [-1, 1]$

$$\left(1 - s^2\right) \frac{d}{dt} G(L; s, t)\Big|_{t=s-0}^{t=s+0} = -1 \ .$$

(iii) *(symmetry)* for all $s, t \in [-1,1]$

$$G(L;t,s) = G(L;s,t) \ .$$

(iv) *(normalization)* for every $s \in [-1,1]$

$$\int_{-1}^{1} G(L;s,t) P_n(t)\, dt = 0 \ .$$

Green's function $G(L;\cdot,\cdot)$ is uniquely determined by the conditions (i)-(iv). $G(L;\cdot,\cdot)$ has the following bilinear expansion in terms of Legendre polynomials P_n:

$$G(L;s,t) = \sum_{k=1}^{\infty} \frac{1}{-n(n+1)} \frac{2n+1}{2} P_n(t) P_n(s).$$

The bilinear expansion admits a representation in terms of elementary functions (see e.g. W. Magnus, F. Oberhettinger (1948))

$$G(L;s,t) = -\frac{1}{2}\left(2\ln 2 - 1 - \ln((1-s)(1+t)\right),$$

$-1 < s \leq t < 1$. The Legendre polynomials P_n of degree n are the eigenfunctions of the Green function with respect to L in the sense of the integral equation

$$P_n(t) = -n(n+1) \int_{-1}^{1} G(L;t,u) P_n(u)\, du,$$

$n = 1, 2, ...$ Let $G\left(L^2;\cdot,\cdot\right)$ be introduced by the convolution

$$G\left(L^2;s,t\right) = \int_{-1}^{1} G(L;s,u) G(L;u,t)\, du.$$

Then $G\left(L^2;\cdot,\cdot\right)$ is the Green function with respect to the iterated operator L^2. $G\left(L^2;s,t\right)$, $(s,t) \in [-1,1]^2$, has the uniformly convergent bilinear expansion in terms of Legendre polynomials

$$G\left(L^2;s,t\right) = \sum_{n=1}^{\infty} \frac{1}{(n(n+1))^2} \frac{2n+1}{2} P_n(s) P_n(t).$$

The standard system of $\mathcal{L}^2(\Omega)$–orthonormal spherical harmonics used in all geosciences reads as follows (see e.g. W. Magnus, F. Oberhettinger (1948)):

$$Y_{n,0}(\xi) = c_{n0}\ P_n(t_\xi) \tag{1}$$

and

$$Y_{n,k}(\xi) = c_{nk}\ P_{n,k}(t_\xi)\ \cos(k\varphi_\xi), \tag{2}$$

$$Y_{n,-k}(\xi) = c_{nk}\ P_{n,k}(t_\xi)\ \sin(k\varphi_\xi), \tag{3}$$

where

$$c_{n0} = \sqrt{\frac{2n+1}{4\pi}} \tag{4}$$

and

$$c_{nk} = \sqrt{\frac{2(2n+1)}{4\pi}\frac{(n-k)!}{(n+k)!}} \tag{5}$$

$n = 1, 2, \ldots;\ k = 1, \ldots, n$. The so–called *associated Legendre functions* are given by

$$P_{n,k}(t) = (1-t^2)^{k/2}\left(\frac{d}{dt}\right)^k P_n(t)\ .$$

Furthermore, for $t \in (-1,+1)$, we have (cf. W. Magnus, F. Oberhettinger (1948))

$$P_{n,k}(t) = (-1)^k\frac{(n+k)!}{(n-k)!}(1-t^2)^{-k/2}\int_t^1 \ldots \int_t^1 P_n(s)(ds)^k\ .$$

The associated Legendre functions change their sign $n-k$ times in the interval $(-1,+1)$. The functions $\varphi \mapsto \cos(k\varphi)$, $\varphi \mapsto \sin(k\varphi)$, $\varphi \in [0, 2\pi)$ have $2k$ zeros in $[0, 2\pi)$.

The geometrical representation of this system of spherical harmonics is useful: $Y_{n,k}$ with $k = 0$ divide the sphere into zones, hence, they are called *zonal harmonics*; $Y_{n,k}$ with $k \neq 0$ divide the sphere into components in which they are alternately positive and negative, they are called *tesseral harmonics.* In particular, for $k = n$, they degenerate into functions that divide the sphere into positive and negative sectors, hence, they are called *sectorial harmonics.*
The close connection between the orthogonal invariance and the addition theorem is established by the *Funke–Hecke formula* for $H \in \mathcal{L}^1[-1,+1]$

$$\int_\Omega H(\xi\cdot\eta)P_n(\zeta\cdot\eta)\ d\omega(\eta) = (LT)(H)(n)\ P_n(\xi\cdot\zeta),$$

where the *Legendre transform* $(LT)(H)$ is given by

$$(LT)(H)(n) = 2\pi \int_{-1}^{1} H(t)P_n(t)\ dt,$$

$n = 0, 1, \ldots$ ($d\omega$ denotes the surface element on Ω). For more details about the theory of spherical harmonics the reader is referred, for example, to C. Müller (1966) and W. Freeden et al. (1998).

We let $\mathcal{X}(\Omega)$ stand either for the space $C(\Omega)$ or $\mathcal{L}^p(\Omega)$, $1 \leq p < \infty$ (with corresponding norm $\|\cdot\|_{\mathcal{X}(\Omega)}$). In what follows we are mainly interested, however, in results for the Hilbert space $(\mathcal{L}^2(\Omega), (\cdot,\cdot)_{\mathcal{L}^2(\Omega)})$. Any function of the form $H_\xi : \Omega \to \mathbb{R}$, $\eta \mapsto H_\xi(\eta) = H(\xi\cdot\eta)$, $\eta \in \Omega$, is called *ξ–zonal function on Ω*. Zonal functions are constant on the set of all $\eta \in \Omega$ with $\xi \cdot \eta = h$ for a fixed $h \in [-1, +1]$. The set of ξ–zonal functions is isomorphic to the set of functions $H : [-1, +1] \to \mathbb{R}$. This gives rise to interpret the spaces $C[-1, +1]$ and $\mathcal{L}^p[-1, +1]$ with norms defined correspondingly as subspaces of $C(\Omega)$ and $L^p(\Omega)$. We let $\mathcal{X}[-1, +1]$ stand either for the space $C[-1, +1]$ or $\mathcal{L}^p[-1, +1]$ (with corresponding norm $\|\cdot\|_{\mathcal{X}[-1,+1]}$). In other words,

$$\|H\|_{\mathcal{X}[-1,+1]} = \|H(\xi\cdot)\|_{\mathcal{X}(\Omega)}$$

for all $\xi \in \Omega$. The *spherical Fourier transform* $H \mapsto (FT)(H)$, $H \in \mathcal{X}(\Omega)$, is given by

$$((FT)(H))(n,k) = (H, Y_{n,k})_{\mathcal{L}^2(\Omega)} = \int_\Omega H(\eta)Y_{n,k}(\eta)\ d\omega(\eta)\ \ .$$

(FT) forms a mapping from $\mathcal{L}^2(\Omega)$ into the space $l^2(\mathcal{N})$ of all sequences $\{H_{n,k}\}$ satisfying

$$\sum_{(n,k)\in\mathcal{N}} H_{n,k}^2 < \infty,$$

where

$$\mathcal{N} = \{(n,k) \mid n = 0, 1, \ldots; k = -n, \ldots, +n\}\ \ .$$

An example of a zonal function is the *Green function* $G(\Delta^*; \cdot, \cdot)$ with respect to the Beltrami operator Δ^*. $G(\Delta^*; \cdot, \cdot)$ is uniquely defined by the following properties (cf. W. Freeden (1979)):

(i) *(differential equation)* for every $\xi \in \Omega$, $\eta \mapsto G(\Delta^*; \xi, \eta)$ is twice continuously differentiable on the set $\{\eta \in \Omega : -1 \leq \xi \cdot \eta < 1\}$, and

$$\Delta^*_\eta G(\Delta^*; \xi, \eta) = -\frac{1}{4\pi} P_0(\xi \cdot \eta) = -\frac{1}{4\pi},$$

$-1 \leq \xi \cdot \eta < 1$, where Δ^*_η means that the operator Δ^* is applied to the variable η.

(ii) *(characteristic singularity)* for every $\xi \in \Omega$,

$$\eta \mapsto G\left(\Delta^*; \xi, \eta\right) - \frac{1}{4\pi} \ln(1 - \xi \cdot \eta)$$

is continuously differentiable on Ω.

(iii) *(rotational symmetry)* for all orthogonal transformations $\mathbf{t}$,

$$G\left(\Delta^*; \mathbf{t}\xi, \mathbf{t}\eta\right) = G\left(\Delta^*; \xi, \eta\right).$$

(iv) *(normalization)* for every $\xi \in \Omega$

$$\int_\Omega G\left(\Delta^*; \xi, \eta\right) \, d\omega(\eta) = 0.$$

An easy calculation (cf. W. FREEDEN (1979)) shows that

$$G\left(\Delta^*; \xi, \eta\right) = \frac{1}{4\pi} \ln(1 - \xi \cdot \eta) + \frac{1}{4\pi} - \frac{1}{4\pi} \ln 2,$$

$-1 \leq \xi \cdot \eta < 1$, satisfies all the defining properties (i)-(iv) of Green's function with respect to Δ^*. Furthermore, $G\left(\Delta^*; \xi, \eta\right)$ admits the following bilinear expansion in terms of spherical harmonics

$$G\left(\Delta^*; \xi, \eta\right) = \sum_{n=1}^{\infty} \frac{2n+1}{4\pi} \frac{1}{-n(n+1)} P_n(\xi \cdot \eta),$$

$-1 \leq \xi \cdot \eta < 1$. The spherical harmonics of degree n, i.e. the eigenfunctions of the Beltrami operator Δ^* with respect to the eigenvalues $\left(\Delta^*\right)^\wedge(n) = -n(n+1)$, $n = 0, 1, \ldots$, are the eigenfunctions of Green's function in the sense of the integral equation

$$Y_{n,k} = \frac{-n(n+1)}{4\pi} \int_\Omega G\left(\Delta^*; \xi, \eta\right) Y_{n,k}(\eta) \, d\omega(\eta),$$

$n = 1, 2, \ldots$. Let $G\left(\left(\Delta^*\right)^2; \cdot, \cdot\right)$ be defined by the convolution

$$G\left(\left(\Delta^*\right)^2; \xi, \eta\right) = \int_\Omega G\left(\Delta^*; \xi, \zeta\right) G\left(\Delta^*; \zeta, \eta\right) \, d\omega(\zeta).$$

Then $G\left((\Delta^*)^2;\cdot,\cdot\right)$ is the Green function with respect to the iterated Beltrami operator $(\Delta^*)^2$. $G\left((\Delta^*)^2;\cdot,\cdot\right)$ allows a uniformly convergent bilinear expansion in terms of spherical harmonics

$$G\left((\Delta^*)^2;\xi,\eta\right) = \sum_{n=1}^{\infty} \frac{2n+1}{4\pi} \frac{1}{(n(n+1))^2} P_n(\xi\cdot\eta),$$

$(\xi,\eta)\in\Omega^2$.

2.2 Sobolev Spaces

Next we consider the linear space $\mathcal{A}$ consisting of all sequences $\{A_{n,k}\}_{(n,k)\in\mathcal{N}}$ of real numbers $A_{n,k}$, $n=0,1,\ldots;\ k=-n,\ldots,+n$. For simplicity, the notation $\{A_n\}_{(n,k)\in\mathcal{N}}\in\mathcal{A}$ is understood to be equivalent to $\{A_{n,k}\}_{(n,k)\in\mathcal{N}}\in\mathcal{A}$ with $A_{n,k}=A_n$ for $k=-n,\ldots,+n$.

Definition 2.1. Let $A=\{A_{n,k}\}_{(n,k)\in\mathcal{N}}\in\mathcal{A}$ be a sequence. We split $\mathcal{N}$ into two parts such that

$$\begin{aligned}\mathcal{N} &= \mathcal{N}^{(0)}\cup\mathcal{N}^{(1)},\\ \emptyset &= \mathcal{N}^{(0)}\cap\mathcal{N}^{(1)},\end{aligned}$$

($\mathcal{N}^{(1)}$ always being assumed to be non–void), where

$$\begin{aligned}\mathcal{N}^{(0)} &= \{(n,k)\in\mathcal{N}: A_{n,k}=0\},\\ \mathcal{N}^{(1)} &= \{(n,k)\in\mathcal{N}: A_{n,k}\neq 0\}.\end{aligned}$$

The sequence $A=\{A_{n,k}\}_{(n,k)\in\mathcal{N}}$ is called *summable* (with respect to $\mathcal{N}^{(1)}$), briefly $\mathcal{N}^{(1)}$–summable, if

$$\sup_{\xi\in\Omega}\left(\sum_{(n,k)\in\mathcal{N}^{(1)}} \frac{1}{A_{n,k}^2}\left(Y_{n,k}(\xi)\right)^2\right)^{1/2} < \infty\ .$$

A summable sequence $A=\{A_{n,k}\}_{(n,k)\in\mathcal{N}}$ with respect to $\mathcal{N}$ is simply called *summable*.

For a given sequence $A=\{A_{n,k}\}_{(n,k)\in\mathcal{N}}$, consider the linear space $\mathcal{E}^A(\Omega)$ of all functions $F\in C^{(\infty)}(\Omega)$ such that $(F,Y_{n,k})_{\mathcal{L}^2(\Omega)}=0$ for all $(n,k)\in\mathcal{N}^{(0)}$ and

$$\sum_{(n,k)\in\mathcal{N}^{(1)}} A_{n,k}^2\,(F,Y_{n,k})_{\mathcal{L}^2(\Omega)}^2 < \infty. \tag{6}$$

On $\mathcal{E}^A(\Omega)$ we introduce an inner product by letting

$$(F,G)_{\mathcal{H}^A(\Omega)} = \sum_{(n,k)\in\mathcal{N}^{(1)}} A^2_{n,k}\ (F,Y_{n,k})_{\mathcal{L}^2(\Omega)}\,(G,Y_{n,k})_{\mathcal{L}^2(\Omega)}$$

and define the space $\mathcal{H}^A(\Omega)$ to be the completion of $\mathcal{E}^A(\Omega)$ with respect to the topology $(\cdot,\cdot)_{\mathcal{H}^A(\Omega)}$. Then we end up with a Hilbert space:

$$\mathcal{H}^A(\Omega) = \overline{\mathcal{E}^A(\Omega)}^{\|\cdot\|_{\mathcal{H}^A(\Omega)}} .$$

It is obvious that $\mathcal{H}^{\{1\}}(\Omega) = \mathcal{L}^2(\Omega)$.
Sobolev spaces equipped with a reproducing kernel structure are of importance for our considerations. In fact, the Hilbert space $\mathcal{H}^A(\Omega)$ corresponding to an $\mathcal{N}^{(1)}$–summable sequence $A := \{A_{n,k}\}_{(n,k)\in\mathcal{N}}$ possesses a reproducing kernel $K_{\mathcal{H}^A(\Omega)}(\cdot,\cdot)$, since the evaluation functional $F\mapsto F(\xi)$, $F\in\mathcal{H}^A(\Omega)$, is continuous for every $\xi\in\Omega$ (cf. Aronszajn (1950)). It can be easily seen that $K_{\mathcal{H}^A(\Omega)}(\cdot,\cdot)$ admits an absolutely and uniformly convergent series representation of the form

$$K_{\mathcal{H}^A(\Omega)}(\xi,\eta) = \sum_{(n,k)\in\mathcal{N}^{(1)}} Y^A_{n,k}(\xi)Y^A_{n,k}(\eta),\ (\xi,\eta)\in\Omega\times\Omega,$$

where

$$Y^A_{n,k}(\xi) = \frac{1}{A_{n,k}}Y_{n,k}(\xi), \quad \xi\in\Omega. \tag{7}$$

2.3 Product Kernels

Suppose that $A = \{A_{n,k}\}_{(n,k)\in\mathcal{N}} \in \mathcal{A}$ is summable (with respect to $\mathcal{N}^{(1)}$). Any function $\Gamma:\Omega\times\Omega\to\mathbb{R}$ of the form

$$\Gamma(\xi,\eta) = \sum_{(n,k)\in\mathcal{N}^{(1)}} \Gamma^{\wedge_A}(n,k)\ Y^A_{n,k}(\xi)Y^A_{n,k}(\eta), (\xi,\eta)\in\Omega^2, \tag{8}$$

where $\Gamma^{\wedge_A}(n,k)\in\mathbb{R}$ for $(n,k)\in\mathcal{N}^{(1)}$, is called an $\mathcal{H}^A(\Omega)$*–product kernel* (briefly, $\mathcal{H}^A(\Omega)$–kernel) if

$$\sup_{\xi\in\Omega}\sum_{(n,k)\in\mathcal{N}^{(1)}} \left(\Gamma^{\wedge_A}(n,k)Y^A_{n,k}(\xi)\right)^2 < \infty\ .$$

In this case the symbol $\{\Gamma^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}^{(1)}}$ is called $\mathcal{H}^A(\Omega)$–admissible.

$\mathcal{H}^A(\Omega)$–convolutions will be introduced in the following way.

DEFINITION 2.2. Let F be of class $\mathcal{H}^A(\Omega)$, and let Γ be an $\mathcal{H}^A(\Omega)$–kernel of the form (8) with $\mathcal{H}^A(\Omega)$–admissible symbol $\{\Gamma^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}^{(1)}}$. Then the convolution of Γ against F is defined by

$$\begin{aligned}(\Gamma *_A F)(\xi) &= (\Gamma(\xi,\cdot), F)_{\mathcal{H}^A(\Omega)} \\ &= \sum_{(n,k)\in\mathcal{N}^{(1)}} \Gamma^{\wedge_A}(n,k) F^{\wedge_A}(n,k) Y^A_{n,k}(\xi),\end{aligned}$$

where

$$F^{\wedge_A}(n,k) = \left(F, Y^A_{n,k}\right)_{\mathcal{H}^A(\Omega)}, (n,k)\in\mathcal{N}^{(1)} \quad . \tag{9}$$

Moreover, the convolution of $\mathcal{H}^A(\Omega)$–kernels Γ_1 and Γ_2 with $\mathcal{H}^A(\Omega)$–admissible symbols $\{\Gamma_1^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}^{(1)}}$ and $\{\Gamma_2^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}^{(1)}}$, respectively, is defined by

$$(\Gamma_1 *_A \Gamma_2)(\xi,\eta) = (\Gamma_1 *_A (\Gamma_2(\cdot,\eta)))(\xi);$$

$\xi,\eta\in\Omega$.

From (9) we immediately see that

$$(\Gamma *_A F)^{\wedge_A}(n,k) = \Gamma^{\wedge_A}(n,k) F^{\wedge_A}(n,k)$$

for all $(n,k)\in\mathcal{N}^{(1)}$. The convolution of two $\mathcal{H}^A(\Omega)$–product kernels with $\mathcal{H}^A(\Omega)$–admissible symbols leads us to the following result.

THEOREM 2.3. *Let Γ_1 and Γ_2 be $\mathcal{H}^A(\Omega)$–kernels with $\mathcal{H}^A(\Omega)$–admissible symbols $\{\Gamma_1^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}^{(1)}}$ and $\{\Gamma_2^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}^{(1)}}$, respectively. Then*

$$\begin{aligned}(\Gamma_1 *_A \Gamma_2)(\xi,\eta) &= (\Gamma_1(\xi,\cdot), \Gamma_2(\cdot,\eta))_{\mathcal{H}^A(\Omega)} \\ &= \sum_{(n,k)\in\mathcal{N}^{(1)}} \Gamma_1^{\wedge_A}(n,k)\Gamma_2^{\wedge_A}(n,k) Y^A_{n,k}(\xi) Y^A_{n,k}(\eta)\end{aligned}$$

for all $(\xi,\eta)\in\Omega\times\Omega$, and $\{(\Gamma_1 *_A \Gamma_2)^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}^{(1)}}$, given by

$$(\Gamma_1 *_A \Gamma_2)^{\wedge_A}(n,k) = \Gamma_1^{\wedge_A}(n,k)\Gamma_2^{\wedge_A}(n,k), \tag{10}$$

constitutes an $\mathcal{H}^A(\Omega)$–admissible symbol of the $\mathcal{H}^A(\Omega)$–kernel $\Gamma_1 * \Gamma_2$.

3 Orthogonal Wavelets

Throughout this chapter we simply assume that $\mathcal{N}^{(1)}$ consists of all pairs $(n,k) \in \mathcal{N}$, i.e. $\mathcal{N}^{(1)} = \mathcal{N}$. Furthermore, $A = \{A_{n,k}\}_{(n,k)\in\mathcal{N}}$ is always assumed to be summable:

$$\sup_{\xi\in\Omega} \left(\sum_{(n,k)\in\mathcal{N}} \left(Y_{n,k}^A(\xi)\right)^2 \right)^{1/2} < \infty \ . \tag{11}$$

After having explained the convolution between $\mathcal{H}(\Omega)$–kernels with $\mathcal{H}(\Omega)$–admissible symbols ($\mathcal{H}(\Omega) = \mathcal{H}^A(\Omega)$) we are now interested in developing countable families $\{\Gamma_j\}_{j\in\mathbb{N}_0}$ of $\mathcal{H}(\Omega)$–product kernels Γ_j which will be understood as scaling functions in our theory of orthogonal wavelets on the sphere.

As a preparation we first introduce a dilation operator acting on a family $\{\Gamma_j\}_{j\in\mathbb{N}_0}$ of $\mathcal{H}(\Omega)$–product kernels in the following way: Let Γ_j be a member of this family. Then the *dilation operator* $D_k, k \in \mathbb{N}_0$, is defined by $D_k\Gamma_j = \Gamma_{j+k}$. In particular, we have $\Gamma_j = D_j\Gamma_0$. Thus, we refer to Γ_0 as the 'mother kernel'. Moreover, we define a *rotation operator* R_ξ, $\xi \in \Omega$, by $R_\xi\Gamma_j = \Gamma_j(\xi,\cdot)$, $j \in \mathbb{N}_0$. In doing so we consequently get by composition of the operators $\Gamma_j(\xi,\cdot) = R_\xi D_j \Gamma_0$ for all $\xi \in \Omega$ and all $j \in \mathbb{N}_0$. Note that all kernels Γ_j are symmetric, so that $\Gamma_j(\xi,\eta) = \Gamma_j(\eta,\xi)$ for all $(\xi,\eta) \in \Omega\times\Omega$ and all $j \in \mathbb{N}_0$.

Moreover, we call a subdivision $\mathcal{P} = \{\mathcal{M}_l\}_{l\in\mathbb{N}_0}$ of non–empty subsets $\mathcal{M}_l$ of the index set $\mathcal{N}$ a *partition of* $\mathcal{N}$, if the following properties are satisfied:

$$\begin{aligned} \bigcup_{l\in\mathbb{N}_0} \mathcal{M}_l &= \mathcal{N}, \\ \mathcal{M}_l \cap \mathcal{M}_k &= \emptyset,\ l \neq k \ . \end{aligned}$$

For brevity we set

$$\mathcal{N}_l = \bigcup_{r=0}^{l} \mathcal{M}_r, \quad l \in \mathbb{N}_0 \ . \tag{12}$$

Simple examples of partitions $\mathcal{P}$ of $\mathcal{N}$ are given as follows:

(i) horizontal partition $\mathcal{P}$ (briefly: H-partition)
$\mathcal{M}_n = \{(n,k) \in \mathcal{N} : k = -n,\ldots,+n\}$, $n \in \mathbb{N}_0$ (see Figure 1).

(ii) vertical partition $\mathcal{P}$ (briefly: V-partition)
$\mathcal{B}_k = \{(n,k),(n,-k) \in \mathcal{N} : n = k, k+1,\ldots\}$, $k \in \mathbb{N}_0$ (see Figure 2).

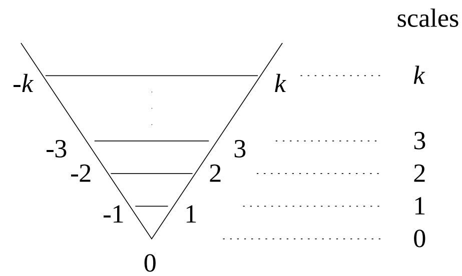

Figure 1: Horizontal partition: the detail space of scale k is the span of all spherical harmonics $Y_{k,-k}, ..., Y_{k,k}$; in the triangle representing the index set $\mathcal{N}$ the multiresolution refers to a horizontal partition.

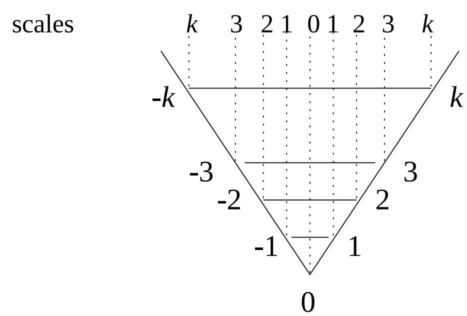

Figure 2: Vertical partition: here, the detail space of scale k is spanned by the infinite system $\{Y_{n,-k}, Y_{n,k} : n = k, k+1, ...\}$; in the triangle representing the index set $\mathcal{N}$ the multiresolution refers to a vertical partition.

But also non–standard partitions may be selected for our wavelet approach. An example is illustrated by Figure 3.

3.1 Scaling Function

We are now in position to introduce the so–called Shannon $\mathcal{H}(\Omega)$–scaling function.

DEFINITION 3.1. Assume that $\mathcal{P}$ is a partition of $\mathcal{N}$. Let the sequence $\{\Phi_j^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}}$, $j \in \mathbb{N}_0$, be defined by

$$\Phi_j^{\wedge_A}(n,k) = \begin{cases} 1 & , \quad (n,k) \in \mathcal{N}_j \\ 0 & , \quad (n,k) \in \mathcal{N}\backslash\mathcal{N}_j \end{cases} .$$

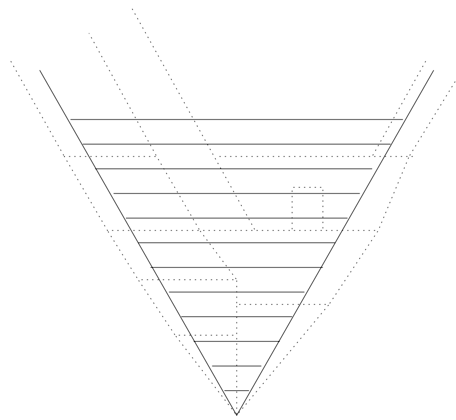

Figure 3: Any arbitrary countable partition of the index set $\mathcal{N}$ generates orthogonal detail spaces.

Then $\{\Phi_j^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}}$ is called the generating function of the $\mathcal{P}$–*Shannon* $\mathcal{H}(\Omega)$–*scaling function*. The family of $\mathcal{H}(\Omega)$–kernels $\{\Phi_j\}_{j\in\mathbb{N}_0}$, given by

$$\Phi_j(\xi,\eta) = \sum_{(n,k)\in\mathcal{N}_j} Y_{n,k}^A(\xi)Y_{n,k}^A(\eta), \quad (\xi,\eta)\in\Omega^2,$$

is called the $\mathcal{P}$–*Shannon* $\mathcal{H}(\Omega)$–*scaling function*.

From the results of the previous section it follows that $\Phi_j(\xi,\cdot)\in\mathcal{H}(\Omega)$. A remarkable property is that Φ_j coincides with its iteration $\Phi_j = \Phi_j *_A \Phi_j$.

Theorem 3.2. *Let* $\{\Phi_j\}_{j\in\mathbb{N}_0}$ *be a* $\mathcal{P}$–*Shannon* $\mathcal{H}(\Omega)$–*scaling function. Then*

$$\lim_{J\to\infty} \|F - F_J\|_{\mathcal{H}(\Omega)} = 0$$

holds for all $F\in\mathcal{H}(\Omega)$, *where* F_J, *given by*

$$F_J = \Phi_J *_A F, \tag{13}$$

is said to be the J–*level approximation of* $F\in\mathcal{H}(\Omega)$.

Proof. We introduce the operator $L_J : \mathcal{H}(\Omega)\to\mathcal{H}(\Omega)$, $J\in\mathbb{N}_0$, by $F_J = L_J F = \Phi_J *_A F$. From the definition of the convolution it follows that

$$L_J F = \sum_{(n,k)\in\mathcal{N}_J} F^{\wedge_A}(n,k) Y_{n,k}^A \ .$$

But this shows us that

$$\begin{aligned}\|L_J F - F\|_{\mathcal{H}(\Omega)} &= \left(\sum_{(n,k)\in\mathcal{N}\setminus\mathcal{N}_J} \left(F^{\wedge_A}(n,k)\right)^2 \right)^{1/2} \\ &\le \left(\sum_{(n,k)\in\mathcal{N}} \left(F^{\wedge_A}(n,k)\right)^2 \right)^{1/2} \\ &= \|F\|_{\mathcal{H}(\Omega)}\end{aligned}$$

for every $J \in \mathbb{N}_0$. We obtain

$$\begin{aligned}&\lim_{J\to\infty} \|L_J F - F\|_{\mathcal{H}(\Omega)} \\ &= \lim_{J\to\infty} \sum_{(n,k)\in\mathcal{N}} \left(1 - \left(\Phi_J^{\wedge_A}(n,k)\right)\right)^2 \left(F^{\wedge_A}(n,k)\right)^2 \\ &= 0 \ .\end{aligned}$$

This is the desired result. □

According to our construction, for any $F \in \mathcal{H}(\Omega)$, each $L_J F$ as defined above provides an approximation of F at scale J. In terms of filtering Φ_J may be interpreted as low–pass filter. L_J is the convolution operator of this low–pass filter. Accordingly we understand the *scale spaces* $\mathcal{V}_J$ to be the image of $\mathcal{H}(\Omega)$ under the operator L_J:

$$\mathcal{V}_J = L_J(\mathcal{H}(\Omega)) = \{\Phi_J *_A F : F \in \mathcal{H}\} \ .$$

3.2 Wavelets

In order to start with the definition of wavelets we introduce a 'refinement (scaling) equation'.

Definition 3.3. Let $\{\Phi_j^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}}$, $j \in \mathbb{N}_0$, be the generating function of the $\mathcal{P}$–Shannon $\mathcal{H}(\Omega)$–scaling function. Then the generating symbol $\{\Psi_j^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}}$, $j \in \mathbb{N}_0$, of the associated *$\mathcal{P}$–Shannon $\mathcal{H}(\Omega)$–wavelet* is defined via the refinement equation

$$\Psi_j^{\wedge_A}(n,k) = \Phi_{j+1}^{\wedge_A}(n,k) - \Phi_j^{\wedge_A}(n,k) \ .$$

The family $\{\Psi_j\}_{j\in\mathbb{N}_0}$ of $\mathcal{H}(\Omega)$–kernels given by

$$\Psi_j(\xi,\eta) = \sum_{(n,k)\in\mathcal{N}} \Psi_j^{\wedge_A}(n,k) Y_{n,k}^A(\xi) Y_{n,k}^A(\eta), \quad (\xi,\eta) \in \Omega\times\Omega,$$

is called the *$\mathcal{P}$–Shannon $\mathcal{H}(\Omega)$–wavelet* associated to the $\mathcal{P}$–Shannon $\mathcal{H}(\Omega)$–scaling function $\{\Phi_j\}_{j\in\mathbb{N}_0}$. The corresponding mother wavelet is denoted by Ψ_0.

Obviously we are able to define the dilation and rotation operator in the same way as we did before. In other words, any member of the family $\{\Psi_j\}_{j\in\mathbb{N}_0}$ can be interpreted as a dilated and rotated copy of the corresponding mother wavelet like $\Psi_j(\xi,\cdot) = R_\xi D_j \Psi_0$.

We easily derive from the refinement equation that

$$\Psi_j^{\wedge_A}(n,k) = \begin{cases} 1 & , \quad (n,k) \in \mathcal{M}_{j+1} \\ 0 & , \quad (n,k) \in \mathcal{N}\backslash\mathcal{M}_{j+1} \end{cases} .$$

Therefore, we see that

$$\Phi_{J+1}^{\wedge_A}(n,k) = \Phi_0^{\wedge_A}(n,k) + \sum_{j=0}^{J} \Psi_j^{\wedge_A}(n,k)$$

for all $(n,k) \in \mathcal{N}$. Similar to the definition of the operators L_J we are now led to convolution operators $B_j : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$, given by

$$B_j F = \Psi_j *_A F, \quad F \in \mathcal{H}(\Omega) \ .$$

Thus, the identity

$$\Phi_{J+1} *_A F = \Phi_0 *_A F + \sum_{j=0}^{J} \Psi_j *_A F \tag{14}$$

can be written in operator formulation as follows:

$$L_{J+1} = L_0 + \sum_{j=0}^{J} B_j \ .$$

The convolution operators B_j characterize the 'detail information' of F at scale j. In terms of filtering, B_j, $j \in \mathbb{N}_0$, may be interpreted as a band–pass filter convolution operator. This fact immediately gives rise to introduce the *detail spaces* as follows:

$$\mathcal{W}_j = B_j(\mathcal{H}(\Omega)) = \{\Psi_j *_A F : F \in \mathcal{H}(\Omega)\} \ .$$

$\mathcal{W}_j$ contains the 'detail information' needed to go from an approximation at level j to an approximation at level $j+1$. In other words, the 'partial reconstruction' $B_j F$ is nothing else than the 'difference of two smoothings' at two consecutive scales: $B_J F = L_{J+1}F - L_J F$.

THEOREM 3.4. *Assume that L_j and B_j, respectively, are the low–pass and band–pass filter convolution operators as defined above. Then the scale spaces $\mathcal{V}_j$ and the detail spaces $\mathcal{W}_j$ satisfy the properties:*

(i) $\mathcal{V}_0 \subset \ldots \subset \mathcal{V}_j \subset \mathcal{V}_{j+1} \subset \ldots \subset \mathcal{H}(\Omega)$

(ii) $\overline{\bigcup_{j\in\mathbb{N}_0} \mathcal{V}_j} = \mathcal{H}(\Omega)$

(iii) $\bigcap_{j\in\mathbb{N}_0} \mathcal{V}_j = \overline{\operatorname{span}_{(n,k)\in\mathcal{M}_0} \{Y^A_{n,k}\}}$

(iv) $\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$ *($\oplus$ means orthogonal direct sum).*

Proof. Clearly we have

$$\mathcal{V}_j = \overline{\operatorname{span}_{(n,k)\in\mathcal{N}_j} \{Y^A_{n,k}\}}, \quad j \in \mathbb{N}_0 ,$$

and

$$\mathcal{W}_j = \overline{\operatorname{span}_{(n,k)\in\mathcal{M}_j} \{Y^A_{n,k}\}}, \quad j \in \mathbb{N}_0 \ .$$

This proves Theorem 3.4. □

If a collection of subspaces $\mathcal{V}_j$ of $\mathcal{H}(\Omega)$ satisfies the conditions of Theorem 3.4 we call them a *$\mathcal{P}$–Shannon $\mathcal{H}(\Omega)$–multiresolution analysis.*

The main result of our *$\mathcal{P}$–Shannon $\mathcal{H}(\Omega)$–wavelet theory* now reads as follows.

THEOREM 3.5. *Let $\{\Phi_j^{\wedge_A}(n,k)\}_{(n,k)\in\mathcal{N}}$, $j \in \mathbb{N}_0$, be the generating symbol of the $\mathcal{P}$–Shannon $\mathcal{H}(\Omega)$–scaling function. Suppose that F is of class $\mathcal{H}(\Omega)$. Then*

$$F_J = \Phi_0 *_A F + \sum_{j=0}^{J-1} \Psi_j *_A F$$

is the J–level approximation of F satisfying

$$\lim_{J\to\infty} \|F - F_J\|_{\mathcal{H}} = 0 \ .$$

3.3 H–Shannon $\mathcal{H}(\Omega)$–wavelets

The horizontal partition is the usual subdivision of $\mathcal{N}$ in rotational invariant wavelet theory (cf. W. FREEDEN (1999)). In the standard case that $A_{n,k} = A_n$ for all $n \in \mathbb{N}_0$ and $k = -n, \ldots, +n$, the addition theorem of spherical harmonics allows to represent the H–Shannon $\mathcal{H}(\Omega)$–scaling functions in univariate form:

$$\Phi_j(\xi,\eta) = \sum_{n=0}^{j} \frac{1}{A_n^2} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta), \quad (\xi,\eta) \in \Omega^2 \ .$$

Correspondingly, we find

$$\Psi_j(\xi,\eta) = \frac{1}{A_{j+1}^2} \frac{2j+3}{4\pi} P_{j+1}(\xi \cdot \eta), \quad (\xi,\eta) \in \Omega^2 \ .$$

All scale spaces and detail spaces are finite–dimensional. More precisely, we have

$$\begin{aligned} \mathcal{V}_j &= \bigoplus_{k=0}^{j} Harm_k, \\ \mathcal{W}_j &= Harm_{j+1}. \end{aligned}$$

Consequently, dim $\mathcal{V}_j = (j+1)^2$ and dim $\mathcal{W}_j = 2j+1$.
Numerous examples of H–Shannon $\mathcal{H}(\Omega)$-wavelets can be given, since a large class of expansions in Legendre polynomials is well-known (see e.g. W. FREEDEN ET AL. (1998)).

Example 3.6.
a) (modified Green's function with respect to $(\Delta^*)^2$) $A_n = \frac{1}{n(n+1)}$; $n = 1, 2, \ldots$; $A_0 = 1$:

$$\lim_{j\to\infty} \Phi_j(\xi,\eta) = \frac{1}{4\pi} + G\left((\Delta^*)^2 ; \xi, \eta \right),$$

where

$$\Phi_j(\xi,\eta) = \frac{1}{4\pi} + \sum_{n=1}^{j} \frac{1}{(n(n+1))^2} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta), \quad (\xi,\eta) \in \Omega^2,$$

and

$$\Psi_j(\xi,\eta) = \frac{1}{(j+1)^2(j+2)^2} \frac{2j+3}{4\pi} P_{j+1}(\xi \cdot \eta), \quad (\xi,\eta) \in \Omega^2.$$

b) (Abel–Poisson kernel) $A_n = h^{-n/2}$ for some fixed $h \in (0,1)$:

$$\lim_{j\to\infty} \Phi_j(\xi,\eta) = \sum_{n=0}^{\infty} h^n \frac{2n+1}{4\pi} P_n(\xi\cdot\eta) = \frac{1}{4\pi}\frac{1-h^2}{(L_h(\xi\cdot\eta))^{3/2}},$$

where

$$\Phi_j(\xi,\eta) = \sum_{n=0}^{j} h^n \frac{2n+1}{4\pi} P_n(\xi\cdot\eta), \quad (\xi,\eta)\in\Omega^2,$$

$$\Psi_j(\xi,\eta) = h^{j+1}\frac{2j+3}{4\pi} P_{j+1}(\xi\cdot\eta), \quad (\xi,\eta)\in\Omega^2,$$

and

$$L_h(\xi\cdot\eta) = 1 + h^2 - 2h(\xi\cdot\eta).$$

c) (Singularity kernel) $A_n = \sqrt{n+\frac{1}{2}}\, h^{-n/2}$ for some fixed $h\in(0,1)$:

$$\lim_{j\to\infty} \Phi_j(\xi,\eta) = \frac{1}{2\pi}\frac{1}{(L_h(\xi\cdot\eta))^{1/2}}.$$

d) (Logarithmic kernel) $A_n = (2n+1)(n+1)h^{-n/2}$ for some fixed $h\in(0,1)$:

$$\lim_{j\to\infty} \Phi_j(\xi,\eta) = \frac{1}{2\pi h}\ln\left(1+\frac{2h}{(L_h(\xi\cdot\eta))^{1/2}+1-h}\right).$$

□

Other types of kernel functions can be found e.g. in W. Freeden et al. (1998).

3.4 V–Shannon $\mathcal{H}(\Omega)$–wavelets

The vertical partition leads us to the Shannon $\mathcal{H}(\Omega)$–scaling function

$$\begin{aligned}\Phi_j(\xi,\eta) \quad = \quad & \sum_{k=0}^{j}\sum_{n=k}^{\infty}\frac{c_{n,k}^2}{A_{n,k}^2} P_{n,k}(t_\xi)P_{n,k}(t_\eta)\cos(k\varphi_\xi)\cos(k\varphi_\eta) \\ & + \sum_{k=1}^{j}\sum_{n=k}^{\infty}\frac{c_{n,k}^2}{A_{n,k}^2} P_{n,k}(t_\xi)P_{n,k}(t_\eta)\sin(k\varphi_\xi)\sin(k\varphi_\eta)\end{aligned}$$

and the $\mathcal{H}(\Omega)$–wavelet

$$\begin{aligned}\Psi_j(\xi,\eta) = & \sum_{n=j+1}^{\infty}\frac{c_{n,j+1}^2}{A_{n,j+1}^2} P_{n,j+1}(t_\xi)P_{n,j+1}(t_\eta)\cos\left((j+1)\varphi_\xi\right)\cos((j+1)\varphi_\eta) \\ & + \sum_{n=j+1}^{\infty}\frac{c_{n,j+1}^2}{A_{n,j+1}^2} P_{n,j+1}(t_\xi)P_{n,j+1}(t_\eta)\sin((j+1)\varphi_\xi)\sin((j+1)\varphi_\eta).\end{aligned}$$

Such kernels can be evaluated numerically by truncating the series. Series–free representations can only be derived for exceptional cases. An example of such a kernel function is discussed in the next section.

4 Calculation of a V-Shannon $\mathcal{H}(\Omega)$-wavelet

This section is concerned with the calculation of a series in $P_{n,k}P_{n,k}$, which can be used as a V-Shannon $\mathcal{H}(\Omega)$-wavelet.

4.1 From a series in P_nP_n to a series in $P_{n,k}P_{n,k}$

Using the connection between P_n and $P_{n,k}$ via integration and differentiation, respectively, we find for $t_\xi, t_\eta \in (-1,1)$

$$
\begin{aligned}
&\sum_{n=k}^{\infty}\frac{c_{n,k}^2}{A_{n,k}}P_{n,k}(t_\xi)P_{n,k}(t_\eta)\cos(k\varphi_\xi)\cos(k\varphi_\eta)\\
&= \sum_{n=k}^{\infty}\frac{1}{A_{n,k}}\,\frac{2(2n+1)}{4\pi}\,\frac{(n-k)!}{(n+k)!}(-1)^k\,\frac{(n+k)!}{(n-k)!}\,\left(1-{t_\xi}^2\right)^{-k/2}\cdot\\
&\quad \int_{t_\xi}^{1}\cdots\int_{t_\xi}^{1}P_n(s)(ds)^k\,\left(1-{t_\eta}^2\right)^{k/2}\left(\frac{d}{dt_\eta}\right)^k P_n(t_\eta)\cos(k\varphi_\xi)\cos(k\varphi_\eta)\\
&= 2(-1)^k\left(\frac{1-{t_\eta}^2}{1-{t_\xi}^2}\right)^{k/2}\sum_{n=k}^{\infty}\frac{1}{A_{n,k}}\,\frac{2n+1}{4\pi}\int_{t_\xi}^{1}\cdots\int_{t_\xi}^{1}P_n(s)(ds)^k\cdot\\
&\quad \left(\frac{d}{dt_\eta}\right)^k P_n(t_\eta)\cos(k\varphi_\xi)\cos(k\varphi_\eta)\\
&= 2(-1)^k\left(\frac{1-{t_\eta}^2}{1-{t_\xi}^2}\right)^{k/2}\int_{t_\xi}^{1}\cdots\int_{t_\xi}^{1}\left(\frac{d}{dt_\eta}\right)^k\sum_{n=k}^{\infty}\frac{2n+1}{4\pi A_{n,k}}\,P_n(s)P_n(t_\eta)(ds)^k\cdot\\
&\quad \cos(k\varphi_\xi)\cos(k\varphi_\eta), \qquad (15)
\end{aligned}
$$

if $A_{n,k}\neq 0$ for every n,k. The last step is allowed, if

$$
A_{n,k} = O\left(n^{3/2+\varepsilon}\right) \qquad (16)
$$

for some $\varepsilon > 0$, since this implies the uniform convergence of a series like ($\sigma \in \{0,1\}$)

$$\begin{aligned}
&\sum_{n=k}^{\infty} \frac{2n+1}{4\pi A_{n,k}} \int_{t_\xi}^{1} \cdots \int_{t_\xi}^{1} P_n(s)(ds)^{l+\sigma} \left(\frac{d}{dt_\eta}\right)^l P_n(t_\eta) \\
&= (-1)^{l+\sigma} \sum_{n=k}^{\infty} \frac{2n+1}{4\pi A_{n,k}} \frac{(n-l-\sigma)!}{(n+l+\sigma)!} \left(1-{t_\xi}^2\right)^{(l+\sigma)/2} P_{n,l+\sigma}(t_\xi) P_{n,l}(t_\eta) \cdot \\
&\qquad \left(1-{t_\eta}^2\right)^{-l/2},
\end{aligned}$$

$\sigma \leq l+\sigma \leq k$, for the following reason. Eq. (87bis) of Ch. V in L. Robin (1958) says that

$$|P_{n,k}(t)| \leq \frac{(n+k)!}{n!}$$

for every $t \in [-1,1]$ and $k = 0,1,...;\ n = k, k+1, ...$ Hence, it follows that

$$\begin{aligned}
&\left| \sum_{n=k}^{\infty} \frac{2n+1}{4\pi A_{n,k}} \frac{(n-l-\sigma)!}{(n+l+\sigma)!} P_{n,l+\sigma}(t_\xi) P_{n,l}(t_\eta) \right| \\
&\leq \left(\sum_{n=k}^{\infty} \left(\frac{2n+1}{4\pi A_{n,k}} \right)^2 \right)^{1/2} \left(\sum_{n=k}^{\infty} \left(\frac{(n-l-\sigma)!}{(n+l+\sigma)!} \right)^2 \left(P_{n,l+\sigma}(t_\xi) P_{n,l}(t_\eta) \right)^2 \right)^{1/2} \\
&\leq \left(\sum_{n=k}^{\infty} \left(\frac{2n+1}{4\pi A_{n,k}} \right)^2 \right)^{1/2} \cdot \\
&\qquad \left(\sum_{n=k}^{\infty} \left(\frac{(n-l-\sigma)!}{(n+l+\sigma)!} \right)^2 \left(\frac{(n+l+\sigma)!}{n!} \right)^2 \left(\frac{(n+l)!}{n!} \right)^2 \right)^{1/2} \\
&\leq \left(\sum_{n=k}^{\infty} \left(\frac{2n+1}{4\pi A_{n,k}} \right)^2 \right)^{1/2} \left(\sum_{n=k}^{\infty} \frac{1}{n^2} \left(\frac{(n+l)\cdots(n+1)}{(n-1)\cdots(n-l-\sigma+1)} \right)^2 \right)^{1/2} \\
&< +\infty,
\end{aligned}$$

if $\sigma = 1$.

Eq. (35) of Ch. IX in L. Robin (1959), which says that there exists a constant $a > 0$, such that

$$\frac{(n-l)!}{(n+l)!} |P_{n,l}(t) P_{n,l}(s)| \leq \frac{a}{\sqrt{n}}$$

for every $s, t \in [-1,1]$ and $l = 0, 1, ...$; $n = l, l+1, ...$, helps us to treat the case $\sigma = 0$, since then

$$\begin{aligned}
&\left| \sum_{n=k}^{\infty} \frac{2n+1}{4\pi A_{n,k}} \frac{(n-l-\sigma)!}{(n+l+\sigma)!} P_{n,l+\sigma}(t_\xi) P_{n,l}(t_\eta) \right| \\
\leq & \left(\sum_{n=k}^{\infty} \left(\frac{2n+1}{4\pi A_{n,k}} \right)^2 n^{\varepsilon} \right)^{1/2} \cdot \\
& \left(\sum_{n=k}^{\infty} \left(\frac{(n-l-\sigma)!}{(n+l+\sigma)!} \right)^2 \left(P_{n,l+\sigma}(t_\xi) P_{n,l}(t_\eta) \right)^2 n^{-\varepsilon} \right)^{1/2} \\
= & \left(\sum_{n=k}^{\infty} \left(\frac{2n+1}{4\pi A_{n,k}} \right)^2 n^{\varepsilon} \right)^{1/2} \left(\sum_{n=k}^{\infty} \left(\frac{(n-l)!}{(n+l)!} P_{n,l+\sigma}(t_\xi) P_{n,l}(t_\eta) \right)^2 n^{-\varepsilon} \right)^{1/2} \\
\leq & \left(\sum_{n=k}^{\infty} \left(\frac{2n+1}{4\pi A_{n,k}} \right)^2 n^{\varepsilon} \right)^{1/2} \left(\sum_{n=k}^{\infty} \frac{a^2}{n^{1+\varepsilon}} \right)^{1/2} \\
< & +\infty.
\end{aligned}$$

Hence, we can use kernels of order $k = 0$, i.e. series expansions in $P_{n,k} P_{n,k}$, where $k = 0$, to develop kernels for general $k = 0, 1, ...$
Note that

$$\left(\frac{d}{dt_\eta} \right)^k \sum_{n=k}^{\infty} \frac{2n+1}{4\pi A_{n,k}} P_n(s) P_n(t_\eta) = \left(\frac{d}{dt_\eta} \right)^k \sum_{n=p}^{\infty} \frac{2n+1}{4\pi A_{n,k}} P_n(s) P_n(t_\eta)$$

for every $p = 0, ..., k$.
We now discuss a particular series of order $k = 0$.
The point of departure is the Green function $G(L; \cdot, \cdot)$ with respect to the Legendre operator L. For brevity we set

$$F(s, t_\eta) = -2\, G(L; s, t_\eta),$$

i.e.

$$F(s, t_\eta) = \begin{cases} 2\ln 2 - 1 - \ln((1-s)(1+t_\eta)), & \text{if} -1 < s \leq t_\eta < 1 \\ 2\ln 2 - 1 - \ln((1-t_\eta)(1+s)), & \text{if} -1 < t_\eta < s < 1 \end{cases}.$$

Note that $A_{n,k} = n(n+1)$ satisfies (16). Obviously,

$$\frac{d}{dt_\eta} F(s, t_\eta) = \begin{cases} -\frac{1-s}{(1-s)(1+t_\eta)} = -\frac{1}{1+t_\eta}, & \text{if } -1 < s < t_\eta < 1 \\ +\frac{1+s}{(1-t_\eta)(1+s)} = +\frac{1}{1-t_\eta}, & \text{if } -1 < t_\eta < s < 1 \end{cases}.$$

More general,

$$\left(\frac{d}{dt_\eta}\right)^k F(s,t_\eta) = \begin{cases} (-1)^k \frac{(k-1)!}{(1+t_\eta)^k}, & \text{if } -1 < s < t_\eta < 1 \\ \frac{(k-1)!}{(1-t_\eta)^k}, & \text{if } -1 < t_\eta < s < 1 \end{cases},$$

$k = 1, 2, \ldots$. Now let t_η be fixed. If $t_\xi \geq t_\eta$, the first integration yields

$$\int_{t_\xi}^{1} \left(\frac{d}{dt_\eta}\right)^k F(s,t_\eta)\, ds = \int_{t_\xi}^{1} \frac{(k-1)!}{(1-t_\eta)^k}\, ds = \frac{(k-1)!}{(1-t_\eta)^k}(1-t_\xi).$$

The case $t_\xi < t_\eta$ needs more attention. We obtain

$$\begin{aligned} \int_{t_\xi}^{1} \left(\frac{d}{dt_\eta}\right)^k F(s,t_\eta)\, ds &= \int_{t_\xi}^{t_\eta} (-1)^k \frac{(k-1)!}{(1+t_\eta)^k}\, ds + \int_{t_\eta}^{1} \frac{(k-1)!}{(1-t_\eta)^k}\, ds \\ &= \left(\frac{-1}{1+t_\eta}\right)^k (k-1)!\,(t_\eta - t_\xi) + \frac{(k-1)!}{(1-t_\eta)^k}(1-t_\eta) \\ &= (-1)^k (k-1)! \frac{t_\eta - t_\xi}{(1+t_\eta)^k} + \frac{(k-1)!}{(1-t_\eta)^{k-1}}, \end{aligned}$$

$k = 1, 2, \ldots$. In the case $t_\xi \geq t_\eta$ we get the further results

$$\int_{t_\xi}^{1} \frac{(k-1)!}{(1-t_\eta)^k}(1-s)\, ds = \frac{(k-1)!}{(1-t_\eta)^k} \int_{0}^{1-t_\xi} \tau\, d\tau = \frac{(k-1)!}{(1-t_\eta)^k} \frac{1}{2}(1-t_\xi)^2$$

and

$$\frac{(k-1)!}{(1-t_\eta)^k} \frac{1}{2} \int_{t_\xi}^{1} (1-s)^2\, ds = \frac{(k-1)!}{(1-t_\eta)^k} \frac{1}{2} \int_{0}^{1-t_\xi} \tau^2\, d\tau = \frac{(k-1)!}{(1-t_\eta)^k} \frac{1}{2 \cdot 3}(1-t_\xi)^3.$$

Consequently, our elementary calculations lead us to

$$\int_{t_\xi}^{1} \cdots \int_{t_\xi}^{1} \frac{(k-1)!}{(1-t_\eta)^k} (ds)^k = \frac{(k-1)!}{(1-t_\eta)^k} \frac{1}{k!}(1-t_\xi)^k = \frac{1}{k}\left(\frac{1-t_\xi}{1-t_\eta}\right)^k; \qquad (17)$$

$k = 1, 2, \ldots$; which can easily be verified by induction.
In the case $t_\xi < t_\eta$ we find

$$\begin{aligned}
&\int_{t_\xi}^{1}\int_{t_\xi}^{1} \left(\frac{d}{dt_\eta}\right)^k F(s, t_\eta)\,(ds)^2 \\
&= \int_{t_\xi}^{t_\eta} (-1)^k (k-1)!\,\frac{t_\eta - s}{(1+t_\eta)^k} + \frac{(k-1)!}{(1-t_\eta)^{k-1}}\,ds + \int_{t_\eta}^{1} \frac{(k-1)!}{(1-t_\eta)^k}(1-s)\,ds \\
&= (-1)^k (k-1)! \int_0^{t_\eta - t_\xi} \frac{\tau}{(1+t_\eta)^k}\,d\tau + (k-1)!\,\frac{t_\eta - t_\xi}{(1-t_\eta)^{k-1}} \\
&\quad + \frac{(k-1)!}{(1-t_\eta)^k}\,\frac{1}{2}(1-t_\eta)^2 \\
&= (-1)^k (k-1)!\,\frac{1}{2}\,\frac{(t_\eta - t_\xi)^2}{(1+t_\eta)^k} + (k-1)!\,\frac{t_\eta - t_\xi}{(1-t_\eta)^{k-1}} + \frac{1}{2}\,\frac{(k-1)!}{(1-t_\eta)^{k-2}}\,,
\end{aligned}$$

provided that $2 \leq k$. By induction we are able to show that ($l \leq k$)

$$\begin{aligned}
&\int_{t_\xi}^{1} \cdots \int_{t_\xi}^{1} \left(\frac{d}{dt_\eta}\right)^k F(s, t_\eta)(ds)^l \\
&= (-1)^k\,\frac{(k-1)!}{l!}\,\frac{(t_\eta - t_\xi)^l}{(1+t_\eta)^k} + \sum_{j=0}^{l-1} \frac{(k-1)!}{(j+1)!\,(l-j-1)!}\,\frac{(t_\eta - t_\xi)^{l-j-1}}{(1-t_\eta)^{k-j-1}}\,,
\end{aligned}$$

since it is easily seen that for $l \geq 2$:

$$\begin{aligned}
&\int_{t_\xi}^{1} \cdots \int_{t_\xi}^{1} \left(\frac{d}{dt_\eta}\right)^k F(s, t_\eta)\,(ds)^l = \int_{t_\xi}^{t_\eta} (-1)^k \frac{(k-1)!}{(l-1)!}\,\frac{(t_\eta - s)^{l-1}}{(1+t_\eta)^k}\,ds \\
&\quad + \int_{t_\xi}^{t_\eta} \sum_{j=0}^{l-2} \frac{(k-1)!}{(j+1)!\,(l-j-2)!}\,\frac{(t_\eta - s)^{l-1-j-1}}{(1-t_\eta)^{k-j-1}}\,ds \\
&\quad + \int_{t_\eta}^{1} \frac{(k-1)!}{(1-t_\eta)^k}\,\frac{1}{(l-1)!}\,(1-s)^{l-1}\,ds \\
&= (-1)^k \frac{(k-1)!}{(l-1)!} \int_0^{t_\eta - t_\xi} \frac{\tau^{l-1}}{(1+t_\eta)^k}\,d\tau
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{j=0}^{l-2} \frac{(k-1)!}{(j+1)!\,(l-j-2)!} \frac{1}{(1-t_\eta)^{k-j-1}} \int_0^{t_\eta - t_\xi} \tau^{l-j-2}\, d\tau \\
& + \frac{(k-1)!}{(1-t_\eta)^k} \frac{1}{(l-1)!} \int_0^{1-t_\eta} \tau^{l-1}\, d\tau \\
= \quad & (-1)^k \frac{(k-1)!}{l!} \frac{(t_\eta - t_\xi)^l}{(1+t_\eta)^k} + \sum_{j=0}^{l-2} \frac{(k-1)!}{(j+1)!\,(l-j-1)!} \frac{(t_\eta - t_\xi)^{l-j-1}}{(1-t_\eta)^{k-j-1}} \\
& + \frac{(k-1)!}{(1-t_\eta)^k} \frac{1}{l!} (1-t_\eta)^l \\
= \quad & (-1)^k \frac{(k-1)!}{l!} \frac{(t_\eta - t_\xi)^l}{(1+t_\eta)^k} + \sum_{j=0}^{l-1} \frac{(k-1)!}{(j+1)!\,(l-j-1)!} \frac{(t_\eta - t_\xi)^{l-j-1}}{(1-t_\eta)^{k-j-1}} \,.
\end{aligned}
$$

Hence, it is not hard to see that for $t_\xi < t_\eta$

$$
\begin{aligned}
& \int_{t_\xi}^{1} \cdots \int_{t_\xi}^{1} \left(\frac{d}{dt_\eta}\right)^k F(s, t_\eta)\,(ds)^k = \qquad (18) \\
= \quad & (-1)^k \frac{1}{k} \left(\frac{t_\eta - t_\xi}{1+t_\eta}\right)^k + \sum_{j=0}^{k-1} \frac{(k-1)!}{(j+1)!\,(k-j-1)!} \left(\frac{t_\eta - t_\xi}{1-t_\eta}\right)^{k-j-1} \\
= \quad & (-1)^k \frac{1}{k} \left(\frac{t_\eta - t_\xi}{1+t_\eta}\right)^k + \frac{1}{k} \left(\sum_{j=1}^{k} \binom{k}{j} \left(\frac{t_\eta - t_\xi}{1-t_\eta}\right)^{k-j} \right) \\
= \quad & \frac{(-1)^k}{k} \left(\frac{t_\eta - t_\xi}{1+t_\eta}\right)^k + \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{t_\eta - t_\xi}{1-t_\eta}\right)^{k-j} 1^j - \frac{1}{k} \left(\frac{t_\eta - t_\xi}{1-t_\eta}\right)^k \\
= \quad & \frac{1}{k} \left(\left(\frac{-1}{1+t_\eta}\right)^k - \frac{1}{(1-t_\eta)^k} \right) (t_\eta - t_\xi)^k + \frac{1}{k} \left(1 + \frac{t_\eta - t_\xi}{1-t_\eta}\right)^k \\
= \quad & \frac{1}{k} \left(\left(\frac{-1}{1+t_\eta}\right)^k - \frac{1}{(1-t_\eta)^k} \right) (t_\eta - t_\xi)^k + \frac{1}{k} \left(\frac{1-t_\xi}{1-t_\eta}\right)^k ,
\end{aligned}
$$

$k = 1, 2, \ldots$. Summarizing our results we obtain in connection with (15), (17), and (18)

$$
K_k(\xi, \eta) = \sum_{n=k}^{\infty} \frac{c_{n,k}^2}{n(n+1)} P_{n,k}(t_\xi) P_{n,k}(t_\eta) \cos(k\varphi_\xi) \cos(k\varphi_\eta)
$$

$$= \begin{cases} 2(-1)^k \frac{1}{4\pi k} \left(\frac{1-t_\eta{}^2}{(1-t_\eta)^2}\right)^{k/2} \left(\frac{(1-t_\xi)^2}{1-t_\xi{}^2}\right)^{k/2} \cos(k\varphi_\xi)\cos(k\varphi_\eta), \text{if } t_\xi \geq t_\eta \\ \frac{1}{2\pi k}\cos(k\varphi_\xi)\cos(k\varphi_\eta)\left((-1)^k \left(\frac{1+t_\eta}{1-t_\eta}\right)^{k/2}\left(\frac{1-t_\xi}{1+t_\xi}\right)^{k/2}\right. \\ \left. + \left(\left(\frac{1-t_\eta}{1+t_\eta}\right)^{k/2} - (-1)^k\left(\frac{1+t_\eta}{1-t_\eta}\right)^{k/2}\right)\frac{(t_\eta - t_\xi)^k}{\left(1-t_\xi{}^2\right)^{k/2}}\right), \qquad \text{if } t_\xi < t_\eta \end{cases}$$

$$= \begin{cases} (-1)^k \frac{1}{2\pi k}\left(\frac{1+t_\eta}{1-t_\eta}\,\frac{1-t_\xi}{1+t_\xi}\right)^{k/2}\cos(k\varphi_\xi)\cos(k\varphi_\eta), & \text{if } t_\xi \geq t_\eta \\ \frac{1}{2\pi k}\left(\left(\frac{1+t_\eta}{1-t_\eta}\right)^{k/2}(-1)^k\left(\left(\frac{1-t_\xi}{1+t_\xi}\right)^{k/2} - \frac{(t_\eta - t_\xi)^k}{\left(1-t_\xi{}^2\right)^{k/2}}\right)\right. \\ \left. + \left(\frac{1-t_\eta}{1+t_\eta}\right)^{k/2}\frac{(t_\eta - t_\xi)^k}{\left(1-t_\xi{}^2\right)^{k/2}}\right)\cos(k\varphi_\xi)\cos(k\varphi_\eta), & \text{if } t_\xi < t_\eta \end{cases}$$

for $k = 1, 2, \dots$.

4.2 Iteration for $k = 0$

In what follows we now calculate the convolution of the investigated kernel with itself, more precisely we discuss with $A_n = n(n+1)$ the expression

$$\begin{aligned} &\int_\Omega \left(\sum_{n=k}^\infty \frac{1}{A_n} Y_{n,k}(\xi)Y_{n,k}(\eta)\right)\left(\sum_{m=k}^\infty \frac{1}{A_m} Y_{m,k}(\eta)Y_{m,k}(\zeta)\right) d\omega(\eta) \\ &= \sum_{n=k}^\infty \sum_{m=k}^\infty \frac{1}{A_n\, A_m} Y_{n,k}(\xi)Y_{m,k}(\zeta)\int_\Omega Y_{n,k}(\eta)Y_{m,k}(\eta)\, d\omega(\eta) \\ &= \sum_{n=k}^\infty \frac{1}{A_n^2} Y_{n,k}(\xi)Y_{n,k}(\zeta); \quad k = 0, 1, \dots; \end{aligned}$$

where the interchanging of summation and integration is justified by B. Levi's Theorem. Note that $\{A_n\}_{n=0,1,\dots}$ is summable with respect to $\mathcal{N} \setminus \{(0,0)\}$. This is the reason why the iterated kernel above can be used as a V-Shannon $\mathcal{H}(\Omega)$-wavelet.
We have to discuss the case $k = 0$ separately, i.e. the Green function $G\left(L^2; \cdot, \cdot\right)$ with respect to the iterated Legendre operator L^2. Our point of departure is the Green function with respect to L. For brevity we set

$$\begin{aligned} K_0(\xi,\eta) &= \sum_{n=1}^\infty \frac{1}{n(n+1)} Y_{n,0}(\xi)Y_{n,0}(\eta) \\ &= \frac{1}{4\pi}\cdot\begin{cases} 2\ln 2 - 1 - \ln\left((1-t_\xi)(1+t_\eta)\right), & \text{if } -1 < t_\xi \leq t_\eta < 1 \\ 2\ln 2 - 1 - \ln\left((1-t_\eta)(1+t_\xi)\right), & \text{if } -1 < t_\eta < t_\xi < 1 \end{cases}, \end{aligned}$$

i.e.

$$K_0^{(2)}(\xi,\zeta) = \int_\Omega K(\xi,\eta)K(\eta,\zeta)\,d\omega(\eta) = \frac{1}{2\pi}\,G\left(L^2;t_\xi,t_\zeta\right) \quad .$$

Without loss of generality, let $t_\xi \leq t_\zeta$. Then (with $t = t_\eta$)

$$\begin{aligned}
&K_0^{(2)}(\xi,\zeta)\\
&= \frac{2\pi}{4\pi}\int_{-1}^{t_\xi} (2\ln 2 - 1 - \ln((1-t)(1+t_\xi)))\cdot\\
&\qquad (2\ln 2 - 1 - \ln((1-t)(1+t_\zeta)))\;dt\\
&\quad + \frac{2\pi}{4\pi}\int_{t_\xi}^{t_\zeta} (2\ln 2 - 1 - \ln((1-t_\xi)(1+t)))\cdot\\
&\qquad (2\ln 2 - 1 - \ln((1-t)(1+t_\zeta)))\;dt\\
&\quad + \frac{2\pi}{4\pi}\int_{t_\zeta}^{1} (2\ln 2 - 1 - \ln((1-t_\xi)(1+t)))\cdot\\
&\qquad (2\ln 2 - 1 - \ln((1-t_\zeta)(1+t)))\;dt\\
&= \frac{1}{2}(1-2\ln 2)(\ln(1+t_\xi)+\ln(1+t_\zeta)+1-2\ln 2)(t_\xi+1)\\
&\quad + \frac{1}{2}\int_{-1}^{t_\xi}(1-2\ln 2)2\ln(1-t)\,dt\\
&\quad + \frac{1}{2}\int_{-1}^{t_\xi}(\ln(1-t))^2 + (\ln(1+t_\xi)+\ln(1+t_\zeta))\ln(1-t)\,dt\\
&\quad + \frac{1}{2}\ln(1+t_\xi)\ln(1+t_\zeta)(t_\xi+1)\\
&\quad + \frac{1}{2}(1-2\ln 2)(\ln(1-t_\xi)+\ln(1+t_\zeta)+1-2\ln 2)(t_\zeta-t_\xi)\\
&\quad + \frac{1}{2}\int_{t_\xi}^{t_\zeta}(1-2\ln 2)(\ln(1+t)+\ln(1-t))\,dt\\
&\quad + \frac{1}{2}\int_{t_\xi}^{t_\zeta}\ln(1-t)\ln(1+t)+\ln(1-t_\xi)\ln(1-t)+\ln(1+t_\zeta)\ln(1+t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\ln(1-t_\xi)\ln(1+t_\zeta)(t_\zeta-t_\xi)\\
&+\frac{1}{2}(1-2\ln 2)(\ln(1-t_\xi)+\ln(1-t_\zeta)+1-2\ln 2)(1-t_\zeta)\\
&+\frac{1}{2}\int_{t_\zeta}^{1}(1-2\ln 2)2\ln(1+t)\,dt\\
&+\frac{1}{2}\int_{t_\zeta}^{1}(\ln(1+t))^2+(\ln(1-t_\xi)+\ln(1-t_\zeta))\ln(1+t)\,dt\\
&+\frac{1}{2}\ln(1-t_\xi)\ln(1-t_\zeta)(1-t_\zeta).
\end{aligned}
$$

By partial integration it follows that

$$
\begin{aligned}
\int \ln x\,dx &= x(\ln x-1),\\
\int (\ln x)^2\,dx &= x\left((\ln x)^2-2\ln x+2\right).
\end{aligned}
$$

Using these identities and the relation

$$
\ln(1+t)\ln(1-t)=-\sum_{j=1}^{\infty}\frac{t^{2j}}{j}\sum_{n=1}^{2j-1}\frac{(-1)^{n+1}}{n},\quad |t|<1,
$$

(cf. I.S. Gradshteyn, I.M. Ryzhik (1980)), which implies

$$
\int_{t_\xi}^{t_\zeta}\ln(1+t)\ln(1-t)\,dt=-\sum_{j=1}^{\infty}\sum_{n=1}^{2j-1}\frac{(-1)^{n+1}}{n}\left.\frac{t^{2j+1}}{j(2j+1)}\right|_{t_\xi}^{t_\zeta},
$$

we obtain

$$
\begin{aligned}
K_0^{(2)}(\xi,\zeta) &= (2\ln 2-1)^2+\frac{1}{2}(1-2\ln 2)((\ln(1+t_\xi)+\ln(1+t_\zeta))(t_\xi+1)\\
&\quad +2(2(\ln 2-1)-(1-t_\xi)(\ln(1-t_\xi)-1)))\\
&+\frac{1}{2}\left(2((\ln 2)^2-2\ln 2+2)-(1-t_\xi)((\ln(1-t_\xi))^2-2\ln(1-t_\xi)+2)\right)\\
&+\frac{1}{2}\ln(1+t_\xi)(2(\ln 2-1)-(1-t_\xi)(\ln(1-t_\xi)-1))\\
&+\frac{1}{2}\ln(1+t_\zeta)(2(\ln 2-1)-(1-t_\xi)(\ln(1-t_\xi)-1))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\ln(1+t_\xi)\ln(1+t_\zeta)(t_\xi+1)\\
&+\frac{1}{2}(1-2\ln 2)((\ln(1-t_\xi)+\ln(1+t_\zeta))(t_\zeta-t_\xi)\\
&\qquad\qquad +(1+t_\zeta)(\ln(1+t_\zeta)-1)-(1+t_\xi)(\ln(1+t_\xi)-1)\\
&\qquad\qquad +(1-t_\xi)(\ln(1-t_\xi)-1)-(1-t_\zeta)(\ln(1-t_\zeta)-1))\\
&+\frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=1}^{2j-1}\frac{(-1)^{n+1}}{n}\left.\frac{t^{2j+1}}{j(2j+1)}\right|_{t_\zeta}^{t_\xi}\\
&+\frac{1}{2}\ln(1-t_\xi)((1-t_\xi)(\ln(1-t_\xi)-1)-(1-t_\zeta)(\ln(1-t_\zeta)-1))\\
&+\frac{1}{2}\ln(1+t_\zeta)((1+t_\zeta)(\ln(1+t_\zeta)-1)-(1+t_\xi)(\ln(1+t_\xi)-1))\\
&+\frac{1}{2}\ln(1-t_\xi)\ln(1+t_\zeta)(t_\zeta-t_\xi)\\
&+\frac{1}{2}(1-2\ln 2)((\ln(1-t_\xi)+\ln(1-t_\zeta))(1-t_\zeta)\\
&\qquad\qquad +2(2(\ln 2-1)-(1+t_\zeta)(\ln(1+t_\zeta)-1)))\\
&+\frac{1}{2}\Big(2\left((\ln 2)^2-2\ln 2+2\right)-(1+t_\zeta)\left((\ln(1+t_\zeta))^2-2\ln(1+t_\zeta)+2\right)\Big)\\
&+\frac{1}{2}(\ln(1-t_\xi)+\ln(1-t_\zeta))(2(\ln 2-1)-(1+t_\zeta)(\ln(1+t_\zeta)-1))\\
&+\frac{1}{2}\ln(1-t_\xi)\ln(1-t_\zeta)(1-t_\zeta)\\
=\;&(2\ln 2-1)^2-\frac{1}{2}(\ln(1+t_\xi)+\ln(1+t_\zeta)+\ln(1-t_\xi)+\ln(1-t_\zeta))\\
&+\frac{1}{2}(1-2\ln 2)\ln(1+t_\xi)(t_\xi+1-1-(1+t_\xi))\\
&+\frac{1}{2}(1-2\ln 2)\ln(1+t_\zeta)(t_\xi+1-1+t_\zeta-t_\xi+1+t_\zeta-(1+t_\zeta))\\
&+\frac{1}{2}(1-2\ln 2)\ln(1-t_\xi)(t_\zeta-t_\xi-2(1-t_\xi)+1-t_\xi+1-t_\zeta-1)\\
&+\frac{1}{2}(1-2\ln 2)\ln(1-t_\zeta)(-(1-t_\zeta)+(1-t_\zeta)-1)\\
&+\frac{1}{2}(1-2\ln 2)(4\ln 2-4+2-2t_\xi-1-t_\zeta+1+t_\xi-1+t_\xi+1-t_\zeta\\
&\qquad\qquad +4\ln 2-4+2+2t_\zeta)+2\left((\ln 2)^2-2\ln 2+2\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}\ln(1+t_\xi)(-(1-t_\xi)(\ln(1-t_\xi)-1)+(t_\xi+1)\ln(1+t_\zeta) \\
& \qquad\qquad -(1+t_\xi)\ln(1+t_\zeta)) \\
& +\frac{1}{2}\ln(1+t_\zeta)(-(1-t_\xi)(\ln(1-t_\xi)-1)+(1+t_\zeta)\ln(1+t_\zeta) \\
& \quad -(1+t_\zeta)\ln(1+t_\zeta)+2(1+t_\zeta)-(1+t_\zeta)+(1+t_\xi) \\
& \quad -(1+t_\zeta)(\ln(1-t_\xi)+\ln(1-t_\zeta))+(t_\zeta-t_\xi)\ln(1-t_\xi)) \\
& +\frac{1}{2}\ln(1-t_\xi)(-(1-t_\xi)\ln(1-t_\xi)+2(1-t_\xi)+(1-t_\xi)\ln(1-t_\xi) \\
& \qquad\qquad -(1-t_\xi)-(1-t_\zeta)\ln(1-t_\zeta)+1-t_\zeta \\
& \qquad\qquad +(1-t_\zeta)\ln(1-t_\zeta)+1+t_\zeta) \\
& +\frac{1}{2}\ln(1-t_\zeta)(1+t_\zeta)+\frac{1}{2}(-2+2t_\xi-2-2t_\zeta) \\
& +\frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=1}^{2j-1}\frac{(-1)^{n+1}}{n}\;\frac{t^{2j+1}}{j(2j+1)}\Bigg|_{t_\zeta}^{t_\xi} \\
= \;& (2\ln 2-1)^2-\frac{1}{2}\ln\left(\left(1-{t_\xi}^2\right)\left(1-{t_\zeta}^2\right)\right)-\frac{1}{2}(1-2\ln 2)\ln(1+t_\xi) \\
& +\frac{1}{2}(1-2\ln 2)t_\zeta\ln(1+t_\zeta)-\frac{1}{2}(1-2\ln 2)\ln(1-t_\xi) \\
& -\frac{1}{2}(1-2\ln 2)\ln(1-t_\zeta)+\frac{1}{2}(1-2\ln 2)(8\ln 2-4) \\
& +2\left((\ln 2)^2-2\ln 2+2\right)+\frac{1}{2}\ln(1+t_\xi)(\ln(1-t_\xi)(t_\xi-1)+1-t_\xi) \\
& +\frac{1}{2}\ln(1+t_\zeta)(3+t_\zeta-2\ln(1-t_\xi)-(1+t_\zeta)\ln(1-t_\zeta)) \\
& +\frac{3}{2}\ln(1-t_\xi)+\frac{1}{2}\ln(1-t_\zeta)(1+t_\zeta)+t_\xi-t_\zeta-2 \\
& +\frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=1}^{2j-1}\frac{(-1)^{n+1}}{n}\;\frac{t^{2j+1}}{j(2j+1)}\Bigg|_{t_\zeta}^{t_\xi} \\
= \;& 6(\ln 2)^2-8\ln 2+5-\frac{1}{2}\ln\left(\left(1-{t_\xi}^2\right)\left(1-{t_\zeta}^2\right)\right) \\
& +\frac{1}{2}(1-2\ln 2)(-\ln(1+t_\xi)+t_\zeta\ln(1+t_\zeta)-\ln(1-t_\xi)-\ln(1-t_\zeta) \\
& \qquad\qquad +8\ln 2-4)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\ln(1+t_\xi)\ln(1-t_\xi)(t_\xi-1)+(1-t_\xi)\ln(1+t_\xi)\right.\\
&\qquad +(3+t_\zeta)\ln(1+t_\zeta)-2\ln(1+t_\zeta)\ln(1-t_\xi)\\
&\qquad \left.-(1+t_\zeta)\ln(1+t_\zeta)\ln(1-t_\zeta)+3\ln(1-t_\xi)+(1+t_\zeta)\ln(1-t_\zeta)\right)\\
&+t_\xi-t_\zeta-2+\frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=1}^{2j-1}\frac{(-1)^{n+1}}{n}\,\frac{t^{2j+1}}{j(2j+1)}\Bigg|_{t_\zeta}^{t_\xi}\\
= \;&-2(\ln 2)^2+1-\frac{1}{2}\ln\left(\left(1-{t_\xi}^2\right)\left(1-{t_\zeta}^2\right)\right)\\
&+\frac{1}{2}(2\ln 2-1)\ln\left(\left(1-{t_\xi}^2\right)(1-t_\zeta)\right)+\frac{1}{2}(1-2\ln 2)t_\zeta\ln(1+t_\zeta)\\
&+\frac{3}{2}\ln((1+t_\zeta)(1-t_\xi))+\frac{1}{2}t_\zeta\ln\left(1-{t_\zeta}^2\right)+\ln(1-t_\zeta)\\
&-\frac{1}{2}(1+t_\zeta)\ln(1+t_\zeta)\ln(1-t_\zeta)+\frac{1}{2}(1-t_\xi)\ln(1+t_\xi)(1-\ln(1-t_\xi))\\
&-\ln(1+t_\zeta)\ln(1-t_\xi)+t_\xi-t_\zeta\\
&+\frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=1}^{2j-1}\frac{(-1)^{n+1}}{n}\,\frac{t^{2j+1}}{j(2j+1)}\Bigg|_{t_\zeta}^{t_\xi}.
\end{aligned}
$$

The last summand can only be approximated by a truncated series. We develop an upper bound of the approximation error. Since it is known that

$$
\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\ln 2,
$$

we obtain

$$
\begin{aligned}
&\frac{1}{2}\left|\sum_{j=l+1}^{\infty}\left(\sum_{n=1}^{2j-1}\frac{(-1)^{n+1}}{n}\right)\frac{1}{j(2j+1)}\left({t_\xi}^{2j+1}-{t_\zeta}^{2j+1}\right)\right|\\
&\quad\leq\;\frac{1}{2}\sum_{j=l+1}^{\infty}\frac{\ln 2}{j(2j+1)}\cdot 2<\frac{\ln 2}{2}\sum_{j=l+1}^{\infty}\frac{1}{j^2}.
\end{aligned}
$$

The value of the Riemann zeta function at 2 is well–known from classical trigonometric Fourier theory (cf. e.g. E. ZEIDLER (1996))

$$
\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6}.
$$

$l =$	$E \approx$
10	$3.3 \cdot 10^{-2}$
50	$6.9 \cdot 10^{-3}$
100	$3.4 \cdot 10^{-3}$
200	$1.7 \cdot 10^{-3}$
500	$6.9 \cdot 10^{-4}$
1000	$3.5 \cdot 10^{-4}$
5000	$6.9 \cdot 10^{-5}$
10000	$3.5 \cdot 10^{-5}$

Table 1: Upper bound of the approximation error for different truncations

Table 1 shows the upper bound

$$E = \frac{\ln 2}{2} \sum_{j=l+1}^{\infty} \frac{1}{j^2}$$

of the approximation error in dependence of some values of l.
The obtained iterated kernel is the scaling function of scale 0.

$$K_0^{(2)}(\xi,\zeta) = \Phi_0(\xi,\zeta) = \Psi_{-1}(\xi,\zeta); \quad \xi,\zeta \in \Omega \setminus \left\{\pm\varepsilon^3\right\} \ .$$

Note that the derived representations of the wavelets are only valid for $t_\xi, t_\zeta \in (-1,1)$. However, the knowledge of the values of the wavelets at the poles is not needed in our approach. For our considerations we simply have to know that the series also converges within this 'boundary', which is certainly true here.

$$\left| \sum_{n=k}^{\infty} \frac{1}{n^2(n+1)^2} Y_{n,k}(\xi) Y_{n,k}(\zeta) \right| \leq \sum_{n=k}^{\infty} \frac{2n+1}{n^2(n+1)^2} < +\infty.$$

4.3 Iteration for $k \neq 0$

Now we can start calculating the iterated kernels for $k \neq 0$. We can represent our kernel in the form

$$K_k(\xi,\eta) = \sum_{n=k}^{\infty} \frac{1}{A_n} Y_{n,k}(\xi) Y_{n,k}(\eta)$$

as

$$K_k(\xi,\eta) = \frac{1}{2\pi k}\, G_k(t_\xi, t_\eta)\cos(k\varphi_\xi)\cos(k\varphi_\eta),$$

if $k > 0$, where G_k denotes the part of K depending on the polar distances t_ξ and t_η. The determination of the convolution yields ($\varphi := \varphi_\eta$, $t := t_\eta$)

$$\begin{aligned}
&\int_\Omega K_k(\xi,\eta)K_k(\eta,\zeta)\, d\omega(\eta) \\
&= \left(\frac{1}{2\pi k}\right)^2 \int_0^{2\pi} \cos(k\varphi_\xi)\cos(k\varphi_\eta)\cos(k\varphi_\eta)\cos(k\varphi_\zeta)\, d\varphi_\eta \cdot \\
&\qquad \int_{-1}^{1} G_k(t_\xi, t)G_k(t, t_\zeta)\, dt \\
&= \left(\frac{1}{2\pi k}\right)^2 \cos(k\varphi_\xi)\cos(k\varphi_\zeta)\pi \int_{-1}^{1} G_k(t_\xi, t)G_k(t, t_\zeta)\, dt.
\end{aligned}$$

Analogously, we have to discuss the series involving spherical harmonics $Y_{n,k}$ with negative order k, where we find

$$\begin{aligned}
&\int_\Omega K_k(\xi,\eta)K_k(\eta,\zeta)\, d\omega(\eta) \\
&= \left(\frac{1}{2\pi k}\right)^2 \int_0^{2\pi} \sin(k\varphi_\xi)\sin(k\varphi_\eta)\sin(k\varphi_\eta)\sin(k\varphi_\zeta)\, d\varphi_\eta \cdot \\
&\qquad \int_{-1}^{1} G_k(t_\xi, t)G_k(t, t_\zeta)\, dt \\
&= \left(\frac{1}{2\pi k}\right)^2 \sin(k\varphi_\xi)\sin(k\varphi_\zeta)\pi \int_{-1}^{1} G_k(t_\xi, t)G_k(t, t_\zeta)\, dt.
\end{aligned}$$

The part G_k, representing the dependence of the kernel K_k on the polar distance, is identical for $Y_{n,k}$ and $Y_{n,-k}$, such that the following calculations are valid in both cases.

Without loss of generality, let $t_\xi \le t_\zeta$. Then we get

$$\int_{-1}^{1} G_k(t_\xi, t) G_k(t, t_\zeta)\, dt$$

$$= \int_{-1}^{t_\xi} (-1)^k \left(\frac{1+t}{1-t}\,\frac{1-t_\xi}{1+t_\xi}\right)^{k/2} \left((-1)^k \left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2} \left(\frac{1-t}{1+t}\right)^{k/2}\right.$$

$$\left. + \left(\left(\frac{1-t_\zeta}{1+t_\zeta}\right)^{k/2} - (-1)^k \left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}\right) \frac{(t_\zeta - t)^k}{(1-t^2)^{k/2}}\right) dt$$

$$+ \int_{t_\xi}^{t_\zeta} \left((-1)^k \left(\frac{1+t}{1-t}\right)^{k/2} \left(\frac{1-t_\xi}{1+t_\xi}\right)^{k/2} + \left(\left(\frac{1-t}{1+t}\right)^{k/2} - (-1)^k \cdot\right.\right.$$

$$\left.\left.\left(\frac{1+t}{1-t}\right)^{k/2}\right) \frac{(t-t_\xi)^k}{(1-{t_\xi}^2)^{k/2}}\right) \left((-1)^k \left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2} \left(\frac{1-t}{1+t}\right)^{k/2}\right.$$

$$\left. + \left(\left(\frac{1-t_\zeta}{1+t_\zeta}\right)^{k/2} - (-1)^k \left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}\right) \frac{(t_\zeta - t)^k}{(1-t^2)^{k/2}}\right) dt$$

$$+ \int_{t_\zeta}^{1} \left((-1)^k \left(\frac{1+t}{1-t}\right)^{k/2} \left(\frac{1-t_\xi}{1+t_\xi}\right)^{k/2} + \left(\left(\frac{1-t}{1+t}\right)^{k/2} - (-1)^k \cdot\right.\right.$$

$$\left.\left.\left(\frac{1+t}{1-t}\right)^{k/2}\right) \frac{(t-t_\xi)^k}{(1-{t_\xi}^2)^{k/2}}\right) (-1)^k \left(\frac{1+t_\zeta}{1-t_\zeta}\,\frac{1-t}{1+t}\right)^{k/2} dt$$

$$= \int_{-1}^{t_\xi} \left(\frac{1-t_\xi}{1+t_\xi}\,\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}$$

$$+ \left((-1)^k \left(\frac{1-t_\zeta}{1+t_\zeta}\right)^{k/2} - \left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}\right) \left(\frac{1-t_\xi}{1+t_\xi}\right)^{k/2} \left(\frac{t_\zeta - t}{1-t}\right)^k dt$$

$$+ \int_{t_\xi}^{t_\zeta} \left(\frac{1-t_\xi}{1+t_\xi}\,\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}$$

$$+\left(\frac{1-t_\xi}{1+t_\xi}\right)^{k/2}\left((-1)^k\left(\frac{1-t_\zeta}{1+t_\zeta}\right)^{k/2}-\left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}\right)\left(\frac{t_\zeta-t}{1-t}\right)^k$$

$$+\left(\frac{1+t_\zeta}{(1-{t_\xi}^2)\,(1-t_\zeta)}\right)^{k/2}\left((-1)^k\left(\frac{1-t}{1+t}\right)^k(t-t_\xi)^k-(t-t_\xi)^k\right)$$

$$+\left(\left(\frac{1-t_\zeta}{1+t_\zeta}\right)^{k/2}-(-1)^k\left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}\right)\frac{1}{(1-{t_\xi}^2)^{k/2}}\cdot$$

$$\left(\left(\frac{(t-t_\xi)(t_\zeta-t)}{1+t}\right)^k-(-1)^k\left(\frac{(t-t_\xi)(t_\zeta-t)}{1-t}\right)^k\right)$$

$$+\int_{t_\zeta}^{1}\left(\frac{1-t_\xi}{1+t_\xi}\,\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}+\left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}\frac{1}{(1-{t_\xi}^2)^{k/2}}\cdot$$

$$\left((-1)^k\left(\frac{(1-t)(t-t_\xi)}{1+t}\right)^k-(t-t_\xi)^k\right)dt$$

$$=:\quad I_1+I_2+I_3.$$

We have to discuss the appearing types of integrals. Let $a,b,c,d\in\mathbb{R}$. Then we get by partial integration

$$\int_a^b\left(\frac{c-t}{1-t}\right)^k dt=\frac{1}{k-1}\;\frac{(c-t)^k}{(1-t)^{k-1}}\bigg|_a^b+\int_a^b\frac{k}{k-1}\,\frac{(c-t)^{k-1}}{(1-t)^{k-1}}\,dt=\ldots$$

$$=\sum_{j=1}^{p}\frac{k}{(k-j+1)(k-j)}\;\frac{(c-t)^{k-j+1}}{(1-t)^{k-j}}\bigg|_a^b+\int_a^b\frac{k}{k-p}\left(\frac{c-t}{1-t}\right)^{k-p}dt$$

for $1\le p\le k-1$. Thus it follows that

$$\int_a^b\left(\frac{c-t}{1-t}\right)^k dt=\sum_{j=1}^{k-1}\frac{k}{(k-j+1)(k-j)}\;\frac{(c-t)^{k-j+1}}{(1-t)^{k-j}}\bigg|_a^b+k\int_a^b\frac{c-t}{1-t}\,dt$$

$$=\sum_{j=1}^{k-1}\frac{k}{(k-j+1)(k-j)}\;\frac{(c-t)^{k-j+1}}{(1-t)^{k-j}}\bigg|_a^b+k(c-1)\ln\frac{|1-a|}{|1-b|}+(b-a)k,$$

provided that $a, b \neq 1$. Hence, the first integral reads as follows

$$
\begin{aligned}
I_1 = & \left(\frac{(1-t_\xi)(1+t_\zeta)}{(1+t_\xi)(1-t_\zeta)} \right)^{k/2} (t_\xi + 1) \\
& + \left((-1)^k \left(\frac{1-t_\zeta}{1+t_\zeta} \right)^{k/2} - \left(\frac{1+t_\zeta}{1-t_\zeta} \right)^{k/2} \right) \left(\frac{1-t_\xi}{1+t_\xi} \right)^{k/2} \cdot \\
& \left(\sum_{j=1}^{k-1} \frac{k}{(k-j+1)(k-j)} \left(\frac{(t_\zeta - t_\xi)^{k-j+1}}{(1-t_\xi)^{k-j}} \right. \right. \\
& \left. \left. - \frac{(t_\zeta + 1)^{k-j+1}}{2^{k-j}} \right) + k(t_\zeta - 1) \ln \frac{2}{1-t_\xi} + (t_\xi + 1)k \right) .
\end{aligned}
$$

Every further non–trivial integral belongs to the type

$$
\int_a^b \left(\frac{(c-t)(d-t)}{1+t} \right)^k dt,
$$

which can be seen by taking into account that

$$
\begin{aligned}
\int_a^b \left(\frac{(c-t)(d-t)}{1-t} \right)^k dt &= \int_{-b}^{-a} \left(\frac{(c+\tau)(d+\tau)}{1+\tau} \right)^k d\tau \\
&= \int_{-b}^{-a} \left(\frac{((-c)-\tau)((-d)-\tau)}{1+\tau} \right)^k d\tau.
\end{aligned}
$$

For $n \in \mathbb{N}$, $n < k$, partial integration yields

$$
\begin{aligned}
\int_a^b \frac{t^n}{(1+t)^k} dt &= -\frac{1}{k-1} \left. \frac{t^n}{(1+t)^{k-1}} \right|_a^b + \int_a^b \frac{n}{k-1} \frac{t^{n-1}}{(1+t)^{k-1}} dt = \ldots \\
&= -\sum_{j=0}^{p} \frac{n!}{(n-j)!} \frac{(k-2-j)!}{(k-1)!} \left. \frac{t^{n-j}}{(1+t)^{k-1-j}} \right|_a^b \\
& \quad + \int_a^b \frac{n!(k-2-p)!}{(n-p-1)!\,(k-1)!} \frac{t^{n-p-1}}{(1+t)^{k-1-p}} dt,
\end{aligned}
$$

provided that $0 \le p \le n-1$, i.e.

$$\begin{aligned}\int_a^b \frac{t^n}{(1+t)^k}\,dt &= -\sum_{j=0}^{n-1} \frac{n!}{(n-j)!}\,\frac{(k-2-j)!}{(k-1)!}\,\frac{t^{n-j}}{(1+t)^{k-1-j}}\Bigg|_a^b \\ &\quad + n!\,\frac{(k-n-1)!}{(k-1)!}\int_a^b \frac{t^0}{(1+t)^{k-n}}\,dt \\ &= -\sum_{j=0}^{n-1} \frac{n!}{(n-j)!}\,\frac{(k-2-j)!}{(k-1)!}\,\frac{t^{n-j}}{(1+t)^{k-1-j}}\Bigg|_a^b \\ &\quad + n!\,\frac{(k-n-1)!}{(k-1)!}\,\gamma,\end{aligned}$$

where we have used the abbreviation

$$\gamma = \begin{cases} \frac{1}{k-n-1}\left(\frac{1}{(1+a)^{k-n-1}} - \frac{1}{(1+b)^{k-n-1}}\right), & \text{if } k-n>1 \\ \ln\frac{|1+b|}{|1+a|}, & \text{if } k-n=1 \end{cases}.$$

If $n \ge k$, we analogously find

$$\begin{aligned}\int_a^b \frac{t^n}{(1+t)^k}\,dt &= -\sum_{j=0}^{k-2} \frac{n!}{(n-j)!}\,\frac{(k-2-j)!}{(k-1)!}\,\frac{t^{n-j}}{(1+t)^{k-1-j}}\Bigg|_a^b \\ &\quad + \int_a^b \frac{n!}{(n-k+1)!\,(k-1)!}\,\frac{t^{n-k+1}}{1+t}\,dt.\end{aligned}$$

For $a, b > -1$ we get

$$\begin{aligned}\int_a^b \frac{t^{n-k+1}}{1+t}\,dt &= t^{n-k+1}\ln|1+t|\Big|_a^b - \int_a^b (n-k+1)t^{n-k}\ln|1+t|\,dt \\ &= t^{n-k+1}\ln|1+t|\Big|_a^b - (n-k+1)\int_{1+a}^{1+b} (\tau-1)^{n-k}\ln\tau\,d\tau \\ &= t^{n-k+1}\ln|1+t|\Big|_a^b - (n-k+1)\sum_{p=0}^{n-k}\binom{n}{p}(-1)^{n-k-p}\int_{1+a}^{1+b}\tau^p\ln\tau\,d\tau \\ &= t^{n-k+1}\ln|1+t|\Big|_a^b\end{aligned}$$

$$-(n-k+1)\sum_{p=0}^{n-k}\left(\begin{array}{c} n \\ p \end{array}\right)(-1)^{n-k-p}\left(\tau^{p+1}\left(\frac{\ln\tau}{p+1}-\frac{1}{(p+1)^2}\right)\right)\Big|_{1+a}^{1+b},$$

where the last integration can be performed by partial integration. Consequently, it follows that

$$\int_a^b \frac{t^n}{(1+t)^k}\,dt = -\sum_{j=0}^{k-2}\frac{n!}{(n-j)!}\,\frac{(k-2-j)!}{(k-1)!}\,\frac{t^{n-j}}{(1+t)^{k-1-j}}\Big|_a^b$$
$$+\left(\begin{array}{c} n \\ k-1 \end{array}\right)\left(t^{n-k+1}\ln|1+t|\Big|_a^b\right.$$
$$\left.-(n-k+1)\sum_{p=0}^{n-k}\left(\begin{array}{c} n \\ p \end{array}\right)(-1)^{n-k-p}\left(\tau^{p+1}\left(\frac{\ln\tau}{p+1}-\frac{1}{(p+1)^2}\right)\right)\Big|_{1+a}^{1+b}\right).$$

Consequently,

$$\int_a^b\left(\frac{(c-t)(d-t)}{1+t}\right)^k dt =$$
$$= \sum_{l=0}^{k}\sum_{m=0}^{k}\left(\begin{array}{c} k \\ l \end{array}\right)\left(\begin{array}{c} k \\ m \end{array}\right)c^{k-l}d^{k-m}(-1)^{l+m}\int_a^b\frac{t^{l+m}}{(1+t)^k}\,dt$$
$$= \sum_{l=0}^{k-1}\sum_{m=0}^{k-1-l}\left(\begin{array}{c} k \\ l \end{array}\right)\left(\begin{array}{c} k \\ m \end{array}\right)c^{k-l}d^{k-m}(-1)^{l+m+1}\cdot$$
$$\left(\sum_{j=0}^{l+m-1}\frac{(l+m)!}{(l+m-j)!}\,\frac{(k-2-j)!}{(k-1)!}\,\frac{t^{l+m-j}}{(1+t)^{k-1-j}}\Big|_a^b\right)$$
$$+\sum_{l=0}^{k-2}\sum_{m=0}^{k-2-l}\left(\begin{array}{c} k \\ l \end{array}\right)\left(\begin{array}{c} k \\ m \end{array}\right)c^{k-l}d^{k-m}(-1)^{l+m}\,\frac{(l+m)!}{(k-1)!}\cdot$$
$$(k-l-m-2)!\left(\frac{1}{(1+a)^{k-l-m-1}}-\frac{1}{(1+b)^{k-l-m-1}}\right)$$
$$+\sum_{l=0}^{k-2}\left(\begin{array}{c} k \\ l \end{array}\right)\left(\begin{array}{c} k \\ k-1-l \end{array}\right)c^{k-l}d^{l+1}(-1)^{k-1}\ln\frac{|1+b|}{|1+a|}$$
$$+kcd^k(-1)^{k-1}\ln\frac{|1+b|}{|1+a|}$$

$$+\sum_{l=0}^{k-1}\sum_{m=k-l}^{k}\binom{k}{l}\binom{k}{m}c^{k-l}d^{k-m}(-1)^{l+m}\cdot$$

$$\left(-\sum_{j=0}^{k-2}\frac{(l+m)!}{(l+m-j)!}\frac{(k-2-j)!}{(k-1)!}\frac{t^{l+m-j}}{(1+t)^{k-1-j}}\bigg|_a^b\right.$$

$$+\binom{l+m}{k-1}\left(t^{l+m-k+1}\ln|1+t|\Big|_a^b-(l+m-k+1)\cdot\right.$$

$$\left.\left.\sum_{p=0}^{l+m-k}\binom{l+m}{p}(-1)^{l+m-k-p}\left(\tau^{p+1}\left(\frac{\ln\tau}{p+1}-\frac{1}{(p+1)^2}\right)\right)\bigg|_{1+a}^{1+b}\right)\right)$$

$$+\sum_{m=0}^{k}\binom{k}{m}d^{k-m}(-1)^{k+m}\cdot$$

$$\left(-\sum_{j=0}^{k-2}\frac{(k+m)!\,(k-2-j)!}{(k+m-j)(k-1)!}\frac{t^{k+m-j}}{(1+t)^{k-1-j}}\bigg|_a^b\right.$$

$$+\binom{k+m}{k-1}\left(t^{m+1}\ln|1+t|\Big|_a^b-(m+1)\cdot\right.$$

$$\left.\left.\sum_{p=0}^{m}\binom{k+m}{p}(-1)^{m-p}\left(\tau^{p+1}\left(\frac{\ln\tau}{p+1}-\frac{1}{(p+1)^2}\right)\right)\bigg|_{1+a}^{1+b}\right)\right)$$

$$=:\ F_k(a,b,c,d).$$

This result allows us to calculate the integrals I_2 and I_3, where we use the introduced abbreviation F_k. The V-Shannon $\mathcal{H}(\Omega)$-wavelet of scale $k-1$, $k\geq 1$, with $A_{n,k}=n(n+1)$ can, therefore, be represented by ($t_\xi, t_\zeta \in (-1,1)$)

$$\Psi_{k-1}(\xi,\zeta)=\frac{1}{4\pi k^2}\left(\cos(k\varphi_\xi)\cos(k\varphi_\zeta)+\sin(k\varphi_\xi)\sin(k\varphi_\zeta)\right)\cdot$$

$$\left(2\left(\frac{(1-t_\xi)(1+t_\zeta)}{(1+t_\xi)(1-t_\zeta)}\right)^{k/2}\right.$$

$$\left.+\left((-1)^k\left(\frac{1-t_\zeta}{1+t_\zeta}\right)^{k/2}-\left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}\right)\left(\frac{1-t_\xi}{1+t_\xi}\right)^{k/2}\right.\ .$$

$$\left(\sum_{j=1}^{k-1} \frac{k}{(k-j+1)(k-j)} \frac{-(t_\zeta+1)^{k-j+1}}{2^{k-j}} \right.$$
$$\left. + k(t_\zeta - 1)\ln\frac{2}{1-t_\zeta} + (t_\zeta+1)k\right)$$
$$+ \left(\frac{1+t_\zeta}{(1-{t_\xi}^2)(1-t_\zeta)}\right)^{k/2} \left(F_k(t_\xi,1,1,t_\xi) - \frac{1}{k+1}(1-t_\xi)^{k+1}\right)$$
$$+ \left(\left(\frac{1-t_\zeta}{1+t_\zeta}\right)^{k/2} - (-1)^k\left(\frac{1+t_\zeta}{1-t_\zeta}\right)^{k/2}\right)\frac{1}{(1-{t_\xi}^2)^{k/2}}\cdot$$
$$\left.\Big((-1)^k F_k(t_\xi,t_\zeta,t_\xi,t_\zeta) - F_k(-t_\zeta,-t_\xi,-t_\xi,-t_\zeta)\Big)\right),$$

provided that $t_\xi \le t_\zeta$. The representation of $\Psi_{k-1}(\xi,\zeta)$ in the case $t_\xi > t_\zeta$ is obtained by interchanging t_ξ and t_ζ in the above result.
The corresponding scaling function is determined by summation of the wavelets. In other words,

$$\Phi_j(\xi,\zeta) = \sum_{k=-1}^{j-1} \Psi_k(\xi,\zeta).$$

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