On the Connectedness of Efficient Solutions in Combinatorial Optimization Problems and Ordered Graphs

Matching and Partial Orders

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Abstract

In *multicriteria optimization problems* the connectedness of the *set of efficient solutions (pareto set)* is of special interest since it would allow the determination of the efficient solutions without considering non-efficient solutions in the process. In the case of the multicriteria problem to minimize matchings the set of efficient solutions is not connected. The set of minimal solutions E_{pot} with respect to the *power ordered set* contains the pareto set. In this work theorems about connectedness of E_{pot} are given. These lead to an automated process to detect all efficient solutions.

1 Introduction

In *multicriteria optimization problems* the connectedness of the *set of efficient solutions (pareto set)* is of special interest since it would allow the determination of the efficient solutions without considering non-efficient solutions during the process.
Since in general the pareto set is not connected we require a connected set containing the pareto set. The set of minimal solutions E_{pot} with respect to the *power ordered set* contains the pareto set. We introduce the term *reflecting the power order*. The binary relation $+$ used in the weighted sum is set in relation to the power order. This helps to characterize the relation between E_{pot} and the pareto set.
The multicriteria matching problem plays an important role in combinatorial optimization. Regard a bipartit, weighted graph with bipartition (X,Y). The vertices of X can be interpreted as factories while the vertices of Y are considered as sites. The weights of the edges e.g. then symbolize environmental control, occupation and traffic accessibility criteria. We search a perfect matching of minimum weight.
In the case of the multicriteria problem to minimize matchings the set of efficient solutions is not connected. We introduce *regular edge exchanges* and set them into relation to the reflection of the power order. This enables us to fix theorems about connectedness of E_{pot} and the pareto set. Therefore we can give an algorithm for an automated process to detect all efficient solution.

2 Power Ordered Sets and Efficient Solutions

We may consider a multicriteria optimization problem on a complete bipartit graph. Without loss of generality an ordered set of edges is regarded. Each edge e is assigned a weight w(e)=$(w_1(e),\ldots,w_k(e))$ by a weighting function w. For each feasible solution M = $\{e_1,\ldots,e_l\}$ the weight w(M) is defined as
w(M)=$(\sum_{i=1}^{l} w_1(e_i),\ldots,\sum_{i=1}^{l} w_k(e_i))$.
We distinguish between weights with equal values which are associated to different edges. Such weights are separately listed in the weighting set as so called multiple weights. Therefore the weighting sets of feasable matchings have the equal cardinality.
In the following we set efficient solutions in relation to minimal solutions of power ordered sets. Therefore we introduce the term of *reflecting the power order.*

Definition 2.1 (Efficient Solution) *A feasible solution M is called* **efficient** *if no further feasible solution M' exists with w(M') $<$ w(M).*

For the definition of the power ordered set the descriptions of [1] are used. Note that the subsequent relation $\leq$ is an order relation.

Definition 2.2 (Power Ordered Set) *Let $\mathcal{E}$ = (E; $\leq$) be a finite partially ordered set. $\mathcal{P}(\mathcal{E})$ = (P(E), $\leq_p$) is called* **power ordered set** *of E, if P(E) = Pot(E) $\setminus \emptyset$ and the relation $\leq_p$ of $\mathcal{P}(\mathcal{E})$ is defined in the following way:*
For all $a_1,\ldots,a_n,b_1,\ldots,b_m \in E$; n, m $\in \mathbb{N}$, $n \leq m$ is valid.
$\{ a_1,\ldots,a_n \} \leq_p \{b_1,\ldots,b_m \}$ if and only if there exists an injective mapping
π: $\{ a_1,\ldots,a_n \} \to \{b_1,\ldots,b_m \}$ such that $a_i \leq \pi(a_i)$ for i = 1,...,n on (E; $\leq$).
$\leq_p$ is called **power order**.

Definition 2.3 (Minimal Solution) *Let $\mathcal{E}$ = (E; $\leq$) be a finite partially ordered set and $\mathcal{P}(\mathcal{E})$ = (P(E), $\leq_p$) be the power ordered set of E. Then an element e $\in$ P(E) is called minimal with respect to the power order if there exists no element e' $\in$ P(E) with e' $<_p$ e.*

In the following we will say minimal instead of minimal with respect to the power order.

Proposition 2.4 *Let w be a weighting function on a set E which defines a partial order on E by componentwise ordering. Then each efficient solution of E is minimal with respect to the power order.*

In the following the *reflecting of the power order* is defined which is essential to set it in relation to weighting sums.

Definition 2.5 (Reflecting the Power Order) *Let (E, $\leq$) be a partially ordered set closed with respect to a binary operator +: $E \times E \to E$.*
+ **reflects** *the power order on $A \subseteq E$, if for all a, b, c, d $\in$ A follows that*
a+b $\leq$ c+d if and only if $\{a, b\} \leq_p \{c, d\}$.

Proposition 2.6 *Let (E, $\leq$) be a partially ordered set closed under the binary operator $+ : \mathrm{E} \times \mathrm{E} \to \mathrm{E}$, + reflecting the power order on $A \subseteq E$. Then a+b $\not\leq$ c holds for all a, b, c $\in$ A.*

With this definition we are able to set the power order in relation to the concept of weighting. We get the following theorem immediately.

Theorem 2.7 *Let $E \subseteq \mathbb{R}_+^k$ be closed under the componentwise addition* $+ : \mathbb{R}_+^k \times \mathbb{R}_+^k \to \mathbb{R}_+^k$ *and partially ordered with respect to the componentwise order. Let + reflect the power order on $A \subseteq E$. Furthermore let w: $A \to \mathbb{R}_+^k$ be a weighting function on A. Then for each two subsets E_1 and E_2 of A follows that $w(E_1) \leq w(E_2)$ if and only if*
$\{\ w(e) \mid e \in E_1\} \leq_p \{w(e) \mid e \in E_2\ \}$.

Corollary 2.8 *Let $E \subseteq \mathbb{R}_+^k$ be closed under the componentwise additio* $+ : \mathbb{R}_+^k \times \mathbb{R}_+^k \to \mathbb{R}_+^k$ *and partially ordered with respect to the componentwise order. Let + reflect the power order on E. Furthermore let w: $E \to \mathbb{R}_+^k$ be a weighting function on E. Then a feasible solution F is efficient if and only if F is minimal on E .*

In the case of the above theorem the weighting function is said to reflect the power order and vice versa. Of course, the power order need not to "be reflected" naturally.

Proposition 2.9 *Let + be the addition on $\mathbb{N}^2$ induced by addition on $\mathbb{N}$ as follows.*
$+((x_1,x_2),(y_1,y_2)) = (x_1+y_1, x_2+y_2)$ for $(x_1,x_2),(y_1,y_2) \in \mathbb{N}^2$.
Since $+((9,0),(0,7)) = (9,7) \geq (8,3) = +((7,1),(1,2))$, but (9,0) is incomparable to (7,1) and (1,2), the operation + does not reflect the power order on $\mathbb{N}^2$.

3 Efficient Matchings and Power Order

In this section we regard the matching problem and introduce *regular edge exchanges*. With that term we set the edge exchanges in relation to mappings reflecting power orders and therefore also to the structure of power orders.

Definition 3.1 (Efficient Matching) *A perfect matching M of a vector weighted, complete, bipartit graph G is called efficient if no further perfect matching M' exists with w(M') < w(M).*

Definition 3.2 (Minimal Matching) *A perfect matching M of a vector weighted, complete, bipartit graph G is called minimal if no further perfect matching M' exists with M' $<_p$ M.*

The following example shows that the pareto graph is not connected which means that the efficient matchings can not be detected only by edge exchanges.

Proposition 3.3 *Let the $K_{4,4}$ with the weighting function w be given (See Figure 1 and Table 1).*
$\{\ a_1,\ b_2,\ c_3,\ d_4\ \}$, $\{\ b_1,\ c_2,\ a_3,\ d_4\ \}$ are efficient perfect matchings. The pareto graph is not connected which means not every efficient solution can be reached via exchange operations.

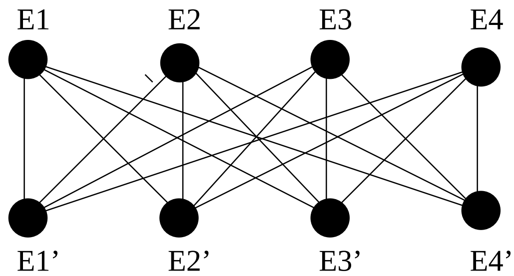

Figure 1

Table 1

Vertices	Edge	Weight
E1, E1'	a1	(0,1)
E1, E2'	a2	(10,10)
E1, E3'	a3	(1,0)
E1, E4'	a4	(10,0)
E2, E1'	b1	(1,0)
E2, E2'	b2	(0,1)
E2, E3'	b3	(10,10)
E2, E4'	b4	(10,0)
E3, E1'	c1	(10,10)
E3, E2'	c2	(1,0)
E3, E3'	c3	(0,1)
E3, E4'	c4	(10,0)
E4, E1'	d1	(0,10)
E4, E2'	d2	(10,10)
E4, E3'	d3	(10,10)
E4, E4'	d4	(1,2)

The following statements result immediately.

Proposition 3.4 *Let M_1, M_2 be matchings of a complete vector weighted graph where $M = M_1 \cap M_2 \neq \emptyset$.*
Then the following holds:

(i) $$w(M_1) \leq w(M_2)\ (w(M_1) \geq w(M_2))$$
$$\Leftrightarrow w(M_1 \setminus M) \leq w(M_2 \setminus M)\ (\ w(M_1 \setminus M) \geq w(M_2 \setminus M)\ )$$

(ii) $$w(M_1) \text{ incomparable to } w(M_2)$$
$$\Leftrightarrow w(M_1 \setminus M) \text{ incomparable to } w(M_2 \setminus M).$$

In order to get a characterization of the pareto set we introduce regular edge exchanges.

Definition 3.5 (Regular Edge Exchange) *Let the $K_{n,n}$ with a weighting function w be given and p, q, p', $q' \in E(K_{n,n})$.*
An exchange of edges p, q with edges p', q' is called **regular** *if*
$\{p, q\} \leq_p \{p', q'\}$ *or* $\{p, q\} \geq_p \{p', q'\}$.
We say an edge exchange p, q with p', q' is **regular minoring** *if*
$\{p, q\} >_p \{p', q'\}$ *and* **regular majoring** *if* $\{p, q\} <_p \{p', q'\}$.

We get immediately statement to get a neighboured minimal matching from an initial one.

Proposition 3.6
Let M_1, M_2 be matchings where $M_1 \setminus (M_1 \cap M_2) = \{p, q\}$, $M_2 \setminus (M_1 \cap M_2) = \{r, t\}$.
Then
a) $M_1 \leq_p M_2$ $(M_1 \geq_p M_2) \Leftrightarrow$ *the edge exchange of p, q with r, t is regular majoring, i.e.* $\{p, q\} \leq_p \{r, t\}$ *(the edge exchange of p, q with r, t is regular minoring, i.e.* $\{p, q\} \geq_p \{r, t\}$*)*
b) If $\{p, q\}$ *is not comparable to* $\{r, t\}$ *then M_1 is not comparable to M_2 (the edge exchange of p, q with r, t is not regular).*

Analogously we achieve a statement to get from an initial efficient solution to neighboured efficient solution.

Proposition 3.7
Let M_1, M_2 be matchings where $M_1 \setminus (M_1 \cap M_2) = \{p, q\}$, $M_2 \setminus (M_1 \cap M_2) = \{r, t\}$.

Let + reflect the power order on $M_1 \cup M_2$. Then
a) $w(M_1) \le w(M_2)$ ($w(M_1) \ge w(M_2)$) $\Leftrightarrow$ the edge exchange of p, q with r, t is regular majoring, i.e. $\{p, q\} \le_p \{r, t\}$ (the edge exchange of p, q with r, t is regular minoring, i.e. $\{p, q\} \ge_p \{r, t\}$)
b) If $\{p, q\}$ is not comparable to $\{r, t\}$ then $w(M_1)$ is not comparable to $w(M_2)$ is efficient (the edge exchange of p, q with r, t is not regular).

4 On the Connectedness of the pareto graph

Example 3.3 shows that the pareto graph need not to be connected. In the former sections the binary relation + was set in relation to the power order. This enables us to fix theorems about the connectedness of the pareto set.

The following statements about $K_{(3,3)}$ lead to a sufficient condition.

Lemma 4.1 *Let w:$E(K_{3,3}) \to \mathbb{R}_+^k$ be a weighting function of $K_{3,3}$, where each two different edges have different weights. Let + reflect the power order on the completion of the set of edge weights with respect to the componentwise order. Then the pareto graph of $K_{3,3}$ is connected.*

proof 1 *Assume there exist two efficient matchings M_1, M_2, that are not neighboured. M_1 and M_2 can be transferred only by regular edge exchanges which are minoring or majoring (otherwise there would exist an efficient matching which at each time could be transferred by one edge exchange to M_1 and M_2). These matchings have no common edge.*
Let $M_1 = \{1, 2, 3\}$ and $M_2 = \{1', 2', 3'\}$, $G[M_1, M_2]$ be isomorph to the following graph (Figure 2):

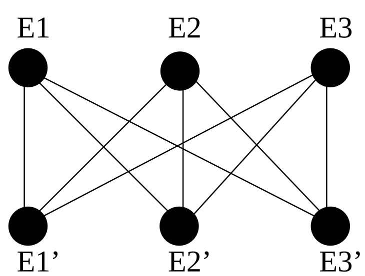

Figure 2

Table 2

Vertices	*Edge*
E1, E1'	*3'*
E1, E2'	*1*
E1, E3'	*h2*
E2, E1'	*2*
E2, E2'	*h1*
E2, E3'	*1'*
E3, E1'	*h3*
E3, E2'	*2'*
E3, E3'	*3*

(In the following, the edges are identified with their edge weights.)
Then it follows
a) $\{1, 2\} \le_p \{3', h_1\}$, i.e. ($1 \le 3'$ and $2 \le h_1$) or ($2 \le 3'$ and $1 \le h_1$)
and
b) $\{1', 2'\} \le_p \{3, h_1\}$, i.e. ($1' \le 3$ and $2' \le h_1$) or ($2' \le 3$ and $1' \le h_1$)
Moreover is
a2) ($\{1, 3\} \le_p \{h_3, 2'\}$) or ($\{2, 3\} \le_p \{h_2, 1'\}$)
and
b3) ($\{2', 3'\} \le_p \{1, h_2\}$) or $\{1', 3'\} \le_p \{2, h_3\}$)

This leads to a contradiction. q.e.d.

Theorem 4.2 *Let w:$E(K_{n,n}) \to \mathbb{R}_+^k$ ($n \in \mathbb{N}$) be a weighting function of the $K_{n,n}$, where each two different edges have different weights. Let + reflect the power order*

on the completion of the set of edge weights with respect to the componentwise order. Let M_1 and M_2 be efficient matchings, which have an edge e in common. Let K' = G[V($K_{n,n}$ \ { a, b })] the graph induced by the vertices of $K_{n,n}$ except the vertices a and b with the incidence function φ where φ(e)=ab.
M_1 is combined with M_2 through a way in the efficiency graph of $K_{n,n}$, if and only if M_1\ { e} is combined with M_2\ { e } through a way in the pareto graph.

proof 2 *We show that M_1\ { e} is combined with M_2\ { e } through a way in the pareto graph if M_1 is combined with M_2 through a way in the pareto graph of $K_{n,n}$.*
Case 1
It exists a way $\gamma = V_1, \dots, V_l$ ($l \in \mathbb{N}$, $l \le n$) from $M_1 \setminus \{e\}$ to $M_2 \setminus \{e\}$ in the pareto graph of K', such that $V_j \cup$ { e } ($j \in \mathbb{N}$, $j \le l$) is an efficient solution in $K_{n,n}$ $\surd$
Case 2
There is no way $\gamma = V_1, \dots, V_l$ ($l \in \mathbb{N}$, $l \le n$) in the pareto graph from $M_1 \setminus \{e\}$ to $M_2 \setminus \{e\}$ in the pareto graph of K', such that $V_j \cup$ { e } ($j \in \mathbb{N}$, $j \le l$) is an efficient solution in $K_{n,n}$. Then M_1 and M_2 are no neighbours.
May $\gamma = V_1, \dots, V_l$ ($l \in \mathbb{N}$, $l \le n$) be a way in the pareto graph of $M_1 \setminus \{e\}$ to $M_2 \setminus \{e\}$ in K'. The subsequent algorithm provides us a way in the pareto graph of $K_{n,n}$.
step 1
May $V_1, \dots, V_j$ with $V_j \cup \{e\}$, j ∈ {1,...,l} be efficient on $K_{n,n}$, but $V_{j+1} \cup \{e\}$ not efficient on $K_{n,n}$.
Then there exist e_1, e_2 such that $E(K_{n,n}) \setminus E(K')$ with $\varphi(e_1) = av_1$, $\varphi(e_2)$=bv_2, $\varphi(e_3) = v_1v_2, e_3 \in V_{j+1} \setminus V_j$ such that $\{w(e_1), w(e_2)\} <_p \{w(e), w(e_3)\}$.
Without loss of generality let $w(e_1) \le w(e)$, $w(e_2) \le w(e_3)$.

This is illustrated in Figure 3.

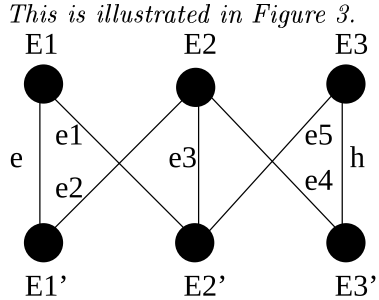

Figure 3

Table 3

Vertices	*Edge*
E1, E1'	*e*
E1, E2'	*e1*
E2, E1'	*e2*
E2, E2'	*e3*
E2, E3'	*e4*
E3, E2'	*e5*
E3, E3'	*h*

Therefore $V_j \cup$ {e } and V'=$V_{j+1} \setminus \{e_3, e\} \cup \{e_1, e_2\}$ are efficient. The above graph can be embedded in the $K_{3,3}$. The theorems 3.7, 4.1 deduce that there exists a way in the pareto graph of the $K_{n,n}$ from V_j to V'. M_2 does not contain e_3, otherwise it would be dominated by M_2\ $\{e_3, e\} \cup \{e_1, e_2\}$.
It follows, that M_2 contains h (otherwise it would contain e_4 and e_5 and the reversed exchange does not occur because of the definition of the way).
Therefore, replace the part of the way V_jV_{j+1} through the way from V_j to V'.
step 2
e_3 is exchanged together with the edge e' during the edge exchange from V_j to V_{j+1} (l > j+1) with edges $\tilde{e_5}\tilde{e_6}$ (Figure 4).

Case 2a:
Between V' and V_{j+2} there exist no $\tilde{V}$ with $\tilde{V} \cup$ {e} not efficient and e'$\notin \tilde{V}$.
Such an edge replacement would lead to an efficient solution in K' since this replacement must be not regular (because of the efficiency of the solutions)
V_{j+2} is efficient either in K' or there exist edges $\tilde{e_1}, \tilde{e_2}$ incident to e

with $\{\tilde{e_1}, \tilde{e_2}, e_3\} <_p \{\tilde{e_5}, \tilde{e_6}, e\}$.
Change the way from V' to

$$\tilde{V} = \begin{cases} V_{j+2} & : \; V_{j+2} \;\; efficient \\ V' \setminus \{e_1, e_2, e'\} \cup \{\tilde{e_1}\tilde{e_2}, e_3\} & : \; else \end{cases}$$

and delete V_{j+2} *from the way (as soon as the adjacent edges to this) and replace all succeeding* V_k *to* $V_k \setminus \{e_3, \tilde{h}, e\} \cup \{h1, h2\}$.

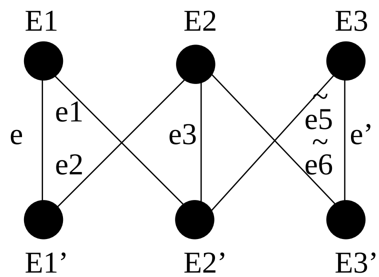

Figure 4

Table 4

Vertices	*Edge*
E1, E1'	*e*
E1, E2'	*e1*
E2, E1'	*e2*
E2, E2'	*e3*
E2, E3'	$\tilde{e}6$
E3, E2'	$\tilde{e}5$
E3, E3'	*e'*

Case 2b:
Between V' and V_{j+2} *there exists a matching* $\tilde{V}$ *with* $\tilde{V} \cup \{e\}$ *not efficient and* $e' \notin \tilde{V}$.
Then there exists an edge $\tilde{h}$ *with* $\tilde{V} \setminus \{\tilde{h}, e\} \cup \{h_1, h_2\}$ *and* $\tilde{V} \setminus \{e_3, e\} \cup \{e_1, e_2\}$ *efficient.*
This is illustrated graphically (Figure 5).

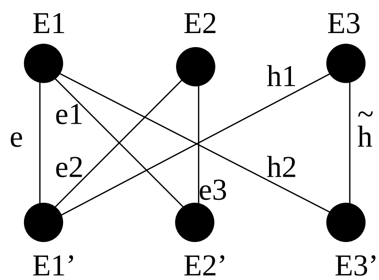

Table 5

Vertices	*Edge*
E1, E1'	*e*
E1, E2'	*e1*
E1, E3'	*h2*
E2, E1'	*e2*
E2, E2'	*e3*
E3, E1'	*h1*
E3, E3'	$\tilde{h}$

With theorem 3.7, 4.1 there exists a way in the pareto graph, which combines both. Replace the corresponding part of the way, set e_3*=*h_3 *and replace all succeeding* V_k *to* $V_k \setminus \{e_3, \tilde{h}, e\} \cup \{h1, h2\}$.
Repeat step 1.) until each matching M on the way is also efficient in $K_{n,n}$. q.e.d.

Theorem 4.3 *Let w:E(*$K_{n,n}$*)* $\rightarrow \mathbb{R}^k_+$ *(* $n \in \mathbb{N}$*) be a weighting function* $K_{n,n}$*, where each two different edges have different weights. Let + reflect the power order on the completion of the set of edge weights with respect to the componentwise order. It follows that the pareto graph is connected.*

proof 3 *by induction.*
For n=1, n=2 and n=3 the theorem is deduced by the assertion and theorem 4.1.
Let us carry out the induction step from n-1 to n.
Assume the pareto graph ist not connected.
Without loss of generality let the pareto graph be 2-connected.
Also it exists at least one minimal edge e_1 *which is contained in an efficient matching. May* M_1 *and* M_2 *be efficient matchings of different components of connection,* M_1 *containing* e_1*. Without loss of generality each exchange of an edge pair of* M_1 *with an edge pair of* M_2 *does not lead to an efficient solution. G[*$M_1 \cup M_2$*] induces a generated cycle (since both matchings have no edge in common).*

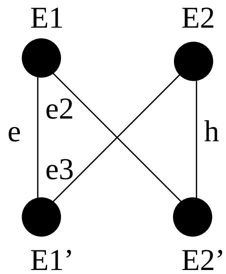

Table 6

Vertices	*Edge*
E1, E1'	*e*
E1, E2'	*e2*
E2, E1'	*e3*
E2, E2'	*h*

Figure 6

e_1 is neighboured to two edges e_2, e_3 of M_2. $h \neq e_1$, $h \in M_1$ is neighboured to e_2, e_3 (Figure 6).
The exchange of e_2, e_3 with e_1, h does not lead to efficient solutions (due to the choice of the matchings), e.g. it is regular (corresponding to the assumption), e.g. $e_2 < e_1$ or $e_3 < e_1$ (as different edges have different weights).
This contradicts to the fact that e_1 is minimal. q.e.d.

Since we used only regular edge exchanges to proof the theorems of this sections the theorems also hold for the adjacency graph of minimal solutions without the condition that + must reflect the power order.

Corollary 4.4 *Let w:$E(K_{n,n}) \rightarrow \mathbb{R}_+^k$ ($n \in \mathbb{N}$) be a weighting function which defines a partial order on the edges by the componentwise order, where each two different edges have different weights. It follows that the adjacency graph of minimal solutions is connected.*

In the case of the examples 3.3 + does not reflect the power order on the edges since (0,1) + (1,0) < (10,10) applies in example 3.3.
However, there are also edge weightings, with which + onto the edges does not reflect the potentially partial order and the pareto graph is connected.

Proposition 4.5 *The edge weights of the following graph correspond to the marked weights. It has two neighboring efficient solutions with the same weight. Therefore, the pareto graph is connected, but the addition on the partially ordered set of the edge weights does not reflect the power order, because (3,0) + (0,3) = (3,3) $\leq$ (3,3) = (1,2) + (2,1),*
but $\{(3,0),(0,3)\} \not\leq_p \{(1,2),(2,1)\}$.

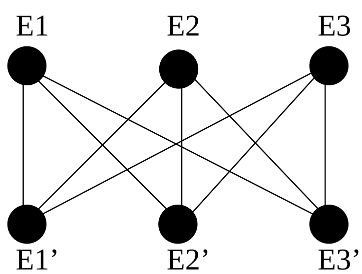

Figure 7

Table 7

Vertices	*Edge*	*Weight*
E1, E1'	*a1*	*(3,0)*
E1, E2'	*a2*	*(1,2)*
E1, E3'	*a3*	*(10,10)*
E2, E1'	*b1*	*(2,1)*
E2, E2'	*b2*	*(0,3))*
E2, E3'	*b3*	*(10,10)*
E3, E1'	*c1*	*(10,10)*
E3, E2'	*c2*	*(10,10)*
E3, E3'	*c3*	*(1,1)*

If + reflects the power order we can construct an algorithm detecting all efficient solutions with proposition 3.7. Another important question is what to do when + does not reflect the power order which usually happens. Then we can detect the minimal solutions the aid of proposition 3.6 and remove the not efficient solutions. The question is how to detect regular minoring edge exchanges.

Corollary 4.6 *Let a,c,b,d* $\in \mathbb{R}^n$*. Then the following holds:*
a) $(sign_{comp}(a\text{-}b) = sign_{comp}(c\text{-}d) = -1)$ *or* $(sign_{comp}(a\text{-}d) = sign_{comp}(c-b) = -1)$
iff $\{ a, c \} <_p \{ b, d \}$
b) $(sign_{comp}(a\text{-}b) = sign_{comp}(c\text{-}d) = 1)$ *or* $(sign_{comp}(a\text{-}d) = sign_{comp}(c\text{-}b) = 1)$
iff $\{ a, c \} >_p \{ b, d \}$
c) $sign_{comp}(a\text{-}b) * sign_{comp}(c\text{-}d) + sign_{comp}(a\text{-}d) * sign_{comp}(c\text{-}b) \leq 0$
iff $\{ a, c \} = \{ d, b \}$ *or* $\{ a, c \}$ *not comparable to* $\{ d, b \}$
where for $(a_1, \ldots, a_n) \in \mathbb{R}^n$

$$sign_{comp}(a_1, \ldots, a_n) = \begin{cases} 1 & : \quad a_1, \ldots, a_n > 0 \\ -1 & : \quad a_1, \ldots, a_n < 0 \\ else & : \quad 0 \end{cases}$$

Therefore we have the possibility to check wether we have got regular edge exchanges or not.

5 Algorithm

In the previous sections we mentioned the power order and examined the pareto graph of the set of minimal solutions with respect to it in the matching problem. Regular edge exchanges were introduced. With the results of the previous sections we can construct an algorithm to detect all minimal matchings and therefore all efficient matchings.

Algorithm: Detecting the efficient solutions

Initialization:
Start with an initial matching M and an initial set of minimal matchings $\mathcal{M} = \emptyset$, the set $\mathcal{E} = \emptyset$ of current not dominated solutions, a set of not regular edge exchanges $\mathcal{I}_M = \emptyset$ with respect to M and the set of matching to be checked M_{check} and the set of removed solutions $\mathcal{M}_{removed}$.

step 1: Getting the initial solution
While there is a regular minoring edge exchange (p, q) with (p', q') possible in M **do** set M := M \ (p, q)∪(p', q').

Set $\mathcal{M} = \{ (M, \mathcal{I}_M) \}, \mathcal{E} = \mathcal{M}$.

od

step 2: Searching in the pareto graph
Repeat
If M is not dominated by an element of $\mathcal{M}$ **then**
Remove all elements of $\mathcal{M}$ dominated by M.
If M is not dominated by an element of $\mathcal{E}$ **then**
Set $\mathcal{E} = \mathcal{E} \cup (M, \mathcal{I}_M)$. Remove all elements of $\mathcal{E}$ dominated by M.

For all not regular or "equal" edge exchanges X exchanging (p, q) of M with (p', q') being no element of $\mathcal{I}_{\mathcal{M}}$ **do**
set M' = M $\setminus$ {(p, q)} $\cup$ {(p', q')}, $\mathcal{I}_{\mathcal{M}'} \cup$ { X }
and $M_{check} = M_{check} \cup$ { (M', $\mathcal{I}_{\mathcal{M}'}$)} $\setminus \mathcal{M}$. **od**

else set $\mathcal{M}_{removed} = \mathcal{M}_{removed} \cup (M, \mathcal{I}_M)$.
fi
If $M_{check} \neq \emptyset$ **then**
M= M' $\in M_{check}$.
While there is a regular minoring edge exchange (p, q) with (p', q') possible in M and $M_{check} \neq \emptyset$ **do** Set $M_{check} = (M_{check} \setminus$ { (M, $\mathcal{I}_{\mathcal{M}}$)}.
Set $\mathcal{M}_{removed} = \mathcal{M}_{removed} \cup$ { (M, $\mathcal{I}_{\mathcal{M}}$)}. M= M' $\in M_{check}$.

od

fi
until $M_{check} = \emptyset$.

step 3:
$\mathcal{E}$ contains all efficient solutions.

To detect wether an edge exchange is regular minoring or not regular we construct the following algorithm depending on Proposition 4.6:

Subalgorithm: Characterization of edge exchanges

Input: a, b, c, d $\in \mathbb{R}^n$
Initialization:
A = $sign_{comp}$(a-b) * $sign_{comp}$(c-d), B = $sign_{comp}$(a-d) * $sign_{comp}(c-b)$.

Characterization step:
if A + B $\leq$ 0 **then**
"Not regular edge exchange, edge having same weights or not regular edge exchange"

else
if ((A=1 and $sign_{comp}$(c-d) = 1) or (B=1 and $sign_{comp}$(c-b) = 1))
then "Regular minoring edge exchange".
else "Regular majoring edge exchange".
fi
fi

6 Example

We look at example 3.3 and get stepwise through the algorithm of the previous section:

Proposition 6.1 *Let* M_1 = { a_1, b_2, c_3, d_4 }, M_2 = { c_1, b_2, a_3, d_4 }, M_3 = { d_1, b_2, c_3, a_4 },
M_4 = { a_1, c_2, b_3, d_4 }, M_5 = { a_1, b_2, d_3, c_4 }, M_6 = { b_1, a_2, c_3, d_4 },
M_7 = { d_1, b_2, a_3, c_4 }, M_8 = { c_1, d_2, a_3, b_4 }, M_9 = { b_1, c_2, a_3, d_4 },
M_{10} = { b_1, d_2, c_3, a_4 }, M_{11} = { d_1, c_2, b_3, a_4 }, M_{12} = { a_1, c_2, d_3, b_4 },
M_{13} = { b_1, d_2, a_3, c_4 }, M_{14} = { d_1, c_2, a_3, b_4 }, M_{15} = { b_1, c_2, d_3, a_4 },
M_{16} = { d_1, a_2, c_3, b_4 }. *Status before step 1:*

Start with a Matching $M = M_1$, $\mathcal{M} = \emptyset$, $\mathcal{E} = \emptyset$, $\mathcal{I}_M = \emptyset$, $\mathcal{M}_{check} = \emptyset$, $\mathcal{M}_{removed} = \emptyset$.

Status after step2:
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), \mathcal{E} = \mathcal{M}, \mathcal{I}_M = \{ [(a_1, b_2), (a_2, b_1)], [(a_1, c_3), (a_3, c_1)], [(a_1, d_4), (d_1, a_4)], [(b_2, c_3), (c_2, b_3)], [(c_3, d_4), (d_3, c_4)] \}$,
$\mathcal{M}_{check} = \{ (M_2, \emptyset), (M_3, \emptyset), (M_4, \emptyset), (M_5, \emptyset), (M_6, \emptyset) \}$, $M = M_2$, $\mathcal{M}_{removed} = \emptyset$.
Remark: M_5 *is not minimal, but there exists no regular minimal edge exchange for a pair of edges of it.*

Status after step2 (2nd iteration):
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), (M_2, \mathcal{I}_{M_2}) \}$, $\mathcal{E} = \{ (M_1, \mathcal{I}_{M_1}) \}$,
$\mathcal{I}_M = \{ [(c_1,a_3),(a_1, c_3)], [(c_1,d_4),(d_4,c_1)], [(b_2,d_4),(d_2,b_4)], [(c_1,b_2),(b_1,c_2)] \}$,
$\mathcal{M}_{check} = \{ (M_3, \emptyset), (M_4, \emptyset), (M_5, \emptyset) \}, (M_7, \emptyset), (M_9, \emptyset), (M_6, \emptyset) \}$,
$\mathcal{M}_{removed} = \{ M_8 \}$, $M = M_3$.

Status after step2 (3rd iteration):
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), (M_2, \mathcal{I}_{M_2}), (M_3, \mathcal{I}_{M_2}) \}$,
$\mathcal{E} = \{ (M_1, \mathcal{I}_{M_1}) \}$, $\mathcal{I}_M = \{ [(d_1,b_2),(b_1, d_2)], [(d_1,a_4),(a_1, d_4)], [(b_2,c_3),(c_2,b_3)], [(c_3,a_4),(a_3,c_4)] \}$, $\mathcal{M}_{check} = \{ (M_4, \emptyset), (M_5, \emptyset) \}, (M_7, \emptyset), (M_9, \emptyset), (M_{10}, \emptyset), (M_{11}, \emptyset), (M_6, \emptyset) \}$, $\mathcal{M}_{removed} = \{ M_8 \}$, $M = M_4$.
Remark: M_{10} *is not minimal, but there exists no regular minimal edge exchange for a pair of edges of it.*

Status after step2 (4th iteration):
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), (M_2, \mathcal{I}_{M_2}), (M_3, \mathcal{I}_{M_3}), (M_4, \mathcal{I}_{M_4}) \}$, $\mathcal{E} = \{ (M_1, \mathcal{I}_{M_1}), (M_4, \mathcal{I}_{M_4}) \}$, $\mathcal{I}_M = \{ [(a_1, b_3),(b_1,a_3)], [(a_1,d_4),(a_4,d_2)], [(c_2,b_3),(b_2,c_3)], [(b_3,d_4),(d_3,b_4)] \}$, $\mathcal{M}_{check} = \{ (M_5, \emptyset), (M_7, \emptyset), (M_9, \emptyset), (M_{10}, \emptyset), (M_{11}, \emptyset), (M_{12}, \emptyset), (M_6, \emptyset) \}$, $\mathcal{M}_{removed} = \{ M_8 \}$ $M = M_7$.

Status after step2 (5th iteration):
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), (M_2, \mathcal{I}_{M_2}), (M_3, \mathcal{I}_{M_3}), (M_4, \mathcal{I}_{M_4}), (M_7, \mathcal{I}_{M_7}) \}$,
$\mathcal{E} = \{(M_1, \mathcal{I}_{M_1}), (M_4, \mathcal{I}_{M_4}) \}$, $\mathcal{I}_M = \{ [(d_1,b_2),(b_1,d_2)], [(d_1,c_4),(c_1, d_4)], [(b_2, c_4),(c_2, b_4)], [(a_3, c_4),(c_3, a_4)] \}$, $\mathcal{M}_{check} = \{ (M_5, \emptyset), (M_9, \emptyset), (M_{10}, \emptyset), (M_{11}, \emptyset), (M_{12}, \emptyset), (M_{14}, \emptyset), (M_6, \emptyset) \}$, $\mathcal{M}_{removed} = \{ M_8, M_{13} \}$,
$M = M_5$.
Status after step2 (6th iteration):
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), (M_2, \mathcal{I}_{M_2}), (M_3, \mathcal{I}_{M_2}), (M_4, \mathcal{I}_{M_4}), (M_7, \mathcal{I}_{M_7}) \}$,
$\mathcal{E} = \{(M_1, \mathcal{I}_{M_1}), (M_4, \emptyset) \}$, $\mathcal{M}_{check} = \{ (M_9, \emptyset), (M_{10}, \emptyset), (M_{11}, \emptyset), (M_{12}, \emptyset), (M_{14}, \emptyset), (M_6, \emptyset) \}$, $\mathcal{M}_{removed} = \{ M_5, M_8, M_{13} \}$, $M = M_9$.

Status after step2 (7th iteration):
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), (M_2, \mathcal{I}_{M_2}), (M_3, \mathcal{I}_{M_2}), (M_4, \mathcal{I}_{M_4}), (M_7, \mathcal{I}_{M_7}), (M_9, \mathcal{I}_{M_9}) \}$, $\mathcal{E} = \{(M_1, \mathcal{I}_{M_1}), (M_9, \mathcal{I}_{M_9}) \}$, $\mathcal{I}_M = \{ [(b_1, c_2),(c_1, b_2)], [(b_1, a_3),(a_1, b_3)], [(b_1, d_4),(d_1, b_4)], [(c_2, a_3),(a_2, c_3)], [(a_3, d_4),(d_3, a_4)] \}$,
$\mathcal{M}_{check} = \{ (M_{10}, \emptyset), (M_{11}, \emptyset), (M_{12}, \emptyset), (M_{14}, \emptyset), (M_6, \emptyset) \}$,
$\mathcal{M}_{removed} = \{ M_5\ M_8, M_{13}, M_{15}\}$, $M = M_{10}$.

Because M_{10}, M_{11} *and* M_{12} *are not minimal, during step 8 and step 10 these matchings are only added to* $\mathcal{M}_{removed}$. M *is set to* M_{14}.

Status after step2 (11th iteration):
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), (M_2, \mathcal{I}_{M_2}), (M_3, \mathcal{I}_{M_3}), (M_4, \mathcal{I}_{M_4}), (M_7, \mathcal{I}_{M_7}), (M_9, \mathcal{I}_{M_9}), (M_{14}, \mathcal{I}_{M_{14}}) \}$, $\mathcal{E} = \{(M_1, \mathcal{I}_{M_1}), (M_9, \mathcal{I}_{M_9}) \}$, $\mathcal{I}_M = \{ [(d_1, a_3),(a_1, d_3)], [(d_1, b_4),(b_1, d_4)], [(c_2, a_3),(a_2, c_3)], [(c_2, b_4),(b_2, c_4)] \}$,
$\mathcal{M}_{check} = \{ (M_{16}, \emptyset), (M_6, \emptyset) \}$, $\mathcal{M}_{removed} = \{ M_5, M_8, M_{10}, M_{11}, M_{12}, M_{13}, M_{15} \}$, $M = M_{16}$.

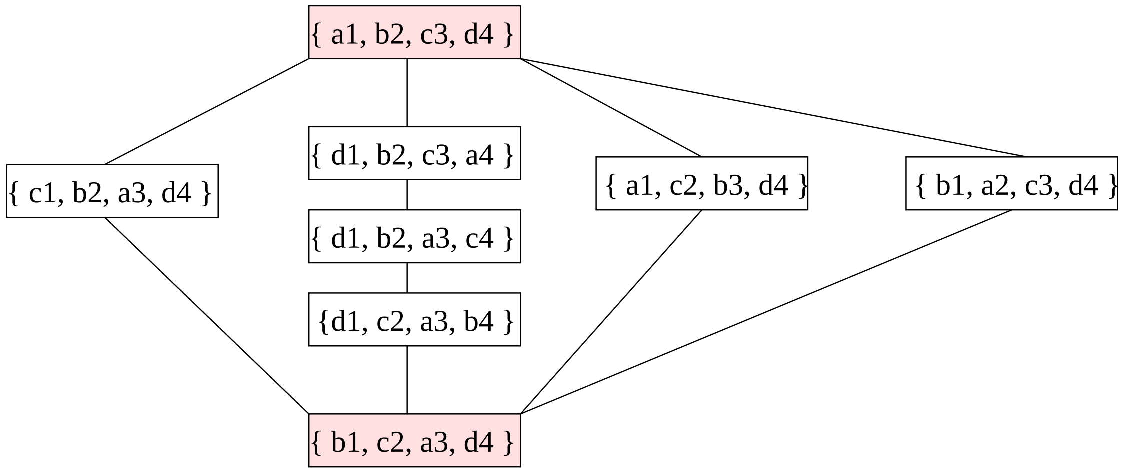

Figure 9: Pareto Graph of Example 3.3

Status after step2 (12th iteration):
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), (M_2, \mathcal{I}_{M_2}), (M_3, \mathcal{I}_{M_3}), (M_4, \mathcal{I}_{M_4}), (M_7, \mathcal{I}_{M_7}),$
$(M_9, \mathcal{I}_{M_9}), (M_{14}, \mathcal{I}_{M_{14}}) \}$, $\mathcal{E} = \{(M_1, \mathcal{I}_{M_1}), (M_9, \mathcal{I}_{M_9})\}$,
$\mathcal{M}_{check} = \{ (M_6, \emptyset) \}$,
$\mathcal{M}_{removed} = \{ M_5, M_8, M_{10}, M_{11}, M_{12}, M_{13}, M_{15}, M_{16} \}$, $M = M_6$.

Status after step2 (13th iteration):
$\mathcal{M} = \{ (M_1, \mathcal{I}_{M_1}), (M_2, \mathcal{I}_{M_2}), (M_3, \mathcal{I}_{M_3}), (M_4, \mathcal{I}_{M_4}), (M_7, \mathcal{I}_{M_7}),$
$(M_9, \mathcal{I}_{M_9}),$
$(M_{14}, \mathcal{I}_{M_{14}}), (M_6, \mathcal{I}_{M_6}) \}$, $\mathcal{E} = \{(M_1, \mathcal{I}_{M_1}), (M_9, \mathcal{I}_{M_9})\}$,
$\mathcal{I}_M = \{ [(b_1,a_2),(a_1,b_2)], [(b_1,d_4),(d_1,b_4)], [(a_2,c_3),(c_2,a_3)], [(a_2,d_4),(d_2,a_4)],$
$[(c_3,d_4),(d_3,c_4)] \}$, $\mathcal{M}_{check} = \emptyset$,
$\mathcal{M}_{removed} = \{ M_5, M_8, M_{10}, M_{11}, M_{12}, M_{13}, M_{15}, M_{16} \}$.

Then the algorithm stops with the set of minimal solutions $\mathcal{M}$ and the set of efficient solutions $\mathcal{E}$.

In Figure 9 the pareto graph is drawn. The efficient solutions are marked with a grey box.

7 Conclusions

We set the concepts of weighting sums and edge exchanges in relation to power orders by introducing the terms of *reflecting the power order* and *regular edge exchanges.*
Using this terms we gave theorems about the pareto graph and an algorithm to detect the efficient solutions in an automatic process.
Further research about the cardinality of the set of minimal solutions of the power ordered set which are not efficient has to be done.

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