A martingale method of portfolio optimization for unobservable mean rate of return.

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June 20, 2000

Abstract

In the Black–Scholes type financial market, the *risky asset* $S_1(\cdot)$ is supposed to satisfy $dS_1(t) = S_1(t)(b(t)dt + \sigma(t)dW(t)$ where $W(\cdot)$ is a Brownian motion. The processes $b(\cdot)$, $\sigma(\cdot)$ are progressively measurable with respect to the filtration generated by $W(\cdot)$. They are known as the *mean rate of return* and the *volatility* respectively. A portfolio is described by a progressively measurable processes $\pi_1(\cdot)$, where $\pi_1(t)$ gives the amount invested in the risky asset at the time t. Typically, the optimal portfolio $\pi_1^*(\cdot)$ (that, which maximizes the expected utility), depends at the time t, among other quantities, on $b(t)$ meaning that the mean rate of return shall be known in order to follow the optimal trading strategy. However, in a real–world market, no direct observation of this quantity is possible since the available information comes from the behavior of the stock prices which gives a *noisy observation* of $b(\cdot)$. In the present work, we consider the optimal portfolio selection which uses only the observation of stock prices.

1 Introduction

The problem of portfolio optimization in continuous time models consists of maximizing the total expected utility of terminal wealth and that of consumption over a given time interval. In the context of complete, standard financial market (in the sense of [7]) the portfolio optimization problem is solved by the *martingale approach* as presented by Karatzas, Lehoczky, Sethi, and Shreve. (See [5], [6], or chapter 3 of [7] or [9]) In this approach (here, for simplicity, we consider the case of only one risky stock) the interest rate, the mean rate of stock return, and the volatility of the stock are described by progressively measurable stochastic processes $(r(t))_{t\in[0,T]}$, $(b(t))_{t\in[0,T]}$, and $(\sigma(t))_{t\in[0,T]}$ respectively on the complete filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ where $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by a Brownian motion $(W(t))_{t\in[0,T]}$ and is augmented by all P–null sets from $\sigma(W(s), s \leq T)$. This filtration represents the information available to the investor. Note that in this modeling all three processes are observable. This is a simplification of the real market. However, in the real–world, the interest rate is obtained from the bond market data. The problem of volatility estimation does not occur in this model since the time is continuous. That is, only the assumption that the mean rate of return is known seems to be a serious disadvantage. In this work, we give a martingale method of portfolio optimization for a model where the mean rate of stock return is not observed directly. In our setting, all decisions made by an investor are based on the restricted information which comes from the observation of stock prices and does not include the certain knowledge of the actual mean rate of return.

Let us brief the common portfolio optimization method as presented in [7] or in [9] to explain the main idea of our approach. Given the processes $(r(t))_{t\in[0,T]}$, $(b(t))_{t\in[0,T]}$ and $(\sigma(t))_{t\in[0,T]}$ on $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in[0,T]})$ as above, the progressively measurable wealth processes $(X^{x,\pi,c}(t))_{t\in[0,T]}$ is defined. It depends on the initial endowment $x\in[0,\infty[$, on the portfolio process $\pi(\cdot)$, and on the consumption process $c(\cdot)$. Due to natural requirements (self–financing condition, non-negativeness of the wealth) the pairs $(\pi(\cdot),c(\cdot))$ are restricted to some set $A(x)$ of admissible pairs. The attitudes towards risk are described by the *utility functions* $\{U_1(t,\cdot),U_2(\cdot):t\in[0,T]\}$ which define for each endowment x

$$V(x):=\sup_{(\pi,c)\in A'(x)} E(\int_0^T U_1(t,c(t))dt+U_2(X^{x,\pi,c}(T))), \tag{1}$$

where $A'(x):=\{(\pi,c)\in A(x):E(\int_0^T \min(0,U_1(t,c(t)))dt+\min(0,U_2(X^{x,\pi,c}(T))))>-\infty\}$. The problem of portfolio optimization is, for a given $x\in[0,\infty[$ to calculate a pair $(\pi^*(\cdot),c^*(\cdot))\in A'(x)$ where $V(x)$ is reached. In the martingale approach first the optimal consumption $c^*(\cdot)$ and the optimal terminal wealth $X^{x,\pi^*,c^*}(T)$ are determined. After that, one computes the corresponding portfolio π^*. The reason why this method works is that the set of all possible terminal wealths $\{X^{x,\pi,c}(T):(\pi,c)\in A(x)\}$ is described by the martingale representation theorem which states that each centered $(\mathcal{F}_t)_{t\in[0,T]}$–martingale is represented as a stochastic integral with respect to $(W(t))_{t\in[0,T]}$. However, other martingale representation theorems are known, for example that of Fujisaki (see Theorem 16.22 in [1]). The theorem of Fujisaki considers a Brownian motion $(\mathcal{V}(t))_{t\in[0,T]}$ which is a martingale with respect to some filtration $(\mathcal{F}_t)_{t\in[0,T]}$. In this theorem, the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ may be strictly larger then the augmentation of $(\sigma(\mathcal{V}(s),s\le t))_{t\in[0,T]}$ by all null sets of $\sigma(\mathcal{V}(s),s\le T)$ but each centered $(\mathcal{F}_t)_{t\in[0,T]}$–martingale is still written as a stochastic integral with respect to $(\mathcal{V}(t))_{t\in[0,T]}$. We apply the theorem of Fujisaki instead of the usual martingale representation theorem in the following setting: The investor's knowledge is based only on the the past stock prices and that gives him a noisy observation of the mean rate of return. Using tools from stochastic filtering theory, we define the innovation process of this noisy observation which is a Brownian motion with respect to the filtration generated by investor's stock price observation. Moreover, the martingale representation theorem of Fujisaki holds, and the martingale method works also in our setting.

This paper is organized as follows: In the second section we recall the basic usual setting and cite the main result on completeness of the market with known mean rates of stock return. Sections 3 and 4 present the modifications of the martingale method needed to solve the portfolio optimization problem under restricted information (unknown mean rate of return). We conclude considering some numerical examples in Sections 5 and 6.

2 Observed mean rates of return and completeness of the market

All processes under consideration are indexed by $[0,T]$. Let $W(\cdot)$ be the standard Brownian motion on the complete probability space $(\Omega,\mathcal{F},P)$. We denote by $\mathcal{N}$ all P-null sets from $\sigma(W(s),s\le T)$. The continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is defined as $\mathcal{F}_t:=\sigma(\sigma(W(s),s\le t)\cup\mathcal{N})$ for all $t\in[0,T]$. A share of the money market has the price $S_0(t)$ at the time t and is given by $dS_0(t):=S_0(t)r(t)dt$, $S_0(0)=1$, where the *risk–free rate process* $r(\cdot)$ is progressively measurable with

$$\text{(2)} \qquad \int_0^T |r(s)|ds < \infty \qquad \text{almost surely.}$$

The stock price per share $S_1(t)$ at the time $t \in [0,T]$ is described by

$$dS_1(t) := S_1(t)(b(t)dt + \sigma(t)dW(t)), \qquad S_1(0) \in]0,\infty[,$$

Here, the progressively measurable *mean rate of return* $b(\cdot)$ and the progressively measurable *volatility process* $\sigma(\cdot)$ satisfy

$$\text{(3)} \qquad \int_0^T |b(s)|ds < \infty, \qquad \int_0^T |\sigma(s)|^2 ds < \infty \text{ almost surely.}$$

In the following we suppose that $(\theta(t) := \sigma(t)^{-1}(b(t) - r(t)))_{t\in[0,T]}$ fulfills

$$\text{(4)} \qquad \int_0^T \theta(s)^2 ds < \infty \text{ almost surely.}$$

Let us also define the processes $Z_0(\cdot)$, $H_0(\cdot)$ by

$$\begin{aligned} Z_0(t) &:= \exp(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds) \\ H_0(t) &:= Z_0(t)S_0(t)^{-1} \qquad \forall t \in [0,T]. \end{aligned} \tag{5}$$

A *portfolio process* $\pi(\cdot)$ is a pair of progressively measurable processes $\pi_0(\cdot)$, $\pi_1(\cdot)$ with

$$\int_0^T |\pi_0(t) + \pi_1(t)||r(t)|dt < \infty \tag{6}$$

$$\int_0^T |\pi_1(t)(b(t) - r(t)|dt < \infty \tag{7}$$

$$\int_0^T |\pi_1(t)\sigma(t)|^2 dt < \infty \tag{8}$$

A *consumption process* $c(\cdot)$ is a progressively measurable process with

$$\text{(9)} \qquad c(t) \geq 0 \quad t \in [0,T], \qquad \int_0^T c(t)dt < \infty \qquad \text{almost surely.}$$

As usual, the wealth process $X^{x,\pi,c}(\cdot)$ corresponding to the initial endowment $x \in [0,\infty[$, to the portfolio $\pi(\cdot)$ and to the consumption $c(\cdot)$ is defined as

$$\begin{aligned} X^{x,\pi,c}(t) &= x + \int_0^t (\pi_0(s) + \pi_1(s))r(s)ds + \int_0^t \pi_1(s)(b(s) - r(s))ds \\ &+ \int_0^t \pi_1(s)\sigma(s)dW(s) - \int_0^t c(s)ds, \qquad \forall t \in [0,T]. \end{aligned}$$

We rewrite the above equation using the process $(W_0(t) := \int_0^t \theta(s)ds + W(t))_{t\in[0,T]}$ as

$$\text{(10)} \qquad X^{x,\pi,c}(t) = x + \int_0^t (\pi_0(s) + \pi_1(s))r(s)ds + \int_0^t \pi_1(s)\sigma(s)dW_0(s) - \int_0^t c(s)ds.$$

Given a consumption $c(\cdot)$ and $x \in [0,\infty[$, the portfolio $\pi(\cdot)$ is called $c(\cdot)$-financed, if

$$\text{(11)} \qquad X^{x,\pi,c}(t) = \pi_0(t) + \pi_1(t) \qquad \forall t \in [0,T].$$

For the $c(\cdot)$-financed portfolio $\pi(\cdot)$ we have

$$(12) \qquad \frac{X^{x,\pi,c}(t)}{S_0(t)} = x - \int_0^t \frac{c(u)}{S_0(u)} du + \int_0^t \frac{1}{S_0(u)} \pi_1(u)\sigma(u) dW_0(u), \qquad \forall t \in [0,T].$$

Let $x \in [0,\infty[$. A consumption and portfolio process pair $(\pi(\cdot), c(\cdot))$ is called admissible at x (written $(\pi, c) \in A(x)$), if $\pi(\cdot)$ is $c(\cdot)$-financed and the wealth process $X^{x,c,\pi}(\cdot)$ corresponding to $x, c(\cdot)$, and $\pi(\cdot)$ fulfills $X^{x,c,\pi}(t) \geq 0$ almost surely for all $t \in [0,T]$. The next theorem (cited from [7], Theorem 3.5, p. 93) describes all terminal wealths that are attainable from a given initial endowment x and a given consumption $c(\cdot)$ by using portfolios π such that $(\pi, c) \in A(x)$. In financial mathematics language it states that the market model under consideration is complete.

Proposition 1. *Let $x \in [0,\infty[$ be given, let $c(\cdot)$ be a consumption process, and let ξ be a nonnegative, $\mathcal{F}_T$–measurable random variable such that*

$$(13) \qquad E(\int_0^T H_0(u)c(u)du + H_0(T)\xi) = x.$$

Then there exists a portfolio process $\pi(\cdot)$ such that $(\pi(\cdot), c(\cdot)) \in A(x)$ and $\xi = X^{x,c,\pi}(T)$.

This result is the main reason why the martingale method of portfolio optimization works. In the following section we consider a financial market where the mean rate is not observable, but a similar result holds.

3 A financial market with unobserved mean rate of return

Let $(\Omega, \mathcal{G}, P, (\mathcal{G}_t)_{t\in[0,T]})$ be a complete filtered probability space where the right continuous filtration $(\mathcal{G}_t)_{t\in[0,T]}$ contains all P-null sets from $\mathcal{G}$. Let $(W(t), \mathcal{G}_t)_{t\in[0,T]}$ be a standard Brownian motion. First, we define the stock price process. Suppose that we are given the $(\mathcal{G}_t)_{t\in[0,T]}$–adapted *mean rate of return* $b(\cdot)$ which is a RCLL (right continuous with left limits) process and a Borel measurable *volatility function* $\sigma : [0,T] \times [0,\infty[\to \mathbb{R}$. Using theorem 5.1.1, p. 97 of [4] we impose the integrability condition

$$(14) \qquad E(\int_0^T |b(s)|^2 ds) < \infty$$

and the Lipschitz condition

$$(15) \qquad \exists\ K \in]0,\infty[: \qquad |\sigma(t,x) - \sigma(t,y)| \leq K|x-y| \qquad \forall x,y \in \mathbb{R}, \quad t \in [0,T]$$

to ensure the existence and uniqueness of the strong solution $L(\cdot)$ of

$$L(t) = \int_0^t b(s)ds + \int_0^t \sigma(s, L(s))dW(s).$$

We also suppose that the volatility function is bounded from below:

$$(16) \qquad \inf\{\sigma(t,x) : t \in [0,T], x \in \mathbb{R}\} > 0.$$

The stock price process $S_1(\cdot)$ is defined by $dS_1(t) = S_1(t)dL(t)$ with initial condition $S_1(0) \in]0,\infty[$. The unique solution of this equation is

$$S_1(t) = S_1(0)\exp(\int_0^t b(s)ds + \int_0^t \sigma(s)dW(s) - \frac{1}{2}\int_0^t \sigma(s)^2 ds) \qquad \forall t \in [0,T].$$

Here $(\sigma(t) := \sigma(t, L(t)))_{t\in[0,T]}$ is to be considered as the *volatility* of the stock. Note that both processes $L(\cdot)$ and $S_1(\cdot)$ contain the same information since $L(t) = \int_0^t S_1(u)^{-1} dS_1(u)$ for all $t \in [0,T]$. The observation of stock prices is described by the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, where

$$\mathcal{F}_t := \sigma(\sigma(S_1(u), u \le t) \cup \mathcal{N}) = \sigma(\sigma(L(u), u \le t) \cup \mathcal{N}) \qquad \forall t \in [0,T],$$

and $\mathcal{N}$ denotes all P-null sets from $\sigma(S_1(u), u \le T) = \sigma(L(u), u \le T)$ as usual. Let us write $\widehat{b}(\cdot)$ to denote the measurable modification of $(E(b(t)|\mathcal{F}_t))_{t\in[0,T]}$ (it's existence is shown by optional projection arguments, see [12], p. 319). Clearly, $\widehat{b}(\cdot)$ is interpreted as an estimation of the mean rate of return based on its noisy observation through stock prices. We also define the innovation process $\mathcal{V}(\cdot)$ as

$$\mathcal{V}(t) := \int_0^t \sigma(u)^{-1} dL(u) - \int_0^t \sigma(u)^{-1} \widehat{b}(u) du = \int_0^t \sigma(u)^{-1} (b(u) - \widehat{b}(u)) du + W(t) \tag{17}$$

for all $t \in [0,T]$.

The theorem of Fujisaki for L^2–martingales is found in [3] or [1], p. 231, but for our applications we will need it for not necessarily square integrable martingales . For this reason, we include the proof.

Proposition 2. *The process $(\mathcal{V}(t), \mathcal{F}_t)_{t\in[0,T]}$ is a Brownian motion. For each centered $(\mathcal{F}_t)_{t\in[0,T]}$–martingale $N(\cdot)$ there exists a $(\mathcal{F}_t)_{t\in[0,T]}$–progressively measurable $\phi(\cdot)$ with $\int_0^T \phi(s)^2 ds < \infty$ almost surely such that*

$$N(t) = \int_0^t \phi(s) d\mathcal{V}(s) \qquad \forall t \in [0,T].$$

Proof. The process $(\mathcal{V}(t), \mathcal{F}_t)_{t\in[0,T]}$ is a continuous martingale since for all $0 \le t < t+h \le T$ the following holds

$$\begin{aligned} E(\mathcal{V}(t+h) - \mathcal{V}(t)|\mathcal{F}_t) &= E(\int_t^{t+h} (b(u) - \widehat{b}(u))\sigma(u)^{-1} du + W(t+h) - W(t)|\mathcal{F}_t) \\ &= \int_t^{t+h} E(E((b(u) - \widehat{b}(u))\sigma(u, L(u))^{-1}|\mathcal{F}_u))|\mathcal{F}_t) du = 0. \end{aligned}$$

The quadratic variation is calculated as

$$\begin{aligned} <\mathcal{V}>(t) &:= \lim_{n\to\infty} \sum_{i=0}^{n-1} |\mathcal{V}(t_{i+1}^n) - \mathcal{V}(t_i^n)|^2 \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} |\int_{t_i^n}^{t_{i+1}^n} (b(u) - \widehat{b}(u))\sigma(u)^{-1} du + W(t_{i+1}^n) - W(t_i^n)|^2 = t, \end{aligned}$$

where the partitions $0 = t_0^n <, \dots, < t_n^n = t$ $(n \in \mathbb{N})$ of $[0,t]$ satisfy

$$\lim_{n\to\infty} \max_{i=0}^{n-1} |t_{i+1}^n - t_i^n| = 0.$$

Thus, Lévy's martingale characterization yields the first assertion. By the theorem of Girsanov,

$$\tilde{W}(t) := \int_0^t b(u)\sigma(u)^{-1} du + W(t) = \int_0^t \widehat{b}(u)\sigma(u)^{-1} du + \mathcal{V}(t)$$

is a Brownian motion with respect to the measure $\tilde{P}$, where

$$d\tilde{P} := \exp(-\int_0^T \widehat{b}(u)\sigma(u)^{-1} d\mathcal{V}(u) - \frac{1}{2}\int_0^T (\widehat{b}(u)\sigma(u)^{-1})^2 du) dP.$$

Since $L(\cdot)$ is the unique solution of $L(t) = \int_0^t \sigma(u, L(u)) d\tilde{W}(u)$, we conclude that

$$\sigma(L(u), u \leq t) \subset \sigma(\tilde{W}(u), u \leq t) \qquad \forall t \in [0, T]$$

On the other hand, we have $\tilde{W}(\cdot) = \int_0^{\cdot} \sigma(u, L(u))^{-1} dL(u)$ showing the converse inclusion. That is, $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmentation of the Brownian filtration $(\sigma(\tilde{W}(u), u \leq t))_{t\in[0,T]}$ by all null sets from $\sigma(\tilde{W}(u), u \leq T)$. The martingale representation theorem (see [8], theorem 3.4.15 and problem 3.4.16) implies that if $\tilde{N}(\cdot)$ is a centered $(\mathcal{F}_t)_{t\in[0,T]}$–martingale with respect to $\tilde{P}$, then it admits the representation $(\tilde{N}(t) = \int_0^t \tilde{\phi}(u) d\tilde{W}(u))_{t\in[0,T]}$ where the uniquely determined progressively $(\mathcal{F}_t)_{t\in[0,T]}$–measurable $\tilde{\phi}(\cdot)$ satisfies $\int_0^T \tilde{\phi}(u)^2 du < \infty$ almost surely. This means

$$d\tilde{N}(t) = \tilde{\phi}(t) d\mathcal{V}(t) + \tilde{\phi}(t)\widehat{b}(t)\sigma(t)^{-1} dt, \qquad \tilde{N}(0) = 0. \tag{18}$$

Write

$$\Lambda(t) := \frac{d\tilde{P}|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} := \exp(-\int_0^t \widehat{b}(u)\sigma(u)^{-1} d\mathcal{V}(u) - \frac{1}{2}\int_0^t (\widehat{b}(u)\sigma(u)^{-1})^2 du) \qquad \forall t \in [0, T].$$

Let $N(\cdot)$ be a centered $(\mathcal{F}_t)_{t\in[0,T]}$–martingale with respect to P, then $\tilde{N}(\cdot) := N(\cdot)\Lambda(\cdot)^{-1}$ is a centered $(\mathcal{F}_t)_{t\in[0,T]}$–martingale with respect to $\tilde{P}$. From the stochastic differentials (18) and

$$d\Lambda(t) = -\widehat{b}(u)\sigma(u)^{-1} d\mathcal{V}(u) \qquad \Lambda(0) = 1$$

we verify by Ito's formula that we have the desired integral representation of $N(\cdot)$ by $\mathcal{V}(\cdot)$:

$$dN(t) = d(\tilde{N}(t)\Lambda(t)) = \Lambda(t)(\tilde{\phi}(t) - \tilde{N}(t)\widehat{b}(t)\sigma(t)^{-1}) d\mathcal{V}(t), \qquad N(0) = 0.$$

□

A share of the money market has the price process $(S_0(t) = \exp(\int_0^t r(u)du)_{t\in[0,T]}$ where the *risk–free rate process* $r(\cdot)$ is $(\mathcal{F}_t)_{t\in[0,T]}$–progressively measurable and satisfies (2). In accordance to (4) we suppose that the $(\mathcal{F}_t)_{t\in[0,T]}$–progressively measurable process $\widehat{\theta}(\cdot)$ defined by

$$\widehat{\theta}(t) := \sigma(t)^{-1}(\widehat{b}(t) - r(t)) \qquad \forall t \in [0, T]$$

satisfies

$$\int_0^T \widehat{\theta}(s)^2 ds < \infty \text{ almost surely.} \tag{19}$$

Let us now define

$$\begin{aligned} Z_0(t) &:= \exp(-\int_0^t \widehat{\theta}(s) d\mathcal{V}(s) - \frac{1}{2}\int_0^t \widehat{\theta}(s)^2 ds) \\ H_0(t) &:= Z_0(t) S_0(t)^{-1} \qquad \forall t \in [0, T]. \end{aligned}$$

Although in our setting these processes differ from the ones with the same name in Section 2, we will reuse these names here as the processes will play the same role in the following as their counterparts in Section 2. All portfolios, consumptions, and wealths are defined almost as before, keeping in mind that they shall be $(\mathcal{F}_t)_{t\in[0,T]}$–progressively measurable: A *portfolio* *process* $\pi(\cdot)$ is a pair $(\pi_0(\cdot), \pi_1(\cdot))$ of $(\mathcal{F}_t)_{t\in[0,T]}$–progressively measurable processes which satisfy

$$\int_0^T |\pi_0(t) + \pi_1(t)||r(t)|dt \;<\; \infty \tag{20}$$

$$\int_0^T |\pi_1(t)(\widehat{b}(t) - r(t)|dt \;<\; \infty \tag{21}$$

$$\int_0^T |\pi_1(t)\sigma(t)|^2 dt \;<\; \infty \tag{22}$$

A *consumption* $c(\cdot)$ is a $(\mathcal{F}_t)_{t\in[0,T]}$–progressively measurable process with (9). Furthermore, we introduce the *wealth process* $X^{x,\pi,c}(\cdot)$ which corresponds to the initial endowment $x \in [0,\infty[$, to the portfolio $\pi(\cdot)$ and to the consumption $c(\cdot)$ by (10). Note that $X^{x,\pi,c}(\cdot)$ is well defined and automatically $(\mathcal{F}_t)_{t\in[0,T]}$–progressively measurable since (17) implies that for all $t \in [0,T]$

$$\begin{aligned} W_0(t) &= \int_0^t \theta(s)ds + W(t) = \int_0^t \sigma(s)^{-1}(b(s) - r(s))ds + W(t) \\ &= \int_0^t \sigma(s)^{-1}(\widehat{b}(s) - r(s))ds + \mathcal{V}(t) = \int_0^t \widehat{\theta}(s)ds + \mathcal{V}(t). \end{aligned}$$

Given a consumption $c(\cdot)$ and $x \in [0,\infty[$, the portfolio $(\pi_0(\cdot), \pi_1(\cdot))$ is called $c(\cdot)$-financed, if (11) holds. If $X^{x,\pi,c}(\cdot)$ corresponds to the $c(\cdot)$-financed portfolio $\pi(\cdot)$, then (11) is fulfilled. Let $x \in [0,\infty[$. A consumption and portfolio process pair $(\pi(\cdot), c(\cdot))$ is called admissible at x (written $(\pi, c) \in A(x)$), if $\pi(\cdot)$ is $c(\cdot)$-financed and the wealth process $X^{x,c,\pi}(\cdot)$ corresponding to $x, c(\cdot)$, and $\pi(\cdot)$ fulfills $X^{x,c,\pi}(t) \geq 0$ almost surely for all $t \in [0,T]$.

Starting from an arbitrary process $\pi_1(\cdot)$, a consumption $c(\cdot)$ and an initial endowment x, we may check if there exists $\pi(\cdot) = (\pi_0(\cdot), \pi_1(\cdot))$ such that $(\pi, c) \in A(x)$ by using

Lemma 1: *Let $\pi_1(\cdot)$ be a $(\mathcal{F}_t)_{t\in[0,T]}$–progressively measurable process satisfying (21) and (22), $c(\cdot)$ be a consumption, and $x \in [0,\infty[$. If*

$$X(t) := S_0(t)(x - \int_0^t \frac{c(u)}{S_0(u)}du + \int_0^t \frac{1}{S_0(u)}\pi_1(u)\sigma(u)dW_0(u)) \geq 0 \text{ almost surely}$$

for all $t \in [0,T]$ then $(\pi_0(t) := X(t) - \pi_1(t))_{t\in[0,T]}$ fulfills (20), $(\pi, c) \in A(x)$ with corresponding wealth $X^{x,\pi,c}(\cdot) = X(\cdot)$.

Proof. By definition, $(\pi_0(t) + \pi_1(t) = X(t))_{t\in[0,T]}$ is continuous, and (20) holds due to (2). Moreover, $X(\cdot)$ satisfies

$$\begin{aligned} dX(t) &= \frac{X(t)}{S_0(t)}dS_0(t) + \pi_1(t)\sigma(t)dW_0(t) - c(t)dt \\ &= (\pi_0(t) + \pi_1(t))r(t)dt + \pi_1(t)\sigma(t)dW_0(t) - c(t)dt \end{aligned}$$

with initial condition $X(0) = x$ showing that $X^{x,\pi,c}(\cdot) = X(\cdot)$ and $(\pi, c) \in A(x)$. □

Now we state a result similar to Proposition 1.

Proposition 3. *Let $x \in [0,\infty[$ be given, let $c(\cdot)$ be a consumption process, and let ξ be a nonnegative, $\mathcal{F}_T$–measurable random variable such that*

$$E(\int_0^T H_0(u)c(u)du + H_0(T)\xi) = x. \tag{23}$$

Then there exists a portfolio process $\pi(\cdot)$ such that $(\pi(\cdot), c(\cdot)) \in A(x)$ and $\xi = X^{x,c,\pi}(T)$.

Proof. Essentially, we shall copy the proof from [7], Theorem 3.5, p. 93. The only difference is that we use the innovation process $\mathcal{V}(\cdot)$ instead of the Brownian motion $W(\cdot)$ replacing $b(\cdot)$ by $\widehat{b}(\cdot)$ and $\theta(\cdot)$ by $\widehat{\theta}(\cdot)$ everywhere in the proof. Furthermore, the martingale representation theorem of Fujisaki is used instead the usual one.

Let us define $J(t) := \int_0^t H_0(u)c(u)du$ and consider the nonnegative $(\mathcal{F}_t)_{t\in[0,T]}$–martingale $M(\cdot)$

$$M(t) := E(J(T) + H_0(T)\xi|\mathcal{F}_t), \qquad t \in [0,T].$$

According to the martingale representation theorem of Fujisaki, there is a $(\mathcal{F}_t)_{t\in[0,T]}$–progressively measurable process $\psi(\cdot)$ satisfying $\int_0^T \psi(u)^2 du < \infty$ almost surely and

$$M(t) = x + \int_0^t \psi(u)d\mathcal{V}(u), \qquad \forall t \in [0,T].$$

In particular, $\max_{t\in[0,T]} M(t) < \infty$ almost surely since $M(\cdot)$ has continuous paths. Similarly, $\max_{t\in[0,T]} S_0(t)$, $\max_{t\in[0,T]} Z_0(t)^{-1}$ are finite almost surely. Define a nonnegative process $X(\cdot)$ by

$$\frac{X(t)}{S_0(t)} := \frac{1}{Z_0(t)} E(\int_t^T H_0(u)c(u) + H_0(T)\xi|\mathcal{F}_t) = \frac{M(t) - J(t)}{Z_0(t)} \tag{24}$$

so that $X(0) = M(0) = x$. Ito's rule implies

$$d(\frac{X(t)}{S_0(t)}) = -\frac{c(t)}{S_0(t)}dt + \frac{1}{S_0(t)}\pi(t)\sigma(t)dW_0(t), \tag{25}$$

where

$$\pi_1(t) = \frac{1}{H_0(t)\sigma(t)}[\psi(t) + (M(t) - J(t))\widehat{\theta}(t)].$$

In fact, we have:

$$\begin{aligned}
d(M(t) - J(t)) &= \psi(t)d\mathcal{V}(t) - H_0(t)c(t)dt \\
Z_0(t)^{-1} &= \exp(\int_0^t \widehat{\theta}(s)d\mathcal{V}(s) + \frac{1}{2}\int_0^t \widehat{\theta}(s)^2 ds) \\
d(Z_0(t)^{-1}) &= Z_0(t)^{-1}(\widehat{\theta}(t)d\mathcal{V}(t) + \widehat{\theta}(t)^2 dt) \\
d(M(t) - J(t))d(Z_0(t)^{-1}) &= \psi(t)\frac{\widehat{\theta}(t)}{Z_0(t)}dt,
\end{aligned}$$

so that

$$\begin{aligned}
d(\frac{X(t)}{S_0(t)}) &= \frac{M(t) - J(t)}{Z_0(t)}(\widehat{\theta}(t)d\mathcal{V}(t) + \widehat{\theta}(t)^2 dt) \\
&\quad + \frac{1}{Z_0(t)}(\psi(t)d\mathcal{V}(t) - H_0(t)c(t)dt) + \psi(t)\frac{\widehat{\theta}(t)}{Z_0(t)}dt \\
&= -\frac{H_0(t)}{Z_0(t)}c(t)dt + \frac{M(t) - J(t)}{Z_0(t)}\widehat{\theta}(t)(\underbrace{\widehat{\theta}(t)dt + d\mathcal{V}(t)}_{dW_0(t)}) \\
&\quad + \frac{\psi(t)}{Z_0(t)}(\underbrace{\widehat{\theta}(t)dt + d\mathcal{V}(t)}_{dW_0(t)}) \\
&= -\frac{H_0(t)}{Z_0(t)}c(t)dt + \frac{1}{Z_0(t)}[\psi(t) + (M(t) - J(t))\widehat{\theta}(t)]dW_0(t) \\
&= -\frac{H_0(t)}{Z_0(t)}c(t)dt + \frac{1}{S_0(t)}\underbrace{\frac{\psi(t) + (M(t) - J(t))\widehat{\theta}(t)}{H_0(t)\sigma(t)}}_{\pi_1(t)}\sigma(t)dW_0(t).
\end{aligned}$$

We check that π_1 fulfills (21) :

$$\begin{aligned}\int_0^T |\pi_1(t)(\widehat{b}(t)-r(t))|dt &= \int_0^T |\frac{1}{H_0(t)\sigma(t)}[\psi(t)+(M(t)-J(t))\widehat{\theta}(t)](\widehat{b}(t)-r(t))|dt \\ &= \int_0^T |\frac{1}{H_0(t)}[\psi(t)+(M(t)-J(t))\widehat{\theta}(t)]\widehat{\theta}(t)|dt < \infty\end{aligned}$$

Since almost all paths of $\psi(\cdot)$, $\widehat{\theta}(\cdot)$ are square integrable and almost all paths of $M(\cdot)-J(\cdot)$, $H_0(\cdot)^{-1}$ are bounded (in fact, they are are continuous). Similarly, $\pi_1(\cdot)$ fulfills (22)

$$\int_0^T |\sigma(t)\pi(t)|^2 dt = \int_0^T \frac{S_0(t)^2}{Z_0(t)^2}|\psi(t)-(M(t)-J(t))\widehat{\theta}(t)|^2 dt < \infty \quad \text{almost surely.}$$

¿From (24) and (25) we see that

$$X(t) = S_0(t)(x - \int_0^t \frac{c(u)}{S_0(u)}du + \int_0^t \frac{1}{S_0(u)}\pi_1(u)\sigma(u)dW_0(u)) \geq 0, \qquad \forall t \in [0,T]$$

almost surely. By the previous lemma, this shows that $(\pi, c) \in A(x)$ with corresponding wealth $X^{x,\pi,c}(\cdot) = X(\cdot)$. □

4 Optimal portfolio selection

In the next step of portfolio optimization the optimal consumption $c^*(\cdot)$ and the optimal terminal wealth $X^{x,\pi^*,c^*}(T)$ have to be determined. We shall see that the usual technique needed for this (see [7], chapter 3 or [9], chapter 3) applies to our situation without changes. However, let us stress the fact that now problem (1) is an optimal portfolio problem in an incomplete market as no longer all $(\mathcal{G}_t)_{t\in[0,T]}$-measurable claims can be hedged via a portfolio and consumption process only based on the observation of the stock price. Let us mention first that for all $(\pi, c) \in A(x)$ the following *budget constraint* holds:

$$E(\int_0^T H_0(u)c(u)du + H_0(T)X^{x,\pi,c}(T)) \leq x \tag{26}$$

It is implied by the fact that

$$(H_0(t)X^{x,\pi,c}(t) + \int_0^t H_0(u)c(u)du = x + \int_0^t H_0(u)(\sigma(u)\pi(u) - X^{x,\pi,c}(u)\widehat{\theta}(u))d\mathcal{V}(u))_{t\in[0,T]}$$

is a non-negative local martingale and hence a supermartingale, which is easily derived from

$$\begin{aligned} dX^{x,\pi,c}(t) &= X(t)r(t)dt + \pi_1(t)\sigma(t)dW_0(t) - c(t)dt \\ &= X(t)r(t)dt + \pi_1(t)(\widehat{b}(t)-r(t))dt + \pi_1(t)\sigma(t)d\mathcal{V}(t) - c(t)dt \\ dW_0(t) &= \widehat{\theta}(t)dt + d\mathcal{V}(t) \\ dH_0(t) &= H_0(t)(-\widehat{\theta}(t)d\mathcal{V}(t) - r(t)dt). \end{aligned}$$

The function $U \in C^1(]0,\infty[,]0,\infty[)$ is called a *utility function* if it is strictly increasing and strictly concave and U' is strictly decreasing with $\lim_{z\to 0} U'(z) = +\infty$, $\lim_{z\to\infty} U'(z) = 0$. The inverse function $I := U'^{\,-1}$ maps $]0,\infty[$ onto $]0,\infty[$. We also have the inequality

$$U(I(b)) \geq U(a) + b(I(b)-a) \qquad \forall a, b \in]0,\infty[. \tag{27}$$

Given utility functions $\{U_1(t,\cdot),U_2(\cdot) : t\in[0,T]\}$ such that U_1 is Borel measurable, we denote by $I_1(t,\cdot)$ and $I_2(\cdot)$ the inverse functions of $U_1(t,\cdot)'$ and of $U_2(\cdot)'$ for all $t\in[0,T]$ respectively. Let

$$\mathcal{X} :]0,\infty[\to]0,\infty] \qquad y\mapsto E(\int_0^T H_0(t)I_1(t,yH_0(T))dt+H_0(T)I_2(yH_0(T)))$$

We suppose that

$$\mathcal{X}(y)<\infty \qquad \forall y\in]0,\infty[\tag{28}$$

From (28) monotone convergence implies that $\mathcal{X}$ is continuous and strictly decreasing with $\lim_{y\to 0}\mathcal{X}(y)=\infty$ and $\lim_{y\to\infty}\mathcal{X}(y)=0$ implying the existence of the inverse mapping $\mathcal{Y}:=\mathcal{X}^{-1}$.

Proposition 4. *Let*

$$E(\int_0^T H_0(t)dt+H_0(T))<\infty \tag{29}$$

be satisfied. For each initial endowment $x\in]0,\infty[$, the solution $(\pi^*,c^*)\in A'(x)$ of the optimization problem (1) with unobservable mean rate of return is given by

$$c^*(t)=I_1(t,\mathcal{Y}(x)H_0(t)), \qquad X^{x,\pi^*,c^*}(T)=I_2(\mathcal{Y}(x)H_0(T)).$$

Proof. In view of (27) we have for all $(\pi,c)\in A(x)$:

$$\begin{aligned} U_1(t,c^*(t)) &\geq U_1(t,c(t))+\mathcal{Y}(x)H_0(t)(I_1(t,\mathcal{Y}(x)H_0(t))-c(t))\\ U_2(X^{x,\pi^*,c^*}(T)) &\geq U_2(X^{x,\pi,c}(T))+\mathcal{Y}(x)H_0(T)(I_2(\mathcal{Y}(x)H_0(T))-X^{x,\pi,c}(T)). \end{aligned}$$

Choosing deterministic consumption and terminal wealth as

$$c(t)=X^{x,\pi,c}(T)=xE(\int_0^T H_0(t)dt+H_0(T))^{-1}$$

in the above inequalities, (29) implies that that

$$E(\int_0^T \min(0,U_1(t,c^*(t))dt)>\infty, \qquad E(\min(0,U_2(X^{x,\pi^*,c^*}(T))))>\infty.$$

That means $(\pi^*,c^*)\in A'(x)$ since

$$E(\int_0^T H_0(t)I_1(t,\mathcal{Y}(x)H_0(t))dt+H_0(T)I_2(\mathcal{Y}(x)H_0(T))=\mathcal{X}(\mathcal{Y}(x))=x. \tag{30}$$

For all $(\pi,c)\in A(x)$ we have

$$\begin{aligned} E(\int_0^T U_1(t,c^*(t))dt+H_0(T)X^{x,\pi^*,c^*}(T)) \geq\ & E(\int_0^T U_1(t,c(t))dt+U_2(X^{x,\pi,c}(T)))\\ &+\mathcal{Y}(x)E(\int_0^T H_0(t)I_1(t,\mathcal{Y}(x)H_0(t))dt+H_0(T)I_2(\mathcal{Y}(x)H_0(T)))\\ &-\mathcal{Y}(x)E(\int_0^T H_0(t)c(t)dt+H_0(T)X^{x,\pi,c}(T)). \end{aligned}$$

Applying budget constraint (26) and (30) we are led to optimality of $(\pi^*,c^*)\in A'(x)$.

$$E(\int_0^T U_1(t,c^*(t))dt+H_0(T)X^{x,\pi^*,c^*}(T))\geq E(\int_0^T U_1(t,c(t))dt+U_2(X^{x,\pi,c}(T))) \quad \forall(\pi,c)\in A'(x).$$

□

5 Estimating the mean rate

The estimation of the mean rate of return from the observation of the stock prices is a filtering problem. There are essentially two cases where the general filtering theory provide recursive finite dimensional filter of practical interest: The case of Kalman filtering and that of hidden Markov models. In both cases the hidden process $b(\cdot)$ is modeled as a linear functional $\langle\beta,\cdot\rangle$ of a not observed $\mathbb{R}^n$–valued *system process* $x(\cdot)$, that is $b(t) := \langle\beta, x(t)\rangle$ for all $t \in [0,T]$ with $\beta \in \mathbb{R}^n$. In the first case the processes are modeled by differential equations

$$\begin{aligned} x(t) &= x(0) + \textstyle\int_0^t A(s)x(s)ds + \int_0^t B(s)dW_1(s) \\ L(t) &= \textstyle\int_0^t \langle\beta, x(s)\rangle ds + \int_0^t \sigma dW(s) \end{aligned}$$

with m–dimensional Brownian motion $W_1(\cdot)$ independent of $W(\cdot)$ and deterministic coefficients $A(\cdot)$, $B(\cdot)$ of appropriated dimensions (see [4], p. 252), $\sigma > 0$. The initial value $x(0)$ is Gaussian and independent of $W_1(\cdot), W(\cdot)$. In the second case the Markov process $x(\cdot)$ takes its values in the set $\{e_1, \ldots, e_m\}$ of orthogonal unit vectors in $\mathbb{R}^n$, and is supposed to be independent from $W(\cdot)$. The observation $L(\cdot)$ is the same as in the first case. For complete discussion of the mean rate estimation by the methods of Hidden Markov models we refer the reader to [2] which also includes parameter re–estimation by EM–algorithm. Finally, let us consider two examples. For simplicity, we suppose that the volatility $\sigma > 0$ is constant meaning that the observation is

$$z(t) := L(t) = \int_0^t S_1(u)dS_1(u) = \int_0^t b(u)du + \sigma W(t) \qquad \forall t \in [0,T].$$

Example 1: (A mean-reverting drift rate) Here, we assume an unobservable drift-rate process of the form

$$\int_0^t b(u)du = b(0) + \int_0^t (a_0 + a_1 b(u))du + \beta dW_1(u) \qquad \forall t \in [0,T]$$

with known constants, $b(0), a_0, a_1, \beta \in \mathbb{R}$ with $a_1 < 0$ and $W_1(\cdot)$ a Brownian motion independent from $W(\cdot)$. This model might be particularly suited for a stock which is regarded as one having an intrinsic drift rate but where short time effects cause the real drift rate to fluctuate around this intrinsic value. We obtain (see [4]) the following stochastic differential equation for $\widehat{b}(t)$:

$$d\widehat{b}(t) = (a_0 + a_1\widehat{b}(t))dt + \frac{P(t)}{\sigma^2}(dz(t) - \widehat{b}(t)dt), \qquad \widehat{b}(0) = b(0)$$

where $P(t)$ is the unique solution of the Riccati equation

$$P'(t) = 2a_1 P(t) + \beta^2 - (\frac{P(t)}{\sigma})^2, \qquad P(0) = 0.$$

Example 2: (A random jump of the mean rate) Here we consider the situation where the drift of a stock changes from a low value to a high value after some exponentially distributed time. This is a possible model for a stock which is seen as some future winner. It more or less fluctuates around a constant value for some time but then, when the market has realized its potential, grows at a high rate. It is of course important to realize the start of this growth as soon as possible. Again, learning the unobserved drift rate gives an invaluable advantage. As an example, we recall the following situation as given in [10], section 9.4 adapting the parameters to our situation: Let $b(\cdot)$ be a Markov process starting with a value of zero and changing to $b \in [0,\infty[$ after some unobservable random time which is exponentially distributed with parameter $\lambda > 0$. We obtain the following stochastic differential equation for $\widehat{b}(\cdot)$:

$$d\widehat{b}(t) = \lambda(b - \widehat{b}(t))dt + \widehat{b}(t)(b - \widehat{b}(t))\frac{(dz(t) - \widehat{b}(t)dt)}{\sigma^2}$$

with an initial value of $\widehat{b}(0) = 0$.

6 An example: Constant unknown mean rate.

To give an application of our main result of Section 4 and to highlight its consequences by some figures, we will concentrate on the choice of the log-utility function, i.e. the case of

$$U_1(t,x) = U_2(x) = \ln(x).$$

In this setting we can present very explicit. We first realize that by performing the same calculations as in [9], page 71, we obtain for the given initial endowment $x \in]0,\infty[$ the optimal consumption, the optimal final wealth, and the optimal portfolio as

$$c^*(t) = \frac{x}{T+1}\frac{1}{H_0(t)}, \qquad \xi^* = \frac{x}{T+1}\frac{1}{H_0(T)}, \qquad \pi^*(t) = \frac{\widehat{b}(t) - r(t)}{\sigma(t)^2}X^{x,\pi^*,c^*}(t) \qquad \forall t \in [0,T].$$

Hence, we obtain the optimal portfolio in our setting by replacing $b(t)$ by its conditional expectation $\widehat{b}(t)$ in the optimal strategy with observable mean rate of return. It should also be noted that the above result directly generalizes to the n-stock situation giving an optimal (relative) portfolio of $\pi^*(t)/X^{x,\pi^*,c^*}(t) = (\sigma(t)\sigma(t)')^{-1}(\widehat{b}(t) - r(t)\underline{1})$. Note in particular that although the utility function in the above portfolio problem is typically non-quadratic, we still have a separation result of estimation and control. To highlight the consequences of the above form of the optimal (relative) portfolio process, we look at the special case where the mean rate equals an unknown constant. More precisely, we have

$$db(t) = 0, \qquad b(0) = b$$

where b is an unknown ("unobservable") constant which is therefore modeled as a Gaussian random variable with given moments $E(b)$ and $Var(b)$. Let us assume a constant volatility σ meaning that $(L(t) := \int_0^t S_1(u)^{-1}dS_1(u) = tb + \sigma W(t))_{t\in[0,T]}$ and consider as observation process

$$z(t) = L(t) - E(b)t = t(b - E(b)) + \sigma W(t) \qquad \forall t \in [0,T].$$

Moreover, we suppose that b and $W_1(\cdot)$ are independent. Then, by applying standard filtering results (see [4]) we obtain

$$\widehat{b}(t) = E(b) + \frac{Var(b)}{\sigma^2 + Var(b)t}z(t) \qquad \forall t \in [0,T].$$

An interesting consequence of the explicit form of $\widehat{b}(\cdot)$ is that the estimate $\widehat{b}(t)$ converges towards the true value b with increasing time t even if the moments of b (i.e. the terms the investor has to specify as input for the equation defining $\widehat{b}(t)$) are totally misspecified (i.e. if the "initial guess" $E(b)$ for b is far away from the real value). This is seen from the law of large numbers:

$$\widehat{b}(t) = E(b) + \frac{Var(b)}{\sigma^2/t + Var(b)}((b - E(b)) + \sigma\frac{1}{t}W_t) \qquad \forall t \in [0,T].$$

This is particularly important for applications as then an exact (!) specification of $E(b)$ and $Var(b)$ is important, but not indispensable. For producing the Figures 1–5 below we have chosen the following data: $b = 0.2$, $r = 0.05$, $\sigma = 0.4$, initial endowment $x = 500$, $S_1(0) = 100$. The chosen discretized step-size is 0.04, i.e. there is a rebalancing of the holdings each day (working with a year of 250 trading days).

Figure 1 shows three different wealth processes all underlying the same Brownian motion. One wealth process is based on the full knowledge of the drift parameter ("known wealth"), one using the learning drift $\widehat{b}(t)$ with $E(b) = 0$, $Var(b) = 0.2$, and the final one uses the initial guess of $\bar{b} = 0$ ("guessed wealth"). As the initial guess is far from the real constant $b = 0.2$ the corresponding wealth process performs worse than the other two. Further, it is remarkable that the learning wealth mimics the "known wealth" nearly perfectly. The drift process seems to learn quite quickly. Figures 2 — 7 depict the mean value evolution of the

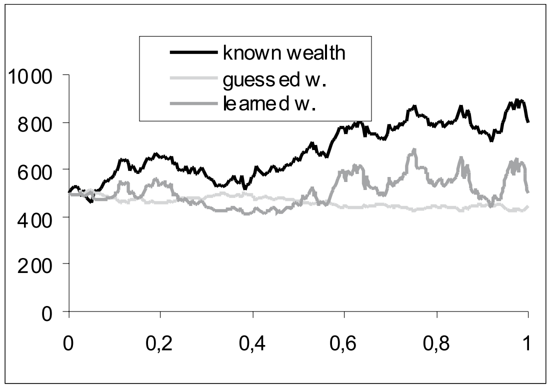

Figure 1: Simulation of wealth processes based on guessed, learned and known drift

known, guessed and learned drift and wealth, respectively. Each mean value evolution is obtained from 5000 independently generated samples. In the Figures 2 — 5, we have chosen $b = E(b) = 0.15, Var(b) = 0.01$. The time horizon is $T = 1$ and $T = 25$ respectively. To highlight the problems with a high variance of the learning process we give the pictures 6 and 7 where we have changed $Var(b)$ to 0.05 and σ to 0.2, the time horizon is $T = 10$.

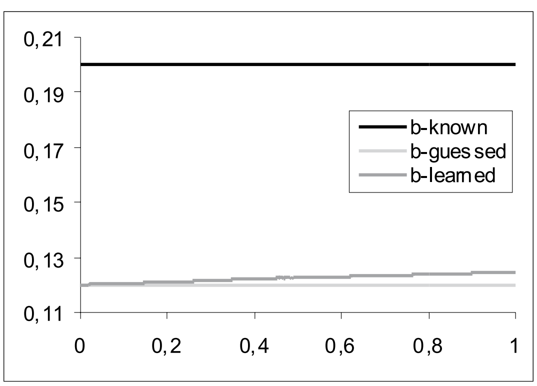

Figure 2: Mean-evolution of drift rates

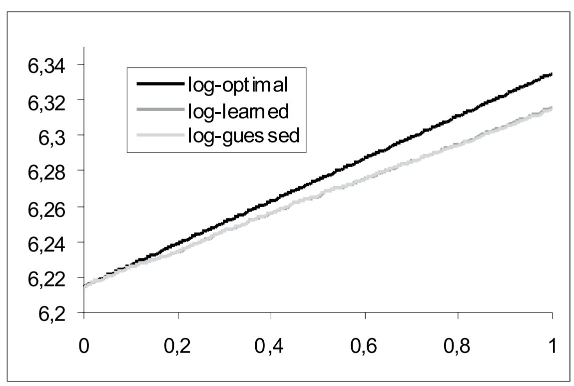

Figure 3: Mean-evolution of the wealth

The Figures 1–7 show that we have to balance out some problems:

- Guessing a drift rate, i.e. choosing a constant and sticking to it, is not a bad strategy over a short time period. If its value is not too far away from the real value then the

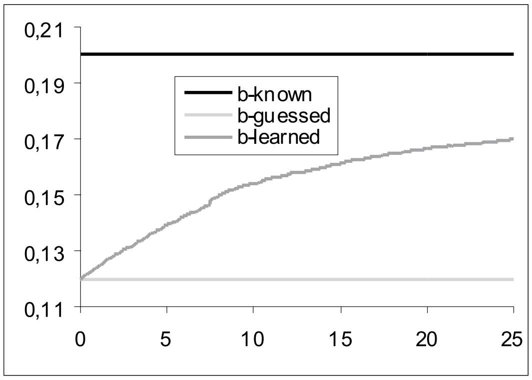

Figure 4: Mean-evolution of drift rates

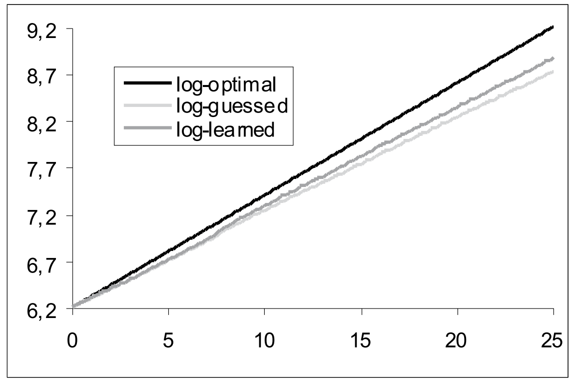

Figure 5: Mean-evolution of the wealth

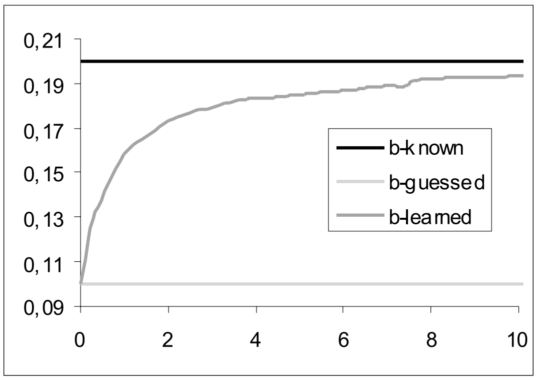

Figure 6: Mean-evolution of drift rates

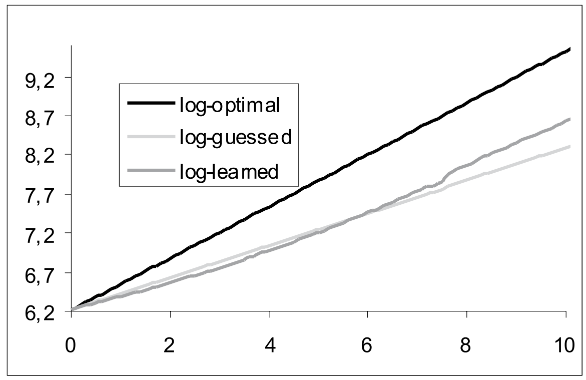

Figure 7: Mean-evolution of the wealth.

corresponding expected final log-wealth is close to the optimal one in the model with perfect knowledge (see Figure 3).

- Using the above guess as starting value in our equations for $\widehat{b}(t)$ (or equivalently using the expected value $E(b)$ as initial guess) does not necessarily lead to a better performance (at least not over a short time period) than the above strategy of sticking to the initial guess (see Figure 3).

- The performance of the learning strategy corresponding to the use of $\widehat{b}(t)$ in the optimal portfolio process depends heavily on both the time horizon and the value of $Var(b)$. If one looks at the realistic situation that this variance is unknown then the investor has to use an estimate (or a guess ...) of it. Here, we are faced with a typical balance problem: If the chosen value is very small then the learning process is slow, and it takes a long time until (the mean of) $\widehat{b}(t)$ is close to b. On the other hand, for such choices the expected final log-wealth seems to beat the above guessed constant strategy even on the short time scale. A high value of (the estimate for) $Var(b)$ speeds up the learning process in the mean (see Figure 6) but also leads to a high variance of the corresponding portfolio process. This typically leads to an underperformance of this strategy compared to the above constant one measured in terms of the expected final log-wealth. However, in the long run it easily outperforms the constant one (see Figure 7).

- If the initial guess of the constant strategy is far from the real value of b then the learning strategy clearly outperforms the constant one even if the chosen estimate for $Var(b)$ is large (see Figure 1).

The picture changes dramatically if we increase the time horizon to $T = 25$ (see Figures 4 and 5). Here, the learning process has grown up to (a mean of) 0.17, and as a consequence the (mean of the) learned wealth is visibly above the guessed wealth. If the variance is high, (pictures 6 and 7) then, until $t = 5$ the learned wealth is below the guessed one, but then due to the well-learned drift overtakes the guessed one and is consistently better in the end.

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