A stochastic control approach to portfolio problems with stochastic interest rates

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Abstract: We consider investment problems where an investor can invest in a savings account, stocks and bonds and tries to maximize her utility from terminal wealth. In contrast to the classical Merton problem we assume a stochastic interest rate. To solve the corresponding control problems it is necessary to prove a verification theorem without the usual Lipschitz assumptions.

Keywords: optimal portfolios, stochastic interest rate, verification theorem

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1 Introduction

The continuous-time portfolio problem has its origin in the pioneering work of Merton (1969, 1971). It is concerned with finding the optimal investment strategy of an investor. More precisely, the investor looks for an optimal decision on how many shares of which security she should hold at every time instant between now and a time horizon T to maximize her expected utility from wealth at the time horizon. In the classical Merton problem the investor can allocate her money into a riskless savings account and d different risky stocks. By describing the actions of the investor via the the portfolio process (i.e. the percentages of wealth invested in the different securities) Merton was able to reduce the portfolio problem to a control problem which could be solved by using standard stochastic control methodology.

A drawback of this approach, however, is the assumption of a deterministic interest rate.[1] Our main objective in the current paper is to overcome this restriction. We assume that the interest rate follows an Ito process and particularly consider the case of the Ho-Lee model and the Vasicek model for the short rate. Such problems are treated rarely in the literature.[2] Further, our theory will enable us to consider mixed bond and stock portfolio problems. We give explicit solutions for both the value functions and the optimal strategies in Section 2.

On the theoretical side, the introduction of stochastic interest rates into the portfolio problem has the consequence that the stochastic differential equation describing the wealth process does not satisfy the usual Lipschitz assumptions needed to apply standard verifcation theorems. However, due to the special structure of this equation, the wealth equation, we are able to prove a suitable verification result in the Appendix. This is possible as some assumptions of the standard verification results as e.g. given in Fleming/Soner (1993) can be weakened substantially via proving some special estimates.

2 Two Portfolio Problems

We consider an economy with $d+1$ assets which are continuously traded on a frictionless market. All traders are assumed to be price takers. The uncertainty is modelled by a probability space $(\Omega, \mathcal{F}, P)$. On this space an m-dimensional Brownian motion $\{(W(t), \mathcal{F}_t)\}_{t\geq 0}$ is defined where $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the Brownian filtration. One of the assets is a savings account following the differential equation

$$dB(t) = B(t)r(t)dt$$

with $B(0) = 1$. Here r denotes the short rate which can be interpreted as the annualized interest for the infinitesimal period $[t, t+dt]$.

In contrast to Merton's classical model[3] we assume a short rate modelled by the SDE

$$dr(t) = a(t)dt + b\,dW(t),$$

$t \in [0, T^*]$, $b > 0$, with initial data $r(0) = r_0$. As explicit examples we will consider the Ho-Lee model given by $a(t) = \tilde{a}(t) + b\zeta(t)$ and a Vasicek approach with $a(t) = \theta(t) - \alpha r(t) + b\zeta(t)$, $\alpha > 0$, respectively. The risk premium ζ is assumed to be

[1] The other main approach to optimal portfolios, the martingal method, plays no role in this paper. We refer to Korn (1997) for an introduction to it.

[2] For related problems see Klüppelberg/Korn(1998), Canestrelli/Pontini (1998) and Sørensen (1999).

[3] See Merton (1969, 1971, 1990), Fleming/Rishel (1975), pp. 160f, Duffie (1992), pp.145ff, Fleming/Soner (1993), pp. 174ff, Korn (1997), pp. 48ff.

deterministic and continuous which implies the progressive measurability of ζ. This assumption particularly guarantees that ζ is bounded on each compact interval. Furthermore let the initial forward rate curve $f^*(0,T)$, $0 \leq T \leq T^*$, be continuously differentiable which leads to $\tilde{a}(t) = f_T^*(0,t) + b^2 t$ and $\theta(t) = f_T^*(0,t) + \alpha f^*(0,t) + \frac{b^2}{2\alpha}(1 - e^{-2\alpha t})$.[4] The price processes of the remaining d assets which can be stocks and/or (discount) bonds are assumed to follow Ito processes of the form

$$dP_i(t) = P_i(t)\Big[\mu_i(t)dt + \sigma_i(t)dW(t)\Big]$$

with $P_i(0) = p_i > 0$ and where $\mu(\cdot)$ is $I\!R^d$-valued and $\sigma_i(\cdot)$ denotes the i th row of the $d \times m$-matrix $\sigma(\cdot)$.

We consider an investor who starts with an initial wealth $x_0 > 0$ at time $t = 0$. In the beginning this initial wealth is invested in the different assets and she is allowed to adjust her holdings continuously up to a fixed planning horizon T. Her investment behaviour is modelled by a portfolio process $\pi = (\pi_1, \ldots, \pi_d)$ which is progressively measurable (with respect to $\{\mathcal{F}_t\}_{t \geq 0}$). Here, $\pi_i(t)$, $i = 0, \ldots, d$, denotes the percentage of total wealth invested in the i-th asset at time t. Obviously, the percentage invested in the savings account is given by $1 - \pi'\underline{1}$ where $\underline{1} := (1, \ldots, 1)' \in I\!R^d$.

If we restrict our considerations to self-financing portfolio processes, her wealth process follows the stochastic differential equation (SDE)

$$(1) \qquad dX(t) = X(t)\Big[\big(\pi(t)'(\mu(t) - r(t) \cdot \underline{1}) + r(t)\big)dt + \pi(t)'\sigma(t)dW(t)\Big]$$

with $X(0) = x_0$ [5].

The wealth equation can be interpreted as a controlled SDE with the control being the portfolio process $\pi(\cdot)$. In this setting the investor chooses a portfolio process to maximize her utility. We assume that her preferences can be represented by the utility function $U(x) = x^\gamma$, $x \geq 0$, $0 < \gamma < 1$. Furthermore, the investor is only allowed to pick out a portfolio process which is admissible in the sense of Definition 3.1 and leads to a *positive* wealth process X^π. Now we are in the position to formulate her optimization problem:[6]

$$(2) \qquad \max_{\pi(\cdot) \in \mathcal{A}^*(0,x_0)} \mathrm{E}(X^\pi(T))^\gamma$$

with

$$\begin{aligned} dX^\pi(t) &= X^\pi(t)\Big[(\pi(t)'(\mu(t) - r(t) \cdot \underline{1}) + r(t))dt + \pi(t)\sigma(t)dW(t)\Big], \\ X^\pi(0) &= x_0 \end{aligned}$$

and

$$\mathcal{A}^*(0, x_0) := \Big\{\pi(\cdot) \in \mathcal{A}(0, x_0) \;:\; X^\pi(s) \geq 0 \;\; P-\text{f.s. for } s \in [0,T]\Big\}.$$

We emphasize that applying optimal control methods to this problem does not automatically yield a positive state process. However, Corollary 3.1 and the special form of the coefficients in the wealth equation (1) will indeed guarantee the positivity of $X^\pi(t)$. Therefore, we obtain $\mathcal{A}^*(0, x_0) = \mathcal{A}(0, x_0)$.

[4] See for example Musiela/Rutkowski (1997), pp. 323f.

[5] See for example Korn (1997), pp. 23f.

[6] Here $\mathcal{A}(0, x_0)$ denotes the set of all admissible controls corresponding to the initial condition $(0, x_0)$. See Definition 3.1 in the appendix.

2.1 A bond portfolio problem

We start in considering a portfolio problem where the investor can split up his wealth in a savings account and a (zero) bond with maturity $T_1 > T$. We assume that the asset price processes can be represented by the Ito processes

$$\begin{aligned} dB(t) &= B(t)r(t)dt, \\ dP(t,T_1) &= P(t,T_1)\Big[\underbrace{(r(t)+\zeta(t)\sigma(t))}_{=:\mu(t)}dt + \sigma(t)dW(t)\Big], \end{aligned}$$

where W is a one-dimensional Brownian motion. In the Ho-Lee and the Vasicek models the volatility of the bond is given by $\sigma(t) = -b(T_1 - t)$ and $\sigma(t) = \frac{b}{\alpha}(\exp(-\alpha(T_1-t))-1)$, respectively.[7] Let $\pi(t)$ be the percentage invested in the bond. This leads to a wealth equation of the form

$$\begin{aligned} (3) \qquad dX(t) &= X(t)\Big[(\pi(t)\mu(t) + (1-\pi(t))r(t))dt + \pi(t)\sigma(t)dW(t)\Big] \\ &= X(t)\Big[(\pi(t)\zeta(t)\sigma(t) + r(t))dt + \pi(t)\sigma(t)dW(t)\Big] \end{aligned}$$

with initial data $X(0) = x_0$.

As in contrast to the classical Merton problem, we assume a stochastic short rate, the drift coefficient includes the additional stochastic term $r(t)$. Thus, to solve the portfolio problem (2) by stochastic control methods we have to look at a two-dimensional state process $Y = (X, r)$. Note that the second component cannot be controlled via $\pi(\cdot)$. Using the notation of (16) in the appendix we get[8]

$$\begin{aligned} Y(t) &= (X(t), r(t))', \\ \Lambda(t,x,r,\pi) &= (x(\pi\zeta\sigma + r), a)', \\ \Sigma(t,x,r,\pi) &= (x\pi\sigma, b)', \\ \Sigma^*(t,x,r,\pi) &= \begin{pmatrix} x^2\pi^2\sigma^2 & bx\pi\sigma \\ bx\pi\sigma & b^2 \end{pmatrix}, \\ A^\pi G(t,x,r) &= G_t + 0,5(x^2\pi^2\sigma^2 G_{xx} + 2x\pi b\sigma G_{xr} + b^2 G_{rr}) \\ &\quad + x(\pi\zeta\sigma + r)G_x + aG_r. \end{aligned}$$

Hence, the following Hamilton-Jacobi-Bellman equation (HJB) has to be solved

$$\begin{aligned} \sup_{|\pi|\le\delta} A^\pi G(t,x,r) &= 0, \\ G(T,x,r) &= x^\gamma, \end{aligned}$$

where $\delta > 0$ will be specified later.

Note that due to the presence of the product rx in the above setting usual verification theorems which require Lipschitz conditions are not applicable to our situation as both the wealth process and the short rate are unbounded processes. We therefore give a suitable verifcation result (Corollary 3.2) in the appendix. This result then allows us to solve HJB with the usual three step procedure. By this, we would like to emphasize our opinion that the third step, verification of all assumptions of both Corollary 3.2 and those made to perform the following calculations, is an essential part of the solution.

[7] See for example Musiela/Rutkowski (1997), pp. 323ff.

[8] For simplicity we often neglect the functional dependencies with respect to t, x and r.

We start with the calculation of the optimal bond position $\pi(\cdot)$.

1st step: Assuming $G_{xx} < 0$ we get the following candidate for the optimal bond position

$$\pi^* = -\frac{\zeta}{\sigma}\frac{G_x}{xG_{xx}} - \frac{b}{\sigma}\frac{G_{xr}}{xG_{xx}}. \tag{4}$$

2nd step: Inserting $\pi^*(t,x,r;G)$ into HJB leads to the PDE

$$\begin{aligned} 0 \;=\;& G_tG_{xx} - 0,5\zeta^2G_x^2 - 0,5b^2G_{xr}^2 + 0,5b^2G_{rr}G_{xx} \\ & -b\zeta G_xG_{xr} + xrG_xG_{xx} + aG_rG_{xx} \end{aligned} \tag{5}$$

with the terminal condition $G(T,x,r) = x^\gamma$. Note that $\zeta = (\mu - r)/\sigma$.

The form of this condition recommends the following separation ansatz

$$G(t,x,r) = f(t,r) \cdot x^\gamma \quad \text{with } f(T,r) = 1 \text{ for all } r.$$

This leads to a second-order PDE for f of the form

$$\begin{aligned} 0 \;=\;& (\gamma-1)ff_t - 0,5b^2\gamma f_r^2 - 0,5\zeta^2\gamma f^2 + 0,5b^2(\gamma-1)ff_{rr} \\ & -b\zeta\gamma ff_r + r\gamma(\gamma-1)f^2 + a(\gamma-1)ff_r \end{aligned}$$

with terminal condition $f(T,r) = 1$. Using the ansatz

$$f(t,r) = g(t) \cdot \exp(\beta(t) \cdot r)$$

with terminal conditions $\beta(T) = 0$ and $g(T) = 1$ and simplification yields

$$\begin{aligned} 0 \;=\;& (\gamma-1)\cdot g' + (\gamma-1)\,(\gamma+\beta')\cdot rg \\ & -\left(0,5\zeta^2\gamma + 0,5b^2\beta^2 + b\zeta\gamma\beta\right)\cdot g + a(\gamma-1)\beta\cdot g. \end{aligned} \tag{6}$$

Our ansatz for f will only be meaningful, if we get an ordinary differential equation (ODE) for g which does not include the short rate r.

In the Ho-Lee model the drift a of the short rate is a function of t, whereas in the Vasicek model it is a function of t and r. Therefore we treat the two interest rate models separately.

Ho-Lee model: In our Ho-Lee setting PDE (6) has the form

$$\begin{aligned} 0 \;=\;& (\gamma-1)\cdot g' + (\gamma-1)\,(\gamma+\beta')\cdot rg \\ & + \underbrace{\left(-0,5\zeta^2\gamma - 0,5b^2\beta^2 - b\zeta\gamma\beta + a(\gamma-1)\beta\right)}_{=:h_1(t)}\cdot g. \end{aligned} \tag{7}$$

Since $a(t) = f_T^*(0,t) + b^2t + b\zeta(t)$ and ζ is assumed to be deterministic and continuous, h_1 is a continuous and deterministic function. Choosing $\beta(t) = \gamma(T-t)$ we infer from (7) the following first-order ODE for g

$$0 = (\gamma-1)\cdot g' + h_1(t)\cdot g$$

with $g(T) = 1$. Separation of variables leads to

$$g(t) = \exp\left(\tfrac{1}{1-\gamma}(H_1(t) - H_1(T))\right),$$

where H_1 is a primitive of h_1. Hence we obtain

$$G(t,x,r) = x^\gamma \cdot \exp\left(\tfrac{1}{1-\gamma}(H_1(t) - H_1(T)) + \gamma(T-t)r\right)$$

as a candidate for the value function. Inserting into (4) gives the corresponding control

$$\begin{aligned}\pi^*(t) &= \frac{1}{1-\gamma}\cdot\frac{\zeta(t)+b\beta(t)}{-\sigma(t)}\\ &= \frac{1}{1-\gamma}\cdot\frac{\zeta(t)+b(T-t)\gamma}{-b(T_1-t)}.\end{aligned}$$

Obviously, $\pi^*(\cdot)$ is continuous, deterministic and therefore bounded.

Vasicek model: With the Vasicek specification of a the PDE (6) has the following form

$$\begin{aligned}0 &= (\gamma-1)\cdot g' + \underbrace{(\gamma-1)(\beta'-\alpha\beta+\gamma)}_{(*)}\cdot rg\\ &\quad+\underbrace{(\theta(\gamma-1)\beta-b\zeta\beta-0,5b^2\beta^2-0,5\zeta^2\gamma)}_{=:h_2(t)}\cdot g.\end{aligned}$$

Our ansatz for f is only meaningful, if β can be calculated so that the factor $(*)$ becomes zero. As a result we have to solve an inhomogeneous ODE for β which has the following form

$$\beta'(t)=\alpha\beta(t)-\gamma$$

with $\beta(T)=0$ leading to

$$\beta(t)=\tfrac{\gamma}{\alpha}(1-\exp(\alpha(t-T))).$$

Choosing β as calculated we again get a first-order homogeneous ODE for g

$$0=(\gamma-1)\cdot g'+h_2(t)\cdot g$$

with $g(T)=1$. Hence

$$g(t)=\exp(\tfrac{1}{1-\gamma}(H_2(t)-H_2(T))),$$

where H_2 is a primitive of h_2. Therefore

$$G(t,x,r)=x^\gamma\cdot\exp\left(\tfrac{1}{1-\gamma}(H_2(t)-H_2(T))+\tfrac{\gamma}{\alpha}(1-\exp(\alpha(t-T)))r\right).$$

The corresponding control reads as follows

$$\begin{aligned}\pi^*(t) &= \frac{1}{1-\gamma}\cdot\frac{\zeta(t)+b\beta(t)}{\sigma(t)}\\ &= \frac{1}{1-\gamma}\cdot\frac{\zeta(t)+b\cdot\frac{\gamma}{\alpha}(1-\exp(\alpha(t-T)))}{\frac{b}{\alpha}(\exp(-\alpha(T_1-t))-1)}.\end{aligned}$$

Again $\pi^*(\cdot)$ is continuous, deterministic and therefore bounded.

In both cases one can choose δ in an appropriate way so that the optimal bond position fulfils the condition $\pi(\cdot)\le\delta$. Moreover the respective $\pi^*(\cdot)$ is of the form

$$\pi^*(t)=\underbrace{\frac{1}{1-\gamma}\cdot\frac{\zeta(t)}{\sigma(t)}}_{\text{Merton result}}-\underbrace{\frac{\gamma}{1-\gamma}\cdot\kappa(t)}_{\text{correction term}}$$

with $\kappa(t)=\frac{T-t}{T_1-t}$ in the Ho-Lee model and $\kappa(t)=\frac{1-e^{-\alpha(T-t)}}{1-e^{-\alpha(T_1-t)}}$ in the Vasicek model. The first term coincides with the classical optimal one in Merton (1969, 1971) when the coefficients are deterministic. The second term can be interpreted as a correction term which is positive and monotonously decreasing to zero up to the terminal date T. Thus, we first have a bigger, negative deviation from the classical result which vanishes at the time horizon. This correction results from the fact that the volatiltiy of the bond decreases as time goes by and hence becomes less risky. Moreover the correction term increases with the investor's risk aversion because the less risky savings account will become more attractive if her risk aversion increases.

3rd step: At first we justify our use of Corollary 3.2 although the state process $Y = (X, r)'$ is two-dimensional: Note that the short rate process does not include the control $\pi(\cdot)$. Therefore one can prove condition (i) and (iii) in Definition 3.1 independently of a specified control. Consider the SDE

$$dr(t) = a(t)dt + b\,dW(t) \tag{8}$$

of the short rate r with $r(0) = r_0$. The coefficients meet the growth and Lipschitz conditions of the existence and uniqueness theorem for SDE.[9] Hence (8) has a unique solution. Using a theorem of Krylov (1980, p. 85) we get

$$\mathrm{E}\Big(\max_{0 \le s \le T} |r(s)|^\rho \Big) < +\infty \tag{9}$$

with $\rho \in I\!N$. Therefore, independently of the control under consideration, the conditions (i) and (iii) are fulfiled by the second component of the state process Y. As a result we can treat our problem as if the state process only consists of X. Note that then the wealth equation is a linear controlled SDE.

We can apply Corollary 3.2 if we are able to prove the following assumptions:

1) $\pi^*(\cdot)$ is progressively measurable,

2) $\pi^*(\cdot)$ meets condition (ii) in definition 3.1,

3) $\pi^*(\cdot)$ meets condition (iii) in definition 3.1,

4) G is a $C^{1,2}$-solution of the HJB,

5) condition (27) is met,

Furthermore, the portfolio process has to lead to a positive wealth process, so

6) $X^{\pi^*} \ge 0$.

Proof of 1): The respective solution $\pi^*(\cdot)$ is continuous and deterministic, hence progressively measurable.

Proof of 2): Property (ii) of an admissible control is met, because the respective $\pi^*(\cdot)$ is bounded.

Proof of 3): By Corollary 3.1 the wealth equation (3) for $\pi^*(\cdot)$ has the solution

$$\begin{aligned} X^*(t) \;&=\; x_0 \exp\left(\int_0^t \pi^*(s)\zeta(s)\sigma(s) + r(s) - 0,5(\pi^*(s)\sigma(s))^2\, ds \right. \\ &\qquad \left. + \int_0^t \pi^*(s)\sigma(s)\, dW(s) \right). \end{aligned} \tag{10}$$

Note that (9) implies

$$\mathrm{E}\left(\left| \int_0^T r(s)\, ds \right| \right) \le T \cdot \mathrm{E}\Big(\max_{0 \le s \le T} |r(s)| \Big) < +\infty$$

[9]See Fleming/Soner (1993, pp. 397f).

and hence

$$\int_0^T r(s)\,ds < +\infty, \quad P - \text{f.s.}.$$

The other assumptions of Corollary 3.1 are obviously met.

With an appropriate constant $K > 0$ we obtain the following estimate. (Be aware of the fact that $\pi^*(\cdot)$, $\sigma(\cdot)$ and $\zeta(\cdot)$ are bounded and that $|uv| \leq u^2 + v^2$ for $u, v \in I\!R$.):

$$\begin{aligned}
(11) \quad X^*(t)^k &= x_0^k \cdot \exp\left(k \int_0^t \pi^*(s)\zeta(s)\sigma(s) + r(s) - 0,5(\pi^*(s)\sigma(s))^2\,ds \right. \\
&\qquad \left. + k \int_0^t \pi^*(s)\sigma(s)\,dW(s)\right) \\
&\leq K \cdot \exp\left(k \int_0^t r(s)\,ds + k \int_0^t \pi^*(s)\sigma(s)\,dW(s)\right) \\
&\leq K \cdot \exp\left(2k \int_0^t r(s)\,ds\right) + K \cdot \exp\left(2k \int_0^t \pi^*(s)\sigma(s)\,dW(s)\right).
\end{aligned}$$

Now consider the integral $\int_0^t r(s)\,ds$. With the form of the short rate process, in the *Ho-Lee model* we get [10]

$$\begin{aligned}
(12) \quad \int_0^t r(s)\,ds &= \int_0^t \left(r_0 + \int_0^s a(u)\,du + \int_0^s b\,dW(u)\right)\,ds \\
&= r_0 t + \int_0^t \int_0^s a(u)\,du ds + b \int_0^t \int_0^s dW(u) ds \\
&= \ldots + b \int_0^t (t-u)\,dW(u).
\end{aligned}$$

The dots represent a term which is deterministic and bounded on $[0,T]$. Using the variation of constants formula for SDE[11] in the *Vasicek model* we obtain

$$r(t) = e^{-\alpha t}\left(r_0 + \int_0^t e^{\alpha u}\Big(\theta(u) + b\zeta(u)\Big)\,du + \int_0^t b e^{\alpha u}\,dW(u)\right).$$

Hence

$$\begin{aligned}
(13) \quad \int_0^t r(s)\,ds &= \int_0^t e^{-\alpha s}\left(r_0 + \int_0^s e^{\alpha u}\Big(\theta(u) + b\zeta(u)\Big)\,du\right)\,ds \\
&\quad + b \int_0^t \int_0^s e^{\alpha(u-s)}\,dW(u) ds \\
&= \ldots + b \int_0^t \int_u^t e^{\alpha(u-s)}\,ds dW(u).
\end{aligned}$$

The dots represent a term which is deterministic and bounded on $[0,T]$.

In both cases the problem is reduced to find an estimate for terms of the form $\exp(\int_0^t h(s)\,dW(s))$ with a deterministic and bounded function h, namely

$$\exp\left(\int_0^t h(s)\,dW(s)\right) = \underbrace{\exp\left(\int_0^t 0,5h^2(s)\,ds\right)}_{=const.} \cdot \underbrace{\exp\left(-\int_0^t 0,5h^2(s)\,ds + \int_0^t h(s)\,dW(s)\right)}_{=:Z(t)}$$

[10] See Ikeda/Watanabe (1981, pp. 117ff) for the interchange of Lebesgue and Ito integral.

[11] See Korn (1997), p. 313.

with

$$\begin{aligned} dZ(t) &= Z(t)h(t)dW(t), \\ Z(0) &= 1. \end{aligned}$$

Using Krylov (1980, p. 85) we find that

$$\mathrm{E}\Big(\max_{0\le t\le T} Z(t)\Big) < +\infty.$$

Because of (11) and (12) or (13), respectively, $(X^*)^k$ can be estimated by processes of the same form as Z in both models. Therefore property 3) is proved.

Proof of 4): Since the condition $G_{xx} < 0$ is met in both models, G is obviously a $C^{1,2}$-solution of the HJB.

Proof of 5): It is sufficent to prove that (27) is met by all *bounded* admissible bond positions $\pi(\cdot)$. Then the respective $\pi^*(\cdot)$ dominates all admissible bond positions. Let $(t', x', r') \in [0,T] \times I\!R^2_+ := \{y \in I\!R^2 : y > 0\}$ and $t' \le t \le T$. We consider the models separately.

Ho-Lee model: The candidate for the value function is

$$G(t,x,r) = x^\gamma \cdot \exp\left(\tfrac{1}{1-\gamma}(H_1(t) - H_1(T)) + \gamma(T-t)r\right),$$

where H_1 denotes a deterministic function which is continuously differentiable. Let K_i, $i = 1,2,3$, be appropriate constants. As H_1, π, ζ, σ and a are bounded functions, an application of Ito's formula yields

$$\begin{aligned}
&G(t, X(t), r(t)) \\
&= X(t)^\gamma \cdot \exp\left(\tfrac{1}{1-\gamma}(H_1(t) - H_1(T)) + \gamma(T-t)r(t)\right) \\
&= (x')^\gamma \exp\left(\gamma \int_{t'}^t \pi(s)\zeta(s)\sigma(s) + r(s) - 0,5(\pi(s)\sigma(s))^2\, ds \right.\\
&\qquad\qquad\qquad \left. +\gamma \int_{t'}^t \pi(s)\sigma(s)\, dW(s)\right) \\
&\quad \cdot \exp\left(\tfrac{1}{1-\gamma}(H_1(t) - H_1(T))\right) \cdot \exp\left(r(t)\gamma(T-t)\right) \\
&\le K_1 \cdot \exp\left(\gamma \int_{t'}^t r(s)\, ds + \gamma \int_{t'}^t \pi(s)\sigma(s)\, dW(s)\right) \cdot \exp(\gamma T r(t)) \cdot \exp(-\gamma t r(t)) \\
&= K_1 \cdot \exp\left(\gamma \int_{t'}^t r(s)\, ds + \gamma \int_{t'}^t \pi(s)\sigma(s)\, dW(s)\right) \cdot \exp\left(\gamma T \int_{t'}^t dr(s)\right) \\
&\quad \cdot \exp\left(-\gamma \int_{t'}^t s\, dr(s) - \gamma \int_{t'}^t r(s)\, ds\right) \\
&= K_1 \cdot \exp\left(\gamma \int_{t'}^t \pi(s)\sigma(s)\, dW(s)\right) \cdot \exp\left(\gamma \int_{t'}^t (T-s)(a(s)\, ds + b\, dW(s))\right) \\
&\le K_2 \cdot \exp\left(\gamma \int_{t'}^t \pi(s)\sigma(s) + b(T-s)\, dW(s)\right) \\
&\le K_3 \cdot \exp\left(\gamma \int_{t'}^t \pi(s)\sigma(s) + b(T-s) dW(s) \right.\\
&\qquad\qquad\qquad \left. -0,5\gamma^2 \int_{t'}^t \Big(\pi(s)\sigma(s) + b(T-s)\Big)^2 ds\right) \\
&=: K_3 \cdot Z(t),
\end{aligned}$$

where Z is the unique solution of

$$dZ(t) = Z(t)\Big(\gamma(\pi(t)\sigma(t) + b(T-t))\Big)dW(t) \quad \text{mit} \quad Z(t') = 1.$$

Using Krylov (1980, p. 85) we arrive at

$$\mathrm{E}\Big(\sup_{t\in[t',T]} |G(t,X(t),r(t))|^2\Big) \leq K_3 \cdot \mathrm{E}\Big(\sup_{t\in[t',T]} |Z(t)|^2\Big) < \infty.$$

Hence we have just proved (27) in the Ho-Lee model.

Vasicek model: Our candidate for the value function is

$$G(t,x,r) = x^\gamma \cdot \exp\left(\tfrac{1}{1-\gamma}(H_2(t) - H_2(T)) + \tfrac{\gamma}{\alpha}(1-\exp(\alpha(t-T)))r\right),$$

where H_2 is a continuously differentiable and deterministic function. With appropriate constants K_i, $i = 1,\ldots,6$, we find that

$$\begin{aligned}
&G(t,X(t),r(t))\\
&= X(t)^\gamma \cdot \exp\left(\tfrac{1}{1-\gamma}(H_2(t) - H_2(T)) + \tfrac{\gamma}{\alpha}(1-\exp(\alpha(t-T)))r(t)\right)\\
&\leq K_1 \cdot X(t)^\gamma \cdot \exp\left(\tfrac{\gamma}{\alpha}(1-\exp(\alpha(t-T)))r(t)\right)\\
&\leq K_2 \cdot \exp\left(\gamma \int_{t'}^t \pi(s)\zeta(s)\sigma(s) + r(s) - 0,5(\pi(s)\sigma(s))^2\,ds\right.\\
&\qquad\qquad \left. + \gamma\int_{t'}^t \pi(s)\sigma(s)\,dW(s)\right) \cdot \exp\left(\tfrac{\gamma}{\alpha}(1-\exp(\alpha(t-T)))\cdot r(t)\right)\\
&\leq K_3 \cdot \exp\left(\gamma\int_{t'}^t r(s)\,ds + \gamma\int_{t'}^t \pi(s)\sigma(s)\,dW(s)\right) \cdot \exp\left(\tfrac{\gamma}{\alpha}r(t)\right)\\
&\quad \cdot \exp\left(-\tfrac{\gamma}{\alpha}\exp(\alpha(t-T))\cdot r(t)\right).
\end{aligned}$$

With the definition $f^h(t,r) := \exp(\alpha(t-T))\cdot r$ an application of Ito's formula yields

$$\begin{aligned}
&f^h(t,r(t))\\
&= f^h(t',r') + \int_{t'}^t \alpha\exp(\alpha(s-T))r(s)\,ds + \int_{t'}^t \exp(\alpha(s-T))\,dr(s)\\
&= f^h(t',r') + \int_{t'}^t \exp(\alpha(s-T))\cdot(\theta(s) + b\zeta(s))\,ds + \int_{t'}^t b\,\exp(\alpha(s-T))\,dW(s).
\end{aligned}$$

Hence, by virtue of the stochastic integral equation of the short rate, we have

$$\begin{aligned}
&G(t,X(t),r(t))\\
&\qquad \leq K_4 \cdot \exp\left(\gamma\int_{t'}^t r(s)\,ds + \gamma\int_{t'}^t \pi(s)\sigma(s)\,dW(s)\right)\cdot\exp\left(\tfrac{\gamma}{\alpha}r(t)\right)\\
&\qquad\quad \cdot\exp\left(-\tfrac{\gamma}{\alpha}\int_{t'}^t b\,\exp(\alpha(s-T))\,dW(s)\right)\\
&\qquad = K_4\cdot\exp\left(\gamma\int_{t'}^t r(s)\,ds + \gamma\int_{t'}^t \pi(s)\sigma(s)\,dW(s)\right)\\
&\qquad\quad \cdot\exp\left(\tfrac{\gamma}{\alpha}r' + \tfrac{\gamma}{\alpha}\int_{t'}^t(\theta(s) - \alpha r(s) + b\zeta(s))\,ds + \tfrac{\gamma}{\alpha}\int_{t'}^t b\,dW(s)\right)\\
&\qquad\quad \cdot\exp\left(-\tfrac{\gamma}{\alpha}\int_{t'}^t b\,\exp(\alpha(s-T))\,dW(s)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq K_5 \cdot \exp\left(\int_{t'}^{t} \gamma\pi(s)\sigma(s) + \tfrac{\gamma}{\alpha} b\Big(1-\exp(\alpha(s-T))\Big)\, dW(s)\right) \\
&\leq K_6 \cdot \exp\left(\int_{t'}^{t} \gamma\pi(s)\sigma(s) + \tfrac{\gamma}{\alpha} b\Big(1-\exp(\alpha(s-T))\Big)\, dW(s)\right. \\
&\qquad \left. - \int_{t'}^{t} 0,5\left[\gamma\pi(s)\sigma(s) + \tfrac{\gamma}{\alpha} b\Big(1-\exp(\alpha(s-T))\Big)\right]^2 ds\right) \\
&=: K_6 \cdot \tilde{Z}(t).
\end{aligned}
$$

Since the process $\tilde{Z}$ has the same properties as Z in the Ho-Lee model an analogous argument leads to (27).

Proof of 6): By virtue of (10), we have $X^* \geq 0$.

The following theorem summerizes our results.

Theorem 2.1 (Bond portfolio problem) *The optimal portfolio processes in the above bond portfolio problems are given by*

$$\pi^*(t) = \frac{1}{1-\gamma} \cdot \frac{\zeta(t)}{\sigma(t)} - \frac{\gamma}{1-\gamma} \cdot \kappa(t)$$

with

a) Ho-Lee case: $\kappa(t) = \frac{T-t}{T_1-t}$,

b) Vasicek case: $\kappa(t) = \frac{1-e^{-\alpha(T-t)}}{1-e^{-\alpha(T_1-t)}}$.

2.2 A mixed stock and bond portfolio problem

In this subsection we assume that the investor can put his money on a savings account, in a stock or in a bond with maturity $T_1 > T$. The dynamics of these assets are given by

$$
\begin{aligned}
dB(t) &= B(t)r(t)dt, \\
dS(t) &= S(t)\Big[\mu_S(t)dt + \sigma_S(t)dW_S(t) + \sigma_{SB}(t)dW_B(t)\Big], \\
dP(t) &= P(t)\Big[\underbrace{(r(t) + \zeta_B(t)\sigma_B(t))}_{=:\mu_B(t)}dt + \sigma_B(t)dW_B(t)\Big],
\end{aligned}
$$

where (W_S, W_B) is a two-dimensional Brownian motion and where, for ease of notation, we write $P(t)$ instead of $P(t, T_1)$. In our model the stock price depends on two risk factors: The first factor W_S contains the specific risk of the stock, and the second W_B comes from the stochastic interest rate model.

In Merton's portfolio problem we can split up the (deterministic) drift μ_S of the stock into a liquidity premium (LP) and an excess return, which should be interpreted as risk premium (RP) in this context:[12]

$$\mu_S = \underbrace{r}_{\text{LP}} + \underbrace{\mu_S - r}_{\text{RP}}.$$

The drift of the stock S under consideration can also be

$$\mu_S(t) = r(t) + \underbrace{\mu_S(t) - r(t)}_{=:\lambda_S(t)},$$

[12]There is no uniform use of the words excess return, risk premium and market price of risk. Apart from the above interpretation of the drift, throughout the paper we denote $\lambda = \mu - r$ as excess return, $\frac{\lambda}{\sigma}$ as risk premium and $\frac{\lambda}{\sigma^2}$ as market price of risk.

where λ_S denotes the risk premium of the stock

In the following, we assume that the excess return $\lambda_S(\cdot)$ of the stock is deterministic and continuous. This implies that $\lambda_S(\cdot)$ is progressively measurable and bounded on $[0,T]$. Furthermore, assume that the coefficients $\sigma_S(\cdot)$, $\sigma_{SB}(\cdot)$ and $\sigma_B(\cdot)$ are deterministic and continuous. In addition, let $\sigma_S(\cdot)$ and $\sigma_B(\cdot)$ be bounded away from zero.

As before we consider both a Ho-Lee and a Vasicek model:

$$dr(t) = a(t)dt + bdW_B(t)$$

with $a(t) = \tilde{a}(t) + b\zeta(t)$ in the Ho-Lee model and $a(t) = \theta(t) - \alpha r(t) + b\zeta(t)$ in the Vasicek model.

Moreover we have $\sigma_B(t) = -b(T_1 - t)$ in the Ho-Lee model and $\sigma_B(t) = \frac{b}{\alpha}(\exp(-\alpha(T_1 - t)) - 1)$ in the Vasicek model.

In this framework the wealth equation (1) has the following form

$$\begin{aligned} dX(t) &= X(t)\Big[(\pi_S(t)\lambda_S(t) + \pi_B(t)\lambda_B(t) + r(t))dt \\ &\quad + \pi_S(t)\sigma_S(t)dW_S(t) + (\pi_S(t)\sigma_{SB}(t) + \pi_B(t)\sigma_B(t))dW_B(t)\Big], \end{aligned}$$

where $\lambda_B(t) := \mu_B(t) - r(t)$ und $\pi := (\pi_S, \pi_B)$.

Using the notations of (16) in the appendix we have

$$\begin{aligned} Y(t) &= (X(t), r(t))', \\ \Lambda(t,x,r,\pi) &= (x(\pi_S\lambda_S + \pi_B\lambda_B + r), a)', \\ \Sigma(t,x,r,\pi) &= \begin{pmatrix} x\pi_S\sigma_S & x(\pi_S\sigma_{SB} + \pi_B\sigma_B) \\ 0 & b \end{pmatrix}, \\ \Sigma^*(t,x,r,\pi) &= \begin{pmatrix} x^2(\pi_S^2\sigma_S^2 + (\pi_S\sigma_{SB} + \pi_B\sigma_B)^2) & bx(\pi_S\sigma_{SB} + \pi_B\sigma_B) \\ bx(\pi_S\sigma_{SB} + \pi_B\sigma_B) & b^2 \end{pmatrix}, \\ A^\pi G(t,x,r) &= G_t + 0,5x^2(\pi_S^2\sigma_S^2 + (\pi_S\sigma_{SB} + \pi_B\sigma_B)^2)G_{xx} + 0,5b^2G_{rr} \\ &\quad + bx(\pi_S\sigma_{SB} + \pi_B\sigma_B)G_{xr} + x(\pi_S\lambda_S + \pi_B\lambda_B + r)G_x + aG_r. \end{aligned}$$

Hence we have to solve the following HJB

$$\begin{aligned} \sup_{|\pi|\le\delta} A^\pi G(t,x,r) &= 0, \\ G(T,x,r) &= x^\gamma. \end{aligned}$$

This will again be done by the 3-step-algorithm.

1st step: Assuming $G_{xx} < 0$ we calculate the canditates for the optimal portfolio positions

$$\pi_S^* = -\underbrace{(\eta_S - \tfrac{\sigma_{SB}}{\sigma_B}\eta_{BS})}_{=:\hat{\eta}_S} \cdot \frac{G_x}{xG_{xx}}, \tag{14}$$

$$\pi_B^* = -\underbrace{\Big((1 + \tfrac{\sigma_{SB}^2}{\sigma_S^2})\eta_B - \tfrac{\sigma_{SB}}{\sigma_B}\eta_S\Big)}_{=:\hat{\eta}_B} \cdot \frac{G_x}{xG_{xx}} - \frac{b}{\sigma_B} \cdot \frac{G_{xr}}{xG_{xx}} \tag{15}$$

with $\eta_S := \lambda_S/\sigma_S^2$, $\eta_B := \lambda_B/\sigma_B^2$ and $\eta_{BS} := \lambda_B/\sigma_S^2$.

2nd step: Inserting $\pi_S^*(t,x,r;G)$ and $\pi_B^*(t,x,r;G)$ in the HJB yields the PDE

$$\begin{aligned} 0 &= G_t G_{xx} + \underbrace{(0,5\sigma_S^2\hat{\eta}_S^2 + 0,5(\sigma_{SB}\hat{\eta}_S + \sigma_B\hat{\eta}_B)^2 - \lambda_S\hat{\eta}_S - \lambda_B\hat{\eta}_B)}_{=:\tilde{\zeta}(t)} G_x^2 \\ & \quad -0,5b^2G_{xr}^2 + 0,5b^2G_{rr}G_{xx} - b\tfrac{\lambda_B}{\sigma_B}G_xG_{xr} + xrG_xG_{xx} + aG_rG_{xx} \end{aligned}$$

with $G(T,x,r) = x^\gamma$. This PDE is of the same form as the corresponding PDE (5) above.[13] Note that $\tilde{\zeta}$, in analogy to ζ in (5), is a continuous and deterministic function. Therefore in the Ho-Lee model we get

$$G(t,x,r) = x^\gamma \cdot \exp\left(\tfrac{1}{1-\gamma}(H_3(t) - H_3(T)) + \gamma(T-t)r\right)$$

and in the Vasicek model

$$G(t,x,r) = x^\gamma \cdot \exp\left(\tfrac{1}{1-\gamma}(H_4(t) - H_4(T)) + \tfrac{\gamma}{\alpha}(1-\exp(\alpha(t-T)))r\right),$$

with appropriate continuously differentiable functions H_3 and H_4, respectively. Insertion into (14) and (15) yields in *both* models for the optimal stock and bond position

$$\begin{aligned} \pi_S^*(t) &= \frac{1}{1-\gamma} \cdot \left(\eta_S(t) - \tfrac{\sigma_{SB}(t)}{\sigma_B(t)}\eta_{BS}(t)\right) \\ &= \frac{1}{1-\gamma} \cdot \hat{\eta}_S(t), \\ \pi_B^*(t) &= \frac{1}{1-\gamma} \cdot \left(\left(1 + \tfrac{\sigma_{SB}^2(t)}{\sigma_S^2(t)}\right)\eta_B(t) - \tfrac{\sigma_{SB}(t)}{\sigma_B(t)}\eta_S(t) - \gamma \cdot \kappa(t)\right) \\ &= \frac{1}{1-\gamma} \cdot (\hat{\eta}_B(t) - \gamma \cdot \kappa(t)), \end{aligned}$$

where $\kappa(t) = \frac{T-t}{T_1-t}$ in the *Ho-Lee model* and $\kappa(t) = \frac{1-e^{-\alpha(T-t)}}{1-e^{-\alpha(T_1-t)}}$ in the *Vasicek model.* Both positions are continuous and deterministic processes, hence bounded.

3rd step: With the same argument as in subsection 2.1 we can apply corollary 3.2. Therefore in both models we must check the following assumptions

1) $\pi^*(\cdot)$ is progressively measurable,

2) $\pi^*(\cdot)$ meets condition (ii) in definition 3.1,

3) $\pi^*(\cdot)$ meets condition (iii) in definition 3.1,

4) G is a $C^{1,2}$-solution of the HJB,

5) condition (27) is met,

6) $X^{\pi^*} \geq 0$.

Note that $\pi^* := (\pi_S^*, \pi_B^*)'$.

Conditions 1) and 2) are met, because in both models $\pi^*(\cdot)$ is a continuous and deterministic process. Obviously 4) is fulfiled. Condition 6) is met since variation of constants leads to

$$\begin{aligned} X(t) = \\ & x_0 \exp\Bigg(\int_0^t \pi_S(s)\lambda_S(s) + \pi_B(s)\lambda_B(s) + r(s) \\ & \qquad - 0,5\Big((\pi_S(s)\sigma_S(s))^2 + (\pi_S(s)\sigma_{SB}(s) + \pi_B(s)\sigma_B(s))^2\Big)\, ds \\ & \qquad + \int_0^t \pi_S(s)\sigma_S(s)\, dW_S(s) + \int_0^t \pi_S(s)\sigma_{SB}(s) + \pi_B(s)\sigma_B(s)\, dW_B(s)\Bigg), \end{aligned}$$

[13] One will obtain the PDE (5), if $\lambda_S \equiv 0$, $\sigma_S \equiv 0$ and $\sigma_{SB} \equiv 0$.

for a admissable control $\pi(\cdot)$. Furthermore, since the wealth process has the same properties as in subsection 2.1 we can prove 3) and 5) using the analogous arguments.

The following theorem summerizes our results.

Theorem 2.2 (Mixed portfolio problem) *The optimal portfolio processes in the above mixed portfolio problem are given by*

$$\begin{aligned} \pi_S^*(t) &= \frac{1}{1-\gamma} \cdot \Big(\underbrace{\eta_S(t) - \tfrac{\sigma_{SB}(t)}{\sigma_B(t)} \eta_{BS}(t)}_{=:\hat{\eta}_S} \Big), && \textit{(stock)} \\ \pi_B^*(t) &= \frac{1}{1-\gamma} \cdot \Big(\underbrace{\Big(1 + \tfrac{\sigma_{SB}^2(t)}{\sigma_S^2(t)}\Big) \eta_B(t) - \tfrac{\sigma_{SB}(t)}{\sigma_B(t)} \eta_S(t)}_{=:\hat{\eta}_B} - \gamma \cdot \kappa(t) \Big) && \textit{(bond)} \end{aligned}$$

with

a) Ho-Lee case: $\kappa(t) = \frac{T-t}{T_1-t}$,

b) Vasicek case: $\kappa(t) = \frac{1-e^{-\alpha(T-t)}}{1-e^{-\alpha(T_1-t)}}$.

Considering the optimal positions the analogy to the pure bond problem becomes clear: The variables $\hat{\eta}_S$ and $\hat{\eta}_B$ can be interpreted as modified market prices of risk, where both are weighted differences of η_S and η_{BS} or η_B and η_S, respectively. In the optimal stock position the market price of risk of the stock is corrected by η_{BS}, which stands for the market price of risk of the bond with respect to the stock.

Similarly, the market price of risk of the bond contains a correction of the optimal bond position by the market price of risk of the stock. Both these corrections are plausible ones as an increase of the market price of risk of the bond makes stock investment less attractive and vice versa. Apart from this remark the interpretation of the bond part as given in Section 2.1 remains valid.

Furthermore, we will get the optimal bond position of subsection 2.1 if we choose $\sigma_S \equiv 0$ and $\sigma_{SB} \equiv 0$ in $\pi_B(\cdot)$.

3 Appendix

In this appendix we will present the technical results and details which enabled us to solve the foregoing portfolio problems by stochastic control methods. Let therefore be $(\Omega, \mathcal{F}, P)$ a complete probability space. Assume that on this space an m-dimensional Brownian motion $\{(W(t), \mathcal{F}_t)\}_{t\in[0,\infty)}$ is defined with $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ being the Brownian filtration. All adapted or progressively measurable processes are adapted or progressively measurable with respect to the Brownian filtration. Let further $|\cdot|$ denote the Euclidean norm or the operator norm, respectively.

As usual we will look at a state process given by a controlled SDE of the form

$$dY(t) = \Lambda(t, Y(t), u(t))dt + \Sigma(t, Y(t), u(t))dW(t) \tag{16}$$

with initial value of $Y(t_0) = y_0$ and a d-dimensional control process $u(\cdot)$. Let $[t_0, t_1]$ with $0 \le t_0 < t_1 < \infty$ be the relevant time intervall. A control strategy $u(\cdot)$ (for short: control) is a progressively measurable process with $u(t) \in U$ for all $t \in [t_0, t_1]$ where the set $U \subset I\!R^d$, $d \in I\!N$, is assumed to be closed. Further let

$Q_0 := [t_0, t_1) \times I\!R^n$, $n \in I\!N$. The coefficient functions

$$\begin{aligned} \Lambda &: \bar{Q}_0 \times U \to I\!R^n, \\ \Sigma &: \bar{Q}_0 \times U \to I\!R^{n,m}, \end{aligned}$$

$m \in I\!N$, are all assumed to be continuous. Further, for all $v \in U$ let $\Lambda(\cdot,\cdot,v)$ and $\Sigma(\cdot,\cdot,v)$ be in $C^1(\bar{Q}_0)$. We then define

Definition 3.1 (Admissible control) *A control $\{(u(t), \mathcal{F}_t)\}_{t\in[t_0,t_1]}$ will be called admissible* [14] *if*

(i) for all $y_0 \in I\!R^n$ the corresponding controlled SDE (16) with initial condition $Y(t_0) = y_0$ admits a pathwise unique solution $\{Y^u(t)\}_{t\in[t_0,t_1]}$,

(ii) for all $k \in I\!N$ the integrability condition

$$\mathrm{E}\left(\int_{t_0}^{t_1} |u(s)|^k \, ds\right) < \infty$$

is satisfied and

(iii) the corresponding state process Y^u satisfies

$$\mathrm{E}^{t_0,y_0}\left(\sup_{t\in[t_0,t_1]} |Y^u(t)|^k\right) < \infty.$$

Let $\mathcal{A}(t_0, y_0)$ denote the set of all admissible controls corresponding to the initial condition $(t_0, y_0) \in Q$.

In the following the above definition will prove to be extremely useful when we have to overcome some technical difficulties which have their origin in the fact that the wealth equation does not satisfy the usual Lipschitz conditions needed in the standard verification theorems of stochastic control.

To ensure existence and uniqueness of the solution of the controlled SDE (16) one typically requires the following Lipschitz and growth conditions for the coefficient functions which imply that controls with property (ii) are already admissible (i.e. they also satisfy properties (i) and (iii)).[15] With a constant $C > 0$ these conditions are:

$$\begin{aligned} &(17) & |\Lambda_t| + |\Lambda_y| &\leq C, \\ & & |\Sigma_t| + |\Sigma_y| &\leq C, \\ &(18) & |\Lambda(s,y,v)| &\leq C(1 + |y| + |v|), \\ & & |\Sigma(s,y,v)| &\leq C(1 + |y| + |v|) \end{aligned}$$

for all $s \in [t_0, t_1]$, $y \in I\!R$ and $v \in U$.

Typically, in our applications the conditions (17) and (18) will not be satisfied. On the other hand we only have to deal with linear controlled SDEs. This will imply that requirement (ii) on an admissible control already ensures requirement (i), too:

[14] This definition is more restrictive than the usual one as e.g. given in Fleming/Rishel (1975, p. 156). However, due to the special form of our control problem all (optimal) controls in this paper will satisfy the more restrictive requirements of our definition.

[15] See Fleming/Soner (1993), p. 398.

Corollary 3.1 (Variation of constants) *Let* $(t_0, y_0) \in Q$ *and let* $A_1^{(j)}$, $j = 1, \ldots,$ d, A_2, $B_1^{(i,j)}$, $i = 1, \ldots m$, $j = 1, \ldots d$, $B_2^{(i)}$, $i = 1, \ldots m$ *be progressively measurable real valued processes satisfying the integrability conditions*

$$\begin{aligned}
\int_{t_0}^{t_1} |A_2(s)|\, ds &< \infty \quad P\text{-f.s.},\ t \geq 0, \\
\int_{t_0}^{t_1} \left(\sum_{j=1}^{d} A_1^{(j)}(s)^2 + \sum_{i=1}^{m} B_2^{(i)}(s)^2 \right) ds &< \infty \quad P\text{-f.s.},\ t \geq 0, \\
\int_{t_0}^{t_1} \left(\sum_{i=1}^{m} \sum_{j=1}^{d} B_1^{(i,j)}(s)^4 \right) ds &< \infty \quad P\text{-f.s.},\ t \geq 0.
\end{aligned}$$

Further, let $u(\cdot)$ *be a control with property (ii) of Definition 3.1. Then the linear controlled SDE*

$$dY^u(t) = Y^u(t) \Big[(A_1(t)'u(t) + A_2(t))dt + (B_1(t)u(t) + B_2(t))'dW(t) \Big] \tag{19}$$

admits the Lebesgue $\bigotimes P$ *unique solution*

$$\begin{aligned}
Y^u(t) &= y_0 \cdot \exp\left(\int_{t_0}^{t_1} \left(A_1(s)'u(s) + A_2(s) - 0,5|B_1(s)u(s) + B_2(s)|^2 \right) ds \right. \\
&\qquad \left. + \int_{t_0}^{t_1} \Big(B_1(s)u(s) + B_2(s) \Big)' dW(s) \right).
\end{aligned}$$

If we only consider bounded admissible controls then the following conditions are sufficient:

$$\begin{aligned}
\int_{t_0}^{t_1} \left(\sum_{j=1}^{d} |A_1^{(j)}(s)| + |A_2(s)| \right) ds &< \infty \quad P\text{-f.s.},\ t \geq 0, \\
\int_{t_0}^{t_1} \left(\sum_{i=1}^{m} \sum_{j=1}^{d} B_1^{(i,j)}(s)^2 + \sum_{i=1}^{m} B_2^{(i)}(s)^2 \right) ds &< \infty \quad P\text{-f.s.},\ t \geq 0.
\end{aligned}$$

Proof of Corollary 3.1: The integrability assumptions together with property (ii) of an admissible control imply the requirements of the variation of constants formula given in Korn (1997). Applying it implies all assertions of the corollary. □

Consequently, for our applications it will be enough to verify properties (ii) and (iii) to obtain admissibility of a control. From now on, controlled SDEs (19) with coefficients satisfying the conditions of Corollary (3.1) will be referred to as linear controlled SDEs.

We will now formulate a standard verifcation theorem and afterwards derive a version suitable for our applications by modifying the relevant parts of the proof of the standard theorem. Therefore, we look at the following setting: Let $\mathcal{O} \subset I\!R^n$ be an open subset of $I\!R^n$. In the case of $\mathcal{O} \neq I\!R^n$ we additionally assume that its boundary $\partial\mathcal{O}$ is a compact $(n-1)$-dimensional C^3-manifold. In analogy to Q_0 we define $Q := [t_0, t_1) \times \mathcal{O}$. Further, let

$$\tau := \inf\{t \in [t_0, t_1] : (t, Y(t)) \notin Q\}$$

denote the exit time of Y from $\mathcal{O}$. Hence, we have

$$(\tau, Y(\tau)) \in \partial^* Q := ([t_0, t_1) \times \partial\mathcal{O}) \cup (\{t_1\} \times \bar{\mathcal{O}}).$$

We now consider continuous, real valued functions L and Ψ that satisfy the polynomial growth conditions

$$\begin{aligned}
(20) \qquad |L(t,y,v)| &\leq C(1+|y|^k+|v|^k), \\
(21) \qquad |\Psi(t,y)| &\leq C(1+|y|^k)
\end{aligned}$$

on $\bar{Q}\times U$ and $\bar{Q}$ for suitable constants $k \in \mathbb{N}$ and $C > 0$. Here, L and Ψ model the running and the terminal utility resulting from the control and the position of the controlled process, respectively. It will be our goal to determine an admissible control $u(\cdot)$ such that for each initial value (t_0, y_0) the utility functional

$$J(t_0,y_0;u) := \mathrm{E}^{t_0,y_0}\left(\int_{t_0}^{\tau} L(s,Y^u(s),u(s))\,dt + \Psi(\tau,Y^u(\tau))\right)$$

will be maximised, i.e. we want to solve $\max\limits_{u\in\mathcal{A}(t_0,y_0)} J(t_0,y_0;u)$.

Therefore, define the value function

$$V(t,y) := \sup_{u\in\mathcal{A}(t,y)} J(t,y;u), \quad (t,y)\in Q.$$

For each function $G \in C^{1,2}(Q)$ and $(t,y)\in Q$, $v\in U$, we consider the following differential operator

$$A^vG(t,y) := G_t(t,y) + 0,5\sum_{i,j=1}^{n}\Sigma^*_{ij}(t,y,v)\cdot G_{y_iy_j}(t,y) + \sum_{i=1}^{n}\Lambda_i(t,y,v)\cdot G_{y_i}(t,y)$$

with $\Sigma^* := \Sigma\Sigma'$. Then, we have:[16]

Theorem 3.1 (Verification theorem) *Let the conditions (17) and (18) on the coefficient functions of the controlled SDE (16) be satisfied. Further assume conditions (20) and (21). Let G be a function with the following properties:*

a) We have:

$$\begin{aligned}
(22) \qquad G &\in C^{1,2}(Q)\cap C(\bar{Q}), \\
(23) \qquad |G(t,y)| &\leq K(1+|y|^k)
\end{aligned}$$

for suitable $K > 0$ and $k \in \mathbb{N}$.

b) G solves the Hamilton-Jacobi-Bellman equation(HJB):

$$\begin{aligned}
(24) \qquad \sup_{v\in U}\left\{A^vG(t,y)+L(t,y,v)\right\} &= 0, \qquad (t,y)\in Q, \\
(25) \qquad G(t,y) &= \Psi(t,y), \quad (t,y)\in\partial^*Q.
\end{aligned}$$

Then we obtain the following result:

(i) $G(t,y) \geq J(t,y;u)$ for all $(t,y)\in Q$ and $u(\cdot)\in\mathcal{A}(t,y)$.

(ii) If for $(t,y)\in Q$ there exists a control $u^*(\cdot)\in\mathcal{A}(t,y)$ with

$$(26) \qquad u^*(s) \in \arg\max_{v\in U}\left(A^vG(s,Y^*(s)) + L(s,Y^*(s),v)\right)$$

for all $s\in[t,\tau]$ where Y^* is the solution of the controlled SDE corresponding to $u^*(\cdot)$ then we have

$$G(t,y) = V(t,y) = J(t,y;u^*),$$

i.e. $u^*(\cdot)$ is an optimal control and G coincides with the value function.

[16] See Fleming/Soner (1993), p. 163.

Besides conditions (17) and (18) the growth condition (23) is not satisfied in our applications, too. Thus, we need to modify the above verification result in a suitable way:

Corollary 3.2 (to the verification theorem) *Consider a linear controlled SDE with coefficients satisfying the assumptions of Corollary 3.1. Assume further that the functions L and ψ satisfy the conditons (20) and (21). Finally, let the function $G \in C^{1,2}(Q) \cap C(\bar{Q})$ be a solution to the Hamilton-Jacobi-Bellman equation (24) with boundary condition (25). Assume that for all $(t, y) \in Q$ and all admissible controls $u(\cdot) \in \mathcal{A}(t, y)$ there exists a $\rho > 1$ such that we have*

$$\mathrm{E}\Big(\sup_{s\in[t,t_1]} |G(s, Y(s))|^{\rho}\Big) < \infty. \tag{27}$$

Then assertions (i) and (ii) of the verfication theorem are valid.

Proof of Corollary 3.2:
Looking at the proof of the verfication theorem as given in Fleming/Soner (1993, pp. 163f) we realize the following:

(i) Conditions (17) and (18) ensure the existence and uniqueness of a solution of the controlled SDE for controls with property (ii) of Definition 3.1. We can then apply the Ito formula to obtain

$$\begin{aligned} G(\theta, Y(\theta)) - G(t, y) - \int_t^\theta A^{u(s)} G(s, Y(s))\, ds \\ = \int_t^\theta G_y(s, Y(s)) \cdot \Sigma(s, Y(s), u(s))\, dW(s) \end{aligned} \tag{28}$$

which corresponds to relation (3.9) in Fleming/Soner (1993).

(ii) The growth condition (18) is used to prove the relation

$$\mathrm{E}^{t,y}\left(\int_t^\theta G_y(s, Y(s)) \cdot \Sigma(s, Y(s), u(s))\, dW(s)\right) = 0 \tag{29}$$

for bounded $\mathcal{O}$. (This corresponds to $\mathrm{E}_{tx} M(\theta) = 0$ for bounded $\mathcal{O}$ in the notation of Fleming/Soner (1993))

(iii) The growth condition (23) is used to show the uniform integrability of $\{G(\theta_p, Y(\theta_p))\}_p$ where θ_p are stopping times with $t \le \theta_p \le t_1$ (In notation of Fleming/Soner (1993) this corresponds to the uniform integrability of $\{W(\theta_p, x(\theta_p))\}_p$. There, one also finds the exact definition of the stopping times θ_p which is irrelevant for our argumentation.)

We now demonstrate that we also have these three properties under the assumptions of our Corollary:

(i) For admissible controls the linear controlled SDE admits a unique solution which is explicitly given in Corollary 3.1. Of course, we can apply the Ito formula to such solutions. Thus, relation (28) remains valid.

(ii) To show property (29) note that the diffusion coefficient of the linear controlled SDE is $\Sigma(t, y, v) = y(B_1(t)v + B_2(t))$. As in Fleming/Soner (1993) we look at a bounded set $\mathcal{O}$ and obtain the following estimate for an admissible control $u(\cdot)$

$$\begin{aligned}
\int_{t_0}^{t_1} |\Sigma(s, Y(s), u(s))|^2\, ds &= \int_{t_0}^{t_1} |Y(s)(B_1(s)u(s) + B_2(s))|^2\, ds \\
&\leq \sup_{s\in[t,t_1]} |Y(s)|^2 \int_{t_0}^{t_1} (|B_1(s)u(s)| + |B_2(s)|)^2\, ds \\
&\leq 2\,\mathrm{diam}(\mathcal{O}) \int_{t_0}^{t_1} |B_1(s)u(s)|^2 + |B_2(s)|^2\, ds \\
&\leq 2\,\mathrm{diam}(\mathcal{O}) \int_{t_0}^{t_1} |B_1(s)|^4 + |u(s)|^4 + |B_2(s)|^2\, ds.
\end{aligned}$$

Here, we have made multiple use of $2|vw| \leq v^2 + w^2$ for $v, w \in I\!R$. Due to property (ii) of an admissible control and the integrability conditions of the coefficients of the linear controlled SDE we obtain

$$\mathrm{E}^{t,y}\left(\int_t^\theta |G_y(s, Y(s)) \cdot \Sigma(s, Y(s), u(s))|^2\, ds\right) < \infty$$

and thus (29).

(iii) Condition (27) implies uniform integrability of $\{G(\theta_p, Y(\theta_p))\}_p$. □

References

Canestrelli, E. (1998): Inquiries on the Applications of Multidimensional Stochastic Processes to Financial Investments, *Working Paper*, Dipartimento di Matematica Applicata ed Informatica, Università Ca' Foscari, Venezia.

Duffie, D. (1992): *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton.

Fleming, W. H.; R. W. Rishel (1975): *Deterministic and Stochastic Optimal Control*, Springer, New York.

Fleming, W. H.; H. M. Soner (1993): *Controlled Markov Processes and Viscosity Solutions*, Springer, New York.

Ho, T. S. Y., S.-B. Lee (1986): Term structure movements and pricing interest contingent claims, *Journal of Finance* 41, 1011-1029.

Ikeda, N., S. Watanabe (1981): *Stochastic Differential Equations and Diffusion Processes*, North Holland, New York.

Karatzas, I.; S. E. Shreve (1991): *Brownian Motion and Stochastic Calculus*, 2nd ed., Springer, New York.

Klüppelberg, C.; R. Korn (1998): Optimal Portfolios with Bounded Value-at-Risk, *Berichte zur Stochastik und verwandten Gebieten*, Johannes Gutenberg-Universität Mainz.

Korn, R. (1997): *Optimal Portfolios*, World Scientific, Singapore.

Krylov, N. (1980): *Controlled Diffusion Processes*, Springer, Berlin.

Merton, R. C. (1969): Lifetime portfolio selection under uncertainty: the continuous case, *Reviews of Economical Statistics* 51, 247-257.

Merton, R. C. (1971): Optimal consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3, 373-413. Erratum: ebenda 6 (1973), 213-214.

Merton, R. C. (1990): *Continuous-time finance*, Basil Blackwell, Cambridge MA.

Musiela, M.; M. Rutkowski (1997): *Martingale Methods in Financial Modelling*, Springer, Berlin.

Sørensen, C. (1999): Dynamic Asset Allocation and Fixed Income Management, *Journal of Financial and Quantitative Analysis* 34, 513-531.

Vasicek, O. (1977): An equilibrium characterisation of the term structure. *Journal of Financial Economics* 5, 177-188.