Some Applications of Impulse Control in Mathematical Finance

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Abstract: We consider three applications of impulse control in financial mathematics, a cash management problem, optimal control of an exchange rate, and portfolio optimisation under transaction costs. We sketch the different ways of solving these problems with the help of quasi-variational inequalities. Further, some viscosity solution results are presented.

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1. Introduction

The use of continuous-time stochastic processes for modelling problems of finance has attracted the interest of many mathematicians over the last decades. There is a vast amount of literature on applications of stochastic calculus and stochastic control techniques to various problems of financial mathematics such as the problem of derivative pricing or that of optimal portfolio selection. However, most parts of the literature lack some reality as the optimal strategies consist of trading or intervention actions at every time instant. The main reason for that lies in the application of stochastic control methods that assume costless trading. In contrast to that, transaction costs cannot be neglected in reality and therefore trading strategies in reality are typically piecewise constant ones where the action times are not predetermined but dependent on the development of the economy.

The appropriate mathematical framework to cover these aspects lies in the theory of impulse control (see Bensoussan and Lions (1984)). The main difference between impulse control and instantaneous stochastic control of diffusions is given by the structure of the optimal control strategies. While in problems of instantaneous stochastic control it is often optimal to control the underlying stochastic process at every time instant, it is an essential part of the definition of an admissible impulse control strategy that the intervention times do not accumulate (see Definition 2.1). This is mostly due to the fact that in an impulse control problem every action of the controller results in costs that are bounded from below by a positive constant (i.e. there exists a fixed cost component). Therefore, an impulse control problem has the additional feature of the choice of a sequence of intervention times and not only that of the choice of optimal actions in every time instant. Due to that additional difficulty, applications of impulse control methods in mathematical finance are still rare examples. On the other hand this means that there is still a wide field of unsolved mathematical problems from both the theoretical and applicational point of view.

The aim of the present paper is to give a (not necessarily complete) survey over some popular examples of impulse control methods in finance. The examples consist of impulse control models for optimal cash management and index tracking under transactions (see e.g. Constantinides and Richard (1978), Buckley and Korn (1998)), models for the optimal control of exchange rates between different currencies (see e.g. Jeanblanc-Piqué (1993), Korn (1997b)), Mundaca and Oksendal (1997), Cadenillas and Zapatero (1999)) and portfolio optimisation problems in the presence of transaction costs (see e.g. Eastham and Hastings (1988), Korn (1998), Bielecki and Pliska (1998) or Morton and Pliska (1995)). The foregoing examples are ordered in increasing degree of difficulty. While the cash management problem is a direct application of usual impulse control methods, the exchange rate model requires some refinements and the portfolio problem exhibits some very special and new features. It is thus not surprising that the expliciteness of the solution of these models decreases with increasing degree of difficulty.

Section 2 of the paper contains a brief review of some basic facts of impulse control models and methods. The above mentioned examples form Section 3 while in Section 4 we present some new technical remarks on viscosity solutions of quasi-variational inequalities and numerical methods for solving impulse control problems. As the main purpose of this paper is to give a survey of applications of impulse control methods in mathematical finance we often leave aside technical details and refer the interested reader to the original sources.

2. Some basic facts of impulse control models and methods

For simplicity and to gain some insight, let us start by looking at a one-dimensional model. We assume that between intervention times our fundamental process is given as the solution of the following stochastic differential equation

(2.1) $$dX(t) = b(X(t))dt + \sigma(X(t))dW(t).$$

The functions $b:\mathbf{R}\to\mathbf{R}$ and $\sigma:\mathbf{R}\to\mathbf{R}$ are assumed to satisfy Lipschitz conditions guaranteeing the existence of a unique, non-exploding solution of (2.1) (see e.g. Karatzas and Shreve (1988), Section 5.2) for every initial condition $X(0) = \xi$ on some probability space (Ω, F, P) equipped with a filtration $(F_t)_{t\geq 0}$ satisfying the usual conditions. $W(t)$ is a one-dimensional F_t-Brownian motion, ξ a real valued random variable independent of $W(t)$, $t\geq 0$, with existing second moment. The controller is now allowed to choose intervention times θ_i where he can shift the process $X(t)$ to another value

(2.2) $$X(\theta_i) = X(\theta_i -) - \Delta x_i .$$

Here, Δx_i is a real number chosen by the controller. Note the striking difference to the usual stochastic control setting (see e.g. Fleming and Soner (1993)) where the controller can influence the drift and/or diffusion term of the fundamental process, but the resulting controlled diffusion remains a continuous process. In the above impulse control situation, the controlled process has a jump of $-\Delta x_i$ at the control instant but the local characteristics of the process (i.e. the drift and diffusion terms of Equation (2.1)) remain unchanged. In Section 3, we will also deal with a situation where the controller can change the local parameters of the fundamental at an intervention time, too.

For the moment, we concentrate on the control problem where the controller faces both intervention and running costs on an infinite time intervall which is given by

$$(2.3)\qquad \min_{\{(\theta_i,\Delta x_i),i\in N\}\in Z} E_x^S\left(\int_0^\infty e^{-\alpha t} f(X(t))dt + \sum_{i=1}^\infty e^{-\alpha\theta_i}(K+k|\Delta x_i|)1_{\{\theta_i<\infty\}}\right)$$

where K, k are positive constants, $f:\mathbf{R}\to[0,\infty)$ is a continuous function. $E_x^S(.)$ denotes the expectation when the process X(t) starts with initial value x and the strategy $S=\{(\theta_i,\Delta x_i), i\in\mathbf{N}\}$ is chosen by the controller. For notational convenience, we will omit the dependence of the process X(t) on the strategy S. Let Z be the set of admissible impulse control strategies which will be defined in

Definition 2.1

An *impulse control strategy* $S=\{(\theta_i,\Delta x_i), i\in\mathbf{N}\}$ is a sequence of *intervention times* θ_i and *control actions* Δx_i with

i) $0\le\theta_i\le\theta_{i+1}$ a.s. $\forall\, i\in\mathbf{N}$

ii) θ_i is a stopping time with respect to the filtration $f_t := \sigma\{X(s-),\ s\le t\},\ t\ge 0$

(2.4) iii) Δx_i is measurable with respect to f_{θ_i}

iv) $X(\theta_i)=X(\theta_i-)-\Delta x_i$,

An impulse control strategy will be called *admissible* if we also have

v) $P(\lim_{i\to\infty}\theta_i\le T)=0 \quad \forall\, T\ge 0$.

Remark 2.2

We will look at problems different from (2.3) such as problems with a finite horizon or with an implicit cost structure in the examples in Section 3 but will demonstrate the main principles of impulse control by considering problem (2.3).

To formulate an analogue to the Bellman principle of stochastic control we define the value function v(x) corresponding to (2.3) by

$$(2.5)\qquad v(x) := \inf_{S\in Z} E_x^S\left(\int_0^\infty e^{-\alpha t} f(X(t))dt + \sum_{i=1}^\infty e^{-\alpha\theta_i}(K+k|\Delta x_i|)1_{\{\theta_i<\infty\}}\right)$$

and the minimum operator M via

$$(2.6)\qquad Mv(x) := \inf_{\Delta x\in R}\,[\,v(x-\Delta x)+k\,|\Delta x|+K\,].$$

Note, that the minimum operator Mv(x) represents the value of the strategy that consists of doing the best immediate action and behaving optimally afterwards (for simplicity, we assume that the minimum in this minimisation will be attained). But considering that there could be states x where an immediate action is not at all optimal yields the inequality

$$v(x) \leq Mv(x). \tag{2.7}$$

On the other hand, at the first time (after the start in x) when it is optimal to intervene, v and Mv must coincide (the optimal action is then equal to the optimal immediate action). Therefore, we conjecture the following variant of the Bellman principle to hold :

$$v(x) = \inf_{\tau\in\Sigma} E_x\left(e^{-\alpha\tau} Mv(X(\tau-)) + \int_0^{\tau} e^{-\alpha t} f(X(t))dt\right) =: G\,v(x) \tag{2.8}$$

where Σ is the set of finite stopping times and E_x is the expectation if the non-intervention strategy is used. Observe that equation (2.8) reduces an impulse control problem to an optimal stopping problem. But this is only a formal success because the reward function of the optimal stopping problem is unknown (more precisely, $Mv(X(\tau-))$ is unknown). An iterative algorithm to solve this formal optimal stopping problem is given in Korn (1997b). The above Bellman principle will serve us to derive the analogue to the HJB-Equation of stochastic control, the quasi-variatonal inequalities, heuristically. For this reason, let us assume that there exists an optimal stopping time τ^* for which the infimum in the Bellman principle (2.8) will be attained and further assume that v(x) is sufficiently smooth and regular to perform all the following manipulations. Under these assumptions, combination of (2.8) and (2.7) together with Itô's formula imply

$$\begin{aligned} v(x) &= \inf_{\tau\in\Sigma} E_x\left(e^{-\alpha\tau} v(X(\tau-)) + \int_0^{\tau} e^{-\alpha t} f(X(t))dt\right) \\ &= E_x\Big(v(x) + \int_0^{\tau^*} e^{-\alpha t}\big(f(X(t)) - \alpha v(X(t))\big)dt + \int_0^{\tau^*} e^{-\alpha t} v'(X(t))\sigma(X(t))dW(t) \\ &\qquad + \int_0^{\tau^*} e^{-\alpha t}\big(v'(X(t))b(X(t)) + \tfrac{1}{2} v''(X(t))\sigma^2(X(t))\big)dt\Big) \\ &\leq E_x\Big(v(x) + \int_0^{s} e^{-\alpha t}\big(f(X(t)) - \alpha v(X(t))\big)dt + \int_0^{s} e^{-\alpha t} v'(X(t))\sigma(X(t))dW(t) + \\ &\qquad + \int_0^{s} e^{-\alpha t}\big(v'(X(t))b(X(t)) + \tfrac{1}{2} v''(X(t))\sigma^2(X(t))\big)dt\Big) \end{aligned} \tag{2.9}$$

for a fixed but otherwise arbitrary positive constant s. Now assume that the expectation of the stochastic integral vanishes, subtract v(x) from both sides of (2.9), divide them by s, and apply the mean value theorem to the integrals. Then, let s converge to zero and assume that this limit can be interchanged with the expectation. Thus, we arrive at the inequality

$$L\,v(x) + f(x) := \tfrac{1}{2}\sigma^2(x)\, v_{xx}(x) + b(x)v_x(x) - \alpha\, v(x) + f(x) \leq 0\,. \tag{2.10}$$

If the optimal stopping time τ^* would be identical to zero then we would have equality in (2.7). However, if τ^* would be positive (at least with positive probability) then by the second equality in (2.9) and the Feynman-Kac representation theorem (see e.g. Karatzas and Shreve (1988), Section 5.7) we would have equality in (2.10). Hence, both inequalities cannot be strict ones simultaneously, i.e. we must have

$$(v(x) - Mv(x))(L\,v(x) + f(x)) = 0\,. \tag{2.11}$$

This gives rise to the following

Definition 2.3
The three relations (2.7), (2.10), and (2.11) are called the *quasi-variational inequalities* (for short: qvi) for problem (2.3).

In Definition 2.3 the derivatives occurring in inequality (2.10) are only assumed to exist as left hand derivatives. v(x) need not necessarily be an element of C^2. It will become clear in a moment that the qvi will play a role similar role to that of the HJB-Equation in stochastic control. To formulate the corresponding result, we introduce a special impulse control strategy which will be constructed with the help of a solution of the qvi.

Definition 2.4
Let v be a continuous solution of the qvi. Then the following impulse control strategy is called a *qvi-control* (if it exists):

$$\begin{aligned} &\text{i)} \quad (\theta_0, \Delta x_0) := (0,0), \\ (2.12) \quad &\text{ii)} \quad \theta_i \;:= \inf\{t \ge \theta_{i-1} : v(X(t-)) = M\,v(X(t-))\}, \\ &\text{iii)} \quad \Delta x_i := \arg\min_{\Delta x}[\, v(X(\theta_i -) - \Delta x) + k\,|\Delta x| + K\,]. \end{aligned}$$

Hence, at every time instant where v and Mv coincide a controller using a qvi-control intervenes. He then chooses the action that is the minimiser of the optimisation problem corresponding to Mv(x) (compare the formal argument leading to the Bellman principle (2.8)). The justification for considering qvi and qvi-controls is given by Theorem 2.5 below (see Korn (1997a) for a proof).

Theorem 2.5
If there exists a solution $v^* \in C^2$ (or better: a "sufficiently regular solution", see the remarks below) of the qvi to problem (2.3) that satisfies the growth conditions

$$E_x^S \int_0^\infty (e^{-\alpha t}\sigma(X(t))\, v^*_x(X(t)))^2 dt < \infty, \tag{2.13}$$

$$E_x^S\,(e^{-\alpha T} v^*(X(T))) \xrightarrow{T\to\infty} 0 \tag{2.14}$$

for every X(t) corresponding to an admissible impulse control strategy S then we have

$$v(x) \ge v^*(x) \quad \forall\, x \in \mathbf{R}. \tag{2.15}$$

Further, if the qvi-control corresponding to v* is admissible then it is an optimal impulse control and v* is identical to the value function v.

Remark 2.6: *"Regularity and the smooth pasting principle"*
The above result can be seen as a verification result for a regular solution of the qvi (and is thus only a sufficient but not necessary result). However, it is the required degree of regularity that causes problems. One can in general not expect to get a C^2-solution of the qvi (a typical example for a C^2-solution of the qvi would be the case where the non-intervention strategy would be optimal). Inspection of the proof of the theorem shows that the C^2-assumption can be substantially weakened. It is in fact only needed to apply Itô's formula. Therefore, by using some generalised versions of Itô's formula that require weaker regularity assumptions we could also weaken the regularity requirements on the solution v* of the qvi in Theorem 2.5. The simplest but often relevant case in the one-dimensional situation is v* being a C^1-function which is C^2 up to a finite number of points. The most prominent example has the following structure:
The real line can be divided into a "continuation set" (i.e. a region where it is optimal not to intervene if the controlled process stays inside this region) and an "action set" (a set where it is optimal to do an immediate control action if the process is inside this set). See also Figure 1 for an illustration.

Insert Figure 1 here.

If this is the case then the controller has to determine:

- the endpoints a, b of the continuation set
- the optimal restarting points α, β inside the continuation set (i.e. the points to where the process should be shifted by the controller when the process has reached an endpoint of the continuation set) or equivalently the optimal actions

Thus, the controller is left with four unknown parameters. Now, his main tool is to use the principle of *smooth pasting*. We will demonstrate what is meant by this. On the continuation set the differential inequality (2.10) must be satisfied as an equality (as there, the controlled process behaves as a diffusion) while on the action set inequality (2.7) has to be an equality (due to the definition of the action set an immediate action is optimal). We first solve the differential equation (2.10) on the continuation set which yields the general form of the solution of this equation. However, as it is a second order equation, the general form will contain two further unknown constants. Hence, we have to determine six unknown constants in total. On the other hand, we have a set of conditions that must be satisfied by the value function. *Continuity* of the value function requires

(2.16) $$v(a) = v(\alpha) + K + k\,(\alpha - a), \qquad v(b) = v(\beta) + K + k\,(b - \beta).$$

Necessary conditions for *optimality* of the actions in {a, b} are (note the form of the minimisation problem in the definition of the operator M)

(2.17) $v'(\alpha) = -k\,,\qquad v'(\beta) = k\,.$

Finally, *continuity* of the *derivative* of the value function requires (note equations (2.16))

(2.18) $v'(a) = -k\,,\qquad v'(b) = k\,.$

All in all, we have six equations for six unknowns, a fact that gives rise to the hope to be able to determine all these unknowns in a unique way. We will see the smooth pasting principle in action in first two examples of the next section. In Figure 1, we have also implicitly assumed that the optimal shifts from each point of the action set are to shift the process immediately to either α or β (depending if we are in the left or the right part of the action set).

Remark 2.7: *"Generalisations and uniqueness"*
a) Uniqueness of a "sufficiently regular" solution to the qvi is an immediate consequence of Theorem 2.5 as the value function of an optimisation problem must be unique. Thus, the most regular solution of the qvi automatically coincides with the value function given it is regular enough to meet the above requirements. This fact is also a justification for the above mentioned smooth pasting principle.
b) The above results and in particular Theorem 2.5 can be generalised in various ways:

- *Finite time horizon*

The main differences are the time dependence of the value function and the occurrence of a terminal condition in the qvi. See Example 3 iii) for more details.

- *Multi-dimensional case*

If the scalar process X(t) will be substitued by a vector process with dynamics given by

(2.19) $$dX_i(t) = b_i(X(t))dt + \sum_{j=1}^{k}\sigma_{ij}(X(t))dW_j(t)\,,\ i = 1,\dots,n$$

then the foregoing results and methods are valid if the operator L occurring in the quasi-variational inequalities is now defined as

(2.20) $$Lv(x) = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{k}\sigma_{im}(x)\sigma_{jm}(x)v_{x_i x_k}(x) + \sum_{i=1}^{n}b_i(x)v_{x_i}(x) - \alpha v(x).$$

The main problem of the multi-dimensional case is *not* the proof of verification theorems, it is the *explicit* solution of the qvi.

- *Constraints*

We will only consider the case where the controlled process is required to stay in a given closed interval I. First, we have to modify the definition of an admissible impulse control. Whenever the process reaches the boundary of I a control action has to be taken to prevent it from leaving I. Also, only control actions that keep the process inside the interval I are admissible. Therefore, in Definition 2.1 we make the following modifications:

(2.4) ii*) θ_i is a stopping time with respect to $f_t := \sigma\{\,X(s-),\ s\le t\,\},\ t \ge 0$, with $\theta_i \le \inf\{t \ge \theta_{i-1} \mid X(t)\in\partial I\}$, where ∂I denotes the boundary of I

(2.4) iv*) $X(\theta_i) = X(\theta_i-) - \Delta x_i$, such that $X(\theta_i) \in I$

Further modifications concern the definitions of the operator M (where it is only minmised over such values of Δx such that (2.4) iv*) is satisfied) and that of the qvi (where we have to add the requirement "$v(x) = Mv(x)$ for $x \in \partial I$" in (2.7). Note further that the qvi now only have to hold on I. With the obvious modification of the definition of a qvi-control for the constrained situation Theorem 2.5 remains valid. However, in our constraint situation we only have to require " $v^* \in C^2(I^\circ)$, $v^* \in C(I)$". The growth conditions (2.13/14) are now automatically satisfied and can be dropped.

• *Maximising utility*
If instead of minimising costs we look at the problem of maximising utility then some obvious changes in the directions of inequality signs and substitutions of minimisations by maximisations have to be made. See the Examples i) and iii) in the next section.

3. Applications of impulse control in mathematical finance

i) An impulse control model for cash management and index tracking

In this first example of an application of impulse control to finance we consider a portfolio manager who is trying to passively track a stock index (such as the DAX). He can do so by setting up a portfolio consisting of all the securities that enter the index (of course hold in the same proportion as they enter the index). If there would be no cash inflow to or outflow from his fund then his portfolio would perfectly track the index. However, there are irregular cash inflows/outflows over time which originate from dividends of the shares, new subscriptions to the fund or fund redemptions. If all these inflows/outflows must be immediately dealt with by security transactions this would result in large amounts of transaction costs. It could therefore be advantageous to the portfolio manager to hold a certain amount of cash. If this cash account has reached a sufficiently high level then he will add parts of it to his portfolio by purchasing additional shares (in the appropriate proportions to track the index). If in contrast it has fallen below a critical level he will increase it by selling some of his shares. However, a positive cash position will reduce the (expected) excess return of the portfolio over the riskless rate (as long as the expected rate of return of the portfolio lies above the riskless rate). Moreover, it will surely lead to a tracking error, i.e. of a deviation of the performance of the total holdings (portfolio **and** the cash account) from the performance of the index, because there is no positive cash weight in an index. Thus, the advantages and disadvantages of holding a cash position have to be judged appropriately. Below we will give a rigorous formulation of and a solution to this problem. All our processes considered there will be relative processes with respect to the wealth of the portfolio managers total holdings.

Let the process C_t be the cash weight in a portfolio (i.e. the fraction of the total wealth hold in cash at time t). We assume that it behaves like a Brownian motion with volatility σ and drift rate η between readjustment times θ_i and θ_{i+1}, i.e.

$$C_t = C_0 + \eta\, t + \sigma\, W(t) + \sum_{i=1}^{\infty}\left(C^{(i)} - C_{\theta_i -}\right)\cdot 1_{\{\theta_i \le t\}} \tag{3.1}$$

where $C^{(i)}$ is the value of the process that the portfolio manager has chosen at the readjustment time θ_i, C_0 the initial cash weight and W(t) a one-dimensional Brownian motion. We assume that every readjustment causes costs that consist of a component which is proportional to the wealth $X(\theta_i -)$ of the portfolio (immediately before the readjustment) and of a component that is proportional to the absolute value of the "transacted" amount of cash,

$$\left|C^{(i)} - C_{\theta_i -}\right| X(\theta_i -). \tag{3.2}$$

We will thus look at the costs (relative to X(t)) of the form

$$K + k\left|C^{(i)} - C_{\theta_i -}\right|. \tag{3.3}$$

Further, we assume that the index portfolio we would like to track has a (relative rate of) excess return π over the riskless rate (which is assumed to be zero) and a volatility of τ. By holding a cash weight of C_t in our tracking portfolio we will get an excess return rate of $(1 - C_t)\,\pi$ and a variance of the tracking error of $(C_t\,\tau)^2$. Our goal will be to have a *good excess return* and a *low tracking error variance* with not too much readjustment costs. Also, the portfolio manager's cash position should be ***non-negative.*** All this leads to the constrained optimisation problem given by the value function

$$\psi(c) = \sup_{(\theta_i, C^{(i)})\in Z} E_c\Big\{ \int_0^{\infty} e^{-\rho t}\left[(1 - C_t)\pi - \lambda\tau^2 C_t^{\,2}\right]dt - \sum_{i=1}^{\infty}\left(K + k\left|C^{(i)} - C_{\theta_i -}\right|\right)e^{-\theta_i \rho}\, 1_{\{\theta_i < \infty\}}\Big\}. \tag{3.4}$$

Here, ρ is a positive discount factor, λ a positive coefficient of risk aversion, c the positive initial cash weight, and Z is the set of admissible strategies with the additional requirement to use only such strategies leading to a non-negative cash weight process C_t. This problem is treated in Constantinides and Richard (1978) and in Buckley and Korn (1998).

The following proposition gives a sufficient condition characterising the value function and it also describes the corresponding optimal control strategy. It is a direct application of the smooth pasting principle (see Remark 2.6).

Proposition 3.1

If there exists a continuous function $V:[0,\infty) \to \mathbf{R}$ and a triple (l,u,U), $0 < l < u < U$ with $V\big|_{(0,\infty)} \in C^1$, $V\big|_{(0,U]} \in C^2$ (where $V''(U) := \lim_{c \uparrow U} U''(c)$) satisfying

i) $\frac{1}{2}\sigma^2\, V''(c) + \eta\, V'(c) + [(1-c)\pi - \lambda\,(\tau c)^2] - \rho\, V(c) = 0 \qquad \forall\, c \in [0,U]$

$$\begin{array}{ll} \text{ii)} & V'(u) = V'(U) = -k,\ V'(l) = k \\ \text{(3.5) iii)} & V(U) = V(u) - K - k(U-u),\ V(0) = V(l) - K - k\,l \\ \text{iv)} & V(c) = V(U) - k(c-U) \quad \forall c \in [U, \infty) \end{array}$$

then V coincides with the value function ψ of (3.4). Moreover, in this case the strategy "*Do nothing as long as C_t is in (0, U). If C_t reaches U then decrease the cash weight to u, if it reaches 0 then increase the cash weight to l. If C_0 is bigger than U then decrease the cash weight immediately to u*" is an optimal strategy.

For a proof (which consists of showing that a function V(.) satisfying (3.5) satisfies the corresponding qvi, too) see Buckley and Korn (1998). In view of the proposition we now only have to show the existence of a function V satisfying the system of equations (3.5). We will construct it in several steps. As in Jeanblanc-Piqué (1993) we will work with the derivative v of V and construct it on [0,U] only. The extension of V on (U, ∞) can then be easily obtained. Due to its affine linear form and the continuity of V and V' in U it is completely determined by the values V(U) and V'(U). The derivative v of V has to satisfy

$$\begin{array}{lll} \text{i)} & \tilde{L}\, v(c) := \tfrac{1}{2}\sigma^2 v''(c) + \eta\, v'(c) - \rho\, v(c) - 2\lambda\tau^2 c - \pi = 0 & \forall c \in [0,U], \\ \text{(3.6) ii)} & \int_0^l (v(x) - k)dx = K, \quad \int_u^U (v(x) + k)dx = -K, & \\ \text{iii)} & v(u) = v(U) = -k, \quad v(l) = k. & \end{array}$$

The general solution of the differential (3.6) i) is given by

$$\text{(3.7)} \qquad v(c) = \mu e^{\lambda_1 (c-\Delta)} + \nu e^{\lambda_2 (c-\Delta)} - \gamma_1 c - \gamma_2 \ ,$$

with $\Delta = u - l$,

$$\text{(3.8)} \qquad \lambda_i = -\frac{\eta}{\sigma^2} \mp \sqrt{\frac{\eta^2 + 2\sigma^2\rho}{\sigma^4}}, \qquad \gamma_1 := 2\frac{\lambda\tau^2}{\rho}, \qquad \gamma_2 := \frac{\pi\rho + 2\lambda\tau^2\eta}{\rho^2},$$

(note especially $\lambda_1 < 0 < \lambda_2$) and where the constants μ, ν depend on l and Δ. Obviously, both these constants are positive. The requirements $v(l) = k$, $v(u) = -k$ lead to the following representations for the constants μ, ν:

$$\text{(3.9)} \qquad \mu = \frac{k(e^{\lambda_2\Delta}+1) - con_1\left[\Delta + l(1-e^{\lambda_2\Delta})\right] - con_2\left[1-e^{\lambda_2\Delta}\right]}{e^{\lambda_2\Delta} - e^{\lambda_1\Delta}}$$

$$\text{(3.10)} \qquad \nu = \frac{-k(e^{\lambda_1\Delta}+1) + con_1\left[\Delta + l(1-e^{\lambda_1\Delta})\right] + con_2\left[1-e^{\lambda_1\Delta}\right]}{e^{\lambda_2\Delta} - e^{\lambda_1\Delta}}$$

Still there are three unknowns to determine, U, l, Δ. To do this we use the three remaining equations and show that

- for every $l > 0$ there exists a positive number $\Delta = \Delta(l)$ with

(3.11) $$-K = \phi_1(l, \Delta(l)) := \int_u^U (v(x)+k)dx,$$

- there exists a pair $(l, \Delta(l))$ satisfying (3.11) and also

(3.12) $$K = \phi_2(l, \Delta(l)) := \int_0^l (v(x)-k)dx,$$

- there exists a triple $(U, l_U, \Delta(l_U))$ with U>0 and $(l_U, \Delta(l_U))$ satisfying (3.11), (3.12) and

(3.13) $$-k = \phi_3(U, l_U, \Delta(l_U)) := v(U).$$

The complete existence proof is given in Buckley and Korn (1998).We have thus reduced the solution of our impulse control problem to the solution of the following system of non-linear equations for (l,Δ,U) arising from the system (3.5):

(3.14)
$$\text{i)}\quad \frac{\mu}{\lambda_1}\left(1-e^{-\lambda_1 l}\right)+\frac{\nu}{\lambda_2}\left(1-e^{-\lambda_2 l}\right)-\tfrac{1}{2}\gamma_1 \cdot l^2-(\gamma_2+k)\,l-K=0$$
$$\text{ii)}\quad \mu e^{\lambda_1(U-l)}+\nu e^{\lambda_2(U-l)}-\gamma_1 U-\gamma_2+k=0$$
$$\text{iii)}\quad \frac{\mu}{\lambda_1}\left(e^{\lambda_1(U-l)}-e^{\lambda_1\Delta}\right)+\frac{\nu}{\lambda_2}\left(e^{\lambda_2(U-l)}-e^{\lambda_2\Delta}\right) - \tfrac{1}{2}\gamma_1(U^2-(l+\Delta)^2)-(\gamma_2-k)(U-l-\Delta)+K=0$$

where we additionally require

(3.15) $$0<l<U,\quad 0<\Delta<U-l$$

with μ and ν functions of (Δ, l) given by equations (3.9/10). However, this problem can be solved instantaneously by standard numerical methods. See Buckley and Korn (1998) for numerical examples and some comparative statics analysis.

ii) Optimal control models of exchange rates

Our second example will be the optimal control of the exchange rate between two currencies by a government bank. We will especially look at a so called target zone model. This means that the goal of the policy followed by the government bank is to keep the exchange rate in a prescribed closed interval I=[L, U]. Such a strategy is a good compromise between the two extremes of a free float (i.e. no control action at all) and a fixed exchange rate (which requires control actions at every time instant).

The main idea behind control models for the exchange rate between two currencies is that there exists a process G(t) called the "fundamental" which is responsible for changes in the exchange rates (of one country). It is regarded as a one-dimensional aggregation of certain economic measures like productivity, rate of unemployment, industrial output or domestic interest rates. Assuming a functional relationship between this fundamental and the exchange rate the goal of a government bank is to control the fundamental in a way such

that the exchange rate stays in a given interval, the target zone. Of course, control actions of the bank are more or less limited to the control of some key interest rates or to selling and buying of foreign currencies. For an overview on some methods, models and the economic background we refer the reader to Flood and Garber (1991) or Krugman (1991).
It must be noticed that the target to keep a process inside an interval does not completely determine the correponding control strategy. However, the introduction of a cost criterion to be minimised helps to figure out a unique one. A model where the government bank uses an impulse control strategy is presented in Jeanblanc-Piqué (1993). This approach is generalised to a model with random control consequences in Korn (1997b). We only give a brief description of the situation covered in Jeanblanc-Piqué (1993). There, it is assumed that the *uncontrolled* fundamental process G(t) follows a Brownian motion with drift

(3.16) $$G(t) = \eta\, t + \sigma\, W(t)$$

and that the logarithm of the exchange rate X(t) is given as

(3.17) $$X(t) = g(G(t))$$

with a C^2-function g(.). On the other hand, in the basic log-linear model of the exchange rate (see Jeanblanc-Piqué (1993)) it is assumed that the log of the exchange rate is given as the sum of the value of G(t) plus a term proportional to the expected "percentage change" in X(t) which has the Itô interpretation as a stochastic differential equation of the form

(3.18) $$dX(t) = \theta^{-1}(X(t)-G(t))\, dt + \exp(t/\theta)\phi(t)dW(t)$$

where θ is a positive constant, $\phi(t)$ a progressively measurable process such that $\exp(t/\theta)\phi(t)$ is square integrable. To determine g(.) we apply Itô's formula to (3.17) (note assumption (3.16)) and compare the resulting drift term with that of equation (3.18) delivering the following differential equation for g(G)

(3.19) $$\tfrac{1}{2}\sigma^2 g''(G) + \eta\, g'(G) - \frac{1}{\theta}\,(\, g(G) - G\,) = 0$$

which has the general solution

(3.20) $$g(G) = G + \eta\theta + Ae^{\mu_1 G} + Be^{\mu_2 G}$$

with $\mu_{1/2} = \eta \pm \sigma^{-2}\sqrt{\eta^2 + (2\sigma^2/\theta)}$ and A,B (yet unknown) real constants. These unknown constants are determined by the requirements that the exchange rate has local extrema (as a function of the fundamental) at the boundaries of the target zone [L, U] (see Krugman (1991) to justify this requirement), i.e. we have the boundary conditions

(3.21) $$g(k_0) = L \quad , \quad g(k_1) = U$$

where $k_0 < k_1$ are the roots of g'(G) = 0. As reported in Jeanblanc-Piqué (1993) there exists a unique pair (A, B) of constants satisfying (3.19/20/21). Thus, X(t) is completely determined as a function of the fundamental. The next task is to find a band [a, b] for the fundamental G(t) such that X(t) remains in the given target zone [L,U]. More precisely, we will try to keep the fundamental in [a,b] whilst minimising the intervention costs, i.e. we solve the problem given by the value function

(3.22) $$v(x) = \min_{(\theta_i, \Delta x_i) \in Z(a,b)} E_x^S \left(\sum_{i=1}^{\infty} [K + k|\Delta x_i|] e^{-\alpha\theta_i} 1_{\{\theta_i < \infty\}} \right)$$

where the set Z(a,b) contains all admissible strategies that keep G(t) inside [a, b]. A natural choice would be $[a, b] = [k_0, k_1]$. This problem could now be solved explicitly in the a similar way to the cash management problem using the principle of smooth pasting. Even more, in the sense of Remark 2.6, we could guarantee sufficient regularity of the value function for every choice $[a, b] \subseteq [k_{min}, k_{max}]$ to apply Theorem 2.5 (see also Jeanblanc-Piqué (1993)).

Proposition 3.2

For every given interval $[a, b] \subseteq [k_{min}, k_{max}]$ there exist constants α, β with $a < \alpha \le \beta < b$ such that the value function of problem (3.22) is given as the unique solution of

(3.23) $$Lv(x) = \tfrac{1}{2}\sigma^2 v''(x) + \eta v'(x) - \alpha v(x) = 0, \qquad x \in [a,b],$$

(3.24) $$v(a) = v(\alpha) + K + k\,|\,a - \alpha\,|,$$

(3.25) $$v(b) = v(\beta) + K + k\,|\,b - \beta\,|.$$

The optimal actions consist of waiting until the process reaches the boundary of the interval. Then it is thrown back from a to α and from b to β, respectively.

However, there is *no* possibility of a choice of the band [a, b]. The reason for this is economical background of the problem! The choice of an arbitrary band [a, b] would typically lead to a jump of the exchange rate after an intervention of the government bank (i.e. a shift of the fundamental). But if this intervention would lead to a completely foreseeable jump of the exchange rate, there would be possibilities of making riskless profits for speculators. Hence, the market which is aware of the strategy of the government bank will anticipate the government bank's action and the exchange rate will stay continuous (see Flood and Garber (1991)). Only the fundamental will jump. We thus have to find a band [a, b] with corresponding optimal shifts from a to α and from b to β such that after the shift of the fundamental the value of the exchange rate retains its value. To demonstrate the existence of such a quadruple (a,α,β,b) we use Figure 2 where we have plotted the (log. of the) exchange rate X(t) curve as a function of the fundamental (compare also to Jeanblanc-Piqué (1993)).

Insert Figure 2 here

Note that to keep the exchange rate in [L, U] it suffices to keep the fundamental in the interval $[k_{min}, k_{max}]$. We look for (a, α, β, b) that solve the corresponding problem (3.22) with

(3.26) $$g(a) = g(\alpha), \quad g(\beta) = g(b).$$

The choice of $b=k_{max}$ implies $\beta=k_0$ via (3.26). Hence, the left end a of the band must be smaller then k_0 which leads to a value of $\alpha \le k_0$ and thus to $g(a) > g(\alpha)$. On the other hand a choice of b slightly bigger than k_1 implies a value of β slightly smaller than k_1. Because the distance between b and β is small, the length of the band [a,b] must be small, too (see Korn (1997a) for a justification of this argument). In particular, a must be bigger than k_0 leading to $g(a) < g(\alpha)$. By continuity there must exist a quadruple (a, α, β, b) satisfying (3.26).

Remark 3.3:
a) In Korn (1997b) the above example is generalised to a situation where the action of the government bank has random consequences. More precisely, the government bank's action results in a random shift of the fundamental where the government bank can only control a parameter of the distribution of the shift.
b) In a very readable paper of Cadenillas and Zapatero (1999) a similar situation is treated. The main differences there are that the exchange rate is controlled directly and that there are also running costs for the deviation of the exchange rate from a prescribed level.

iii) Impulse control strategies for portfolio problems with transaction costs

This problem will contain some new features compared to the previous two. It will have a finite time horizon, a two-dimensional controlled process and the control costs affect the controlled process but not the value function. It has its origins in Eastham and Hastings (1988). Further generalisations and modifications can be found in Korn (1998).
We consider a securities market made up of a stock and a bond with price dynamics given by

(3.27) $\quad dP_0(t) = 0, \qquad P_0(0) = 1 \quad$ "Bond",
(3.28) $\quad dP_1(t) = P_1(t)\,(b\,dt + \sigma dW(t)), \qquad P_1(0) = p \quad$ "Stock" .

The trading strategy of an investor is completely described by the process of his bond and stock holdings (B(t), S(t)). As long as he does not rebalance these holdings they evolve as multiples of the relevant securities prices, i.e. we have

(3.29) $\quad dB(t) = 0 \quad$ "Bond holdings",
(3.30) $\quad dS(t) = S(t)\,(b\,dt + \sigma\,dW(t)) \quad$ "Stock holdings".

At every intervention time θ_i (i.e. a time where the investor rebalances his holdings) the investor has to pay a sum of fixed and proportional transaction costs of the form

(3.31) $\quad K + k\,|\,\Delta S_i\,|$

with $0 < K,\ 0 \le k < 1,\ \Delta S_i := S(\theta_i) - S(\theta_{i-1})$. These transaction costs have to be paid from the bond holdings. Therefore, we have the following balance equation:

(3.32) $\quad B(\theta_i) = B(\theta_{i-1}) - \Delta S_i - k\,|\Delta S_i| - K$

Hence, the i th action of an investor can be identified with the change ΔS_i in the stock holdings. By introducing the wealth process $X(t) := B(t)+S(t)$ our goal will be to maximise the expected utility from terminal wealth at a given time horizon T. We restrict ourselves to the treatment of the following problem (for more general problems see Korn (1998))

(3.33) $$\max_{(\theta_i,\Delta S_i)\in Z} E\left(1-e^{-\lambda X(T)}\right)$$

where Z is the set of admissible impulse control settings where in addition the action ΔS_i is constrained by the requirement of yielding non-negative values for $B(\theta_i)$, $S(\theta_i)$. This ensures that the wealth of our holdings after all securities are sold is always bounded from below by $-K$. We could also require to have a non-negative wealth after selling all securities, but for simplicity we drop this condition here.
Before proceeding with the solution of our problem we have to adapt the results of Section 2 to the finite time horizon and the multidimensional setting. First of all the time variable t will appear in the value function which will be defined by

(3.34) $$v(t,B,S) := \sup_{(\theta_i,\Delta S_i)\in Z} E_{t,B,S}\left(1-e^{-\lambda X(T)}\right).$$

The analogue of the operator M of Section 2 is given as

(3.35) $$Mv(t,B,S) := \max_{\Delta S\in A(B,S)} v(t,B-K-\Delta S-k\,|\Delta S|,\, S+\Delta S)$$

where A(B,S) is the above described feasible set for the actions given the holdings of (B,S). Note that the transaction costs enter into the components of v but *not* as an additional term in the maximisation problem as in Section 2! There is no separation between control costs and control gains as in the foregoing examples. The tradeoff between gaining a better rebalanced portfolio and paying transaction costs is not seen explicitely. As in the Section 2 we have (note the change in the inequality sign due to the maximisation !)

(3.36) $$v(t,B,S) \geq Mv(t,B,S).$$

At the first optimal intervention time (after starting in (t,B,S)) v and Mv must coincide. But due to the finite time horizon T it might be optimal not to intervene on [0, T] at all. This leads to the Bellman principle of the form

(3.37) $$v(t,B,S) = \max\{v_0(t,B,S),\ \sup_{\tau\in\Sigma_{t,T}} E_{t,B,S}(Mv(\tau,B,S(\tau)))\} =: \mathcal{G}\, v(t,B,S)$$

where $v_0(t,B,S)$ is the expected utility of the non-intervention strategy starting in (t,B,S), i.e.

(3.38) $$v_0(t,B,S) = E_{t,B,S}\left(1-\exp(-\lambda(B+Se^{(b-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))}))\right).$$

Note that for simplicitly we do not require that the securities have to be sold at time T. The finiteness of the time horizon also affects the qvi in the form of an additional terminal constraint. Further, the differential operator will now include a partial derivative with respect to the time variable t. As the bond price is constant a partial derivative with respect to B will not occur in the qvi. Therefore, the fact that the controlled process is two-dimensional enters the qvi only in an indirect way. The qvi for problem (3.34) are explicitly given as

i) $$L\,v(t,B,S) := \tfrac{1}{2}\sigma^2 S^2 v_{SS}(t,B,S) + bSv_S(t,B,S) + v_t(t,B,S) \leq 0$$

(3.39) ii) $$v(t,B,S) \geq Mv(t,B,S)$$

iii) $(v(t,B,S)-Mv(t,B,S))\, Lv(t,B,S) = 0$

iv) $v(T,B,S) = 1-\exp(-\lambda(B+S))$ for $(t,B,S) \in [0,T] \times [0,\infty)^2$.

Note the change of the inequality sign in i) due to the change from minimisation to maximisation compared to Section 2. With these modifications we obtain a similar verification theorem as in Section 2, i.e. a result of the form "A sufficiently regular solution of the qvi coincides with the value function and the corresponding qvi-control is an optimal one". See Korn (1998) for details. The main difficulty now is the task of *solving* the qvi. In particular, if we try to imitate the method of solution of the two preceeding examples we do not know

- the form of the continuation set NT ("the no-transaction region"),
- the explicit analytical form of the solution of the boundary and terminal value problem

$$Lv(t,B,S) = 0 \text{ on } NT, \quad v(T,B,S) = 1-\exp(-\lambda(B+S)) \text{ on } [0,\infty)^2$$

where the boundary conditions on $\partial(NT)$ are given by a suitable smooth pasting condition (the explicit form of which also has to be determined !).

Therefore, in Korn (1998) an approximation procedure is described which is based on an asymptotic analysis of a transformed version of the above qvi. The idea of this procedure stems from Whalley and Wilmott (1994) where a similar approach is applied to the problem of option pricing under transaction costs. To give an outline of this procedure let us first drop the requirement of non-negative bond and stock holdings. In the case *without* transaction costs the optimal strategy was derived by Pliska (1986). He showed that the optimal stock holdings $S^*(t)$ are constant (in wealth !), in particular

(3.40) $$S^*(t) = \frac{b}{\lambda\sigma^2},$$

and that the corresponding wealth process $X(t)$ is a Brownian motion with a positive drift. If $\psi^*(t)$ denotes the number of shares of the stock hold at time t and $\varphi^*(t)$ the amount of money invested in the bond at time t (recall the constant bond price !) are given by

(3.41) $$\varphi^*(t) = X(t) - \frac{b}{\lambda\sigma^2}, \qquad \psi^*(t) = \frac{b}{\lambda\sigma^2 P_1(t)}.$$

Note that in spite of the simple form of the stock holdings process the investor has to trade at every time instant to follow this strategy !
If $v(.,.,.)$ denotes the value function for the problem with transaction costs but *without* the non-negativity constraint then due to the multiplicity of the exponential function we have

(3.42) $$v(t,B,S) = e^{-\lambda B} v(t,0,S).$$

Hence, one expects an optimal strategy which is independent of the total wealth of the investor. More precisely, it should have the form: " *Rebalance your holdings only if the stock price $P_1(t)$ has moved in such a way that your current number of stocks owned is far away from $\psi^*(t)$.*" The bond holdings are uniquely determined by the self-financing condition. Thus, the optimal strategy should be given by an interval $[\psi^*(t)-y_-(P_1(t)),\psi^*(t)+y_+(P_1(t))]$

with boundaries depending on the stock price and the actual number of shares of the stock. We have to specify the optimal restarting points $\psi^*(t)-\hat{y}_-(P_1(t))$ and $\psi^*(t)-\hat{y}_+(P_1(t))$ after an intervention, too. To determine v(t,0,S) we separate the stock price and the number of shares by introducing the function

$$(3.43) \qquad q(t,p,y) := v(t,0,py)+1 .$$

Then the qvi for v(t,0,py) implies the following qvi for q(t,p,y):

$$\text{i)} \quad Lq(t,p,y) := \tfrac{1}{2}\sigma^2 p^2 q_{pp}(t,p,y) + bpq_p(t,p,y) + q_t(t,p,y) \le 0,$$

$$(3.44) \ \text{ii)} \quad q(t,p,y) \ge M\,q(t,p,y) := \max_{u\in R} \{ e^{-\lambda(up+k|u|p+K)} q(t,p,y+u) \},$$

$$\text{iii)} \quad (q(t,p,y)-M\,q(t,p,y))\,Lq(t,p,y) = 0,$$

$$\text{iv)} \quad q(T,p,y) = -e^{-\lambda py} .$$

These qvi will now be solved approximately via an asymptotic analysis. The idea behind this asymptotic approach is the assumption that both the transformed value function q and the transaction costs are functions of a small parameter $\varepsilon > 0$ where the transaction cost function has to vanish for $\varepsilon = 0$. Then the value function will (formally) be expanded in powers of ε. Substituting this expansion into the equation $Lq = 0$ (which is assumed to be valid on the continuation set) and reordering the terms corresponding to different powers of ε, we obtain an infinite set of equations for the coefficients of the expansion which have to vanish simultaneously. By neglecting coefficients of order higher than $\varepsilon^{1/2}$ we will only consider the first three such equations. We can solve them with the help of appropriate boundary conditions which arise from the usual smooth pasting requirements. Note that this is a heuristical point of view as we cannot prove the required regularity. We will not go into further details but refer the reader to Korn (1998) or Whalley and Wilmott (1994).

Insert Figure 3 here

Figure 3 shows a typical example of an asymptotically optimal strategy as a result of an asymptotic analysis. The data are b=0.1, σ=0.2, λ=0.01, k= 0.005, K=0.01 (which corresponds to ε = 0.000855). The strategy of Figure 3 is as follows: if one starts with a positive number of shares of stock not exceeding the upper dotted line as a function of the initial stock price then one should only sell an optimal number of shares if the share price increases sufficiently, i.e. if the initial number of shares reaches the outer dotted line as a function of the stock price. In that event the optimal action is to decrease the holdings such that the pair $(\psi(t).\ P_1(t))$ lies on the next upper dotted line. Note that starting with positive stock holdings it is never optimal to purchase additional shares ! The inner line is the optimal strategy without transaction costs. If one starts with a negative amount of shares of the stock (i.e. if one has sold shares short), it will only be optimal to buy some shares if the stock prices increases sufficiently where the two lower dotted lines take the roles of the upper ones in the case of positive initial stock holdings. If the inital stock holdings are outside the no transaction region then one has to go immediately to the nearest optimal restarting point.

Remark 3.4: *Comments on the asymptotically optimal strategy*
The form of the above strategy has some striking features. First of all, it is *time independent*. This is surely a surprise. The reasons for this are twofold. One reason is that the bond price is constant, i.e. the time value of money does not change. However, it should still be relevant if we are close to the time horizon or not when we decide about a transaction, because immediately before T the expected gain from the better rebalanced position should not be worth the (strictly positive !) transaction costs. The explanation for not taking this into account in the above strategy lies in the assumption that the transaction costs are small. Neglecting higher order terms in the expansion lead to both the time independence and the *symmetry* of the strategy around the Pliska-solution. A natural question to ask is if the above strategy leads to non-negative pairs (B(t), S(t)) if one starts with positive initial holdings. In the case of positive S(0) this will always be the case if the gain from selling the shares of stock exceeds K which is obviously satisfied. We cannot guarantee a non-negative wealth process when starting with a negative S(0). Even more, for every given B(0) the probability for the wealth process X(t) getting negative is positive. However, this is also the case in the situation *without* transaction costs.

Remark 3.5:
There is more work on portfolio optimisation and impulse control. Some remarkable examples are Pliska and Morton (1995) and Bielecki and Pliska (1998). Both these examples have the common feature that the optimality criterion is based on an average over time and is thus essentially a problem with an inifnite horizon. In such a case one can hope for a stationary optimal strategy. Further, Bielecki and Pliska (1998) include the possibility to model prices depending on economic factors which adds more realism to the model. For details we refer the reader to the above cited sources.

iv) Some genral remarks

The above examples highlight the main problems of impulse control methods applied to mathematical finance. As long as the fundamental process is one-dimensional, the time horizon is infinite, and control costs and gain can be separated there is a good chance of solving the problem. Even in this case we see the need of having an explicit analytical solution of the (partial) differential equation associated to the operator L. Only then we have hope for the smooth pasting principle to work. However, typical problems of mathematical finance (such as the portfolio one) are high dimensional and have a finite horizon. One way out of this problem would be to concentrate on stationary strategies, but even this only eliminates the time difference. Another way is to give up the hope for an explicit solution and to rely on numerical methods. As such methods typically rely of discretisation of the qvi we should be sure that the value function satisfies the qvi at least in some weak sense. This problem is dealt with in the next section.

4. Impulse control and viscosity solutions

The main drawback of the optimal portfolio problem of Example 3 iii) lies in the fact that one cannot guarantee sufficient regularity of the solution of the qvi to apply the standard verification theorem 2.5 (respectively a suitable variant). A celebrated approach to overcome such difficulties in stochastic control was the introduction of so called viscosity solutions by Crandall and Lions (1984). The attractive properties of viscosity solutions regarding uniqueness and stability make them an essential tool to solve stochastic control problems. They are especially important for proving convergence of numerical discretisation schemes. For a general introduction to viscosity solutions we refer to the book of Fleming and Soner (1993) or the "user guide" by Crandall, Ishii and Lions (1992). Since these references do not cover the impulse control case we show at least that the value function of our impulse control problem is a viscosity solution of the qvi. For completeness, let us give a definition of a viscosity solution of the qvi (note that the complementarity conditon (2.11) will be modelled by the maximisation in (4.1/2) below). For the moment, we concentrate on the qvi of Definition 2.3 when we will refer to "the" qvi below.

Defintion 4.1
Let v be a continuous function. v is called a
a) *viscosity subsolution* of the qvi if for all $\phi \in C^2$ with $\phi(\bar{x}) = v(\bar{x})$ and $v \le \phi$ we have

(4.1) $\max\{-(\frac{1}{2}\sigma(\bar{x})^2\phi''(\bar{x}) + b(\bar{x})\phi'(\bar{x}) - \alpha v(\bar{x}) + f(\bar{x})),\, v(\bar{x}) - Mv(\bar{x})\} \le 0.$

b) *viscosity supersolution* of the qvi if for all $\phi \in C^2$ with $\phi(\bar{x})=v(\bar{x})$ and $v \ge \phi$ we have

(4.2) $\max\{-(\frac{1}{2}\sigma(\bar{x})^2\phi''(\bar{x}) + b(\bar{x})\phi'(\bar{x}) - \alpha v(\bar{x}) + f(\bar{x})),\, v(\bar{x}) - Mv(\bar{x})\} \ge 0.$

c) *viscosity subsolution* of the qvi if v is both a viscosity sub- and supersolution.

As a converse to the usual verification result 2.5 we have

Theorem 4.2 "Viscosity property of the value function"
Let the coefficients b(.), σ(.) of (2.1) be Lipschitz-continuous and let the running cost function f(.) be polynomially bounded. Assume further that the value function v of (2.5) is continuous, polynomially bounded and satisfies the following Bellman principle:

$$(4.3)\quad v(x) = \inf_{(\theta_i,\Delta x_i)\in Z} E_x^S\Big(e^{-\alpha\tau}v(X(\tau)) + \int_0^\tau e^{-\alpha s}f(X(s))ds + \sum_{i=1}^{\infty} e^{-\alpha\theta_i}(K + k|\Delta x_i|)1_{\{\theta_i\le\tau\}}\Big)$$

for all finite stopping times τ.

Then v is a viscosity solution of the qvi.

Proof:

a) We first verify the viscosity subsolution property. Let $\phi \in C^2$ with $\phi(x) = v(x)$ and $v \le \phi$. As v is polynomially bounded we can assume that ϕ is at most polynomially bounded. Let S denote the non-intervention strategy. Choosing $\tau = t$ in (4.3) yields

$$(4.4) \qquad \phi(x) = v(x) \le E_x^S\left(e^{-\alpha t} v(X(t)) + \int_0^t e^{-\alpha s} f(X(s))ds\right) \le E_x^S\left(e^{-\alpha t}\phi(X(t)) + \int_0^t e^{-\alpha s} f(X(s))ds\right),$$

where the expectations are finite due to the above assumption and the polynomiality of the functions occurring in (4.4). Moreover, due to the continuity of X(s) on [0, t] we have

$$(4.5) \qquad e^{-\alpha t}\phi(X(t)) = \phi(x) + \int_0^t e^{-\alpha s} L\phi(X(s))\, ds + \int_0^t e^{-\alpha s}\phi(X(s)\, \sigma(X(s))\, dW(s),$$

$$(4.6) \qquad \phi(x) = E_x^S(e^{-\alpha t}\phi(X(t)) - \int_0^t e^{-\alpha s} L\phi(X(s))\, ds).$$

Subtracting (4.6) from (4.4), dividing the resulting inequality

$$(4.7) \qquad 0 \le E_x^S\left(\int_0^t e^{-\alpha s}\left[f(X(s) - L\phi(X(s))\right] ds\right)$$

by t and letting $t \downarrow 0$ yields

$$(4.8) \qquad 0 \le L\phi(x) + f(x) = \tfrac{1}{2}\sigma(x)^2\phi''(x) + b(x)\phi'(x) - \alpha v(x) + f(x)$$

via the mean value theorem and the dominated convergence theorem. As $v(x) \le Mv(x)$ is always satisfied we have thus proved the viscosity subsolution property.

b) To prove the viscosity supersolution property let $\phi \in C^2$ with $\phi(x) = v(x)$ and $v \ge \phi$. We will show this by contradiction. By continuity, the assumption of

$$(4.9) \qquad \max\{-(\tfrac{1}{2}\sigma(x)^2\phi_{xx}(x) + b(x)\phi_x(x) - \alpha v(x) + f(x)),\ v(x) - Mv(x)\} < 0$$

yields the existence of a $\delta > 0$ and a $\gamma > 0$ with

$$(4.10) \qquad v(y) \le Mv(y) - \gamma$$

$$(4.11) \qquad \tfrac{1}{2}\sigma(y)^2\phi_{xx}(y) + b(y)\phi_x(y) - \alpha v(y) + f(y) \ge \gamma$$

for all $y \in B_\delta(x)$. Let $\{(\theta_i, \Delta x_i)\}_{i\in N}$ be an admissible strategy, θ be the first exit time of the corresponding controlled pocess X(t) from $B_\delta(x)$. Then we have

$$(4.12) \qquad e^{-\alpha\theta}\phi(X(\theta)) = \phi(x) + \sum_{k=1}^{\infty}\Big[\int_{\theta_{k-1}\wedge\theta}^{\theta_k\wedge\theta} e^{-\alpha s} L\phi(X(s))ds$$

$$+ \int_{\theta_{k-1}\wedge\theta}^{\theta_k\wedge\theta} e^{-\alpha s}\phi(X(s))\sigma(X(s))dW(s)\Big] + \sum_{k=1}^{\infty} e^{-\alpha\theta_k}(\phi(X(\theta_k)-\phi(X(\theta_k-)))1_{\{\theta_k\le\theta\}}$$

and hence

(4.13) $\phi(x)$

$$=E_x^S\Big(e^{-\alpha\theta}\phi(X(\theta)) - \sum_{k=1}^{\infty}\int_{\theta_{k-1}\wedge\theta}^{\theta_k\wedge\theta} e^{-\alpha s}L\phi(X(s))ds - \sum_{k=1}^{\infty} e^{-\alpha\theta_k}(\phi(X(\theta_k)-\phi(X(\theta_k-)))1_{\{\theta_k\le\theta\}}\Big)$$

$$\le E_x^S\Big(e^{-\alpha\theta}\phi(X(\theta)) - \sum_{k=1}^{\infty}\int_{\theta_{k-1}\wedge\theta}^{\theta_k\wedge\theta} e^{-\alpha s}(\gamma - f(X(s)))ds + \sum_{k=1}^{\infty} e^{-\alpha\theta_k}(K+k|\Delta x_k|)1_{\{\theta_k\le\theta\}}\Big)$$

$$\le E_x^S\Big(e^{-\alpha\theta}\phi(X(\theta)) + \sum_{k=1}^{\infty}\int_{\theta_{k-1}\wedge\theta}^{\theta_k\wedge\theta} e^{-\alpha s}f(X(s))ds + \sum_{k=1}^{\infty} e^{-\alpha\theta_k}(K+k|\Delta x_k|)1_{\{\theta_k\le\theta\}}\Big)$$

$$-\gamma E_x^S\Big(\sum_{k=1}^{\infty}\int_{\theta_{k-1}\wedge\theta}^{\theta_k\wedge\theta} e^{-\alpha s}ds\Big).$$

This implies

$$\text{(4.14)}\quad v(x) \le E_x^S\Big(e^{-\alpha\theta}v(X(\theta)) + \sum_{k=1}^{\infty}\int_{\theta_{k-1}\wedge\theta}^{\theta_k\wedge\theta} e^{-\alpha s}f(X(s))ds$$

$$+ \sum_{k=1}^{\infty} e^{-\alpha\theta_k}(K+k|\Delta x_k|)1_{\{\theta_k\le\theta\}}\Big) - \gamma E_x^S\Big(\sum_{k=1}^{\infty}\int_{\theta_{k-1}\wedge\theta}^{\theta_k\wedge\theta} e^{-\alpha s}ds\Big)$$

$$\le E_x^S\Big(e^{-\alpha\theta}v(X(\theta)) + \sum_{k=1}^{\infty}\int_{\theta_{k-1}\wedge\theta}^{\theta_k\wedge\theta} e^{-\alpha s}f(X(s))ds$$

$$+ \sum_{k=1}^{\infty} e^{-\alpha\theta_k}(K+k|\Delta x_k|)1_{\{\theta_k\le\theta\}}\Big) - \frac{\gamma}{\alpha} E_x^S\big(1-e^{-\alpha(\theta\wedge\theta_1)}\big)$$

Every strategy with positive probability for an intervention before X(t) leaves the ball $B_\delta(x)$ cannot yield the infimum in the Bellman principle due to (4.10). Even more, due to (4.10) there does not exist a sequence of strategies such that the corresponding costs converge against the infimum. On the other hand, for each strategy without an intervention before θ the second expectation in (4.14) is strictly positive and *independent* of that strategy. Also, for such a strategy the corresponding costs are above a lower bound which is bigger than the infimum in the Bellman principle (4.3). In total, assumption (4.9) leads to a contradiction to the Bellman principle. Hence, we have proved viscosity supersolution property.

Remark 4.3 "Generalisations"

a) Making obvious modifications, the proof of the theorem remains valid in

- the general n-dimensional case with $x\in\mathbf{R}^n$.

- the general n-dimensional case with $(t,x)\in[0, T)\times\mathbf{R}^n$, if one additionally considers the occurrence of the time variable t in v, the time derivative in the qvi, the terminal condition $v(T, x) = \Psi(x)$, and the requirement that $\Psi(x)$ is polynomially bounded.

b) In the presence of the constraint $X(t)\subseteq O$ where $O\subseteq\mathbf{R}^n$ is a closed set, the non-intervention strategy as chosen in part a) above need not be admissible ! However, by choosing $\tau:= \inf\{s > 0 \mid X(s)\in\partial^*O\} \wedge t$ the proof remains correct. In part b) of the proof we have to choose δ, γ such that we have $B_\delta(t, x)\subset O$. Note in particular that due to our assumption we have $(t, x)\notin\partial^*O$.

The following uniqueness result states that the value function is the the only (sufficiently regular) viscosity solution of the qvi satisfying the Bellman principle (3). A proof of this result can be found in Korn (1999).

Theorem 4.4 "Uniqueness"
Under the assumptions of Theorem 4.2 and that of a bounded, Lipschitz continuous function f(x) the value function is the unique bounded, continuous function of the qvi.

Remark 4.5:
a) The main purpose of the uniqueness theorem is a substantial weakening of the verification theorem 2.5. If we have a candidate for the value function, and if we are able to show that it satisfies the assumptions of Theorem 4.4 and is a continous, bounded solution of the qvi then we have found the value function. Further applications of the theory of viscosity solutions such as convergence proofs of numerical schemes or stability theorems are currently under research.
b) Conditions for continuity of the value function are given in Chapter 3 of Korn (1997a).

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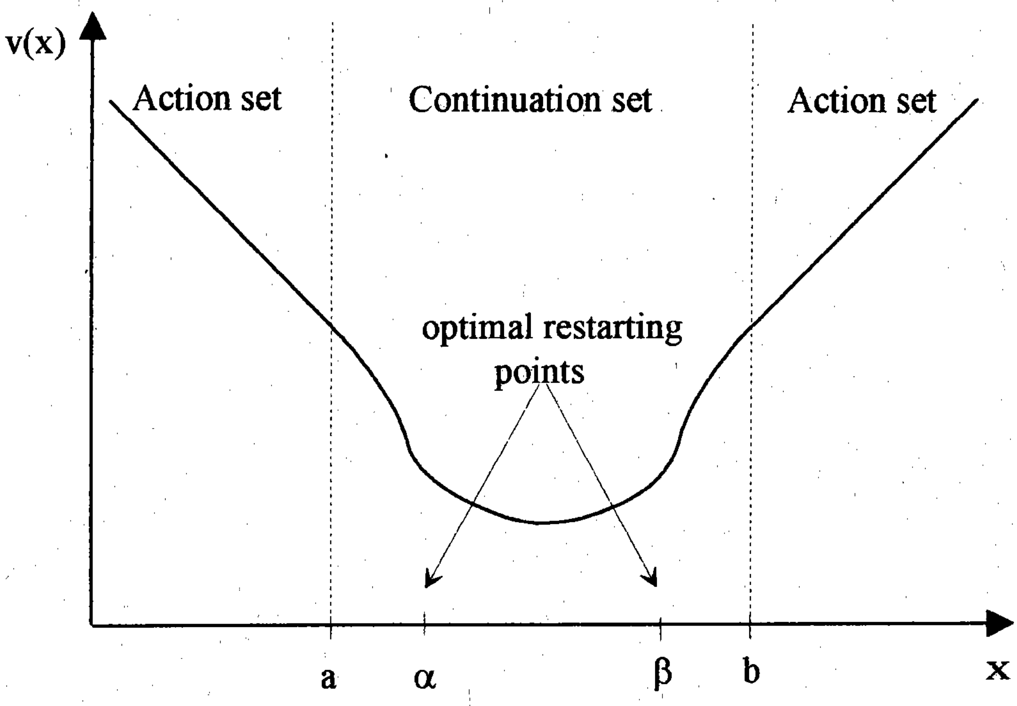

Figure 1: Value function, continuation and action sets

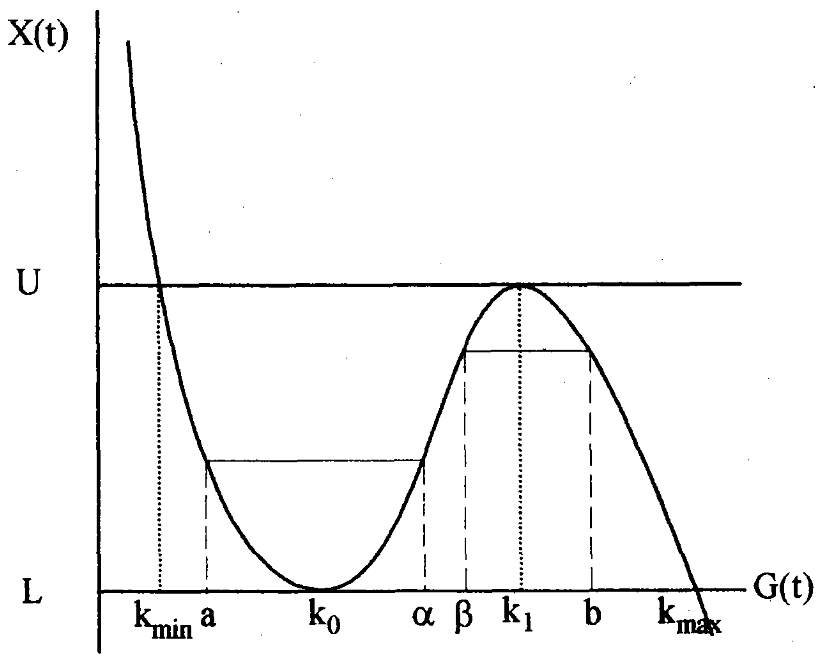

Figure 2: Log. exchange rate and control band

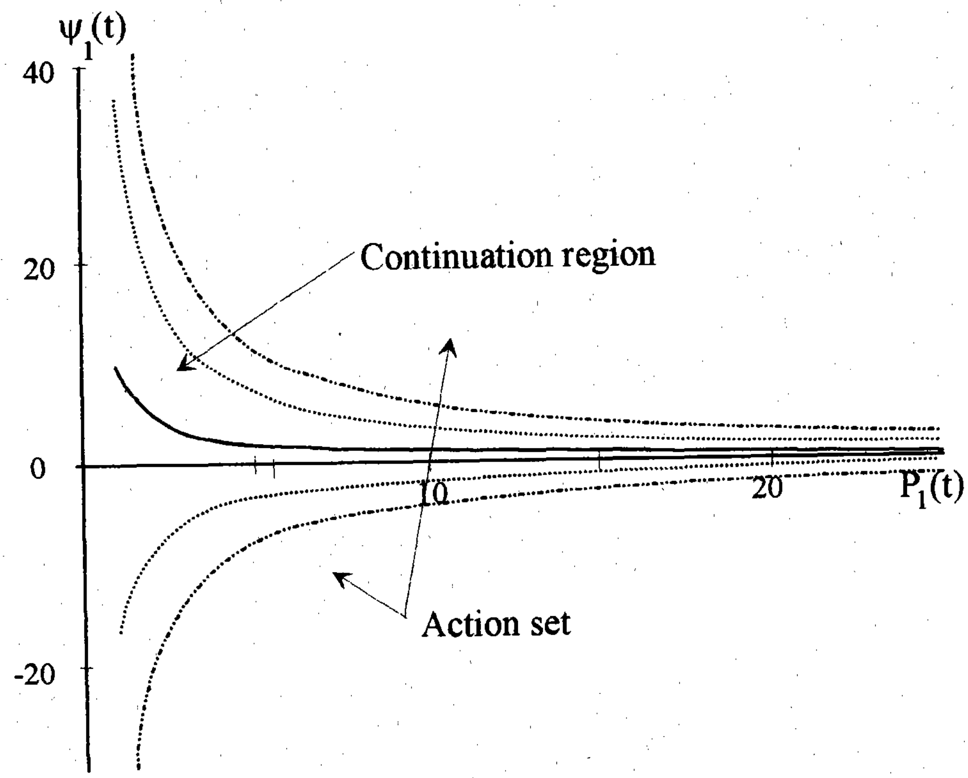

Figure 3: Optimal stock holdings