

# Anchored hyperplane location problems

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## Abstract

The *anchored hyperplane location problem* is to locate a hyperplane passing through some given points  $\mathcal{P} \subseteq \mathbb{R}^n$  and minimizing either the sum of weighted distances (*median problem*), or the maximum weighted distance (*center problem*) to some other points  $\mathcal{Q} \subseteq \mathbb{R}^n$ .

If the distances are measured by a norm, it will be shown that in the median case there exists an optimal hyperplane that passes through at least  $n - k$  affinely independent points of  $\mathcal{Q}$ , if  $k$  is the maximum number of affinely independent points of  $\mathcal{P}$ . In the center case, there exists an optimal hyperplane which is at maximum distance to at least  $n - k + 1$  affinely independent points of  $\mathcal{Q}$ . Furthermore, if the norm is a smooth norm, *all* optimal hyperplanes satisfy these criteria. These new results generalize known results about unrestricted hyperplane location problems.

## 1 Introduction

Approximating a set of given points  $\mathcal{Q}$  in  $\mathbb{R}^n$  by a linear function is known as the *linear fit problem* or the *hyperplane location problem*. The goal is to find a hyperplane minimizing the sum of weighted distances to the points in  $\mathcal{Q}$ , or minimizing the maximum weighted distance to the points in  $\mathcal{Q}$ , respectively. As distance measure any norm is possible, but also gauges and metrics have recently been discussed ([Sch99, PC01]). In this paper a restricted version of the hyperplane location problem — the so called *anchored hyperplane location problem* — is analyzed, namely, the hyperplane is additionally forced to pass through some given points  $p \in \mathcal{P}$ .

Hyperplane location problems appear in several mathematical disciplines, where they have mainly been studied with the Euclidean and the rectangular distance. In *robust statistics* variants of the hyperplane location problem are known as absolute errors regression, median problems,  $L_1$  regression,  $L_\infty$  regression, and orthogonal/vertical  $L_1$ -fit, or  $L_\infty$ -fit problems, respectively, depending on the type of distance measure and on the objective function used. Related investigations are going back to the 18th century (see e.g. [Bos57]). In [RL87], a motivation of these problems is given. In *numerical mathematics*, certain approximation problems, for example the approximation of a given function by a linear function, lead in a natural way to the same type of problems, see, e.g., [Ric64]. In *computational geometry*, hyperplane location problems are known as linear  $L_1$  or  $L_\infty$  approximation problems. Here, in particular, the time complexity of the Euclidean variant of the problem was investigated by several authors, cf. e.g., [KM93, HII<sup>+</sup>93]. The two-dimensional version of the problem has been studied also in *operations research*, known as the *line facility location problem* which is a special case of location problems. Line location problems in the plane were first discussed by [Wes75], and later by many other authors, e.g., [MN80, MT82, LW86]. Recently, line and hyperplane location problems have also been discussed for distance measures apart from the Euclidean and the rectangular distance, namely, for block norms by [Sch96], arbitrary norms by [Sch98, MS98], smooth norms (see [MS]), and also for gauges (see [PC01]).

If distances are measured by a norm, the main results for unrestricted hyperplane location problems are the following incidence criteria. There always exists an optimal hyperplane for the median problem that passes through  $n$  affinely independent points of  $\mathcal{Q}$ , and in the center case, there always exists an optimal hyperplane which is at maximum distance from  $n + 1$  affinely independent points of  $\mathcal{Q}$ . This has first been noted for the Euclidean distance, independently by many authors (see, for example, [Wes75, MN80, LW86, MN83, KM93, HII<sup>+</sup>93]) and has later been generalized to block norms and even to distances derived from arbitrary norms ([Sch98, MS98, Sch99]). A slightly weaker condition for gauges in the case of the median objective function has recently been developed by [PC01]. Furthermore, if and only if the norm is a smooth norm, *all* optimal hyperplanes in the median problem pass through  $n$  affinely independent points of  $\mathcal{Q}$ , and in the center problem *all* optimal hyperplanes are at maximum distance from  $n + 1$  affinely independent points of  $\mathcal{Q}$  ([MS]).

In this paper a new technique is developed to transfer both incidence properties to restricted hyperplane location problems in the following sense. If the hyperplane approximating  $\mathcal{Q}$  is forced to contain  $k$  affinely independent points of a set  $\mathcal{P}$  then there exists a median hyperplane passing through at least  $n - k$  affinely independent points of  $\mathcal{Q}$ , and a center hyperplane which is at maximum distance from  $n - k + 1$  affinely independent points of  $\mathcal{Q}$ . Sharper results for smooth norms will also be developed. For  $k = 0$  this directly yields new proofs for the incidence properties of unrestricted hyperplane location problems.

## 2 The Anchored Hyperplane Location Problems

Let  $d$  be a distance derived from a norm, i.e.,

$$d(x, y) = \gamma(y - x) \text{ for all } x, y \in \mathbb{R}^n$$

for some norm  $\gamma$ . We consider the following problems (AMH) and (ACH):

Given two finite sets  $\mathcal{P}$  and  $\mathcal{Q}$  of points in  $\mathbb{R}^n$ , find a hyperplane  $H$  passing through all points in  $\mathcal{P}$  and minimizing

$$\text{(AMH)} \quad f(H) = \sum_{q \in \mathcal{Q}} w_q d(H, q), \text{ or}$$

$$\text{(ACH)} \quad g(H) = \max_{q \in \mathcal{Q}} w_q d(H, q), \text{ respectively,}$$

where  $w_q \geq 0$  for all  $q \in \mathcal{Q}$  are nonnegative weights and  $d$  is a distance measure derived from a norm.

An optimal hyperplane  $H$  for problem (AMH) will be called an *anchored median hyperplane*, and an *anchored center hyperplane* is an optimal solution of problem (ACH).

First, note that the anchored hyperplane location problem only is feasible, if there exists a hyperplane containing the whole set  $\mathcal{P}$ . In particular, (AMH) and (ACH) are feasible if and only if the maximum number of affinely independent points in  $\mathcal{P}$  is smaller or equal than  $n$ . Let  $k = \dim_{\text{aff}}(\mathcal{P})$  denote the maximum number of affinely independent points in  $\mathcal{P}$ . Without loss of generality, we therefore assume in the following that

$$k \leq n.$$

Furthermore, note that, if  $k + \dim_{\text{aff}}(\mathcal{Q}) \leq n$  then there exists a (clearly optimal) hyperplane containing *all* points in  $\mathcal{P}$  and in  $\mathcal{Q}$ . To avoid this trivial case, we also assume that

$$\dim_{\text{aff}}(\mathcal{Q}) > n - k.$$

As mentioned before, both problems (AMH) and (ACH) have been studied extensively in their unrestricted versions, i.e., with  $\mathcal{P} = \emptyset$  (see, e.g., [MS98, Sch99] and references therein). The restricted version of (AMH) has been discussed by [MN83] in a planar setting, i.e.,  $n = 2$  and  $\mathcal{P}$  consisting of one single point. For the Euclidean distance it has been shown, that all optimal lines for problem (AMH) contain at least one point from the set  $\mathcal{Q}$ . The same has been noted in [KM93].

Anchored line problems, also in  $\mathbb{R}^3$ , have been considered in [Fol95, FSS<sup>+</sup>95], but with another objective function. Motivated by problems in brain surgery, it is discussed how to find a line maximizing the closest distance to the given point set  $\mathcal{Q}$ .

In the following we will generalize the incidence properties of unrestricted hyperplane location problems to the restricted versions of (AMH) and (ACH). The proofs we are going to present will combine the techniques developed in [Sch99] and in [PC01]. They are based on the minimization of quasiconcave functions, which contain the sum, or the maximum of piecewise linear functions. We first turn to the case of median hyperplanes.

### 3 Anchored Median Hyperplanes

**Lemma 1** *Let  $\mathcal{M} = \{1, 2, \dots, M\}$ ,  $M \geq n$  and let  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function, and  $f_m : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $m \in \mathcal{M}$  be affine linear and non-constant functions. Consider the problem to minimize*

$$f(x) = \frac{1}{h(x)} \sum_{m \in \mathcal{M}} |f_m(x)|.$$

*Furthermore, suppose that a minimum of  $f$  exists. Then there exists an optimal solution  $x^*$  and a subset  $\mathcal{M}^* \subseteq \mathcal{M}$  with  $|\mathcal{M}^*| \geq n$  such that*

$$f_m(x^*) = 0 \text{ for all } m \in \mathcal{M}^*.$$

Moreover, if  $h$  is strictly convex, all optimal solutions  $x^*$  satisfy  $f_m(x^*) = 0$  for at least  $n$  of the functions  $f_m$ .

Proof: Since the functions  $f_m$  are non-constant, the following sets

$$H_m := \{x \in \mathbb{R}^n : f_m(x) = 0\}, m \in \mathcal{M}$$

are hyperplanes that partition  $\mathbb{R}^n$  into polyhedral cells. On each cell  $C$  all the functions  $|f_m|$ ,  $m \in \mathcal{M}$  are affine linear, and therefore also  $\sum_{m \in \mathcal{M}} |f_m(x)|$  is an affine linear function. Since  $h$  is convex,  $f$  is a quasiconcave function on each cell. Minimizing over each of the cells separately, and using that a finite optimal solution exists, yields a minimum at a cell vertex  $x^*$ , (see, e.g. [BSS93]). Since  $x^*$  is a cell vertex, it lies on the intersection of at least  $n$  hyperplanes  $H_m$ . Defining

$$\mathcal{M}^* := \{m \in \mathcal{M} : x^* \in H_m\}$$

yields the first result.

For the second part of the lemma, note that an affine linear function divided by a strictly convex function is a strictly quasiconcave function, meaning that on each cell  $C$  the objective  $f$  is quasiconcave and therefore attains its minima only at cell vertices. QED

In the following we will describe a hyperplane by its normal vector  $s \in \mathbb{R}^n \setminus \{0\}$  and its intercept  $b \in \mathbb{R}$ , i.e.,

$$H_{s,b} = \{x \in \mathbb{R}^n : \langle s, x \rangle + b = 0\}.$$

We now can state the main result for anchored median hyperplanes.

**Theorem 1** *Let  $d$  be a distance derived from a norm  $\gamma$ , and let  $k \leq n$  be the number of affinely independent points in  $\mathcal{P}$ . Then there exists an anchored median hyperplane passing through at least  $n - k$  affinely independent points of  $\mathcal{Q}$ .*

Proof: First note that hyperplanes lying too far away from  $\mathcal{Q}$  need not be considered, and since the length of the normal vector  $s$  can be assumed to be bounded, we can restrict the problem to a compact feasible set yielding the existence of a minimum. Furthermore, since  $s \neq 0$  is required for a hyperplane, we conclude that the normal vector  $s^*$  of the optimal solution to

problem (AMH) has at least one non-zero component. Without loss of generality assume that this is the first one, i.e,  $s_1^* \neq 0$ . Dividing all coefficients by  $s_1^*$  yields an optimal solution with first component of the normal vector is equal to 1. Without loss of generality we will therefore assume  $s_1^* = 1$ . According to [Man99] or, independently, [PC01] the distance between a point  $q \in \mathbb{R}^n$  and a hyperplane  $H_{s,b}$  can be calculated by

$$d(q, H_{s,b}) = \frac{|\langle s, q \rangle + b|}{\gamma^0(s)},$$

where  $\gamma^0$  denotes the dual (or polar) norm of  $\gamma$ , defined by

$$\gamma^0(x) = \max\{\langle x, y \rangle : \gamma(y) \leq 1\}.$$

For  $\mathcal{P} = \{p_1, p_2, \dots, p_l\}$  problem (AMH) can now be reformulated as the following non-linear programming problem in  $n + 1$  variables.

$$\min_{s,b} \frac{1}{\gamma^0(s)} \sum_{q \in \mathcal{Q}} w_q |\langle s, q \rangle + b|$$

$$\begin{aligned} \text{such that } \langle s, p_i \rangle + b &= 0 \text{ for } i = 1, 2, \dots, l \\ s_1 &= 1 \end{aligned}$$

Since  $k$  is the maximum number of affinely independent points of  $\mathcal{P}$  we conclude that the linear dimension

$$\dim \left\{ \begin{pmatrix} p_1 \\ 1 \end{pmatrix}, \begin{pmatrix} p_2 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} p_l \\ 1 \end{pmatrix} \right\} = k,$$

where  $\begin{pmatrix} p_i \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$ ,  $i = 1, 2, \dots, l = |\mathcal{P}|$ . Consequently, the coefficient matrix of

$$\langle s, p_i \rangle + b = 0 \text{ for } i = 1, 2, \dots, l$$

has rank  $k$ , meaning that there exist  $k$  variables which can be substituted and eliminated in the objective function. Together with  $s_1 = 1$  (which is linearly independent of the equations  $\langle s, p_i \rangle + b = 0$ ) there remain  $(n + 1) - k - 1 =$

$n - k$  variables, denoted by  $t \in \mathbb{R}^{n-k}$ . This leads to the following equivalent problem in  $\mathbb{R}^{n-k}$ .

$$\min f(t) = \frac{1}{\gamma^0(s(t))} \sum_{q \in \mathcal{Q}} w_q |\langle s(t), q \rangle + b(t)|$$

with  $s$  and  $t$  affine linear functions,

$$\begin{aligned} s : \mathbb{R}^{n-k} &\rightarrow \mathbb{R}^n \text{ and} \\ b : \mathbb{R}^{n-k} &\rightarrow \mathbb{R}. \end{aligned}$$

Since the composition of a convex and an affine linear function still remains convex, and the functions  $f_q(t) = w_q(\langle s(t), q \rangle + b(t))$  are affine linear we can apply Lemma 1 in dimension  $n - k$  and conclude that there exists an optimal solution  $t^*$  and a set  $\mathcal{Q}^* \subseteq \mathcal{Q}$  with  $|\mathcal{Q}^*| \geq n - k$  such that

$$\langle s(t^*), q \rangle + b(t^*) = 0$$

for all  $q \in \mathcal{Q}^*$ . Defining  $s^* = s(t^*)$  and  $b^* = b(t^*)$  it follows that for all  $q \in \mathcal{Q}^*$

$$q \in H_{s^*, b^*},$$

meaning that the optimal hyperplane  $H^* := H_{s^*, b^*}$  passes through at least  $n - k$  points in  $\mathcal{Q}$ . To see that there exists a hyperplane passing through at least  $n - k$  *affinely independent* points, find some affinely dependent points in  $\mathcal{Q}^*$ , delete them from  $\mathcal{Q}$ , and, with the same argumentation, find a new hyperplane until it passes through  $n - k$  affinely independent points. QED

Some remarks should be added.

- Note that the optimal hyperplane may pass through more than  $n - k$  affinely independent points since points in  $\mathcal{Q}$  that lie within the affine hull of  $\mathcal{P}$  will automatically be covered.
- The statement

*Consider problem (AMH) with a distance derived from a norm, and with  $|\mathcal{P}| \leq n$ . Then there exists an anchored median hyperplane passing through at least  $n - |\mathcal{P}|$  affinely independent points of  $\mathcal{Q}$ .*

follows directly from Theorem 1, but only provides a weaker result.

- For  $\mathcal{P} = \emptyset$  a *halving property* has been shown for locating a median hyperplane in normed spaces (see [Sch99]), i.e., if  $Q^+(H)$ ,  $Q^-(H)$ , and  $Q^0(H)$  are the points of  $Q$  lying below, underneath, and on the hyperplane  $H$ , respectively, then *all* median hyperplanes satisfy

$$\left| \sum_{q \in Q^+} w_q - \sum_{q \in Q^-} w_q \right| \leq \sum_{q \in Q^0} w_q.$$

This property does in general not hold true, if  $|\mathcal{P}| \geq 1$ , not even for  $n = 2$  and the Euclidean distance, as the following example demonstrates:

Let  $\mathcal{P} = \{0\}$ ,  $\mathcal{Q} = \{(10, 1), (11, 0), (-10, 1)\}$ , and assume equal weights for the points in  $\mathcal{Q}$ . Then the optimal anchored line  $l^*$  passes through 0 and (10, 1), with the two remaining points of  $\mathcal{Q}$  lying on the same side of  $l^*$ .

## 4 Anchored Center Hyperplanes

In this section, too, we first derive a result for minimizing a quasiconcave function. This time, we deal with a function  $g$ , which is given as the maximum of piecewise linear functions, all divided by the same convex denominator.

**Lemma 2** *Let  $\mathcal{M} = \{1, 2, \dots, M\}$ ,  $M \geq n$  and let  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function, and  $f_m : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $m \in \mathcal{M}$  be affine linear functions. Consider the problem to minimize*

$$g(x) = \frac{1}{h(x)} \max_{m \in \mathcal{M}} |f_m(x)|.$$

*Furthermore, suppose that a minimum of  $g$  exists. Then there exists an optimal solution  $x^*$  and a subset  $\mathcal{M}^* \subseteq \mathcal{M}$  with  $|\mathcal{M}^*| \geq n + 1$  such that*

$$g(x^*) = \frac{|f_m(x^*)|}{h(x^*)} \text{ for all } m \in \mathcal{M}^*.$$

*Moreover, if  $h$  is strictly convex, all optimal solutions  $x^*$  satisfy  $g(x^*) = \frac{|f_m(x^*)|}{h(x^*)}$  for at least  $n + 1$  of the indices  $m \in \mathcal{M}$ .*

**Proof:** First suppose that there exists some  $x \in \mathbb{R}^n$  such that  $f_m(x) = 0$  for all  $m \in \mathcal{M}$ . Then  $x$  is the optimal solution and the lemma is trivially true.



Otherwise, we can assume that  $\max_{m \in \mathcal{M}} |f_m(x)| > 0$  for all  $x \in \mathbb{R}^n$ . Define for all  $m \in \mathcal{M}$  two cells given by

$$\begin{aligned} C_m^+ &= \{x \in \mathbb{R}^n : f_m(x) = \max_{k \in \mathcal{M}} |f_k(x)|\}, \text{ and} \\ C_m^- &= \{x \in \mathbb{R}^n : -f_m(x) = \max_{k \in \mathcal{M}} |f_k(x)|\}. \end{aligned}$$

Then each of the  $C_m^+, C_m^-$  is either empty or a polyhedral set bounded by hyperplanes

$$\begin{aligned} H_k^+ &= \{x \in \mathbb{R}^n : f_m(x) = f_k(x)\} \text{ or} \\ H_k^- &= \{x \in \mathbb{R}^n : f_m(x) = -f_k(x)\} \end{aligned}$$

for  $k \neq m$ . On each cell,  $\max_{k \in \mathcal{M}} |f_k(x)|$  is affine linear and hence,  $g$  is quasiconcave on each cell. Since a minimum exists, we conclude that there exists an optimal cell vertex  $x^*$ , e.g., of the cell  $C_{\bar{m}}^+$  (see, e.g. [BSS93]). Due to the assumption that  $\max_{m \in \mathcal{M}} |f_m(x)| > 0$  for all  $x \in \mathbb{R}^n$ , we know that  $H_k^+ \cap H_k^- \cap C_{\bar{m}}^+ = \emptyset$  for all  $k \in \mathcal{M}$  and hence each cell vertex of  $C_{\bar{m}}^+$  is the intersection of  $n$  of the hyperplanes  $H_k^+, H_k^-, k \neq \bar{m}$ , with pairwise different indices  $k$ . Consequently, the cardinality of

$$\mathcal{M}^* = \{k \in \mathcal{M} : x^* \in H_k^+ \text{ or } x^* \in H_k^-\} \cup \{\bar{m}\}$$

is greater or equal than  $n + 1$ , proving the first part of the lemma.

In case of a strictly convex function  $h$ , we get that  $g$  is strictly quasiconcave on each cell  $C_m$ , and hence *all* optimal solutions are attained at cell vertices. QED

Note that in [Dre82] the following related result for the case of *convex* functions  $f_m, m \in \mathcal{M}$  has been provided: There exists an optimal solution  $x^*$  and a subset  $\mathcal{M}^* \subseteq \mathcal{M}$  with  $|\mathcal{M}^*| \leq n + 1$  such that  $x^*$  is the optimal solution for the reduced problem

$$\min_{m \in \mathcal{M}^*} \max f_m(x),$$

i.e., an optimal solution can be found by looking at all subsets with cardinality smaller or equal than  $n + 1$  separately.

Now the main result for finding anchored center hyperplanes can be shown.

**Theorem 2** *Let  $d$  be a distance derived from a norm  $\gamma$ , and let  $k \leq n$  be the number of affinely independent points in  $\mathcal{P}$ . Then there exists an anchored center hyperplane which is at maximum distance from at least  $n - k + 1$  affinely independent points of  $\mathcal{Q}$ .*

Proof: The proof works along the lines of the proof to Theorem 1. Assuming  $s_1^* = 1$  and using

$$d(q, H_{s,b}) = \frac{|\langle s, q \rangle + b|}{\gamma^0(s)},$$

for the distance between a point  $q$  and a hyperplane  $H_{s,b}$ , (ACP) can be rewritten as

$$\min g(t) = \frac{1}{\gamma^0(s(t))} \max_{q \in \mathcal{Q}} w_q |\langle s(t), q \rangle + b(t)|$$

with  $s$  and  $t$  affine linear functions,  $\gamma^0(s(t))$  convex, and  $t \in \mathbb{R}^{n-k}$ .

Applying Lemma 2 in dimension  $n - k$  yields the existence of an optimal solution  $t^*$  and a set  $\mathcal{Q}^* \subseteq \mathcal{Q}$  with  $|\mathcal{Q}^*| \geq n - k + 1$  such that for  $s^* := s(t^*)$  and  $b^* := b(t^*)$  we get that

$$g(H_{s^*,b^*}) = \frac{1}{\gamma^0(s^*)} \max_{q' \in \mathcal{Q}} w_{q'} |\langle s^*, q' \rangle + b^*| = \frac{w_q}{\gamma^0(s^*)} |\langle s^*, q \rangle + b^*|$$

for all  $q \in \mathcal{Q}^*$ . Hence, the optimal hyperplane  $H^* := H_{s^*,b^*}$  is at maximum distance from at least  $n - k + 1$  points in  $\mathcal{Q}$ , and using the same iterative argument as in the proof of Theorem 1 shows that these points can be assumed to be affinely independent. QED

## 5 Anchored Hyperplanes and Smooth Norms

A smooth norm  $\gamma$  is defined as follows. Consider the unit ball of  $\gamma$ , given by

$$B_\gamma = \{x \in \mathbb{R}^n : \gamma(x) \leq 1\}.$$

The norm  $\gamma$  is called a *smooth norm*, if  $B_\gamma$  is supported by exactly one hyperplane for any point  $x \in \partial B_\gamma$  on its boundary, i.e. the unit ball of  $\gamma$  contains no vertices. In this case we get the following stronger result.

**Theorem 3** *Let  $d$  be a distance derived from a smooth norm  $\gamma$ , and let  $k$  be the number of affinely independent points in  $\mathcal{P}$ . Then **all** anchored median hyperplanes are passing through at least  $n - k$  affinely independent points of  $Q$ , and **all** anchored center hyperplanes are at maximum distance from at least  $n - k + 1$  affinely independent points of  $Q$ .*

Proof: Since  $\gamma$  is a smooth norm, the dual norm  $\gamma^0$  is strictly convex, see for example [Köt66]. Consequently, we derive the following objective functions (analogously to the proof of Theorem 1), but with a strictly convex denominator  $g(s(t))$ .

$$\min f(t) = \frac{1}{\gamma^0(s(t))} \sum_{q \in Q} w_q |\langle s(t), q \rangle + b(t)|$$

for the median problem, and

$$\min g(t) = \frac{1}{\gamma^0(s(t))} \max_{q \in Q} w_q |\langle s(t), q \rangle + b(t)|$$

for the center problem, respectively. As in the proofs of Theorems 1 and 2  $s$  and  $t$  are affine linear functions and  $t \in \mathbb{R}^{n-k}$ . Since the composition of a strictly convex function and an affine linear function still remains *strictly* convex we can apply the second part of Lemma 1 and of Lemma 2, respectively, and conclude the result. QED

## 6 Conclusion

Theorems 1 and 2 provide the basics for polynomial-time algorithms to solve anchored hyperplane location problems with median or center objective function, since an enumeration approach is possible in both cases. If  $k$  is the number of affinely independent points in  $\mathcal{P}$  we have to check

- all  $\binom{n-k}{|Q|}$  hyperplanes passing through all points in  $\mathcal{P}$  and through any (affinely independent) subset of  $n - k$  points of  $Q$  in the median case, and

- all  $\binom{n-k+1}{|\mathcal{Q}|}$  hyperplanes which are at maximum distance from  $n-k+1$  (affinely independent) points of  $\mathcal{Q}$  and contain all points in  $\mathcal{P}$  in the center case.

Since evaluating a hyperplane can be done in  $O(|\mathcal{Q}|)$ , this yields the following complexity results.

**Corollary 1** *If the distance has been derived from a norm, an anchored median hyperplane can be found in  $O(|\mathcal{Q}|^{n-k+1})$  time, and an anchored center hyperplane can be found in  $O(|\mathcal{Q}|^{n-k+2})$  time, assuming that a norm evaluation can be done in constant time.*

*In the case that  $d$  has been derived from a smooth norm, the same time complexity is sufficient to determine **all** optimal anchored hyperplanes.*

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