Absorption Cross Section of Scalar Field in Supergravity Background

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Abstract

It has recently been shown that the equation of motion of a massless scalar field in the background of some specific p branes can be reduced to a modified Mathieu equation. In the following the absorption rate of the scalar by a $D3$ brane in ten dimensions is calculated in terms of modified Mathieu functions of the first kind, using standard Mathieu coefficients. The relation of the latter to Dougall coefficients (used by others) is investigated. The S–matrix obtained in terms of modified Mathieu functions of the first kind is easily evaluated if known rapidly convergent low energy expansions of these in terms of products of Bessel functions are used. Leading order terms, including the interesting logarithmic contributions, can be obtained analytically.

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I. INTRODUCTION

Recently the equations of motion of several cases of massless scalar fields propagating in a supergravity background describing p–brane solitons have been shown to be reducible to a Schrödinger–like equation with a singular potential and hence to a modified Mathieu equation, so that various aspects, such as absorption probabilities, become exactly calculable [1,2], which by AdS/CFT correspondence may yield information on correlation functions in a related world volume effective field theory. The singular potential appearing in the coefficients of the metric is in the case of the $D3$–brane the Coulomb potential in 6 spatial dimensions. In view of the fact, that very few such exactly solvable cases are known and that a Mathieu–type equation arises in a number of such problems as a result of the invariance of the wave equation under various diagonal dimensional reductions on the world volume, these theories are of exceptional importance and deserve to be studied in full detail. The two recent investigations [1,2] of the absorption of partial waves of a massless scalar field by $D3$ branes in 10 dimensions [1] and by a dyonic string in six dimensions (or a $D1/D5$ brane intersection in 10 dimensions or extremal 2–charge black hole in 5 dimensions or $M2/M5$ brane intersection in 11 dimensions) [2] study the resulting modified Mathieu equation in terms of expansion coefficients introduced by Dougall [3] in 1916 and present very few details. It is not possible to follow the calculations of these papers without extensive work of one's own, which is made even more difficult by singularities of expansion coefficients that require additional attention. These studies are, in fact, complicated applications of modified Mathieu functions, which, in our opinion become even more complicated if instead of standärd Mathieu coefficients, i.e. those in modern texts, the coefficients of Dougall are used, for the calculation of which the authors of ref. [1] developed in addition their own algorithm. The significance of the Mathieu equation in such contexts can also be seen from a different angle since the equation occurs also as the appropriately transformed small fluctuation equation in the study of Born–Infeld theory in the bosonic light–brane approximation with only an electric field $\mathbf{E} = -\nabla\phi$ in $p = 3$ dimensions and the remaining

components of the vector potential as massless scalar fields, reduced to only one field y in the simplest case. One can show that finite energy configurations of these fields independent of one another are not stable, but their combination with appropriate boundary condition [4,5] (equivalent to the Dirichlet boundary condition) is (with supersymmetry) a BPS state with corresponding Bogomol'nyi equation. Investigating the stability of this configuration, i.e. the $D3$–brane, with respect to transverse fluctuations of both the throat or fundamental string and the brane, one again arrives at an equation with the singular potential $1/r^4$ [4,6] which can be converted into a modified Mathieu equation [7,8]. Thus in each of these cases a Schrödinger–like equation is obtained with the singular potential $1/r^4$. Such potentials have been the subject of investigation 30 years ago [9] and were motivated by the lack of understanding of weak interactions at that time. Thus the potential $1/r^4$ and the associated scattering problem had also been investigated, and various forms of the S–matrix had been given [7,8,10,11] in terms of modified Mathieu functions or related functions for which – at the latest since the publication of refs. [12] and [13] – widely used definitions and notations exist.

In view of the scarcity of fully solvable examples of theories on a supergravity background we consider it worthwhile to reexamine the case of the propagation of a massless scalar field in the presence of a 3–brane by using modified Mathieu functions with standard Mathieu coefficients and the S–matrix evaluated in terms of these. In our opinion these calculations are more transparent than those using Dougall coefficients and are easier to follow with reference to modern literature on the subject. In view of the complexity of the calculations, due also to the fact that later iterations contribute to earlier lower order terms, we present these in some detail. Our presentation below should therefore also enable others to follow the reasoning, and this particularly since leading order terms can be understood without resorting to numerical methods.

In the following we first formulate the semiclassical gravity problem and reduce it to the modified Mathieu equation. We do not rederive the S–matrix, but recapitulate in Appendix A the main steps in the derivation, and in particular some steps that have not been written

out explicitly in ref. [8], this being the prime reference on which our considerations are based. We then consider briefly the gauge field theory approach in a simplified Born–Infeld version to demonstrate how this also leads to the Mathieu equation. Following this we consider the Floquet exponent associated with Mathieu functions and show how this has to be calculated in singular cases (such as the the cases to be considered here and in the S–wave case already in the dominant approximation). The calculation of coefficients of series expansions of modified Mathieu functions is then considered and the Dougall coefficients used in refs. [1,2] are compared with ordinary, i.e. standard, Mathieu coefficients as in ref. [12]. We calculate examples in singular and asymptotic cases (the latter being those that permit one to ignore the singularities of early coefficients). Higher order contributions are obtained with Mathematica. We show that Dougall coefficients are more difficult to obtain than ordinary coefficients – an observation that may explain why Dougall did not evaluate any of his own coefficients in his work of 1916. We then evaluate the relevant quantities appearing in the S–matrix and hence the absorption probabilities and cross sections. Where comparable, our results can be seen to agree with those of ref. [1]. The treatment presented below makes full use of the well established theory of the Mathieu equation and can therefore point the way to explore other aspects, such as application to double–centered $D3$ branes and to higher energies which have been discussed recently [14].

II. THE SCALAR FIELD IN THE $D3$–BRANE METRIC

The supergravity background for an extremal Dp–brane in the 10–dimensional type IIB theory is [15–17]

$$ds^2 = \frac{1}{\sqrt{H}}(-dt^2 + dx_{\parallel}^2) + \sqrt{H}dx_{\perp}^2 \tag{1}$$

where (r being the radial coordinate in the $SO(5)$ symmetric space orthogonal to the branes)

$$dx_{\parallel}^2 = \sum_{i=1}^{p} dx_i^2, \quad dx_{\perp}^2 = dr^2 + r^2 d\Omega_{(8-p)}^2 \tag{2}$$

and the harmonic function H is given by

$$H = 1 + \frac{R^{(7-p)}}{r^{(7-p)}}, \tag{3}$$

For $p = 3$, the case of interest here, i.e. the $D3$ brane coupled to the 4–form RR–potential [17,18], R with $R^4 = 4\pi g_s N \alpha'^2$ (g_s the string coupling and N the number of $D3$ branes) is the radius of S^5 and AdS_5 in the socalled decoupling limit in which one obtains a duality between $\mathcal{N} = 4\, U(N)$ supersymmetric Yang–Mills theory in 4 dimensions and string theory in the near horizon $AdS_5 \times S^5$ background [19,20]. As pointed out in ref. [1], for a comparison of considerations in terms of supergravity and those in terms of D–branes, one is interested in the domain of small ωR, where ω is the energy of the field incident on the brane.

For a massless scalar fluctuation field ϕ around the dilaton field Φ given by [15]

$$e^{\Phi} = H^{(3-p)/4}(r)$$

(which is constant for $p = 3$) in the background of this metric, the equation of motion is

$$\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \phi = 0 \tag{4}$$

After separation of the S^5 harmonics $Y(\theta_i)$, in particular the Gegenbauer polynomial $C_l(\cos\theta)$, where $x = r\cos\theta$, and a factor $e^{i\omega t}$ the radial wave function $\psi_l(r) = y(r)/r^{\frac{5}{2}}$ of the l–th partial wave of energy ω of the scalar field ϕ is found to satisfy

$$\left[\frac{1}{r^5} \frac{\partial}{\partial r} \left(r^5 \frac{\partial}{\partial r} \right) - \frac{l(l+4)}{r^2} + \omega^2 + \frac{\omega^2 R^4}{r^4} \right] \psi_l(r) = 0,$$
$$\left[\frac{\partial^2}{\partial r^2} + \omega^2 + \frac{\omega^2 R^4}{r^4} - \frac{(l+\frac{3}{2})(l+\frac{5}{2})}{r^2} \right] y = 0 \tag{5}$$

We see that for $R^4 \neq 0$ this is the equation of an attractive singular potential with coupling constant $g_0^2 = \omega^2 R^4$. For $\omega^2 > 0$ an incident wave allows both transmitted and reflected waves, and from the ratio of coefficients one can determine the S–matrix. It is convenient to make the substitutions

$$y = r^{1/2} \phi(r), \quad r = \gamma e^z, \quad \gamma = g_0/h, \quad h^2 = \omega g_o = \omega^2 R^2, \quad \lambda = (l+2)^2, \tag{6}$$

which convert the range of r from 0 to ∞ to that of z from $-\infty$ to $+\infty$. The equation thereby becomes the modified Mathieu equation

$$\frac{d^2\phi}{dz^2} + \left[2h^2 \cosh 2z - \lambda\right]\phi = 0 \tag{7}$$

In view of the principal interest in the relation of our semiclassical gravity consideration with the superconformal limit of the dual theory in the near–horizon domain, we are here interested in waves of low energy, i.e. of small ω, and so in solutions of the modified Mathieu equation around $h^2 = 0$. The modified Mathieu equation allows series expansions of this type in terms of exponential, hyperbolic and cylindrical functions and (surprisingly) in each of the cases with the same coefficients $c_{2r}^{\nu}(h^2)$ where ν is the Floquet exponent and the subscript r a positive or negative integer or zero (not to be confused with the radial coordinate). The solutions in terms of exponentials are written $Me_{\nu}(z, h^2)$, those in terms of hyperbolic functions cosh and sinh Mc_{ν} and Ms_{ν}. The solutions of the i–th kind are those in terms of cylindrical functions and are written $M_{\nu}^{(i)}(z, h^2)$ where $i = 1, 2, 3, 4$ correspond respectively to expansions in terms of Bessel, Neumann or Hankel[1,2] functions. The series of $M_{\nu}^{(i)}(z, h^2)$ converge uniformly only for $|\cosh z| > 1$, whereas the series of $Me_{\nu}(z, h^2)$ is uniformly convergent for all finite complex values of z as shown in ref. [12]. Since $r = 0$ corresponds to $z = -\infty$, the S–matrix is obtained by continuing the solution at $z = -\infty$ to $z = \infty$. This means that a solution $M_{\nu}^{(3)}$ has to be continued, via matching to $Me_{\pm\nu}$ (across the domain $|z| < 1$), to a linear combination of $M_{\nu}^{(3)}$ and $M_{\nu}^{(4)}$ at $+\infty$. A few main steps of this calculation are given in Appendix A. The expression for the S–matrix finally obtained is

$$S = \frac{R^2 - 1}{R^2 - e^{-2i\pi\nu}} . e^{-i\pi\nu} \tag{8}$$

where

$$R = \frac{M_{-\nu}^{(1)}(0, h^2)}{M_{\nu}^{(1)}(0, h^2)} \tag{9}$$

This S–matrix describes the scattering of an incident wave (component of the scalar field) of energy ω off the spherically symmetric potential. One could visualise this scattering as a

spacetime curvature effect or − with black hole event horizon zero − as that of a potential barrier surrounding the horizon. With the horizon shrunk to zero at the origin (implying in the field theory a relation between mass and charge reminiscent of the Bogomol'nyi equation), this extremal case corresponds to that of a BPS state.

The absorptivity is $A = 1 - SS^{\star}$. The absorption cross section differs from this by a multiplicative factor in front. The absorption cross section σ_{abs}^l of the l−th partial wave in n spatial dimensions has been derived in ref. [21] and is given by

$$\sigma_{abs}^l = \frac{2^{n-2}\pi^{n/2-1}}{\omega^{n-1}}(n/2-2)!(l+n/2-1)\binom{l+n-3}{l}(1-|S|^2) \tag{10}$$

For $n = 6$ as in our case this l−wave (here semiclassical) absorption cross section (or socalled greybody factor) is given by

$$\sigma_{abs}^l = \frac{8\pi^2}{3\omega^5}(l+1)(l+2)^2(1-|S|^2) \tag{11}$$

III. THE $D3$–BRANE IN BORN–INFELD THEORY

To supplement the previous section, we consider briefly the simplest version of supersymmetric Born–Infeld electrodynamics for the 3–brane. Our main intention is to recall that the equation of small fluctuations about the $D3$–brane is again a modified Mathieu equation as obtained above. In the simplest such model reduced to the static case we write the Lagrangian

$$L = \int d^p x \mathcal{L}, \ \mathcal{L} = 1 - \left[1 - (\partial_i \phi)^2 + (\partial_i y)^2 + \right.$$
$$\left. (\partial_i \phi . \partial_i y)^2 - (\partial_i \phi)^2 (\partial_j y)^2 \right]^{\frac{1}{2}} - \Sigma_p e \phi \delta(\mathbf{r}) \tag{12}$$

Here $E_i = F_{0i} = -\partial_i \phi, i = 1, ..., p$ and $y(x_i)$ originates from one of the gauge field components A_a for $a = p + 1, ..., d - 1, d =$ dimension, which represent the transverse displacements of the brane (of which we consider only one, e.g. A_9). The source term of the electric field with charge e and $\Sigma_3 = 4\pi$ hints at spherical symmetry. Considering only this case here and

hence that of S–waves, we obtain two Euler–Lagrange equations which we can write

$$\partial_r \left(r^{p-1} \frac{\partial \mathcal{L}}{\partial(\partial_r y)} \right) = 0, \quad r^{p-1} \frac{\partial \mathcal{L}}{\partial(\partial_r y)} = c$$

where c is a constant of integration. Explicitly,

$$\frac{\phi'}{[1 - (\phi')^2 + (y')^2]^{\frac{1}{2}}} = -\frac{e}{r^{p-1}}, \quad \frac{-y'}{[1 - (\phi')^2 + (y')^2]^{\frac{1}{2}}} = \frac{c}{r^{p-1}} \tag{13}$$

so that

$$\frac{\phi'}{y'} = \frac{e}{c} \equiv \frac{1}{a} \tag{14}$$

Then

$$(\phi')^2 = \frac{e^2}{r^{2(p-1)} + e^2(1 - a^2)}, \quad (y')^2 = \frac{(ea)^2}{r^{2(p-1)} + e^2(1 - a^2)} \tag{15}$$

The p–brane and anti-p–branes are now given by

$$y(r) = (\overset{+}{-}) \, ae \int_r^\infty dr \frac{1}{\sqrt{r^{2(p-1)} - r_0^{2(p-1)}}} \tag{16}$$

where $r_0^{2(p-1)} = e^2(a^2 - 1) \geq 1$. In view of the proportionality (14) the Lagrangian can be written

$$\mathcal{L} = 1 - \sqrt{1 - (1 - a^2)(\partial_i \phi)^2} - \Sigma_p e \phi \delta(\mathbf{r}) \tag{17}$$

The contribution to the energy not including the source term is for $p = 3$

$$E_{ns} = \int d^3 x \left\{ \frac{1}{\sqrt{1 - (1 - a^2)(\partial_i \phi)^2}} - 1 \right\}$$

Only for a charge e which is kept fixed under a scale transformation is the energy minimal in the limit $a^2 \to 1$. This is the limit of the Bogomol'nyi bound and hence for this value of a^2 the Born–Infeld configuration, i.e. the Dp–brane or string is classically stable, i.e. a nontopological BPS state. The reason is that for $a = 1$ we have $\phi' = y'$ which in the original context with $y = A_9$ implies $F_{0r} = \partial_r A_9$. This again has implications for the supersymmetry variation of the gaugino χ, the susy partner of the gauge field, which is

$$\delta \chi = \Gamma^{\mu\nu} F_{\mu\nu} \epsilon$$

(μ, ν being the original 10 dimensional indices and $\Gamma^{\mu\nu}$ the appropriate combination of 10 dimensional Dirac matrices). In the case of only the electric field and the one excitation under consideration the gaugino variation is

$$\begin{aligned}
\delta\chi &= -(\Gamma^{0r}\partial_r A_0 + \Gamma^{9r}\partial_r A_9)\epsilon \\
&\stackrel{a=1}{=} -(\Gamma^{0r} + \Gamma^{9r})\partial_r A_0\epsilon
\end{aligned} \tag{18}$$

Thus, since $1 + \Gamma^{0r}\Gamma^{9r}$ is a projection operator it is precisely for $a = 1$ that the variation $\delta\chi$ can be zero for some nonzero ϵ, thus preserving a fraction (half) of the number of supersymmetries.

The energy E_{ns} is infinite, but integrating from $r = \delta$ to infinity so that $y = ae/\delta, dE_{ns}/dy$, the energy per unit length of the string is finite, i.e.

$$\frac{dE_{ns}}{dy} = \frac{1}{2}(1 - a^2)\frac{4\pi e}{a} = const.$$

As shown in [4], unless $a = 1$ supersymmetry is completely broken (i.e. the supersymmetry variation of the gaugino would not be preserved). In this limit the throat radius r_0 becomes smaller and smaller, and the brane pair moves further and further apart. If one considers small fluctuations ξ orthogonal to the brane and the string, one obtains the small fluctuation equation [4]

$$\triangle_r\xi + \Omega^2\left[1 + \frac{e^2(p-2)^2}{r^{2(p-1)}}\right]\xi = 0 \tag{19}$$

where $\Omega^2 \geq 0$ for stability. The radial part of these equations is with $\xi = r^{-\frac{(p-1)}{2}}\psi$ and angular momentum l

$$\frac{d^2\psi}{dr^2} + \left[\frac{1}{r^2}\left\{l(l + p - 2) - \frac{(p-1)(p-3)}{4}\right\} + \Omega^2\left(1 + \frac{e^2(p-2)^2}{r^{2(p-1)}}\right)\right]\psi = 0$$

Thus for $p = 3$ and $x = \Omega r, \kappa = e\Omega^2$ and for S-waves since the string cannot depend on the angular variables of the worldvolume

$$\left(\frac{d^2}{dx^2} + 1 + \frac{\kappa^2}{x^4}\right)\psi = 0 \tag{20}$$

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This equation is an S-wave radial Schrödinger equation for an attractive singular potential $\propto$ x^{-4} but depends only on the single coupling parameter κ with constant positive Schrödinger energy.

IV. CALCULATION OF THE FLOQUET EXPONENT IN SINGULAR AND NONSINGULAR CASES

The Floquet exponent ν enters the discussion of the Mathieu equation in view of the Bloch wave property of the modified Mathieu function $Me_\nu(z, h)$

$$Me_\nu(z + i\pi, h) = e^{i\nu\pi} Me_\nu(z, h) \tag{21}$$

The Floquet exponent can be introduced in several ways. In ref. [12] (p.107) ν is introduced by the relation

$$\cos \pi\nu = y_I(\pi, \lambda, h^2) \tag{22}$$

where $y_I(x)$ is a fundamental solution of the (periodic) Mathieu equation satisfying the boundary conditions $y_I(0) = 1, y_I'(0) = 1$ and λ is the eigenvalue which, of course, is not necessarily an integer. With a perturbation theory ansatz for $y_I(\pi, \lambda, h^2)$ around $h^2 = 0$ the following expansion is then shown to result (cf. ref. [12], p.124):

$$\cos \pi\nu = \cos \pi\sqrt{\lambda} + h^4 \frac{\pi \sin \pi\sqrt{\lambda}}{4\sqrt{\lambda}(\lambda - 1)} +$$
$$+ h^8 \left[\frac{15\lambda^2 - 35\lambda + 8}{64(\lambda - 1)^3(\lambda - 4)\lambda\sqrt{\lambda}} \pi \sin \pi\sqrt{\lambda} - \frac{\pi^2 \cos \pi\sqrt{\lambda}}{32\lambda(\lambda - 1)^2} \right] + O(h^{12}) \tag{23}$$

Alternatively one can apply directly perturbation theory to a trivial periodic solution of the limit $h^2 \to 0$ as shown in ref. [8]. In this case the following expansion is obtained

$$\lambda = \nu^2 + \frac{h^4}{2(\nu^2 - 1)} + \frac{(5\nu^2 + 7)h^8}{32(\nu^2 - 1)^3(\nu^2 - 4)}$$
$$+ \frac{(9\nu^4 + 58\nu^2 + 29)h^{12}}{64(\nu^2 - 1)^5(\nu^2 - 4)(\nu^2 - 9)} + O(h^{16}) \tag{24}$$

This series can be reversed to yield ν, i.e.

$$\nu^2 = \lambda - \frac{h^4}{2(\lambda - 1)} - \frac{(13\lambda - 25)h^8}{32(\lambda - 1)^3(\lambda - 4)}$$
$$- \frac{(45\lambda^3 - 455\lambda^2 + 1291\lambda - 1169)h^{12}}{64(\lambda - 1)^5(\lambda - 4)^2(\lambda - 9)} + O(h^{16}) \tag{25}$$

and so

$$\nu = \sqrt{\lambda} + \frac{h^4}{4(1 - \lambda)\sqrt{\lambda}} - \frac{(8 - 35\lambda + 15\lambda^2)h^8}{64(\lambda - 4)(\lambda - 1)^3\lambda\sqrt{\lambda}} + O(h^{12}) \tag{26}$$

One can easily check the agreement with the above expression for $\cos \pi\nu$ by evaluating $\cos \pi\nu$ with ν of this expansion.

An obvious feature of all of these expansions is that they are singular for integral values of λ. This behaviour is wellknown. It means that these expansions are in these cases really asymptotic expansions for large values of ν or λ and can be used in such cases. However, for other values the expansions can also be convergent for sufficiently small values of h^2 as is shown in ref. [12]. For values of ν close to an integer or $\sqrt{\lambda}$ close to an integer one has to expand around these as is also mentioned in ref. [12] (pp.124-125). We demonstrate this in the case of ν almost equal to 2. Thus we set

$$\nu = 2 + \delta, \quad \sqrt{\lambda} = 2 + \epsilon$$

so that

$$\cos \pi\nu = \cos \pi(2 + \delta) = \cos 2\pi. \cos \pi\delta = 1 - \frac{\pi^2\delta^2}{2} + \cdots \tag{27}$$

and consider the limit $\epsilon \to 0$. Expanding the cosine and sine expressions appearing in eq.(23) about $\lambda = 4$ we have ($\lambda - 4 \approx 4\epsilon$)

$$\cos \sqrt{\lambda}\pi = \cos 2\pi + (\lambda - 4)(-\sin \sqrt{\lambda}\pi)_{\lambda=4}.\frac{\pi}{2\sqrt{\lambda}} + \cdots = 1 - \frac{(\lambda - 4)^2\pi^2}{8} + \cdots$$

and

$$\sin \sqrt{\lambda}\pi = \sin 2\pi + (\lambda - 4)(\cos \sqrt{\lambda}\pi)_{\lambda=4}.\frac{\pi}{2\sqrt{\lambda}} + \cdots = \frac{\pi(\lambda - 4)}{2\sqrt{\lambda}} + \cdots$$

Substitution into eq.(23) and considering the approach $\lambda \to 4$ gives

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$$\cos \pi\nu = \left(1 - \frac{\pi^2}{8}(\lambda-4)^2 + \cdots\right) + \frac{h^4\pi^2(\lambda-4)}{4\sqrt{\lambda}(\lambda-1).2\sqrt{\lambda}}$$
$$+ h^8\left(\frac{(15\lambda^2-35\lambda+8)\pi^2(\lambda-4)}{64(\lambda-1)^3(\lambda-4)\lambda\sqrt{\lambda}.2\sqrt{\lambda}} - \frac{\pi^2.1}{32\lambda(\lambda-1)^2}\right) + \cdots$$
$$= \left(1 - \frac{(\lambda-4)^2\pi^2}{8} + \cdots\right) + \frac{h^4\pi^2}{8\lambda(\lambda-1)}\left((\lambda-4) + \cdots\right)$$
$$+ \frac{h^8\pi^2}{128\lambda^2(\lambda-1)^3}\left(11\lambda^2 - 31\lambda + 8) + \cdots\right) + O(h^{12}) \tag{28}$$

Hence (observe the cancellation of factors $(\lambda-4)$ in the term of $O(h^8)$)in the limit $\epsilon \to 0$

$$\cos \pi\nu = 1 + \frac{h^8\pi^2(11 \times 16 - 31 \times 4 + 8)}{2^7 4^2 3^3} + \cdots = 1 + \frac{5\pi^2 h^8}{2^9 3^2} + \cdots \tag{29}$$

Comparing the expansions (27) and (29) we obtain

$$\delta = \pm i\frac{\sqrt{5}}{3}(h/2)^4$$

We see that although the coefficients of higher order terms of eq.(23) contain factors $(\lambda-4)$ in the denominators and so suggest divergences, the trigonometric factors $\sin \pi\sqrt{\lambda}$ in the numerator always cancel these out and thus yield a regular expansion for ν which is even convergent within a certain domain around $h^2 = 0$. One thus obtains the expansion

$$\nu = 2 - \frac{i\sqrt{5}}{3}(h/2)^4 + \frac{7i}{108\sqrt{5}}(h/2)^8 + \frac{11851i}{31104\sqrt{5}}(h/2)^{12} \cdots \tag{30}$$

This expansion has also been given in ref. [1]. Expansions around other integral values of $\sqrt{\lambda}$ are obtained similarly. Of course, the higher the value of this integer, the more terms at the beginning of the series are identical with those obtainable from the perturbation series (26) above. Thus in the case of $\sqrt{\lambda} = 3, 4$, we obtain from the first two terms of expansion (26)

$$\nu = 3 + \frac{h^4}{4(-8)3} + \cdots = 3 - \frac{(h/2)^4}{6} + \cdots$$

and similarly

$$\nu = 4 - \frac{(h/2)^4}{15} + \cdots$$

in agreement also with results of ref. [1] up to the given order, i.e.

$$\sqrt{\lambda} = 3: \quad \nu = 3 - \frac{1}{6}(h/2)^4 + \frac{133}{4320}(h/2)^8 + \frac{311}{1555200}(h/2)^{12} + \cdots,$$

$$\sqrt{\lambda} = 4: \quad \nu = 4 - \frac{1}{15}(h/2)^4 - \frac{137}{27000}(h/2)^8 + \frac{305843}{680400000}(h/2)^{12} \cdots \tag{31}$$

V. RELATION BETWEEN DOUGALL AND STANDARD MATHIEU COEFFICIENTS

One may wonder how the Mathieu function coefficients of Dougall [3] which are used in refs. [1,2], are related to those in modern standard literature such as ref. [12]. We therefore demonstrate their precise connection here. It is crucial thereby to distinguish between nonsingular or asymptotic cases and singular cases, as we shall see. We begin with the nonsingular case and calculate a coefficient given in ref. [12] (up to the first nonleading contribution) by starting from Dougall's definition of his coefficients. We shall see that the coefficients given in ref. [12] obtained from simple continued fraction solution of the basic recurrence relation of the coefficients are in this case not only easier to derive but have also a simpler form than the coefficients of Dougall.

The modified Mathieu function in terms of exponentials is defined in ref. [12] as the following sum

$$Me_\nu(z, h^2) := \sum_{r=-\infty}^{\infty} c_{2r}^\nu(h^2) e^{(\nu+2r)z} \tag{32}$$

where $\nu \neq \pm 1, \pm 2, \cdots$. In ref. [12](p. 131) the following relation of general validity is given and used

$$Me_{-\nu}(z, h) = Me_\nu(-z, h) \tag{33}$$

This relation implies that

$$c_{2r}^\nu(h^2) = c_{-2r}^{-\nu}(h^2) \quad and \quad \frac{c_{2r}^\nu(h^2)}{c_0^\nu(h^2)} = \frac{c_{-2r}^{-\nu}(h^2)}{c_0^{-\nu}(h^2)} \tag{34}$$

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We shall see explicitly that this relation holds also in what we call the singular case below. Dougall [3] defines in his work the solution corresponding to $Me_\nu(z,h)$ as

$$\frac{J(\nu,z)}{\phi(\nu/2)} = \sum_{r=-\infty}^{\infty} (-1)^r \frac{\phi(r+\nu/2)}{\phi(\nu/2)} e^{(\nu+2r)z} \tag{35}$$

We therefore expect the equivalences

$$\frac{J(\nu,z)}{\phi(\nu/2)} = Me_\nu(z,h^2), \quad (-1)^n \frac{\phi(n+\nu/2)}{\phi(\nu/2)} = \frac{c_{2n}^\nu(h^2)}{c_0^\nu(h^2)} \tag{36}$$

We now verify the latter of these relations for the case $n=1$ in the nonsingular case (i.e. for $\nu \neq$ integer$+O(h^2)$), i.e. we show that

$$-\frac{\phi(\nu/2+1)}{\phi(\nu/2)} \rightarrow \frac{c_2^\nu(h^2)}{c_0^\nu(h^2)} = \frac{h^2}{4(\nu+1)} + \frac{(\nu^2+4\nu+7)h^6}{128(\nu+1)^3(\nu+2)(\nu-1)} + \cdots \tag{37}$$

where the expression on the right is given in ref. [12](p.121). We also show thereby that in leading order for small h^2

$$\frac{\phi(n+\nu/2)}{\phi(\nu/2)} = \frac{(h/2)^{2n}\nu!}{(n+\nu)!n!}\left(1+O(h^4)\right) \tag{38}$$

in agreement with ref. [12] (p.121). The demonstration of agreement requires eqs. (24), (25) (cf. also [12],p. 119), i.e.

$$s = l+2, \quad s^2 = \nu^2 + \frac{h^4}{2(\nu^2-1)} + \frac{(5\nu^2+7)h^8}{32(\nu^2-1)^3(\nu^2-4)} + \cdots \tag{39}$$

or

$$\pm s = \nu + \frac{h^4}{4\nu(\nu^2-1)} + \cdots \tag{40}$$

This general relation is an asymptotic expansion in ν (i.e. for ν large), and can be obtained perturbatively [8]. It is crucial, of course, to deal separately with values of ν close to a singular value like $\nu = 2$ (see below).

Dougall defines his coefficients ϕ by an expansion, of which the leading and next–to– leading contributions are [3]

$$\phi(n + \nu/2) = \frac{(h/2)^{2n+\nu}}{(n + \nu/2 + s/2)!(n + \nu/2 - s/2)!} \left\{ 1 - \right.$$

$$\sum_{p_1=0}^{\infty} \frac{(h/2)^4}{(n + \nu/2 + s/2 + 1 + p_1)}$$

$$\frac{1}{(n + \nu/2 + s/2 + 2 + p_1)(n + \nu/2 - s/2 + 1 + p_1)(n + \nu/2 - s/2 + 2 + p_1)}$$

$$\left. + \cdots \right\}$$

(41)

Taking into account only the leading contribution, we have

$$\frac{\phi(n + \nu/2)}{\phi(\nu/2)} = \frac{(h/2)^{2n}(\nu/2 + s/2))!(\nu/2 - s/2)!}{(n + \nu/2 + s/2)!(n + \nu/2 - s/2)!}$$

(42)

Using

$$(-z)! = \frac{\pi}{(z - 1)! \sin \pi z}$$

(43)

and the approximation $s \approx \nu$ (cf. eq.(40)) we obtain

$$\frac{\phi(n + \nu/2)}{\phi(\nu/2)} = \frac{(h/2)^{2n}\nu!}{(n + \nu)!n!} \left(1 + O(h^4) \right)$$

(44)

in agreement with $c_{2n}^{\nu}(h^2)/c_0^{\nu}(h^2)$ of ref. [12]. We see therefore that the expansion (40) plays an important role in establishing the connection between the coefficients $\phi(r + \nu/2)$ and c_{2r}^{ν} in the nonsingular case.

We now consider the next–to–leading contribution in eq.(41). Setting

$$A_q^{(1)} = \sum_{p_1=0}^{\infty} \frac{1}{(q + s/2 + 1 + p_1)(q + s/2 + 2 + p_1)(q - s/2 + 1 + p_1)(q - s/2 + 2 + p_1)} \quad (45)$$

we can make the partial fraction separation

$$A_q^{(1)} = \sum_{p_1=0}^{\infty} \left[\frac{1}{s(s - 1)} \left(\frac{1}{p_1 + q + 1 + s/2} - \frac{1}{p_1 + q + 2 - s/2} \right) \right.$$

$$\left. - \frac{1}{s(s + 1)} \left(\frac{1}{p_1 + q + 2 + s/2} - \frac{1}{p_1 + q + 1 - s/2} \right) \right]$$

(46)

The sums over individual terms are divergent. Thus the expression depends crucially on taking differences. In order to deal with these we use the formula [22]

$$\sum_{k=0}^{n-1} \frac{1}{k + y} = \psi(n + y) - \psi(y)$$

(47)

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where $\psi(y)$ is the derivative of the log of the gamma function $\Gamma(y)$. Thus

$$\sum_{p_1=0}^{n-1} \left(\frac{1}{p_1 + q + 1 + s/2} - \frac{1}{p_1 + q + 2 - s/2} \right) \overset{n\to\infty}{\longrightarrow} \psi(q + 2 - s/2) - \psi(q + 1 + s/2)$$

and

$$\sum_{p_1=0}^{n-1} \left(\frac{1}{p_1 + q + 2 + s/2} - \frac{1}{p_1 + q + 1 - s/2} \right) \overset{n\to\infty}{\longrightarrow} \psi(q + 1 - s/2) - \psi(q + 2 + s/2)$$

so that

$$\begin{aligned}
A_q^{(1)} = &\frac{1}{s(s-1)} \Big(\psi(q + 2 - s/2) - \psi(q + 1 + s/2) \Big) \\
&- \frac{1}{s(s+1)} \Big(\psi(q + 1 - s/2) - \psi(q + 2 + s/2) \Big)
\end{aligned} \tag{48}$$

We now use eq.(40) in order to reexpress s interms of ν. Then

$$\begin{aligned}
\frac{1}{s(s-1)} &= \frac{1}{\nu(\nu-1)} \left[1 - \frac{(2\nu - 1)h^4}{4\nu^2(\nu-1)^2(\nu+1)} + \cdots \right] \\
\frac{1}{s(s+1)} &= \frac{1}{\nu(\nu+1)} \left[1 - \frac{(2\nu + 1)h^4}{4\nu^2(\nu-1)(\nu+1)^2} + \cdots \right]
\end{aligned} \tag{49}$$

Setting $q = n + \nu/2$ and dealing similarly with the arguments of the functions ψ, we obtain in lowest order of h

$$\begin{aligned}
A_{q=n+\nu/2}^{(1)} = &\frac{1}{\nu(\nu-1)} \Big[\psi(n + 2) - \psi(\nu + n + 1) \Big] \\
&- \frac{1}{\nu(\nu+1)} \Big[\psi(n + 1) - \psi(\nu + n + 2) \Big]
\end{aligned} \tag{50}$$

Again we consider a difference, i.e.

$$\triangle_n := A_{n+\nu/2}^{(1)} - A_{n-1+\nu/2}^{(1)}$$

We now use the formula

$$\psi(n + 1) = -C + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \tag{51}$$

where C is the Euler constant. Then

$$\psi(n + 2) - \psi(n + 1) = \frac{1}{n + 1}, \quad \psi(n + 1) - \psi(n) = \frac{1}{n}$$

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and $\triangle_n$ becomes

$$
\begin{aligned}
\triangle_n &= \frac{1}{\nu(\nu-1)}\left[\frac{1}{n+1} - \psi(\nu+n+1) + \psi(\nu+n]\right] \\
&\quad - \frac{1}{\nu(\nu+1)}\left[\frac{1}{n} - \psi(\nu+n+2) + \psi(\nu+n+1)\right]
\end{aligned}
\tag{52}
$$

We now require yet another formula of the ψ function, i.e.

$$
\psi(x) = -C + \sum_{i=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{x+n}\right)
\tag{53}
$$

With this we obtain

$$
\psi(\nu+n) - \psi(\nu+n+1) = -\frac{1}{\nu+n}, \quad \psi(\nu+n+1) - \psi(\nu+n+2) = -\frac{1}{\nu+n+1}
$$

(where in each case the dummy summation index of the second sum, i.e. i, was renamed $i-1$). We therefore obtain in the dominant approximation

$$
\triangle_n = \frac{1}{\nu(\nu-1)}\left[\frac{1}{n+1} - \frac{1}{\nu+n}\right] - \frac{1}{\nu(\nu+1)}\left[\frac{1}{n} - \frac{1}{\nu+n+1}\right]
\tag{54}
$$

For $n=1$ this implies

$$
\triangle_1 = A_{q=1+\nu/2}^{(1)} - A_{q=\nu/2}^{(1)} = -\frac{1}{2(\nu+1)(\nu+2)}
\tag{55}
$$

in the dominant approximation. This difference will now have to be substituted into the Dougall coefficient

$$
\begin{aligned}
\frac{\phi(\nu/2+1)}{\phi(\nu/2)} &= \frac{\left(\frac{h}{2}\right)^2(\nu/2+s/2)!(\nu/2-s/2)!\left\{1 - \left(\frac{h}{2}\right)^4 A_{\nu/2+1}^{(1)} + \cdots\right\}}{(\nu/2+1+s/2)!(\nu/2+1-s/2)!\left\{1 - \left(\frac{h}{2}\right)^4 A_{\nu/2}^{(1)} + \cdots\right\}} \\
&\approx \left(\frac{h}{2}\right)^2 \frac{\left\{1 - \left(\frac{h}{2}\right)^4\left(A_{\nu/2+1}^{(1)} - A_{\nu/2}^{(1)}\right) + \cdots\right\}}{(\nu/2+1+s/2)(\nu/2+1-s/2)}
\end{aligned}
\tag{56}
$$

Inserting eq.(55) and for s the series of eq.(40) we obtain

$$
\frac{\phi(\nu/2+1)}{\phi(\nu/2)} = \frac{h^2}{4(\nu+1)} + \frac{h^6(\nu^2+4\nu+7)}{128(\nu-1)(\nu+1)^3(\nu+2)} + O(h^{10}) = -\frac{c_2^\nu}{c_0^\nu}
\tag{57}
$$

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in agreement with ref. [12](p.121). To obtain the Dougall coefficients in this form is thus seen to be rather complicated. This may explain why Dougall himself does not evaluate any of his coefficients explicitly.

In the remainder of this section we calculate with the method of Dougall the important coefficients $\phi(\pm\nu/2)$ for the S–wave case, i.e. $s = 2$, and demonstrate the agreement with results of ref. [1]. The results will also be needed in the next section in establishing ratios corresponding to the ratio of eq.(57). From eq.(48) we obtain for the leading term in the limit $h^2 \to 0$ for $q = \nu/2$ and $s = 2$ (i.e. $\nu \approx 2$, a socalled singular case)

$$
\begin{aligned}
A^{(1)}_{\nu/2} &= \left\{ \frac{1}{2} \left[\psi(\nu/2 + 1) - \psi(\nu/2 + 2) \right] - \frac{1}{6} \left[\psi(\nu/2) - \psi(\nu/2 + 3) \right] \right\}_{h^2 \to 0} \\
&= \frac{1}{2} \left[\psi(2) - \psi(3) \right] - \frac{1}{6} \left[\psi(1) - \psi(4) \right]
\end{aligned} \tag{58}
$$

Using eq.(51) we obtain

$$
A^{(1)}_{\nu/2} = \frac{1}{18}
$$

The authors of ref. [1] developed another algorithm in which (cf. their Appendix A)

$$
A^{(1)} = A^{(1)}_z \equiv S[1] \tag{59}
$$

and for $s = 2$ (their $r = 1$)

$$
A^{(1)}_z = \frac{3 + 2z}{3z(2 + z)} + \frac{\psi(z) - \psi(z + 2)}{3} \tag{60}
$$

Here

$$
\psi(z) - \psi(z + 2) = \frac{d}{dz} \left[\ln \Gamma(z) - \ln \Gamma(z + 2) \right] = -\frac{2z + 1}{z(z + 1)} \tag{61}
$$

so that

$$
A^{(1)}_z = \frac{1}{3z(z + 1)(z + 2)} \tag{62}
$$

and for $\nu \approx 2$ one obtains again

$$
A^{(1)}_{\nu/2} = \frac{1}{18}
$$

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as above. It follows that for this case with $\Re\nu > 0$ (cf. eq.(41),

$$\phi(\nu/2) \approx \frac{(h/2)^\nu}{(\nu/2 + s/2)!(\nu/2 - s/2)!}\left\{1 - \frac{(h/2)^4}{18}\right\}$$
$$\approx \frac{1}{2}(h/2)^2\left[1 + O(h^4)\right] \tag{63}$$

Fortunately for this case the expansion (57) does not seem to possess terms which diverge for ν close to a positive integer$\neq 1$, and evidently even if it did – since in our cases $\nu = $ an integer $\neq 1 + O(h^4)$ – would not affect the leading term of eq.(59). This is radically different when ν is close to a negative integer such as -2 in that case. We see from (53) that if $x = -2 + O(h^4)$, $\psi \propto 1/h^4$, and hence $A_q^{(1)} \propto 1/h^4$, and so this term will contribute to the leading factor in the coefficient ϕ of eq.(41).

We now consider the coefficient $\phi(-\nu/2), \Re\nu > 0$, for which

$$s = 2, \quad \nu = 2 - \frac{i\sqrt{5}}{3}(h/2)^4 + \cdots \tag{64}$$

and calculate this first with the method of Dougall. The following steps given explicitly demonstrate clearly how singularities in the limit $h^2 \to 0$ arise and how they have to be handled. Thus with eq.(41):

$$
\begin{aligned}
\phi(-\nu/2) &= \frac{(h/2)^{-\nu}}{(-\nu/2+s/2)!(-\nu/2-s/2)!}\Bigg\{1 \\
&\quad - \sum_{p_1=0}^{\infty} \frac{(h/2)^4}{(-\nu/2+s/2+1+p_1)(-\nu/2+s/2+2+p_1)} \\
&\quad \cdot \frac{1}{(-\nu/2-s/2+1+p_1)(-\nu/2-s/2+2+p_1)} + \cdots \Bigg\} \\
&\approx \frac{(h/2)^{-2}}{0!(-2+\frac{i\sqrt{5}}{6}(h/2)^4)!}\Bigg\{1-(h/2)^4 \cdot \\
&\quad \cdot \Bigg[\frac{1}{(1)(2)(-1+\frac{i\sqrt{5}}{6}(h/2)^4)(\frac{i\sqrt{5}}{6}(h/2)^4)} \\
&\quad + \frac{1}{(2)(3)(\frac{i\sqrt{5}}{6}(h/2)^4)(1+\frac{i\sqrt{5}}{6}(h/2)^4)} \Bigg] \\
&\quad + \sum_{p_1=2}^{\infty}\cdots \Bigg\} \\
&\approx \frac{(h/2)^{-2}}{(-2+\frac{i\sqrt{5}}{6}(h/2)^4)!}\Bigg\{1-(h/2)^4\Bigg[\frac{1}{(-2)(\frac{i\sqrt{5}}{6}(h/2)^4)}+\frac{1}{6(\frac{i\sqrt{5}}{6}(h/2)^4)}\Bigg]+\cdots\Bigg\} \\
&= \frac{(h/2)^{-2}}{(-2+\frac{i\sqrt{5}}{6}(h/2)^4)!}\Bigg\{1+\frac{2}{i\sqrt{5}}\Bigg\} \\
&\overset{eq.(43)}{=} \frac{1}{\pi}(h/2)^{-2}\Bigg\{1+\frac{2}{i\sqrt{5}}\Bigg\}(1)(-1)\pi\frac{i\sqrt{5}}{6}(h/2)^4 \\
&= -(h/2)^2\left(\frac{2+i\sqrt{5}}{6}\right)
\end{aligned}
\tag{65}
$$

The steps above clearly show how the singular terms in the next–to–leading contribution contribute to the dominant order.

The result (65) will now be shown to agree with the calculations of the method of ref. [1]. For this purpose we set in eq.(62) $z = -\nu/2$ and replace ν by the expression in eq.(64). Then

$$
A^{(1)}_{-\nu/2} \approx \frac{-8}{3.2(i\sqrt{5}/3).2.(h/2)^4} = -\frac{2}{i\sqrt{5}(h/2)^4}
$$

and

$$
v \equiv 1 - (h/2)^4 A^{(1)}_{-\nu/2} = \frac{2+i\sqrt{5}}{i\sqrt{5}}
$$

From eq.(41)we obtain therefore

$$
\phi(-\nu/2) = \frac{(h/2)^{-2}}{0!(-2+\frac{i\sqrt{5}}{6}(h/2)^4)!}.v
\tag{66}
$$

Using for the factorial again eq.(43), one obtains

$$\phi(-\nu/2) = -\frac{(h/2)^{-2\frac{i\sqrt{5}}{6}}(h/2)^4\pi}{\pi}.\left\{1 + \frac{2}{i\sqrt{5}}\right\} \tag{67}$$

Thus finally

$$\phi(-\nu/2) = -\frac{2 + i\sqrt{5}}{6}(h/2)^2 \tag{68}$$

in agreement with eq.(65).

VI. CALCULATION OF STANDARD MATHIEU COEFFICIENTS

Our next objective is to compute a Dougall coefficient (i.e. a ratio of two quantities ϕ) and to compare it with a standard Mathieu coefficient in the nontrivial singular case. As a suitable example we choose the coefficient $\phi(-\nu/2 + 1)$ which when divided by $\phi(-\nu/2)$ (calculated above) ought to agree with the Mathieu coefficient $-c_2^{-\nu}/c_0^{-\nu}$ according to eq.(36), i.e. we wish to demonstrate that

$$\phi(-\nu/2 + 1)/\phi(-\nu/2) = -c_2^{-\nu}/c_0^{-\nu} \tag{69}$$

We begin with the calculation of the Dougall coefficient. Using eq.(41), we have

$$
\begin{aligned}
\phi(-\nu/2 + 1) \quad = \quad & \frac{(h/2)^{2-\nu}}{(1 - \nu/2 + s/2)!(1 - \nu/2 - s/2)!}\Bigg\{1 \\
& - \sum_{p_1=0}^{\infty} \frac{(h/2)^4}{(1 - \nu/2 + s/2 + 1 + p_1)(1 - \nu/2 + s/2 + 2 + p_1)} \\
& \cdot \frac{1}{(1 - \nu/2 - s/2 + 1 + p_1)(1 - \nu/2 - s/2 + 2 + p_1)} + \cdots\Bigg\} \\
\overset{eq.(64) for \nu}{=} \quad & \frac{1}{(1)(-1 + \frac{i\sqrt{5}}{6}(h/2)^4)!}\Bigg\{1 - (h/2)^4\Bigg[\frac{1}{(2)(3)(\frac{i\sqrt{5}}{6}(h/2)^4)(1)} \\
& + \frac{1}{(3)(4)(1)(2)}\Bigg] + \sum_{p_1=2}^{\infty}\cdots\Bigg\} \\
\approx \quad & \frac{(0)!\sin\pi(1 - \frac{i\sqrt{5}}{6}(h/2)^4)}{\pi}\left\{1 - \frac{1}{i\sqrt{5}}\right\} \\
\approx \quad & \frac{i\sqrt{5} - 1}{6}(h/2)^4 \tag{70}
\end{aligned}
$$

With eq.(68) we obtain therefore

$$\frac{\phi(-\nu/2+1)}{\phi(-\nu/2)} = (h/2)^2 \left(\frac{1-i\sqrt{5}}{2+i\sqrt{5}}\right)\left[1+O(h^4)\right] \tag{71}$$

Similarly one obtains

$$\phi(\nu/2+1) = \frac{1}{6}(h/2)^4\left[1+O(h^4)\right], \quad \frac{\phi(\nu/2+1)}{\phi(\nu/2)} = \frac{1}{3}(h/2)^2\left[1+O(h^4)\right] \tag{72}$$

Our next step is to derive the corresponding expression from the continued fraction relation of the recurrence relation of the standard Mathieu coefficients. This recurrence relation is given by (cf. ref. [12], p. 117)

$$\frac{c_{2r}^{\nu}}{c_{2r-2}^{\nu}} = \cfrac{1}{h^{-2}[s^2-(\nu+2r)^2] - \cfrac{1}{h^{-2}[s^2-(\nu+2r+2)^2] - \cfrac{1}{h^{-2}[s^2-(\nu+2r+4)^2]\cdots}}} \tag{73}$$

Here we set $r = 1$ and replace ν by $-\nu$. Then we again use (64) and a) set $s = 2$, and b) replace ν by the expansion given in eq.(64). One then has to go as far as the terms explicitly written out in the following continued fraction only to obtain the dominant contribution:

$$\frac{c_2^{-\nu}}{c_0^{-\nu}} = \cfrac{1}{h^{-2}[s^2-(-\nu+2)^2] - \cfrac{1}{h^{-2}[s^2-(-\nu+4)^2] - \cfrac{1}{h^{-2}[s^2-(-\nu+6)^2]\cdots}}} \tag{74}$$

Making the substitutions we obtain

$$\frac{c_2^{-\nu}}{c_0^{-\nu}} = \cfrac{1}{h^{-2}[4] - \cfrac{1}{h^{-2}[-\frac{4i\sqrt{5}}{3}(h/2)^4] - \cfrac{1}{h^{-2}[-12]\cdots}}} \tag{75}$$

This can be seen to reduce to

$$\frac{c_2^{-\nu}}{c_0^{-\nu}} = (h/2)^2 \frac{i\sqrt{5}-1}{i\sqrt{5}+2} \tag{76}$$

which agrees with the negative of the above Dougall coefficient as expected on the basis of eq.(35). One can see that the calculation here is simpler than that of both (68) and (70).

The reciprocal of the continued fraction relation (73) is (cf. ref. [12], p.117)

$$\frac{c_{2r-2}^{\nu}}{c_{2r}^{\nu}} = \cfrac{1}{h^{-2}[s^2 - (\nu + 2r - 2)^2] - \cfrac{1}{h^{-2}[s^2 - (\nu + 2r - 4)^2] - \cfrac{1}{h^{-2}[s^2 - (\nu + 2r - 6)^2] \cdots}}}$$

(77)

Replacing here ν by $-\nu$ we have

$$\frac{c_{2r-2}^{-\nu}}{c_{2r}^{-\nu}} = \cfrac{1}{h^{-2}[s^2 - (-\nu + 2r - 2)^2] - \cfrac{1}{h^{-2}[s^2 - (-\nu + 2r - 4)^2] - \cfrac{1}{h^{-2}[s^2 - (-\nu + 2r - 6)^2] \cdots}}}$$

(78)

Here we put $r = 0$ and again make the replacements of eq.(64). Then

$$\frac{c_{-2}^{-\nu}}{c_0^{-\nu}} = -\frac{h^2}{12}\left[1 + O(h^4)\right]$$

(79)

For $r = 2$ in eq.(78) we obtain

$$\frac{c_4^{-\nu}}{c_2^{-\nu}} = \frac{12}{h^2(1 - i\sqrt{5})}$$

(80)

so that with eq. (76)

$$\frac{c_4^{-\nu}}{c_0^{-\nu}} = \frac{c_4^{-\nu}}{c_2^{-\nu}}\frac{c_2^{-\nu}}{c_0^{-\nu}} = -\frac{3}{2 + i\sqrt{5}}$$

(81)

In a similar way we obtain

$$\frac{c_{-4}^{-\nu}}{c_{-2}^{-\nu}} = -\frac{h^2}{2^5}\left[1 + O(h^4)\right], \quad \frac{c_{-4}^{-\nu}}{c_0^{-\nu}} = \frac{c_{-4}^{-\nu}}{c_{-2}^{-\nu}}\frac{c_{-2}^{-\nu}}{c_0^{-\nu}} = \frac{h^4}{3.2^7}\left[1 + O(h^4)\right]$$

(82)

Summarising we have as leading contributions of standard Mathieu coefficients in the singular case $l = 0$ or $s = 2$:

$$\frac{c_2^{-\nu}}{c_0^{-\nu}} = (h/2)^2 \frac{i\sqrt{5}-1}{i\sqrt{5}+2} \left[1 + O(h^4)\right] = \frac{c_{-2}^{\nu}}{c_0^{\nu}}$$

$$\frac{c_{-2}^{-\nu}}{c_0^{-\nu}} = -\frac{(h/2)^2}{3} \left[1 + O(h^4)\right] = \frac{c_2^{\nu}}{c_0^{\nu}}$$

$$\frac{c_4^{-\nu}}{c_0^{-\nu}} = -\frac{3}{2+i\sqrt{5}} \left[1 + O(h^4)\right] = \frac{c_{-4}^{\nu}}{c_0^{\nu}}$$

$$\frac{c_{-4}^{-\nu}}{c_0^{-\nu}} = \frac{(h/2)^4}{2^3.3} \left[1 + O(h^4)\right] = \frac{c_4^{\nu}}{c_0^{\nu}}$$

$$\frac{c_6^{-\nu}}{c_0^{-\nu}} = (h/2)^2 \frac{1}{2+i\sqrt{5}} \left[1 + O(h^4)\right] = \frac{c_{-6}^{\nu}}{c_0^{\nu}}$$

$$\frac{c_{-6}^{-\nu}}{c_0^{-\nu}} = -\frac{(h/2)^6}{2^3.3^2.5} \left[1 + O(h^4)\right] = \frac{c_6^{\nu}}{c_0^{\nu}} \tag{83}$$

In Appendix B we give several more terms of these expansions calculated with Mathematica.

VII. EVALUATION OF THE QUANTITY R IN THE SINGULAR CASE

Having determined the standard Mathieu coefficients in the singular S–wave case, we can proceed to evaluate the quantity R entering the S–matrix. R was defined in ref. [8](see also Appendix A) as

$$R = \alpha_\nu/\alpha_{-\nu}, \quad \alpha_\nu(h) = Me_\nu(0,h)/M_\nu^{(1)}(0,h) \tag{84}$$

The function $Me_\nu(z,h)$ was defined previously. The functions $M_\nu^{(i)}(z,h)$, for $i = 1,2,3,4$, are corresponding expansions of the modified Mathieu function in terms of cylindrical functions $J_\nu(z), Y_\nu(z), H_\nu^{(1)}(z), H_\nu^{(2)}(z)$ respectively. In particular we have the expansion (cf. ref. [12], p. 178)

$$Me_\nu(0,h)M_\nu^{(1)}(z,h) = \sum_{r=-\infty}^{\infty} c_{2r}^\nu(h^2) J_{\nu+2r}(2h\cosh z) \tag{85}$$

As shown in ref. [12](p. 180), a much better expansion to use in practice for $M_\nu^{(1)}(z,h)$ in view of its rapid convergence, is

$$c_{2r}^{\pm\nu}(h^2)M_{\pm\nu}^{(1)}(z,h) = \sum_{l=-\infty}^{+\infty} (-1)^l c_{2l}^{\pm\nu}(h^2) J_{l-r}(he^{-z}) J_{\pm\nu+l+r}(he^z) \tag{86}$$

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so that in particular

$$c_{2r}^{\pm\nu}(h^2)M_{\pm\nu}^{(1)}(0,h) = \sum_{l=-\infty}^{+\infty}(-1)^l c_{2l}^{\pm\nu}(h^2)J_{l-r}(h)J_{\pm\nu+l+r}(h) \tag{87}$$

This formula is amazing. It implies that one and the same quantity $M_{\pm\nu}^{(1)}(0,h)$ can be obtained from many different expansions (and so different Bessel functions) by allocating different values to r, i.e. e.g. $r = 0$ and 2. An analogous observation has also been made by Dougall [3].

We begin with the evaluation of $Me_\nu(0,h)$, i.e.

$$Me_\nu(0,h) = c_0^\nu(h^2)\sum_r \frac{c_{2r}^\nu(h^2)}{c_0^\nu(h^2)} \tag{88}$$

With the help of the standard Mathieu coefficients evaluated previously we obtain

$$\left[Me_\nu(0,h)\right]_{h^2\to 0} = c_0^\nu(0)\left[1 - \frac{3}{2+i\sqrt{5}}\right] = c_0^\nu(0)\left[\frac{i\sqrt{5}-1}{2+i\sqrt{5}}\right] = \left[Me_{-\nu}(0,h)\right]_{h^2\to 0} \tag{89}$$

The last equality follows also from the general property $Me_{-\nu}(z,h) = Me_\nu(-z,h)$ (cf. ref. [12], p. 131). More terms can be calculated with Mathematica. Thus

$$Me_\nu(0,h) = 1 + \frac{c_2^\nu(h^2)}{c_0^\nu(h^2)} + \frac{c_4^\nu(h^2)}{c_0^\nu(h^2)} + \frac{c_6^\nu(h^2)}{c_0^\nu(h^2)} + \frac{c_{-2}^\nu(h^2)}{c_0^\nu(h^2)} + \frac{c_{-4}^\nu(h^2)}{c_0^\nu(h^2)} + \frac{c_{-6}^\nu(h^2)}{c_0^\nu(h^2)} + \frac{c_{-8}^\nu(h^2)}{c_0^\nu(h^2)} + \frac{c_{-10}^\nu(h^2)}{c_0^\nu(h^2)}$$

We have to take into account also $\frac{c_{-8}^\nu(h^2)}{c_0^\nu(h^2)}$ and $\frac{c_{-10}^\nu(h^2)}{c_0^\nu(h^2)}$ because these contribute to orders h^4 and h^6, i.e.

$$\frac{c_{-8}^\nu(h^2)}{c_0^\nu(h^2)} = \frac{(i-\sqrt{5})}{2^3 3(i+\sqrt{5})}\left(\frac{h}{2}\right)^4 + \frac{202i+35\sqrt{5}}{2^4 3^3 5(i+\sqrt{5})}\left(\frac{h}{2}\right)^8$$

$$\frac{c_{-10}^\nu(h^2)}{c_0^\nu(h^2)} = \frac{-i+\sqrt{5}}{2^3 3^2 5(i+\sqrt{5})}\left(\frac{h}{2}\right)^6$$

respectively. We then obtain the following result correct up to order h^6:

$$Me_\nu(0,h) = \frac{2i}{i+\sqrt{5}} + \frac{4i}{3(i+\sqrt{5})}\left(\frac{h}{2}\right)^2 + \frac{65i+11\sqrt{5}}{2^3 3^1 5(i+\sqrt{5})}\left(\frac{h}{2}\right)^4$$

$$+ \frac{11(83i+89\sqrt{5})}{2^2 3^3 5(19i-5\sqrt{5})}\left(\frac{h}{2}\right)^6 + O(h^7) \tag{90}$$

We note here that for general values of ν not equal to an integer one obtains (up to and including contributions of $O(h^6)$)

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$$Me_\nu(0,h) = 1 + \frac{2}{\nu^2 - 1}\left(\frac{h}{2}\right)^2 + \frac{\nu^2 + 2}{(\nu^2 - 1)(\nu^2 - 4)}\left(\frac{h}{2}\right)^4$$
$$+ \frac{2(\nu^6 + 4\nu^4 - 39\nu^2 - 62)}{(\nu^2 - 1)^3(\nu^2 - 4)(\nu^2 - 9)}\left(\frac{h}{2}\right)^6 + O(h^7) \tag{91}$$

Next we evaluate $M_\nu^{(1)}(0,h)$ with the help of eq.(87) choosing $r = 0$ and then as a check $r = 2$. In the first case we obtain the expansion

$$M_\nu^{(1)}(0,h) = J_0(h)J_\nu(h) - \frac{c_2^\nu(h^2)}{c_0^\nu(h^2)}J_1(h)J_{\nu+1}(h) - \frac{c_{-2}^\nu(h^2)}{c_0^\nu(h^2)}J_{-1}(h)J_{\nu-1}(h)$$
$$+ \frac{c_4^\nu(h^2)}{c_0^\nu(h^2)}J_2(h)J_{\nu+2}(h) + \frac{c_{-4}^\nu(h^2)}{c_0^\nu(h^2)}J_{-2}(h)J_{\nu-2}(h) + \cdots \tag{92}$$

In lowest orders of h^2 this is

$$M_\nu^{(1)}(0,h) = J_0(h)J_\nu(h) + \frac{c_{-4}^\nu(h^2)}{c_0^\nu(h^2)}J_{-2}(h)J_{\nu-2}(h)$$

which when evaluated in lowest orders of h^2 implies

$$\frac{(h/2)^2}{2}.1 + \frac{-3}{2 + i\sqrt{5}}.\frac{(h/2)^2}{2!0!}.1$$

It follows that

$$M_\nu^{(1)}(0,h) = \frac{1}{2}(h/2)^2\left[\frac{-1 + i\sqrt{5}}{2 + i\sqrt{5}}\right] \tag{93}$$

Here, of course, ν has as before the S–wave value, i.e.

$$\nu = 2 - \frac{i\sqrt{5}}{3}(h/2)^4 + \cdots \tag{94}$$

If we set $r = 2$ in eq.(87) we obtain

$$M_\nu^{(1)}(0,h) = \frac{c_0^\nu}{c_4^\nu}\left[\frac{c_0^\nu}{c_0^\nu}J_{-2}J_{\nu+2} - \frac{c_2^\nu}{c_0^\nu}J_{-1}J_{\nu+3}\right.$$
$$\left. - \frac{c_{-2}^\nu}{c_0^\nu}J_{-3}J_{\nu+1} + \frac{c_4^\nu}{c_0^\nu}J_0J_{\nu+4} + \frac{c_{-4}^\nu}{c_0^\nu}J_{-4}J_\nu + \cdots\right] \tag{95}$$

In lowest orders of h^2 this is

$$M_\nu^{(1)}(0,h) = \frac{c_0^\nu}{c_4^\nu}J_{-2}J_{\nu+2} + \frac{c_{-4}^\nu}{c_4^\nu}J_{-4}J_\nu + \cdots$$

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which when evaluated in lowest orders gives

$$\frac{3.2^7}{h^4}\frac{(h/2)^{\nu+4}}{\nu!}\left[\frac{1}{2(\nu+1)(\nu+2)}-\frac{3}{(2+i\sqrt{5})4.3.2}\right]$$

which reduces to

$$\frac{1}{2}(h/2)^2\left[\frac{-1+i\sqrt{5}}{2+i\sqrt{5}}\right]$$

in agreement with the previous result, i.e. eq.(93).

Next we come to $M_{-\nu}^{(1)}(0,h)$. Again we use first the method with $r=0$ in eq. (87).We have

$$M_{-\nu}^{(1)}(0,h) = J_0(h)J_{-\nu}(h) - \frac{c_2^{-\nu}(h^2)}{c_0^{-\nu}(h^2)}J_1(h)J_{-\nu+1}(h) - \frac{c_{-2}^{-\nu}(h^2)}{c_0^{-\nu}(h^2)}J_{-1}(h)J_{-\nu-1}(h)$$

$$+ \frac{c_4^{-\nu}(h^2)}{c_0^{-\nu}(h^2)}J_2(h)J_{-\nu+2}(h) + \frac{c_{-4}^{-\nu}(h^2)}{c_0^{-\nu}(h^2)}J_{-2}(h)J_{-\nu-2}(h) + \cdots \tag{96}$$

In lowest orders of h^2 this is

$$M_{-\nu}^{(1)}(0,h) = J_0(h)J_{-\nu}(h) + \frac{c_4^{-\nu}(h^2)}{c_0^{-\nu}(h^2)}J_2(h)J_{-\nu+2}(h)$$

which when evaluated in lowest orders of h^2 implies (with the help of the power expansion of the Bessel function $J_\mu(2h)$)

$$1.\left(\frac{(h/2)^{-\nu}}{(-\nu)!}+O(h^4)+\frac{(h/2)^{-\nu+4}}{2!(-\nu+2)!}\right)+\frac{1}{2}\frac{c_4^{-\nu}}{c_0^{-\nu}}(h/2)^{4-\nu}$$

and so

$$(h/2)^2\left(\frac{1}{2}+\frac{(1-4i\sqrt{5})}{6(2+i\sqrt{5})}\right)=+\frac{1}{2}(h/2)^2-\frac{1}{6}(h/2)^2(2+i\sqrt{5})$$

It follows that

$$M_{-\nu}^{(1)}(0,h) = \frac{1}{6}(h/2)^2(1-i\sqrt{5}) \tag{97}$$

If we set $r=2$ in eq. (87) and evaluate $M_{-\nu}^{(1)}(0,h)$, we obtain

$$M_{-\nu}^{(1)}(0,h) = \frac{c_0^{-\nu}}{c_4^{-\nu}}\left[J_{-2}J_{-\nu+2} - \frac{c_2^{-\nu}}{c_0^{-\nu}}J_{-1}J_{-\nu+3}\right.$$

$$\left. - \frac{c_{-2}^{-\nu}}{c_0^{-\nu}}J_{-3}J_{-\nu+1} + \frac{c_4^{-\nu}}{c_0^{-\nu}}J_0J_{-\nu+4} + \frac{c_{-4}^{-\nu}}{c_0^{-\nu}}J_{-4}J_{-\nu} + \cdots\right] \tag{98}$$

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In lowest orders of h^2 this is

$$\frac{c_0^{-\nu}}{c_4^{-\nu}}\left[J_{-2}J_{-\nu+2} + \frac{c_4^{-\nu}}{c_0^{-\nu}}J_0 J_{-\nu+4} + \cdots\right]$$

Evaluating this as before we obtain in leading orders

$$M_{-\nu}^{(1)}(0,h) = -\left(\frac{2+i\sqrt{5}}{3}\right)\left[\frac{1}{2}(h/2)^2.1 - \left(\frac{3}{2+i\sqrt{5}}\right).1.\frac{(h/2)^{-\nu+4}}{(-\nu+4)!}\right] \tag{99}$$

Hence in leading order

$$M_{-\nu}^{(1)}(0,h) = (h/2)^2\left[\frac{1-i\sqrt{5}}{6}\right] \tag{100}$$

which is seen to be in agreement with eq. (97). With Mathematica we obtain higher order terms, i.e.

$$\begin{aligned}
M_{\nu}^{(1)}(0,h) &= \frac{1+i\sqrt{5}}{6}\left(\frac{h}{2}\right)^2 + \frac{1+i\sqrt{5}}{9}\left(\frac{h}{2}\right)^4 \\
&+ \frac{1}{2160}\left(-290 + 151i\sqrt{5} + 120\left(5-i\sqrt{5}\right)\left(C + \ln\frac{h}{2}\right)\right)\left(\frac{h}{2}\right)^6 \\
&+ \frac{1}{3240}\left(-514 - 73i\sqrt{5} + 120\left(5-i\sqrt{5}\right)\left(C + \ln\frac{h}{2}\right)\right)\left(\frac{h}{2}\right)^8,
\end{aligned} \tag{101}$$

$$\begin{aligned}
M_{-\nu}^{(1)}(0,h) &= \frac{1-i\sqrt{5}}{6}\left(\frac{h}{2}\right)^2 + \frac{1-i\sqrt{5}}{9}\left(\frac{h}{2}\right)^4 \\
&+ \frac{1}{2160}\left(-290 - 151i\sqrt{5} + 120\left(5+i\sqrt{5}\right)\left(C + \ln\frac{h}{2}\right)\right)\left(\frac{h}{2}\right)^6 \\
&+ \frac{1}{3240}\left(-514 + 73i\sqrt{5} + 120\left(5+i\sqrt{5}\right)\left(C + \ln\frac{h}{2}\right)\right)\left(\frac{h}{2}\right)^8
\end{aligned} \tag{102}$$

With these results we can evaluate α_ν and $\alpha_{-\nu}$. Thus again in the dominant approximation

$$\alpha_\nu = Me_\nu(0,h)/M_\nu^{(1)}(0,h) = c_0^\nu/\left[\frac{1}{2}(h/2)^2\right] \tag{103}$$

and

$$\alpha_{-\nu} = Me_{-\nu}(0,h)/M_{-\nu}^{(1)}(0,h) = -c_0^{-\nu}.\left(\frac{3}{2+i\sqrt{5}}\right)/\left[\frac{1}{2}(h/2)^2\right] \tag{104}$$

It follows that

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$$R = \frac{M_{-\nu}^{(1)}(0,h)}{M_{\nu}^{(1)}(0,h)}$$

$$= -\frac{i+\sqrt{5}}{-i+\sqrt{5}} + \frac{-49 + 80\left(C + \ln\frac{h}{2}\right)}{128\sqrt{5}(2i+\sqrt{5})}h^4$$

$$+ \frac{\left(-49 + 80\left(C + \ln\frac{h}{2}\right)\right)\left(151i - 58\sqrt{5} + 120\left(-i+\sqrt{5}\right)\left(C + \ln\frac{h}{2}\right)\right)}{737280(-7i+\sqrt{5})}h^8$$

$$= -\frac{2+i\sqrt{5}}{3} + \frac{\left(5 - 2i\sqrt{5}\right)\left(-49 + 80\left(C + \ln\frac{h}{2}\right)\right)}{360}\left(\frac{h}{2}\right)^4$$

$$+ \frac{1}{51840}\left\{49\left(449 + 85i\sqrt{5}\right) - 80\left(743 + 232i\sqrt{5}\right)\left(C + \ln\frac{h}{2}\right)\right.$$

$$\left. + 19200\left(2 + i\sqrt{5}\right)\left(C + \ln\frac{h}{2}\right)^2\right\}\left(\frac{h}{2}\right)^8 \tag{105}$$

with $c_0^\nu = c_0^{-\nu} = 1$ (cf.MS, p. 122, eq. (39)).

We can now identify our quantities with those of ref. [1]. Comparison with the Dougall coefficients evaluated previously implies

$$\alpha_\nu = 1/\phi(\nu/2), \quad \alpha_{-\nu} = 1/\phi(-\nu/2) \tag{106}$$

and so in the notation of ref. [1]

$$R = \phi(-\nu/2)/\phi(\nu/2) \tag{107}$$

A remarkable feature of the expression (105) is its unit modulus, as was also observed in ref. [1]. It means that R is a pure phase factor

$$R = e^{i\pi\gamma}, RR^* = 1 \tag{108}$$

When and why this behaviour occurs is discussed at the end of Appendix A.

VIII. THE QUANTITY R IN THE GENERAL CASE

In the general case, or for $s = l+2$ sufficiently large so that no problems with singularities arise, we can evaluate R and so $M_{\pm\nu}^{(1)}(0,h)$ and $Me_{\pm\nu}(0,h)$ by simply using the power series expansions of ν and the standard Mathieu coefficients and, of course, the power series expansion of Bessel functions $J_\mu(2h)$. One then obtains

$$M_{\pm\nu}^{(1)}(0,h) = \frac{1}{(\pm\nu)!}\left(\frac{h}{2}\right)^{\pm\nu}\left[1 + \frac{2}{\nu^2-1}\left(\frac{h}{2}\right)^2 \mp \frac{2(\nu^2 \mp 3\nu - 7)}{(\nu\pm1)^2(\nu\mp1)(\nu^2-4)}\left(\frac{h}{2}\right)^4\right.$$

$$\left.\mp \frac{4(\nu^4 \mp 11\nu^3 - 2\nu^2 \pm 59\nu - 23)}{(\nu\pm1)^2(\nu\mp1)^3(\nu^2-4)(\nu^2-9)}\left(\frac{h}{2}\right)^6 + \cdots\right] \tag{109}$$

This implies for R

$$R = \frac{\nu!}{(-\nu)!\left(\frac{h}{2}\right)^{2\nu}} \cdot \frac{\left[1 + \frac{2}{\nu^2-1}\left(\frac{h}{2}\right)^2 + \frac{2(\nu^2+3\nu-7)}{(\nu-1)^2(\nu+1)(\nu^2-4)}\left(\frac{h}{2}\right)^4 + \cdots\right]}{\left[1 + \frac{2}{\nu^2-1}\left(\frac{h}{2}\right)^2 - \frac{2(\nu^2-3\nu-7)}{(\nu+1)^2(\nu-1)(\nu^2-4)}\left(\frac{h}{2}\right)^4 + \cdots\right]}$$

$$= \frac{\nu!(\nu-1)!\sin\pi\nu}{\pi\left(\frac{h}{2}\right)^{2\nu}} \cdot \left[1 + \frac{4\nu}{(\nu^2-1)^2}\left(\frac{h}{2}\right)^4\right.$$

$$\left.+ \frac{2\nu(4\nu^5 + 15\nu^4 - 32\nu^3 - 12\nu^2 + 64\nu - 111)}{(\nu^2-1)^4(\nu^2-4)^2}\left(\frac{h}{2}\right)^8 + \cdots\right] \tag{110}$$

The first few terms of the expansion of the function $Me_\nu(0,h)$ which is needed for comparison with the results of ref. [1] have been obtained in the previous section.

From eq.(110) we extract for later reference

$$\left(\frac{\sin\pi\nu}{R}\right)^2 = \frac{\pi^2\left(\frac{h}{2}\right)^{4\nu}}{\left\{\nu!(\nu-1)!\right\}^2\left[1 + \frac{4\nu}{(\nu^2-1)^2}\left(\frac{h}{2}\right)^4 + \cdots\right]^2} \tag{111}$$

This expansion will be used below in the low order approximation of the absorptivity for higher partial waves.

IX. CALCULATION OF THE ABSORPTIVITY

We consider the absorptivity in a general case, and hence allow for complex Floquet exponents ν, which we set

$$\nu = n + i(\alpha + i\beta) = (n - \beta) + i\alpha \tag{112}$$

where $n = 2, 3, 4, \ldots$ and α and β are real and of $O(h^4)$. In evaluating the S–matrix for small h^4 one has to be careful to make the expansions in the appropriate way. Thus we write $SS^\star$

$$SS^\star = \frac{(1 - \frac{1}{R^2})(1 - \frac{1}{R^{\star 2}})}{\left(e^{i\pi\nu} - \frac{e^{-i\pi\nu}}{R^2}\right)\left(e^{-i\pi\nu^\star} - \frac{e^{i\pi\nu^\star}}{R^{\star 2}}\right)} \tag{113}$$

which can be rewritten

$$SS^\star = \frac{e^{2\pi\alpha}(1 - \frac{1}{R^2})(1 - \frac{1}{R^{\star 2}})}{\left[1 - \left\{ \cos 2\pi\beta.(\frac{1}{R^2} + \frac{1}{R^{\star 2}}).e^{2\pi\alpha} + i\sin 2\pi\beta.(\frac{1}{R^2} - \frac{1}{R^{\star 2}}).e^{2\pi\alpha} - \frac{e^{4\pi\alpha}}{R^2 R^{\star 2}} \right\} \right]} \tag{114}$$

Here we set

$$e^{2i\pi\beta} \equiv 1 + if, \quad e^{2\pi\alpha} \equiv 1 + g \tag{115}$$

where f is complex and g is real (in the S–wave case $g = -\frac{2\sqrt{5}\pi}{3}(h/2)^4 + O(h^8)$). Then

$$\cos 2\pi\beta = 1 - \Im f, \Im f \approx \frac{1}{2}(2\pi\beta)^2 \approx \frac{1}{2}(\sin 2\pi\beta)^2 \tag{116}$$

and

$$\sin 2\pi\beta = \Re f \tag{117}$$

Then

$$SS^\star = (1 + g)\Bigg[1$$
$$- \frac{\left\{ g(\frac{1}{R^2} + \frac{1}{R^{\star 2}} - \frac{2+g}{R^2 R^{\star 2}}) - \Im f.(1 + g)(\frac{1}{R^2} + \frac{1}{R^{\star 2}}) + (1 + g).\Re f.i(\frac{1}{R^2} - \frac{1}{R^{\star 2}}) \right\}}{(1 - \frac{1}{R^2})(1 - \frac{1}{R^{\star 2}})}\Bigg]^{-1} \tag{118}$$

We now consider two limiting cases.

(i) $\alpha \to 0$ implying $g \to 0$.

In this case $R = R^\star$ and so $1/R^2 \simeq O(h^4)$. This is the case of real Floquet exponents and so excludes the case of S–waves. Here

$$SS^\star = \frac{1}{1 + \frac{\Im f.\frac{2}{R^2}}{(1 - \frac{1}{R^2})^2}} \simeq 1 - \frac{2\Im f}{\frac{R^2}{(1 - \frac{1}{R^2})^2}} \simeq 1 - \frac{4\left(\frac{\sin \pi\beta}{R}\right)^2}{(1 - \frac{1}{R^2})^2} \tag{119}$$

since

$$\Im f \simeq \frac{1}{2}(\sin 2\pi\beta)^2 \simeq 2\sin^2 \pi\beta.$$

The absorptivity A is therefore given by

$$A = 1 - SS^\star \approx \frac{4\left(\frac{\sin \pi\beta}{R}\right)^2}{(1 - \frac{1}{R^2})^2} \approx 4\left(\frac{\sin \pi\beta}{R}\right)^2 \tag{120}$$

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With the help of eq.(111) this can be written

$$
A \approx \frac{4\pi^2 \left(\frac{h}{2}\right)^{4\nu}}{\left\{\nu!(\nu-1)!\right\}^2} \cdot \left[1 + \frac{4\nu}{(\nu^2-1)^2}\left(\frac{h}{2}\right)^4 + O(h^8)\right]^{-2}
$$

$$
= \frac{4\pi^2 \left(\frac{h}{2}\right)^{4(l+2)}}{\left\{(l+1)!(l+2)!\right\}^2} \cdot \left[1 + \frac{4\nu}{(\nu^2-1)^2}\left(\frac{h}{2}\right)^4 + O(h^8)\right]^{-2} \tag{121}
$$

in agreement with ref. [1]. This can be easily evaluated (e.g. with Maple), e.g. already in the case of P–waves (i.e. in spite of singularities in higher order terms here omitted) and yields in this case

$$
A = \frac{\pi^2 h^{12}}{3^2 2^{14}}\left[1 + \left\{\frac{53}{1152} - \frac{1}{24}\log\left(\frac{he^\gamma}{2}\right)\right\}h^4\right] \tag{122}
$$

where γ is the Euler constant (also written C). This result agrees with the result in ref. [1]. We observe, in particular, that logarithmic energy contributions arise in the expansion. The formula (121), of course, does not apply in the case of S–waves.

(ii) $\beta \to 0$ implying $f \to 0$.

In this case $RR^\star = 1$, so that $|R| \sim O(h^0)$, and we cannot expand as in the previous case. However, $g \approx O(h^4)$, so that we can expand in powers of g. Thus

$$
SS^\star = \frac{1+g}{1 + g\left\{1 + \frac{g}{2-R^2-R^{\star 2}}\right\}}
$$

$$
= 1 - \frac{g^2}{2 - R^2 - R^{\star 2}} + O(g^3)
$$

$$
= 1 + O(h^8) \tag{123}
$$

This is the case of complex Floquet exponents as in the S–wave case. In this case

$$
SS^\star = 1 - \frac{9g^2}{20} + O(g^3) \tag{124}
$$

and so

$$
A = \frac{9g^2}{20} + O(g^3) = \pi^2(h/2)^8 + \frac{2\pi^2}{9}\left(7 - 12(\gamma + \ln\frac{h}{2})\right)(h/2)^{12} + O(h^{16}) \tag{125}
$$

in agreement with ref. [1] and the rough lowest order calculation of ref. [16]. We can also compute for this particularly interesting S–wave case the amplitudes of the reflected, transmitted and incident waves A_r, A_t, A_i defined in Appendix A. One finds

$$
A_r = R - \frac{1}{R} = -\frac{2i\sqrt{5}}{3} + O(h^4),
$$
$$
A_t = 2i\sin\nu\pi = \frac{2\sqrt{5}}{3}\pi(h/2)^4 + O(h^8)
$$
$$
A_i = Re^{i\nu\pi} - \frac{1}{Re^{i\nu\pi}} = -\frac{2i\sqrt{5}}{3} - \frac{4\sqrt{5}\pi}{9}(h/2)^4 + O(h^8) \tag{126}
$$

We observe that in the limit $h^4 \to 0$ we have $A_t = 0$ and $A_r = A_i$, i.e. there is only reflection of the fluctuation or disturbance around the D–brane like reflection from a wall and no transmission, which can be interpreted as a vanishing of the disturbance on the brane (implying a Dirichlet boundary condition). On the basis of the analogy with the case of the open fundamental string between brane and antibrane in Born–Infeld theory we can expect that as the energy increases, transmission (i.e. absorption) sets in and becomes the dominant effect at high energies. This is similar to what one finds in quantum mechanics of a potential well of depth $-V_o$ [23]. There the properly normalised transmission and reflection coefficients $T(E), R(E)$, where E is the energy, have the behaviour $T(E) \to 0, R(E) \to 1$ as $E \to 0$, but $T(E) \to 1$ and $R(E) \to 0$ as $E \to \infty$. The high energy behaviour of the effect considered here can presumably be investigated with the help of large–h asymptotic expansions of modified Mathieu functions which we expect to be formally (i.e. apart from sign and complex i changes) similar to those of periodic Mathieu functions with a parameter q defined as the solution of

$$
(l + 2)^2 = -2h^2 + 2hq + O(h^0). \tag{127}
$$

The Floquet exponent is then given by [12](p.210), [25]

$$
\cos\pi\nu + 1 = \frac{\pi e^{4h}}{(8h)^{q/2}}\left[\frac{1 + 3(q^2+1)/64h}{\Gamma((1-q)/4)\Gamma((3-q)/4)} + O\left(\frac{\log h}{h^2}\right)\right]. \tag{128}
$$

One observes that again logarithmic contributions in the energy appear.

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X. CONCLUSIONS

In the above we considered the impingement of a massless scalar field on a $D3$ brane in 10 dimensions and calculated the S matrix and partial wave absorption and reflection amplitudes and rates for this process. Instead of coefficients introduced by Dougall for the expansion of the modified Mathieu functions involved, we used (in the low energy domain) rapidly convergent series in terms of products of Bessel functions. We demonstrated that the Mathieu function coefficients are such that many different expansions in terms of products of Bessel functions all yield the same low energy power series for the modified Mathieu functions of the first kind. We think, this is the best way to evaluate the absorption rates of the problem in the low energy domain. The leading term matching procedures of refs. [15,16] maybe select dominant terms of the expansions considered here. Since the metric considered is extremal, one can visualise the absorption of the partial waves of the scalar field as absorption into the brane or black hole with vanishing event horizon (examples with nonvanishing horizon have for instance been treated in [24]). Since several other string models lead also to the modified Mathieu equation in analogous contexts, the above considerations, which have definite advantages over those involving the coefficients of Dougall, may be of wider interest.

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Appendix A

Here we recapitulate the main steps of the derivation of the S–matrix. We follow ref. [8], but instead of repeating the steps there, we emphasize those which have not been written out explicitly there. For ease of comparison we consider the repulsive potential which means simply that the considerations below employ coupling g as in ref. [8] instead of g_0 used above. The two cases are trivially related through

$$g_0 = ig \tag{A.1}$$

In the repulsive case we have a regular solution y_{reg} of eq.(5)at $r = 0$, i.e. one proportional to $exp(-g/r)$. The variable of the cylindrical functions involved in $M^{(j)}(z,h)$ is $\omega = 2h\cosh z = (ig/r + kr)$. Thus in leading order for small h^2 and $r \to 0 (z \to -\infty)$ we can write

$$y_{reg} = r^{1/2} M_\nu^{(3)}(z,h) \overset{\Re z \to -\infty}{\simeq} r^{1/2} \left[H_\nu^{(1)}(\omega) + O(h^2) \right] \overset{r \to 0}{\simeq} \left(\frac{2}{\pi g} \right)^{\frac{1}{2}} e^{-\frac{g}{r}} e^{-i(\nu+1)\frac{\pi}{2}} \left[1 + O(h^2) \right] \tag{A.2}$$

If we let $Rez \to -\infty$ here and then replace z by $-z$, the solution has the asymptotic behaviour e^{ikr}. The series expansion defining the Bessel function J_ν has the following important property for integers n

$$J_\nu(2h\cosh(z + in\pi)) = exp(in\nu\pi) J_\nu(2h\cosh z) \tag{A.3}$$

so that

$$M_\nu^{(1)}(z + in\pi, h) = exp(in\nu\pi) M_\nu^{(1)}(z,h). \tag{A.4}$$

Since for $Me_\nu(z,h)$ also

$$Me_\nu(z + in\pi, h) = exp(in\nu\pi) Me_\nu(z,h) \tag{A.5}$$

we have the proportionality

$$Me_\nu(z,h) = \alpha_\nu(h) M_\nu^{(1)}(z,h) \tag{A.6}$$

with (e.g.)

$$\alpha_\nu(h) = \frac{Me_\nu(0,h)}{M_\nu^{(1)}(0,h)} \tag{A.7}$$

As mentioned earlier, expansions in terms of cylindrical functions like (85) converge uniformly only in domains $|\cosh z| > 1$, whereas the expansion (32) of $Me_\nu(z,h)$ converges for all finite complex values of z. Hence we match $M_\nu^{(3)}(z,h)$ in the domain $\Re z < 0$ to a linear combination of $M_\nu^{(3)}(z,h)$ and $M_\nu^{(4)}(z,h)$ in the domain $\Re z > 0$ by matching both to a combination of $Me_\nu(z,h)$ and $Me_{-\nu}(z,h)$ in the intermediate domain. We have

$$z = \log \sqrt{k/g}\, r \; (\overset{-}{+}) \; i\pi/4. \tag{A.8}$$

In the domain of r close to zero we set

$$r^{1/2} M_\nu^{(3)} = r^{1/2}\left(\alpha Me_\nu + \beta Me_{-\nu} \right)$$

$$\frac{d}{dr}\left(r^{1/2} M_\nu^{(3)} \right) = \alpha \frac{d}{dr}\left(r^{1/2} Me_\nu \right) + \beta \frac{d}{dr}\left(r^{1/2} Me_{-\nu} \right) \tag{A.9}$$

where α and β have to be determined. In the domain of large r we set, with constants α', β', A, B, which have to be determined

$$r^{1/2}\left(\alpha' Me_\nu + \beta' Me_{-\nu} \right) = r^{1/2}\left(A M_\nu^{(3)} + B M_\nu^{(4)} \right),$$

$$\alpha' \frac{d}{dr}\left(r^{1/2} Me_\nu \right) + \beta' \frac{d}{dr}\left(r^{1/2} Me_{-\nu} \right) = A \frac{d}{dr}\left(r^{1/2} M_\nu^{(3)} \right) + B \frac{d}{dr}\left(r^{1/2} M_\nu^{(4)} \right). \tag{A.10}$$

We match the $Me_\nu, Me_{-\nu}$ combination (variable z) on the left to that on the right (variable $-z$) at $\Re z = 0 (r = \sqrt{g/k})$, so that

$$r^{1/2}\left(\alpha Me_\nu + \beta Me_{-\nu} \right)_{z=+i\pi/4} = r^{1/2}\left(\alpha' Me_\nu + \beta' Me_{-\nu} \right)_{z=-i\pi/4},$$

$$\left[\alpha \frac{d}{dr}\left(r^{1/2} Me_\nu \right) + \beta \frac{d}{dr}\left(r^{1/2} Me_{-\nu} \right) \right]_{z=+i\pi/4} = \left[\alpha' \frac{d}{dr}\left(r^{1/2} Me_\nu \right) + \beta' \frac{d}{dr}\left(r^{1/2} Me_{-\nu} \right) \right]_{z=-i\pi/4} \tag{A.11}$$

Since $Me_\nu(z) = Me_{-\nu}(-z)$ and at $r = \sqrt{g/k}, z = \pm i\pi/4$, also

$$\frac{d}{dr} = \mp \left(\frac{k}{g} \right)^{1/2} \frac{d}{dz},$$

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the latter become

$$\alpha M e_{-\nu} + \beta M e_{\nu} = \alpha' M e_{\nu} + \beta' M e_{-\nu}$$

$$\alpha \frac{d}{dz} M e_{-\nu} + \beta \frac{d}{dz} M e_{\nu} = \alpha' \frac{d}{dz} M e_{\nu} + \beta' \frac{d}{dz} M e_{-\nu} \tag{A.12}$$

From these equations we obtain immediately

$$\alpha' = \beta, \quad \beta' = \alpha \tag{A.13}$$

From eqs.(A.9) we obtain (W meaning Wronskian)

$$\alpha = \frac{W[M_\nu^{(3)}, M e_{-\nu}]}{W[M e_\nu, M e_{-\nu}]}, \quad \beta = -\frac{W[M_\nu^{(3)}, M e_\nu]}{W[M e_\nu, M e_{-\nu}]}, \tag{A.14}$$

From (A.10) we obtain similarly

$$A = \frac{-W[M_\nu^{(3)}, M e_\nu] W[M e_\nu, M_\nu^{(4)}] + W[M_\nu^{(3)}, M e_{-\nu}] W[M e_{-\nu}, M_\nu^{(4)}]}{W[M_\nu^{(3)}, M_\nu^{(4)}] W[M e_\nu, M e_{-\nu}]},$$

$$B = \frac{W[M_\nu^{(3)}, M e_\nu] W[M e_\nu, M_\nu^{(3)}] - W[M_\nu^{(3)}, M e_{-\nu}] W[M e_{-\nu}, M_\nu^{(3)}]}{W[M_\nu^{(3)}, M_\nu^{(4)}] W[M e_\nu, M e_{-\nu}]} \tag{A.15}$$

We now use eq.(A.6) and Wronskians $W[M_\nu^{(i)}, M_\nu^{(j)}] \equiv [i, j]$ given in ref. [12], i.e.

$$[3, 4] = -\frac{4i}{\pi}, \quad [1, 3] = -[1, 4] = \frac{2i}{\pi} \tag{A.16}$$

and the circuit relation ([12], p. 169)

$$M_{-\nu}^{(1)} = e^{i\nu\pi} M_\nu^{(1)} - i \sin \nu\pi M_\nu^{(4)}. \tag{A.17}$$

Then

$$W[M e_\nu, M e_{-\nu}] = -\frac{2 \sin \nu\pi}{\pi} \alpha_\nu \alpha_{-\nu},$$

$$W[M e_{-\nu}, M_\nu^{(3)}] = \frac{2i}{\pi} e^{-i\nu\pi} \alpha_{-\nu} \ , \ \ W[M e_{-\nu}, M_\nu^{(4)}] = -\frac{2i}{\pi} e^{i\nu\pi} \alpha_{-\nu}, \tag{A.18}$$

With these expressions A and B are found to be

$$A = \frac{1}{2i \sin \nu\pi} \left(\frac{\alpha_\nu}{\alpha_{-\nu}} - \frac{\alpha_{-\nu}}{\alpha_\nu} \right), \quad B = \frac{1}{2i \sin \nu\pi} \left(\frac{\alpha_\nu}{\alpha_{-\nu}} - e^{-2i\nu\pi} \frac{\alpha_{-\nu}}{\alpha_\nu} \right) \tag{A.19}$$

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The regular solution thus continued to $r = \infty$ is then

$$
y_{reg} \simeq r^{1/2} \left[A M^{(3)}(z, h) + B M^{(4)}(z, h) \right]
$$

$$
\simeq \left(\frac{2}{k\pi} \right)^{1/2} \left\{ A e^{ikr} e^{-i(\nu + \frac{1}{2})\frac{\pi}{2}} + e^{-i\frac{\pi}{2}} B e^{-ikr} e^{i(\nu + \frac{1}{2})\frac{\pi}{2}} \right\} \tag{A.20}
$$

In terms of the variable z and with $R \equiv \alpha_\nu / \alpha_{-\nu}$ this can be written

$$
\left(\frac{2r}{2h\pi \cosh z} \right)^{\frac{1}{2}} e^{-i(\nu + \frac{1}{2})\frac{\pi}{2}} \left\{ 2i \sin \nu\pi \right\} e^{2ih \cosh z}
$$

$$
\overset{\Re z \to \infty}{\simeq} \left(\frac{2r}{2h\pi \cosh z} \right)^{\frac{1}{2}} e^{-i(\nu + \frac{1}{2})\frac{\pi}{2}} \left\{ (R - \frac{1}{R}) e^{2ih \cosh z} \right.
$$

$$
\left. + i(Re^{i\nu\pi} - \frac{e^{-i\nu\pi}}{R}) e^{-2ih \cosh z} \right\} \tag{A.21}
$$

If we take $A_i = (Re^{i\nu\pi} - \frac{e^{-i\nu\pi}}{R})$ as the amplitude of the incident wave, the amplitudes A_r and A_t of the reflected and transmitted waves are $A_r = R - \frac{1}{R}$ and $A_t = 2i \sin \nu\pi$ respectively in agreement with ref. [1]. With the definition of the S–matrix in the partial wave expansion of the scattering amplitude $f(\theta)$, with $x = \cos \theta$, for n space dimensions (here we have $n = 6$) given by [21]

$$
e^{ikx} + f(\theta) \frac{e^{ikr}}{r^{(n-1)/2}} \simeq \frac{1}{2(ikr)^{(n-1)/2}} \sum_{l=0}^{\infty} \left\{ S e^{ikr} + (-1)^l i^{n-1} e^{-ikr} \right\} \tilde{\mathcal{P}}_l(\cos \theta) \tag{A.22}
$$

where

$$
\tilde{\mathcal{P}}_l(\cos \theta) = \sqrt{\frac{2}{\pi}} 2^{n/2 - 1} \Gamma(n/2 - 1)(l + \frac{n}{2} - 1) C_l(\cos \theta)
$$

and $C_l(\cos \theta)$ is a Gegenbauer polynomial, we obtain for this

$$
S = \frac{R - \frac{1}{R}}{(Re^{i\nu\pi} - \frac{e^{-i\nu\pi}}{R})} . e^{-i\pi l} \tag{A.23}
$$

It is easy to verify that for ν real and $R \equiv e^y$ real, unitarity is preserved, i.e. unity minus reflection probability = transmission probability, i.e.

$$
1 - \frac{|R - \frac{1}{R}|^2}{|Re^{i\nu\pi} - \frac{e^{-i\nu\pi}}{R}|^2} = \frac{|2 \sin \nu\pi|^2}{|Re^{i\nu\pi} - \frac{e^{-i\nu\pi}}{R}|^2} \tag{A.24}
$$

We observe that this relation remains valid if the real quantity $R \equiv e^y$ and the pure phase factor $e^{i\pi\nu}$ exchange their roles, i.e. if R becomes a pure phase factor and $e^{i\pi\nu}$ a real exponential. The latter is precisely what happens in the S–wave case of the attractive potential discussed above.

Appendix B

Below we give the explicit form of the first three terms of the small–h^2 perturbation expansions of the leading coefficients $c_{2r}^{-\nu}$ of expansions of modified Mathieu functions in the S–wave case ($l = 0$). The nonleading terms have been obtained with Mathematica.

$$\frac{c_2^{-\nu}}{c_0^{-\nu}} = \frac{i+\sqrt{5}}{-2i+\sqrt{5}}\left(\frac{h}{2}\right)^2 + \frac{11(5-2i\sqrt{5})}{2^3 5(-2i+\sqrt{5})^2}\left(\frac{h}{2}\right)^6 + \frac{(72311i-17746\sqrt{5})}{2^7 3^3 5(-2i+\sqrt{5})^3}\left(\frac{h}{2}\right)^{10} + O(h^{11}),$$

$$\frac{c_{-2}^{-\nu}}{c_0^{-\nu}} = -\frac{1}{3}\left(\frac{h}{2}\right)^2 - \frac{3+16i\sqrt{5}}{2^3 3^3}\left(\frac{h}{2}\right)^6 + \frac{5146-391i\sqrt{5}}{2^6 3^5 5}\left(\frac{h}{2}\right)^{10} + O(h^{11}),$$

$$\frac{c_4^{-\nu}}{c_0^{-\nu}} = \frac{i-\sqrt{5}}{i+\sqrt{5}} + \frac{11(5i+\sqrt{5})}{2^3 3^1 5(i+\sqrt{5})}\left(\frac{h}{2}\right)^4 + \frac{2966\sqrt{5}-28889i}{2^7 3^5 5(2i+\sqrt{5})}\left(\frac{h}{2}\right)^8 + O(h^9),$$

$$\frac{c_{-4}^{-\nu}}{c_0^{-\nu}} = \frac{1}{2^3 3}\left(\frac{h}{2}\right)^4 + \frac{18+125i\sqrt{5}}{2^6 3^3 5}\left(\frac{h}{2}\right)^8 + \frac{1303i\sqrt{5}-25774}{2^8 3^5 5^2}\left(\frac{h}{2}\right)^{12} + O(h^{13}),$$

$$\frac{c_6^{-\nu}}{c_0^{-\nu}} = \frac{1}{2+i\sqrt{5}}\left(\frac{h}{2}\right)^2 - \frac{290i+49\sqrt{5}}{2^2 3^3 5(i+\sqrt{5})}\left(\frac{h}{2}\right)^6 + \frac{39559i-4597\sqrt{5}}{2^7 3^5 5(-i+\sqrt{5})}\left(\frac{h}{2}\right)^{10} + O(h^{11}),$$

$$\frac{c_{-6}^{-\nu}}{c_0^{-\nu}} = -\frac{1}{2^3 3^2 5}\left(\frac{h}{2}\right)^6 - \frac{19+157i\sqrt{5}}{2^6 3^4 5^2}\left(\frac{h}{2}\right)^{10} + \frac{132035-5159i\sqrt{5}}{2^7 3^6 5^3 7}\left(\frac{h}{2}\right)^{14} + O(h^{15}).$$