Tunneling of Born–Infeld Strings to $D2$–Branes

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Abstract

A Born–Infeld theory describing a $D2$–brane coupled to a 4–form RR field strength is considered, and the general solutions of the static and Euclidean time equations are derived and discussed. The period of the bounce solutions is shown to allow a consideration of tunneling and quantum–classical transitions in the sphaleron region. The order of such transitions, depending on the strength of the RR field strength, is determined. A criterion is then derived to confirm these findings.

PACS Numbers:11.10.Lm, 11.27.+d, 05.70.Fh

I. INTRODUCTION

Recently it was argued [1] that a fundamental string can be viewed as a collapsed brane, or more precisely as a p–brane ($p \geq 2$) with $(p-1)$ spatial directions of its worldvolume (e.g. with spatial part $\mathbf{R} \times S^{p-1}$) collapsed to zero size (S^{p-1} shrunk to zero). This view is

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somehow a reversal of the traditional view of considering D–branes as extended objects on which open strings can put their ends [2], and whose low energy dynamics is described by nonlinear Born–Infeld theory on the worldvolume of the brane [3]. The latter view implies, as has been shown [4] the existence of string–like configurations. In ref. [1] the reversal of the traditional view was considered and the question was asked whether by application of external forces (here an RR–spacetime gauge potential) a string may expand back into a D–brane, and this was – in fact – shown to be possible in the $D2$–brane case considered via a tunneling process.

In the following our objective is to investigate in more generality the problem posed in ref. [1] and its solutions and the tunneling process, and to determine the order of the quantum–classical transition (in analogy with phase transitions) at nonzero temperatures as a function of the external force.

The decay of a Born–Infeld brane–antibrane system by tunneling was discussed earlier in ref. [4] but was in that context discarded as being physically irrelevant. The $D2$–brane model of [1] , however, seems a fascinating and computationally manageable model which allows the investigation of various aspects of tunneling related to strings.

In Section 2 we describe the formulation of the problem and its solutions. In Section 3 we calculate the wavelength and energy of the general static configuration. In Section 4 we determine the Euclidean version of the model for the torus brane (defined by constant radius) and determine the period of the bounce configuration. With this and the sphaleron configuration we can determine the phase diagrams for quantum classical transitions of the string of vanishing radius into a brane of constant radius.

It is found that a weak RR field leads to a sharp transition described as of the first order, whereas a stronger RR field leads to one of the second order, described as smooth. In Sections 5 and 6 we then determine and discuss a criterion for the occurrence of transitions of either order. Finally in Section 7 we summarize our findings in some conclusions.

II. FORMULATION OF THE PROBLEM AND ITS SOLUTIONS

Our starting point is the same Born–Infeld action describing a $D2$–brane coupled to the RR–background spacetime 3–form gauge potential $A_{\mu\nu\rho}$ with field strength $H = dA$ as that of ref. [1]; we also use the same conventions (e.g. $\alpha' = 1$). Thus the action we consider is

$$I = -\frac{1}{4\pi^2 g}\int d^3\xi\left\{\sqrt{-det\left(g^{ind}_{\alpha\beta} + 2\pi\mathcal{F}_{\alpha\beta}\right)} + \frac{1}{3!}\epsilon^{\alpha\beta\gamma}A_{\mu\nu\rho}\partial_\alpha X^\mu\partial_\beta X^\nu\partial_\gamma X^\rho\right\}. \qquad (1)$$

Here $\mu, \nu, \rho = 0, \cdots, 9$ are spacetime indices, and $\alpha, \beta, \gamma = 0, 1, 2$ worldvolume indices, g is the (type IIA) string coupling and the dilaton is taken to be constant. The induced metric is the pullback of the spacetime metric (here assumed to be flat) and thus is $\partial_\alpha X^\mu \partial_\beta X_\mu$. The $U(1)$ gauge field tensor $\mathcal{F}_{\alpha\beta}$ contains in particular the electric field $\mathcal{E}(t, z) := 2\pi(\partial_0 A_z - \partial_z A_0)$. The background spacetime gauge field H is taken to be $H_{0123} = h$=const. As in ref. [1] we choose the world volume to be cylindrical and hence define

$$X^0 = t, \quad X^1 = z, \quad X^2 = R(t, z)\cos\sigma, \quad X^3 = R(t, z)\sin\sigma,$$

and all other X^i = const., and $R > 0$, $0 < \sigma < 2\pi$. We also restrict our considerations here only to those of the electric field component of the gauge field A_α. The action in terms of the worldvolume coordinates t, z, σ with σ integrated out then becomes

$$I = \int dt \int dz\mathcal{L}, \quad \mathcal{L} = -\frac{1}{2\pi g}\left(R\sqrt{1 - \dot{R}^2 - \mathcal{E}^2 + R'^2} - \frac{h}{2}R^2\right) \qquad (2)$$

where dots and primes denote derivatives with respect to t and z respectively. The Lagrangian density $\mathcal{L}(R, A_0, A_z, \dot{R}, \dot{A}_0, \dot{A}_z)$ yields in particular the Gauss law type of equation

$$\frac{\partial D}{\partial z} = 0, \quad D = \frac{\mathcal{E}R}{\sqrt{1 - \dot{R}^2 - \mathcal{E}^2 + R'^2}} = const. \qquad (3)$$

In the static case to be considered below the constant $D \equiv e$ can be looked at as a charge per radian, i.e. $2\pi e\delta(z)A_0$ would be the appropriate source term in the static Lagrangian. Solving eq.(3) for $\mathcal{E}$ one finds

$$\mathcal{E} = \frac{D\sqrt{1 - \dot{R}^2 + R'^2}}{\sqrt{D^2 + R^2}}. \qquad (4)$$

Since D is constant and so unaffected by the form of R it is convenient to express the action in terms of D and regard the result as a functional of R so that the variation of the new Lagrangian density $\tilde{\mathcal{L}}$ with respect to $\mathcal{E}$ is zero. The appropriate Legendre transform of $\mathcal{L}$ is

$$\tilde{\mathcal{L}} = \mathcal{L} - \frac{D\mathcal{E}}{2\pi g} \tag{5}$$

since with eq.(3) $\partial\mathcal{L}/\partial\mathcal{E} = \frac{D}{2\pi g}$. Evaluating $\tilde{\mathcal{L}}$ with the help of eq. (4) one obtains as in [1]

$$\tilde{\mathcal{L}} = -\frac{1}{2\pi g}\left(\sqrt{R^2+D^2}\sqrt{1-\dot{R}^2+R'^2} - \frac{h}{2}R^2\right). \tag{6}$$

Since $\mathcal{L}$ is a functional of R, A_0, A_z, we have conjugate momenta $P = \partial\mathcal{L}/\partial\dot{R}, P_{A_0} = \partial\mathcal{L}/\partial\dot{A}_0 = 0$ and $P_{A_z} = D/g$. As usual the vanishing of P_{A_0} implies the Gauss law as the constraint (3), so that the Hamiltonian H is defined by

$$H = \int dz\left(P\dot{R} + P_{A_z}\dot{A}_z - \mathcal{L} + \frac{1}{g}A_0\frac{\partial D}{\partial z}\right) \tag{7}$$

where A_0 acts as Lagrange multiplier. Evaluating this, one obtains

$$H = \frac{1}{2\pi g}\int dz\left\{\frac{(1+R'^2)\sqrt{D^2+R^2}}{\sqrt{1-\dot{R}^2+R'^2}} - \frac{h}{2}R^2\right\}. \tag{8}$$

The static energy E is therefore

$$E = \frac{1}{2\pi g}\int dz\left\{\sqrt{(1+R'^2)(D^2+R^2)} - \frac{h}{2}R^2\right\}. \tag{9}$$

Variation of E with respect to R yields the equation for the static configuration

$$\sqrt{\frac{R^2+D^2}{1+R'^2}} - \frac{h}{2}R^2 = C \tag{10}$$

where C is an integration constant. We rewrite this equation

$$R' = \overset{-}{(+)}\,\frac{h}{hR^2+2C}\sqrt{(R_+^2-R^2)(R^2-R_-^2)}, \quad R' \neq 0 \tag{11}$$

with

$$R_\pm{}^2 = \frac{2}{h^2}\left[(1-Ch) \pm \sqrt{1-2Ch+h^2D^2}\right] \tag{12}$$

and assume for simplicity that $C \geq D$. We note also for later reference

$$R_+^2 R_-^2 = \frac{4}{h^2}(C^2 - D^2), \quad R_+^2 + R_-^2 = \frac{4}{h^2}(1 - Ch). \tag{13}$$

Many special solutions have already been discussed in ref. [1]. E.g. for $h = 0$ the case $C = D$ implies spikes $R = R_0 \exp(\pm z/C)$ for $z < 0$ or $z > 0$ respectively corresponding to the BIon in $D3$ Born–Infeld theory [5], and the case $C^2 > D^2$ implies catenoids $R = \sqrt{C^2 - D^2}\cosh(z - z_0)$. Here we are interested in the general solution. However, we note the possibility of the two signs of R' in eq.(11). The significance of these is the same as those of the catenoid discussed in [5]. There one sign gives the spike–like solution which minimizes the action, the other sign the solution which maximizes the action. The socalled wormhole solution is the two–spike solution obtained by matching the two at a suitable point.

We now consider the general solution of the static equation (10). Assuming $R_- \leq R \leq R_+$ and $R(z_0) = R_+$, eq.(10) becomes

$$\int_R^{R_+} dR \frac{hR^2 + 2C}{\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}} = -h(z_0 - z). \tag{14}$$

We observe that for real values of z the solution has turning points at $R = R_+$ and R_-. The distance between these therefore defines half a wavelength λ of oscillations between these.

The left hand side of eq. (14) can be evaluated in terms of elliptic integrals. Then the general solution is given by (cf. ref. [6], pp.56, 300-301)

$$hR_+ E(\psi, k) + \frac{2C}{R_+} F(\psi, k) = -h(z_0 - z). \tag{15}$$

Here $F(\psi, k)$ and $E(\psi, k)$ are the incomplete elliptic integrals of the first and second kinds respectively, k is the elliptic modulus given by

$$k = \frac{\sqrt{R_+^2 - R_-^2}}{R_+}$$

and ψ is the angle defined by

$$\psi = \sin^{-1}\sqrt{\frac{R_+^2 - R^2}{R_+^2 - R_-^2}}.$$

We now consider two important limiting cases: a) The limit C=D. In this case

$$R_+ = \frac{2}{h}\sqrt{1-Dh}, \quad R_- = 0, \quad k = 1$$

and

$$\sin\psi = \sqrt{R_+^2 - R^2}/R_+.$$

Using $E(\psi,1) = \sin\psi$, $F(\psi,1) = \ln(\tan\psi + \sec\psi)$ eq.(15) becomes

$$\sqrt{R_+^2 - R^2} + \frac{2D}{hR_+}\ln\frac{R_+ + \sqrt{R_+^2 - R^2}}{R} = -(z_0 - z) \tag{16}$$

which agrees with a result of ref. [1] and describes a BI–string with spheroidal bulge centered at z_0 where $R = R_+$. For $R \to R_- = 0$ the solution becomes string–like (this case may be dubbed 'vacuum solution'). b) The limit $C = (1 + h^2D^2)/2h$ is the limit of the sphaleron configuration, i.e. that with zero separation between the turning points. In this case

$$R_- = R_+ = \frac{\sqrt{2(1-Ch)}}{h}, \quad k = 0.$$

Using

$$E(\psi,0) = F(\psi,0) = \psi$$

eq.(15) becomes

$$\left(hR_+ + \frac{2C}{R_+}\right)\psi = -h(z_0 - z), \quad i.e.\ \frac{2}{\sqrt{1-h^2D^2}}\psi = -h(z_0 - z)$$

which implies

$$R^2 = R_+^2 - (R_+^2 - R_-^2)\sin^2\frac{h\sqrt{1-h^2D^2}(z-z_0)}{2} = R_+^2$$

so that

$$R = R_+ = \frac{\sqrt{2(1-Ch)}}{h} = \frac{1}{h}\sqrt{1-h^2D^2} \equiv R_S$$

where R_S is the sphaleron configuration, in agreement with a result discussed in [1] and of particular interest for $h \neq 0$.

III. CALCULATION OF WAVELENGTH AND ENERGY OF THE GENERAL STATIC CONFIGURATION

Having prepared the ground for our considerations we now proceed to calculate the wavelength λ of the general static solution of eq.(14). As mentioned above, the points R_+, R_- are turning points, so that with R in eq.(14) replaced by R_-, we can put $z - z_0 = \lambda/2$. Then

$$\int_{R_-}^{R_+} dR \frac{hR^2 + 2C}{\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}} = hR_+ E(\frac{\pi}{2}, k) + 2C\frac{1}{R_+}F(\frac{\pi}{2}, k)$$
$$= hR_+E(k) + \frac{2C}{R_+}K(k) = h\frac{\lambda}{2} \quad (17)$$

where $E(k)$ and $K(k)$ are the complete elliptic integrals of the second and first kinds respectively. The wavelength λ can therefore be expressed as

$$\lambda = 2R_+E(k) + \frac{4C}{hR_+}K(k). \quad (18)$$

Using the special values $E(1) = 1, K(1) = \infty, E(0) = K(0) = \pi/2$ (cf. [6], p.10), we can obtain the values of λ of the two limiting cases discussed above. Thus for a) with $C = D$, one obtains $\lambda = \infty$ or frequency $\Omega = 0$, and for b) with $C = (1 + h^2D^2)/(2h)$, the sphaleron limit, one obtains $\lambda = 2\pi/(h\sqrt{1 - h^2D^2})$. With this we can define as sphaleron frequency

$$\Omega_{sph} = h\sqrt{1 - h^2D^2}.$$

Next we compute the energy of the solution (14) by replacing R'^2 in E by the expression obtained from eq.(10), i.e.

$$R'^2 = \frac{4(R^2 + D^2)}{(hR^2 + 2C)^2} - 1. \quad (19)$$

We rewrite the expression (9) with the contribution of the fundamental BI string (the limit $C - D = 0, R = R' = 0$) separated in view of its divergence in the linear case, i.e. we write with a somewhat arbitrary subdivision

$$E = \frac{D}{2\pi g}\int dz + U_0, \quad U_0 = \frac{1}{2\pi g}\int dz\left[\frac{2(R^2 + D^2)}{hR^2 + 2C} - \frac{h}{2}R^2 - D\right]. \quad (20)$$

Here the first term in E divided by the length $\int dz$ gives the tension of the BI string, i.e. $T_{BI} = D/2\pi g$. As shown in ref. [4] the quantization condition on the D–flux is

$$\frac{1}{2\pi}\int_{S^1} D = gn, \quad i.e. \;\; D = gn,$$

where n is the number of fundamental strings, each of tension $1/2\pi\alpha'$, adsorbed by the BI string. Thus the first contribution to E in eq.(20) is the energy of n fundamental strings. Then by explicit calculation and using $\int_0^\lambda dz = -2\int_{R_-}^{R_+} dR/R'$ and again replacing R' by the expression of eq. (19), we obtain (cf. Appendix A)

$$U_0 = \frac{1}{6\pi gh}\Bigg[\frac{12D(D-C) + (1-k^2)h^2R_+^4}{R_+}K(k) + 2R_+\Big\{3\Big[2 - h(C+D)\Big] - (2-k^2)h^2R_+^2\Big\}E(k)\Bigg]. \tag{21}$$

Of course, the R_-^2–dependence is still contained in k^2. Again we consider the two limiting cases of above. In case a), the limit $C = D$ with

$$\lim_{k\to 1}(1-k^2)K(k) = 0,$$

one can show directly that

$$U_0 = \frac{h}{6\pi g}R_+^3, \quad R_+ = \frac{2}{h}\sqrt{1-Dh}, \quad k = 1, \tag{22}$$

which is consistent with eq.(15) of ref. [1] and represents a bulk energy. In case b), the sphaleron limit $C = (1+h^2D^2)/2h$, direct calculation yields

$$U_0 = \frac{(1-hD)^2}{2R_+gh^3}, \quad R_+ = \frac{\sqrt{2(1-Ch)}}{h} = \frac{\sqrt{1-h^2D^2}}{h}, \quad k = 0 \tag{23}$$

To show that this is physically relevant, we insert the constant value $R = R_S = \sqrt{1-h^2D^2}/h$ into the expression for the static energy E of eq.(9) and integrate over a length L. Then E becomes

$$E = \frac{1+h^2D^2}{4\pi gh}L.$$

From eq. (20) we know that this energy has to be

$$E = \frac{DL}{2\pi g} + U_0,$$

i.e. this has to be satisfied. Equating the two expressions we obtain

$$U_0 = \frac{(1 - hD)^2}{4\pi g h} L.$$

If we put here $L = \lambda = 2\pi / h\sqrt{1 - h^2 D^2}$, we obtain

$$U_0 = \frac{(1 - hD)^2}{2R_+ g h^3}$$

in agreement with eq.(23).

IV. THE QUANTUM–CLASSICAL DECAY RATE TRANSITION TO THE TORUS BRANE

In the following we consider first the solutions to the Euclidean version of the action (1), then specialise to the torus brane defined by the condition $R' = 0$, obtain the corresponding periodic bounce solutions, and then the phase diagrams.

We are interested in solutions to the Euclidean version of the action (1), i.e. (cf. eq. (6))

$$\tilde{I}_E(D, R) = \frac{1}{2\pi g} \int d\tau dz \left[\sqrt{(R^2 + D^2)(1 + \dot{R}^2 + R'^2)} - \frac{h}{2} R^2 \right] \tag{24}$$

where $\dot{R}$ now denotes differentiation with respect to Euclidean time $\tau = -it$. The equation of motion derived from (24) is

$$\frac{R(1 + \dot{R}^2 + R'^2)}{\sqrt{(R^2 + D^2)(1 + \dot{R}^2 + R'^2)}} - \frac{\partial}{\partial \tau} \frac{\dot{R}(R^2 + D^2)}{\sqrt{(R^2 + D^2)(1 + \dot{R}^2 + R'^2)}} - \frac{\partial}{\partial z} \frac{R'(R^2 + D^2)}{\sqrt{(R^2 + D^2)(1 + \dot{R}^2 + R'^2)}} - hR = 0. \tag{25}$$

Here we consider only tunneling to the torus brane ($R' = 0$). In this case the equation reduces to

$$\frac{R(1 + \dot{R}^2)}{\sqrt{(R^2 + D^2)(1 + \dot{R}^2)}} - \frac{\partial}{\partial \tau} \frac{\dot{R}(R^2 + D^2)}{\sqrt{(R^2 + D^2)(1 + \dot{R}^2)}} - hR = 0. \tag{26}$$

After some manipulation one can show that eq. (26) is exactly the same as eq. (11) if z there is replaced by τ. In fact this is easily understood from the symmmetry of the Euclidean action (24) under the exchange $\tau \leftrightarrow z$. Hence the periodic bounce solution is in this case obtained directly from eq.(15) by changing $z_0 - z$ into $\tau_0 - \tau$, i.e. from

$$hR_+E(\psi,k) + \frac{2C}{R_+}F(\psi,k) = -h(\tau_0 - \tau). \tag{27}$$

With this expression for Euclidean time $\tau_0 - \tau$ we can calculate the period P. We first consider the Euclidean action for this configuration with $R' = 0$ (as stated earlier) which is

$$\begin{aligned} \tilde{I}_{E,cl}(D,R_{cl}) &= \frac{1}{2\pi g}\int d\tau dz \left[\sqrt{({R_{cl}}^2 + D^2)(1+\dot{R}_{cl}^2)} - \frac{h}{2}{R_{cl}}^2\right] \\ &= L\left[\frac{D}{2\pi g}\int dz + U_0\right] \end{aligned} \tag{28}$$

where U_0 is given by eq. (21) and L is now the circumference of the torus in Euclidean time. We write the part with the contribution of fundamental strings subtracted out

$$\tilde{I}^{(0)}_{E,cl}(D,R_{cl}) = LU_0(C,D,h). \tag{29}$$

As in statistical mechanics the period P of this configuration is to be identified with the reciprocal of temperature T (cf. ref. [7]). Since the wavelength $\lambda = 2\pi/\Omega$, where Ω is the corresponding angular frequency, we can rewrite eq. (27) here

$$P = 2R_+E(k) + \frac{4C}{hR_+}K(k) = \frac{1}{T}. \tag{30}$$

The sphaleron configuration in this tunneling is

$$R = R_S = \frac{\sqrt{1-h^2D^2}}{h}.$$

The barrier height or sphaleron energy is

$$E_{sph} = \frac{L}{4\pi gh}(1-hD)^2 \tag{31}$$

as observed at the end of Section 3. From this we obtain for the sphaleron action

$$I^{(0)}_{sph} = \frac{E_{sph}}{T}. \tag{32}$$

With the formulas at hand we can now plot the phase diagrams.The integration constant C here plays the same role as the energy E of a pseudoparticle in the usual quantum mechanics around such a configuration ($E = 0$ in the latter being its value for the vacuum solution, and $E = E_{sph}$ that for the sphaleron, see e.g. [9] and [10]). In the present case the vacuum solution has $C = D$, and the sphaleron $C = (1 + h^2 D^2)/2h$. In Fig.1(a) we show the behaviour of the period P as a function of C for $D = 1$ and $h = 0.7$. We observe a monotonically decreasing behaviour characteristic of a smooth second order transition. In Fig. 1(b) we plot for the same values of D and h the behaviour of the sphaleron action per unit length $I^{(0)}_{sph}/L$ and the Euclidean action per unit length $\tilde{I}^{(0)}_{E,cl}/L$ versus temperature T. We observe the expected smooth transition from one to the other. In Figs.2(a) and (b) we plot the same quantities again for $D = 1$ but for the smaller value $h = 0.3$ of the RR field. This time we observe a rise of the period beyond a critical value, and a corresponding nonsmooth bifurcation of the action implying a sharp quantum classical transition, generally described as of first order.

V. THE CRITERION FOR A SHARP TRANSITION

A criterion for the occurrence of sharp or first order quantum–classical transitions was developed in ref. [7]. A more explicit criterion useful for practical purposes was obtained in ref. [8]. Our considerations here are based on this latter reference where a detailed explanation of steps can be found. Further applications of the criterion have been given in refs. [9] and [10]. The criterion for occurrence of a first order transition is $\Omega^2 - \Omega_S^2 > 0$, where Ω_S is the sphaleron frequency and Ω the possible frequency different from Ω_S [8].

With some manipulations eq.(26) can be put into the following form

$$\ddot{R} - R\frac{1+\dot{R}^2}{R^2+D^2} + hR(1+\dot{R}^2)\sqrt{\frac{1+\dot{R}^2}{R^2+D^2}} = 0. \qquad (33)$$

We expand this equation around the sphaleron configuration R_S by setting

$$R = R_S + \delta R(\tau), \quad R_S = \frac{1}{h}\sqrt{1-h^2D^2}. \qquad (34)$$

Using the following expansions

$$
\begin{aligned}
\frac{1}{R^2+D^2} &= h^2\Bigg[1-2h\sqrt{1-h^2D^2}\delta R + h^2(3-4h^2D^2)\delta R^2 \\
&\quad +4h^3(2h^2D^2-1)\sqrt{1-h^2D^2}\delta R^3+\cdots\Bigg], \\
\frac{1+\dot{R}^2}{R^2+D^2} &= h^2\Bigg[1-2h\sqrt{1-h^2D^2}\delta R \\
&\quad +\Big\{h^2(3-4h^2D^2)\delta R^2+\delta\dot{R}^2\Big\} \\
&\quad +\Big\{4h^3(2h^2D^2-1)\sqrt{1-h^2D^2}\delta R^3 - 2h\sqrt{1-h^2D^2}\delta\dot{R}^2\delta R\Big\} \\
&\quad +\cdots\Bigg], \\
\sqrt{\frac{1+\dot{R}^2}{R^2+D^2}} &= h\Bigg[1-h\sqrt{1-h^2D^2}\delta R \\
&\quad +\frac{1}{2}\Big\{h^2(2-3h^2D^2)\delta R^2+\delta\dot{R}^2\Big\} \\
&\quad -\frac{h}{2}\sqrt{1-h^2D^2}\Big\{(2-5h^2D^2)h^2\delta R^3+\delta R\delta\dot{R}^2\Big\}+\cdots\Bigg]
\end{aligned} \tag{35}
$$

one then obtains

$$
\begin{aligned}
R\frac{1+\dot{R}^2}{R^2+D^2} &= h\Bigg[\quad \sqrt{1-h^2D^2} - h(1-2h^2D^2)\delta R \\
&\quad +\sqrt{1-h^2D^2}\Big\{h^2(1-4h^2D^2)\delta R^2+\delta\dot{R}^2\Big\} \\
&\quad -h\Big\{h^2(1-8h^2D^2+8h^4D^4)\delta R^3+(1-2h^2D^2)\delta\dot{R}^2\delta R\Big\} \\
&\quad +\cdots\Bigg]
\end{aligned} \tag{36}
$$

and

$$
\begin{aligned}
R(1+\dot{R}^2)\sqrt{\frac{1+\dot{R}^2}{R^2+D^2}} &= \quad \sqrt{1-h^2D^2}+h^3D^2\delta R \\
&\quad +\sqrt{1-h^2D^2}\Big\{-\frac{3}{2}h^4D^2\delta R^2+\frac{3}{2}\delta\dot{R}^2\Big\} \\
&\quad +\Big\{h^3(2h^2D^2-\frac{5}{2}h^4D^4)\delta R^3+\frac{3}{2}h^3D^2\delta R\delta\dot{R}^2\Big\} \\
&\quad +\cdots
\end{aligned} \tag{37}
$$

Eq. (33) can now be expanded as

$$\delta\ddot{R} + h^2(1-h^2D^2)\delta R + G_2(\delta R) + G_3(\delta R) + \cdots = 0 \tag{38}$$

where

$$\begin{aligned} G_2(\delta R) &= \frac{h\sqrt{1-h^2D^2}}{2}\Big[-h^2(2-5h^2D^2)\delta R^2 + \delta\dot{R}^2\Big], \\ G_3(\delta R) &= \frac{h^2}{2}\Big[h^2(2-12h^2D^2+11h^4D^4)\delta R^3 + (2-h^2D^2)\delta\dot{R}^2\delta R\Big]. \end{aligned} \tag{39}$$

In lowest or linear order we have therefore

$$\delta\ddot{R} + h^2(1-h^2D^2)\delta R = 0 \tag{40}$$

with solution

$$\delta R = a\cos\Omega_S\tau, \quad \Omega_S = h\sqrt{1-h^2D^2} \tag{41}$$

where a is a small amplitude of oscillation around sphaleron. We observe that the sphaleron period resulting from eq. (41), i.e.

$$P_{sph} = \frac{2\pi}{\Omega_S} = \frac{2\pi}{h\sqrt{1-h^2D^2}}$$

coincides exactly, as expected, with the wavelength defined after eq.(18).

We proceed to the second order calculation by setting

$$\delta R = a\cos\Omega\tau + a^2\eta_1(\tau), \quad \Omega^2 = \Omega_S^2 + a\triangle_1\Omega^2. \tag{42}$$

with a new or as yet undetermined frequency Ω. Inserting this trial solution into eq.(38) and reexpressing squares of $\cos\Omega\tau$ by $\cos 2\Omega\tau$ we obtain at this level of the approximation

$$\begin{aligned} \eta_1 = \left(\frac{\partial^2}{\partial\tau^2} + \Omega_S^2\right)^{-1}\Big[&\triangle_1\Omega^2\cos\Omega_S\tau \\ +\frac{\Omega_S}{4}\Big\{[h^2(2-5h^2D^2)-\Omega^2] &+ [h^2(2-5h^2D^2)+\Omega^2]\cos 2\Omega\tau\Big\}\Big]. \end{aligned} \tag{43}$$

In order to avoid infinity and have a defined fluctuation, the square bracket on the right hand side must not contain the zero mode of the operator $\left(\frac{\partial^2}{\partial\tau^2} + \Omega_S^2\right)$. Hence we must demand $\triangle_1\Omega^2 = 0$. This condition therefore yields

$$\Omega = \Omega_S \tag{44}$$

to this order of the calculation together with the fluctuation

$$\eta_1 = g_1 + g_2 \cos 2\Omega_S \tau \tag{45}$$

where

$$g_1 = \frac{\Omega_S}{4}\frac{1-4h^2D^2}{1-h^2D^2}, \quad g_2 = -\frac{\Omega_S}{4}\frac{1-2h^2D^2}{1-h^2D^2}. \tag{46}$$

We proceed to the third order by setting

$$\delta R = a\cos\Omega\tau + a^2\eta_1(\tau) + a^3\eta_2(\tau), \quad \Omega^2 = \Omega_S^2 + a\triangle_1\Omega^2 + a^2\triangle_2\Omega^2. \tag{47}$$

Inserting this ansatz into eq.(38) yields with a procedure as before but now at this level

$$\eta_2 = \frac{1}{a^3}\left(\frac{\partial^2}{\partial\tau^2} + \Omega_S^2\right)^{-1}\left[\chi_2^{(0)} + \chi_2^{(1)}\cos\Omega_S\tau + \chi_2^{(2)}\cos 2\Omega_S\tau + \chi_3^{(3)}\cos 3\Omega_S\tau\right], \tag{48}$$

where (only this is needed here)

$$\begin{aligned}\chi_2^{(1)} &= a(\Omega^2 - \Omega_S^2) - a^3\Big[-h^2(2-5h^2D^2)\Omega_S(g_1 + \frac{g_2}{2}) + \Omega_S^3 g_2 \\ &\qquad + \frac{3}{8}h^4(2-12h^2D^2+11h^4D^4) + \frac{h^2}{8}\Omega_S^2(2-h^2D^2)\Big] \\ &= a^3\triangle_2\Omega^2 - a^3\Big[1-4h^2D^2\Big]\frac{h^4}{2}.\end{aligned} \tag{49}$$

Again we set this equal to zero to avoid infinity in (48) and so determine $\triangle_2\Omega^2$. The criterion for a sharp or first order transition $\Omega^2 - \Omega_S^2 > 0$ then implies

$$\triangle_2\Omega^2 > 0, \quad i.e. \quad hD < \frac{1}{2}. \tag{50}$$

We observe that this result agrees with our earlier findings that the smooth second order transition occurs for larger values of the applied force or h. Thus also the bifurcation point in the plot of the action must disappear in the large h domain. We now investigate this point in more detail.

VI. TUNNELING OF THE STRING TO AN ARBITRARY $D2$–BRANE

It is much more difficult to understand the tunneling associated with the sphaleron of the general static solution (15), i.e. for general values of ψ in

$$hR_+E(\psi,k) + \frac{2C}{R_+}F(\psi,k) = -h(z_0 - z) \tag{51}$$

However, we can understand qualitatively this tunneling using the tunneling to the torus investigated in the previous sections. We first compute the barrier height which is equal to the sphaleron energy, i. e.

$$\begin{aligned} E_{sph} &= \frac{1}{2\pi g}\int dz\left[\sqrt{(R^2+D^2)(1+R'^2)} - \frac{h}{2}R^2\right] - c.f.s. \\ &= \frac{1}{6\pi gh}\left[\frac{12D(D-C)+(1-k^2)h^2R_+^4}{R_+}K(k)\right. \\ &\quad \left. +2R_+\left\{3[2-h(C+D)] - (2-k^2)h^2R_+^2\right\}E(k)\right] \end{aligned} \tag{52}$$

where c.f.s. is the contribution of the fundamental strings. In fact, the expression (52) is exactly the same as the classical action for tunneling to the torus divided by L, the circumference of the torus, as one can see by comparison with eq. (21). With this observation we can deduce the following facts.

1) In the region of $hD > \frac{1}{2}$ the energy E_{sph} is minimal for tunneling to the torus ($R' = 0$). Hence tunneling to the torus may be the dominant process.

2) In the region of $hD < \frac{1}{2}$ the energy E_{sph} has a minimum at the tunneling which corresponds to the bifurcation point in the action–versus–temperature diagram (cf. Fig. 2(b)). Thus in this region the dominant tunneling may not be tunneling to the torus but to a wiggly distorted brane.

It is important therefore to understand when the bifurcation point appears. We can study this, for instance, by determining those values of $C = C(D,h)$, which correspond to the bifurcation points in tunneling with $R' = 0$. In order to determine this parameter C as a function of D and h, we recall that the period P of the bounce configuration for a first order transition has an extremum period as can also be seen from Fig.2(a) (or ref. [11]). Thus this $C(D,h)$ is determined by

$$\left.\frac{dP}{dC}\right|_{C=C(D,h)} = 0. \tag{53}$$

Here P is, of course, the same as the wavelength λ of eq. (18). In Fig.3(a) the solution $C(D,h)$ of this equation is plotted for $D=1$ as a function of h. One can see that the maximum of C meets the bifurcation point at $C=1.2$ (cf. Fig.2(a)) at exactly $h=0.5$, the limit set by the criterion eq. (50). Fig. 3(b) shows a corresponding plot for $D=2$. Both of these plots therefore show how the bifurcation point disappears in the large h region, i.e. beyond the limit given by the criterion (50). Finally, in Fig. 4 we show the shape of a brane at a bifurcation point.

VII. CONCLUSIONS

The expansion of a string into a $D2$–brane by tunneling through a barrier provided by a 4–form RR field strength was already shown to be possible in ref. [1]. In the above we have investigated this problem in more detail by deriving the explicit solutions of the classical equation in terms of elliptic functions. These solutions then permitted the derivation of the period of the bounce configurations, as well as a detailed study of the sphaleron at the top of the potential barrier. With these tools it was possible to study the quantum–classical behaviour at nonzero temperatures in the sphaleron domain, and to determine the order of the transitions (in analogy with phase transitions) depending on the magnitude of the applied RR field. We then confirmed these findings with the derivation of a criterion for the occurrence of such transitions in the present case. The case of the $D2$–brane discussed here has the advantage of permitting explicit calculations with relative ease. We expect the method, however, to be applicable also to more complicated cases.

Acknowledgements: D.K. P. and S.T. acknowledge support by the Deutsche Forschungsgemeinschaft (DFG), Y.–G. M acknowledges support by an A. von Humboldt research fellowship.

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Appendix A

Here we indicate briefly how eq.(21) is obtained. Substituting R' of eq.(11) into the expression (20) for U_0 one obtains an expression that can be written

$$U_0 = \frac{2}{\pi g h}\Bigg\{ -\frac{h^2}{4}\int_{R_-}^{R_+} \frac{R^4 dR}{\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}} + D(D-C)\int_{R_-}^{R_+} \frac{dR}{\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}} + \left[1 - \frac{h(C+D)}{2}\right]\int_{R_-}^{R_+} \frac{R^2 dR}{\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}}\Bigg\}. \quad \text{(A.1)}$$

Using formulas 218.01 and 218.00 of ref. [6] one can obtain

$$\int_{R_-}^{R_+} \frac{R^2 dR}{\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}} = R_+ E(k) \quad \text{(A.2)}$$

and

$$\int_{R_-}^{R_+} \frac{dR}{\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}} = \frac{1}{R_+} K(k). \quad \text{(A.3)}$$

Also using formulas 218.06 and 314.04 of ref. [6] one finds

$$\int_{R_-}^{R_+} \frac{R^4 dR}{\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}} = \frac{1}{3} R_+^3 \Big[-(1-k^2)K(k) + 2(2-k^2)E(k)\Big]. \quad \text{(A.4)}$$

Inserting these expressions into eq.(A.1) one obtains eq.(21).

Figure Captions

Fig. 1(a)

The period P of the bounce as a function of the parameter C for $D = 1$ and $h = 0.7$.

Fig. 1(b)

The sphaleron action and classical action per unit length as function of temperature T for $D = 1$ and $h = 0.7$.

Fig. 2(a)

The period P of the bounce as a function of the parameter C for $D = 1$ and $h = 0.3$.

Fig. 2(b)

The sphaleron action and classical action per unit length as function of temperature T for $D = 1$ and $h = 0.3$.

Fig. 3(a)

The parameter C as solution of $dP/dC = 0$, P the period of the bounce, plotted as a function of h for $D = 1$. The dotted line represents a maximum of C at given h. One should observe the disapppearance of the bifurcation point at $C = 1.2$ (cf. Fig. 2(a)) beyond $h = 0.5$ in agreement with the criterion of Section 5.

Fig. 3(b)

The parameter C as solution of $dP/dC = 0$, P the period of the bounce, plotted as a function of h for $D = 2$. The dotted line represents a maximum of C at given h. One should observe the disapppearance of the bifurcation point beyond $h = 0.25$ in agreement with the criterion of Section 5.

Fig. 4

The shape of the brane at the bifurcation point for $D = 1, h = 0.3, C = 1.20434$.

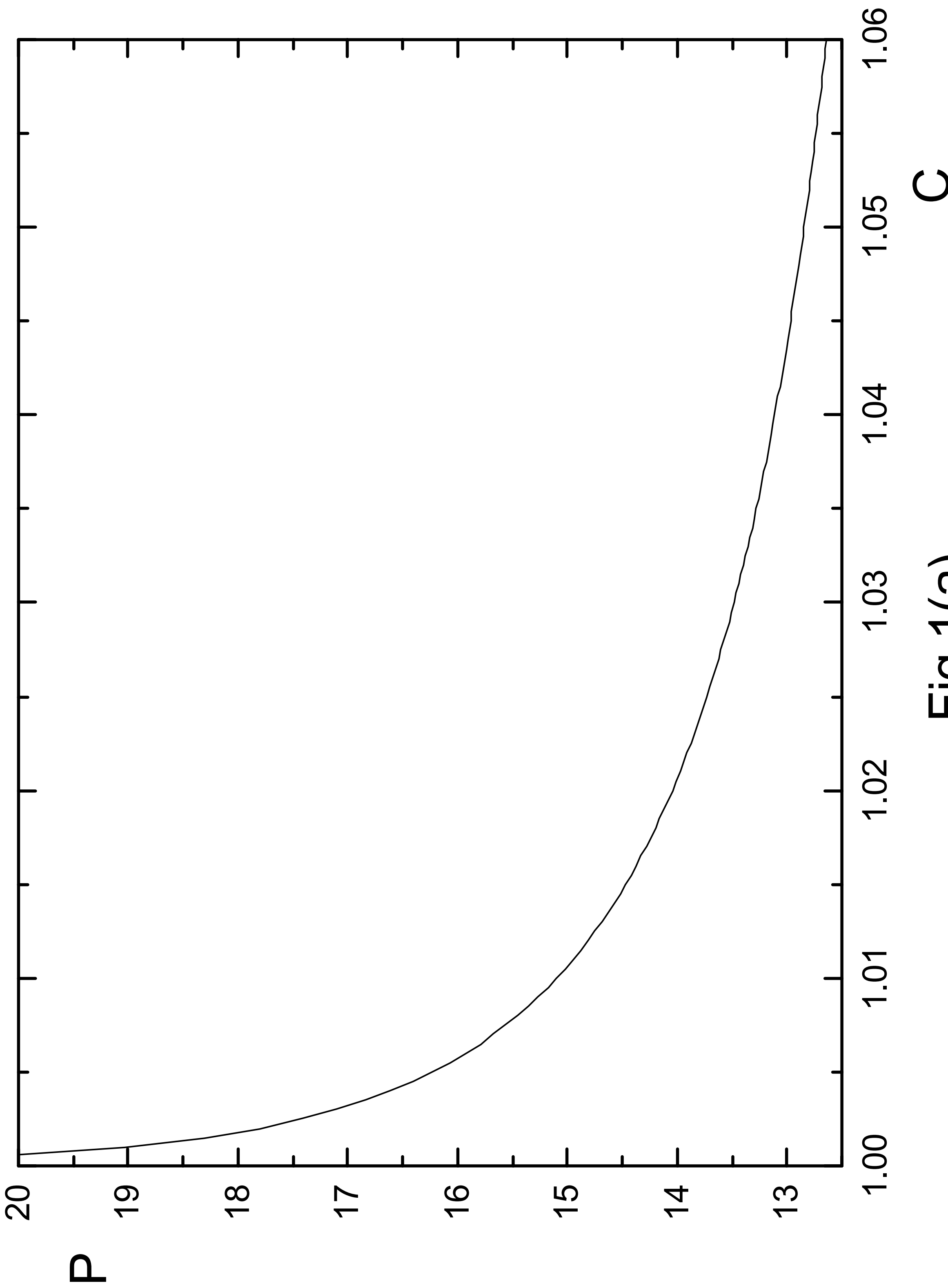
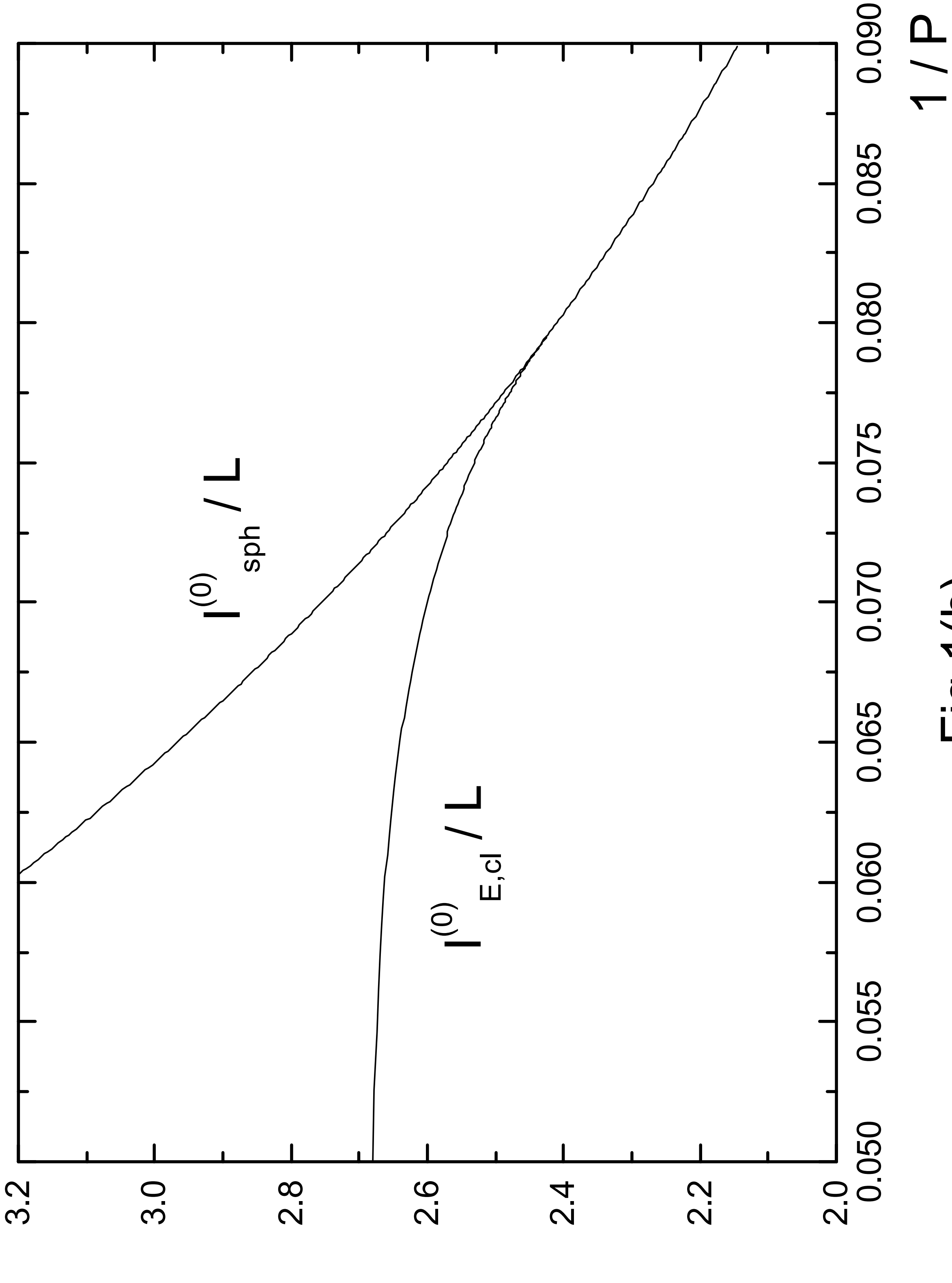


Fig.1(a)



3.2
3.0
2.8
2.6
2.4
2.2
2.0
0.050
0.055
0.060
0.065
0.070
0.075
0.080
0.085
0.090
1 / P
$I^{(0)}_{sph} / L$
$I^{(0)}_{E,cl} / L$

Fig.1(b)

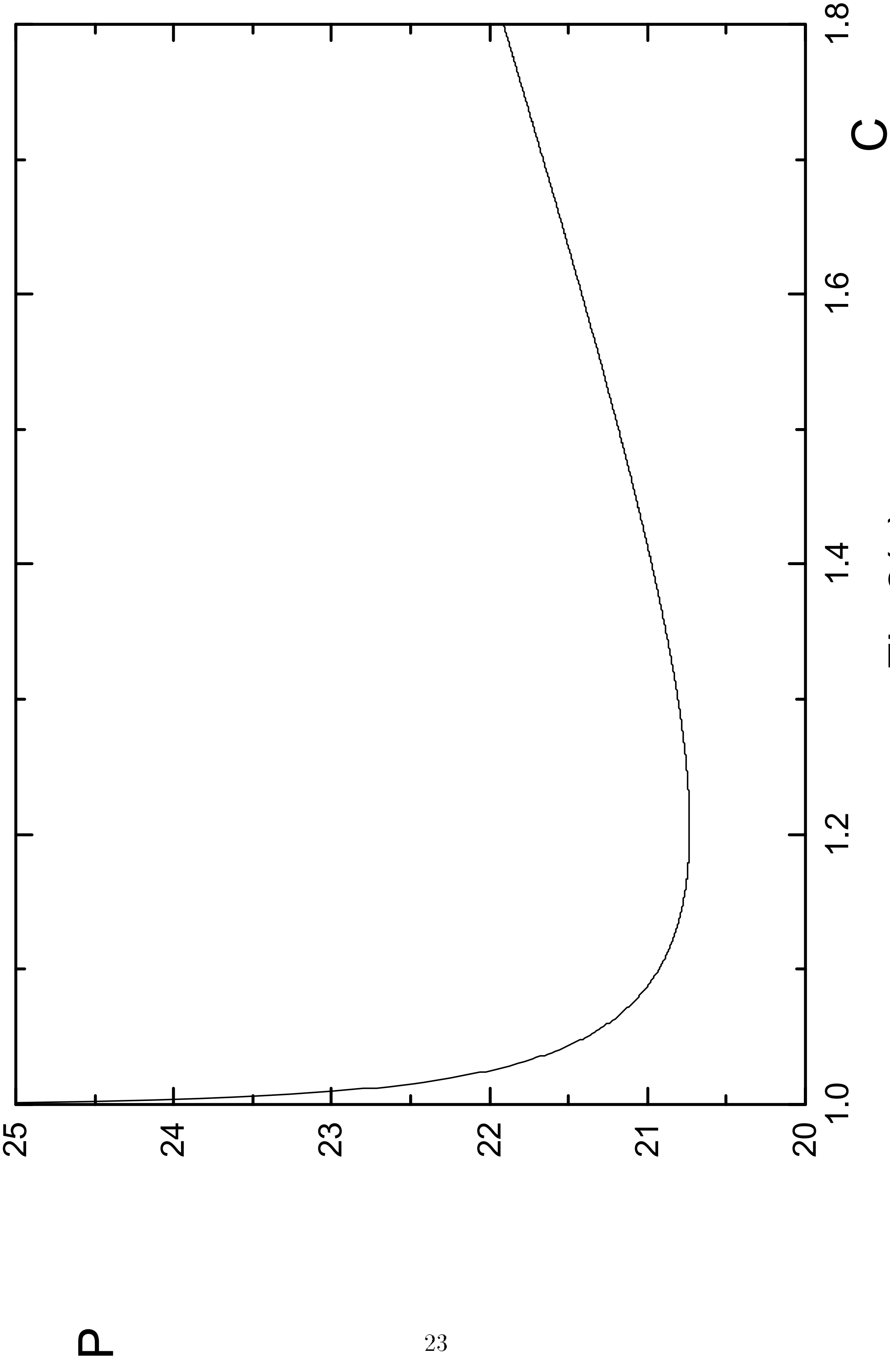
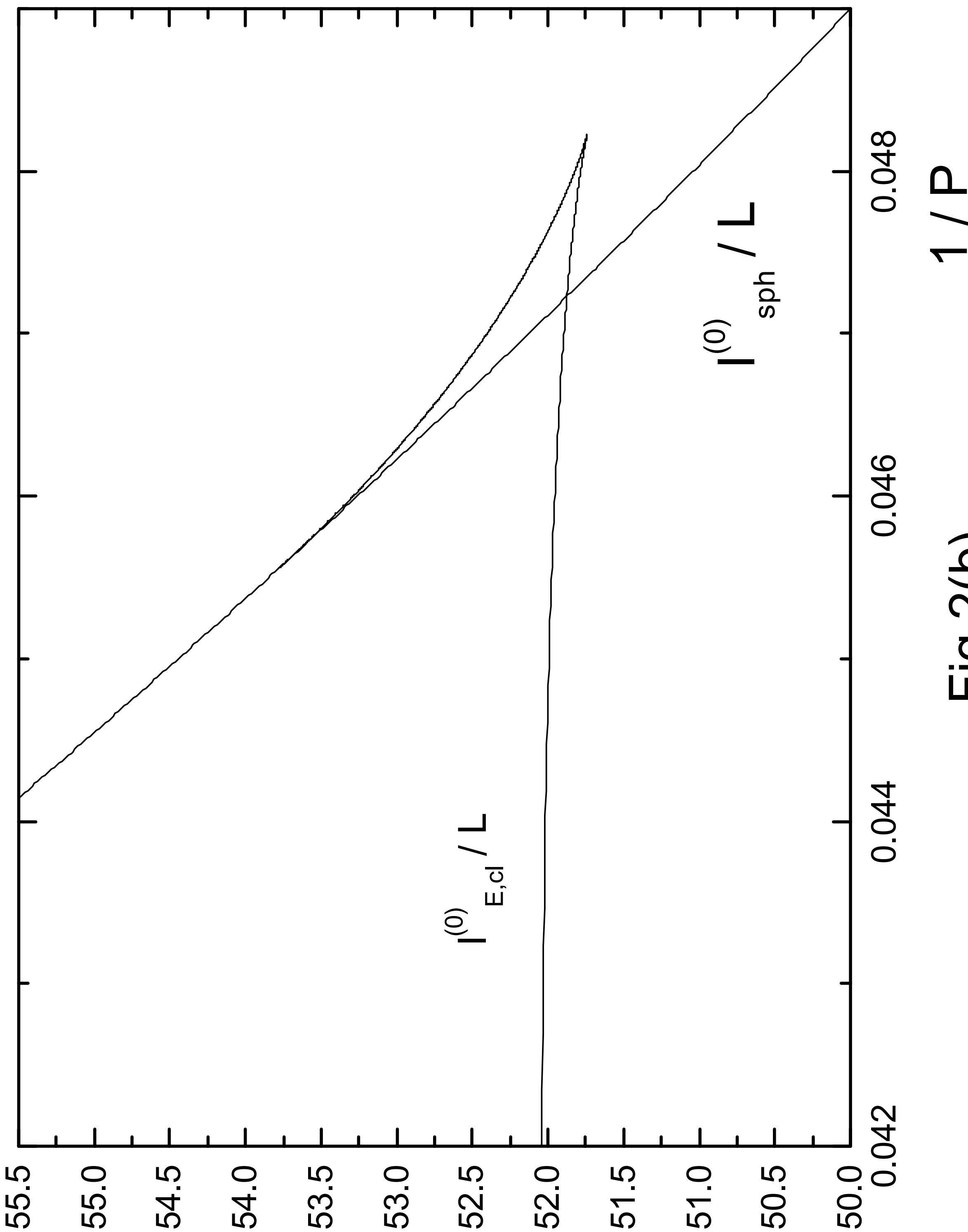


Fig.2(a)



55.5
55.0
54.5
54.0
53.5
53.0
52.5
52.0
51.5
51.0
50.5
50.0
0.042
0.044
0.046
0.048
1 / P
$I^{(0)}_{E,cl}$ / L
$I^{(0)}_{sph}$ / L

Fig.2(b)

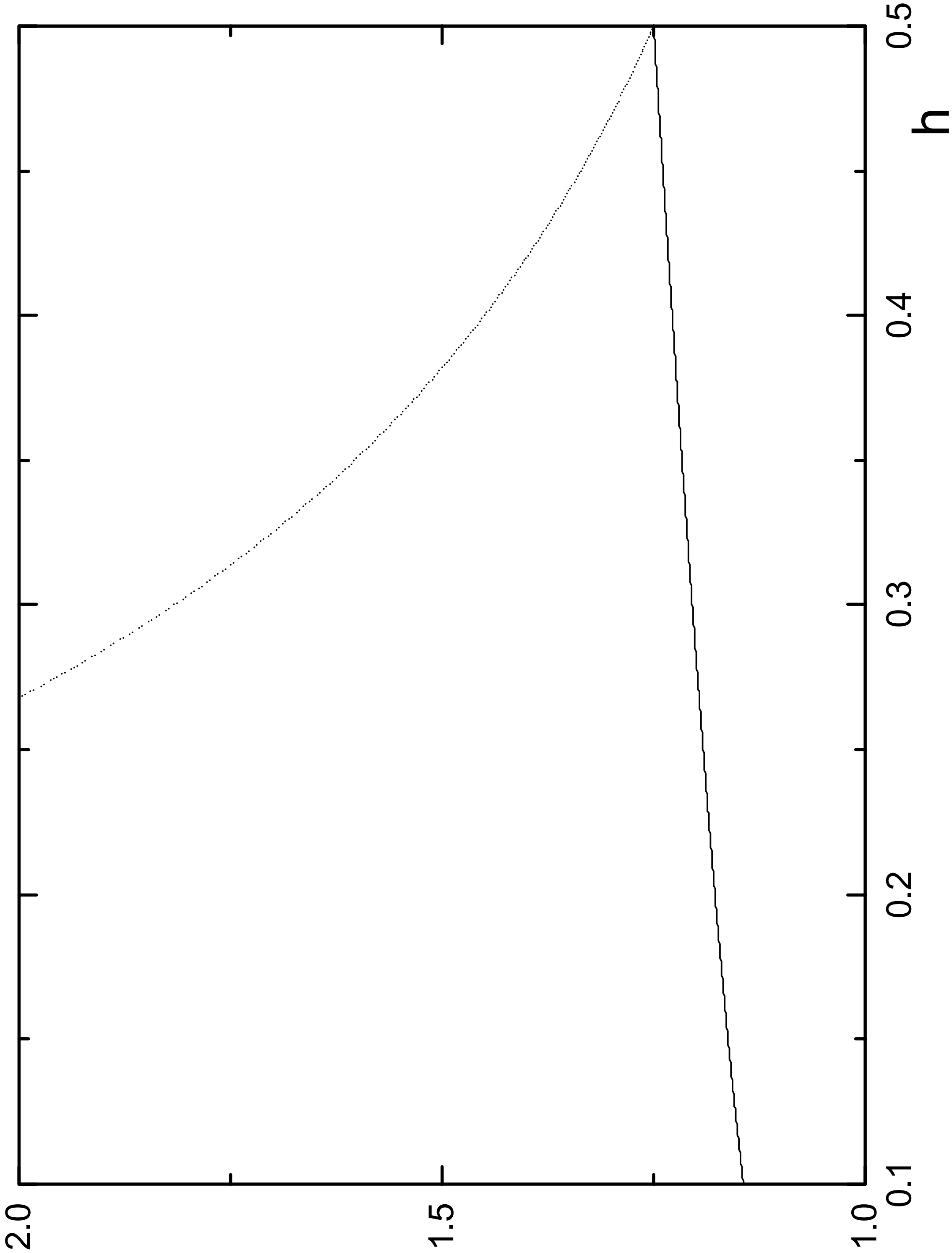


Fig.3(a)

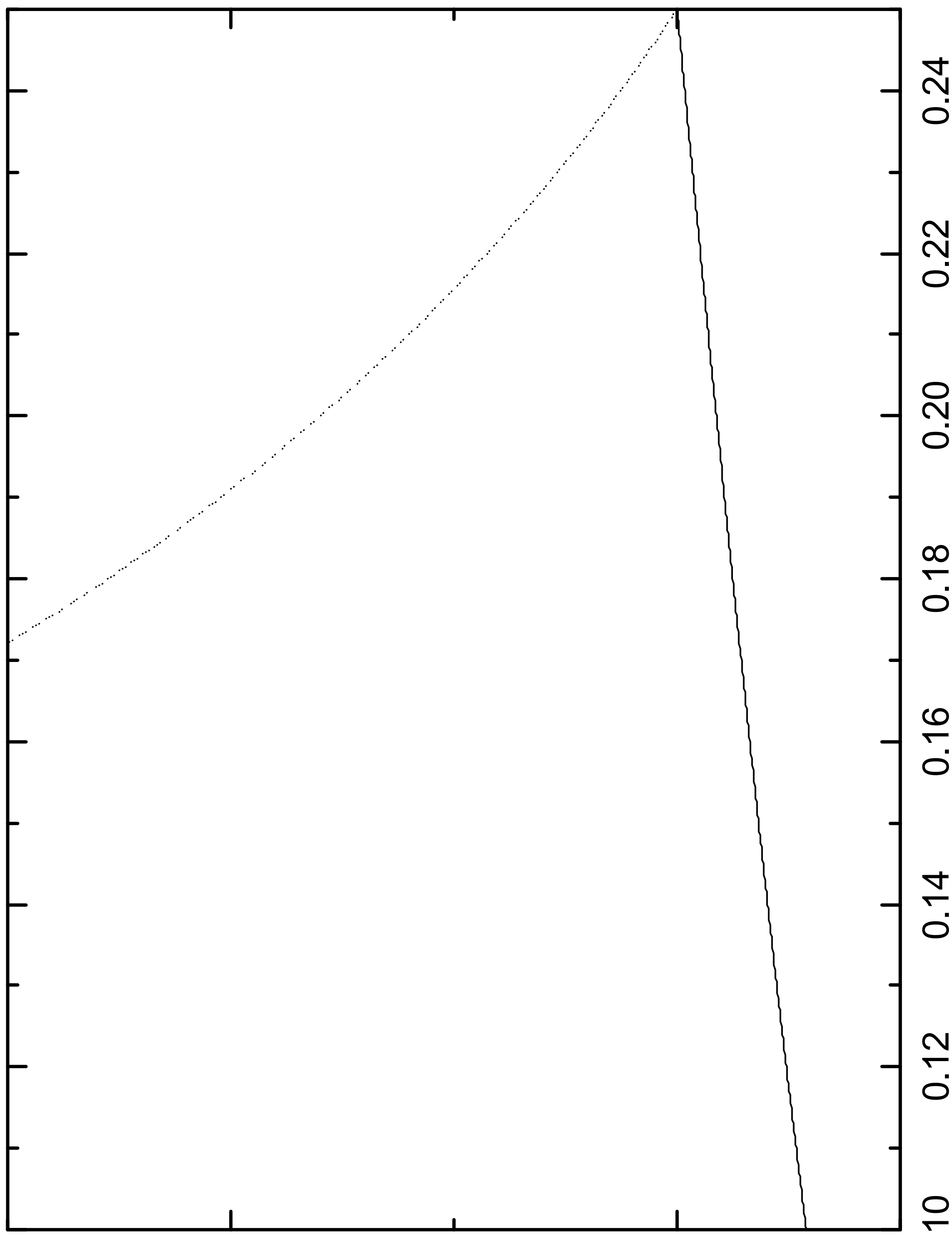


Fig.3(b)

Fig.4