Functional Relation of interquark potential with interquark distance

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Abstract

The functional relation between interquark potential and interquark distance is explicitly derived by considering the Nambu–Goto action in the $AdS_5 \times S^5$ background. It is also shown that a similar relation holds in a general background. The implications of this relation for confinement are briefly discussed.

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AdS/CFT correspondence [1] enables us to understand quantum phenomena in the large N limit of $\mathcal{N} = 4$ super Yang–Mills theory with the classical string description [2,3] in the $AdS_5 \times S^5$ background. In particular, the finite temperature effect on the gauge theory side of the correspondence has been discussed in detail [4,5], and on the AdS side the Wilson loop has been calculated at both zero temperature [6,7] and finite temperature [8,9] from worldsheet areas [10]. The main differences between the finite temperature case and the zero temperature case are, in our opinion, (1) the presence of a maximal separation distance which makes the dependence of the interquark distance on the minimum point of the string a multi–valued function, and (2) the appearance of a cusp (or bifurcation point) in the graph of interquark potential–vs–interquark distance [8]. As in Ref. [7] "quark" and "antiquark" here mean the ends on the AdS_5 boundary of a string describing an infinitely massive W–boson stretching between the horizon and the AdS_5 boundary. The near–extremal N D3–branes are located at the Schwarzschild horizon $U = U_T(r = \alpha' U_T)$.

The two differences referred to strongly suggest that there is a hidden functional relation between these quantities as we know from the study of equations of state in thermal physics.

In the following we derive this relation explicitly in the AdS_5 background. Also, it will be shown that a similar relation[1] holds for an arbitrary background.

We start with the classical Nambu–Goto action for the string worldsheet

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{det G_{MN} \partial_\alpha X^M \partial_\beta X^N} \tag{1}$$

in the near extremal Euclidean Schwarzschild-AdS_5 background [1]

$$ds_E^2 = \alpha' \left[\frac{U^2}{R^2} \left(f(U) dt^2 + dx_i dx_i \right) + \frac{R^2 f(U)^{-1}}{U^2} dU^2 + R^2 d\Omega_5^2 \right]. \tag{2}$$

Here $x_i, i = 1, 2, 3$, are the D3–brane coordinates and R is the radius of both AdS_5 and S^5 and $U = r/\alpha'$. The function $f(U)$ is defined by $f(U) = 1 - U_T^4/U^4$ and the temperature is

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[1]For an arbitrary background this is a relation between the string energy and the separation of string ends at the boundary.

given by $T = U_T/(\pi R^2)$ (see e.g. [9]). After identifying $\tau = t$ and $\sigma = x$, it is easy to show that for the static case the action S_{NG} becomes (setting $\alpha' = 1$)

$$S_{NG} = \frac{\tilde{\tau}}{2\pi} \int dx \sqrt{U'^2 + \frac{U^4 - U_T^4}{R^4}} \tag{3}$$

where the prime denotes differentiation with respect to x, and $\tilde{\tau}$ is the entire Euclidean time interval. U_T is the position of the horizon. Since the Hamiltonian is a constant of motion we have

$$\frac{U^4 - U_T^4}{\sqrt{U'^2 + \frac{U^4 - U_T^4}{R^4}}} = const \equiv R^2 \sqrt{U_0^4 - U_T^4} \tag{4}$$

where the constant solution U_0 is the minimum point of the string configuration (by symmetry at $x = 0$). One can explicitly derive the static solution of S_{NG} by integrating eq.(4) in terms of elliptic functions, i.e., [11,12]

$$x = \frac{R^2 \sqrt{U_0^2 - U_T^2}}{2\sqrt{2}U_0 U_T} \left[F\left(\sin^{-1} \frac{\sqrt{(U^2 - U_0^2)(U^2 - U_T^2)}}{U^2 - U_0 U_T}, \frac{U_0 + U_T}{\sqrt{2(U_0^2 + U_T^2)}} \right) \right. \tag{5}$$
$$\left. -F\left(\sin^{-1} \frac{\sqrt{(U^2 - U_0^2)(U^2 - U_T^2)}}{U^2 + U_0 U_T}, \frac{U_0 - U_T}{\sqrt{2(U_0^2 + U_T^2)}} \right) \right]$$

where $F(\phi, k)$ is the incomplete elliptic integral of the first kind with modulus k of the associated Jacobian elliptic functions. Since the interquark distance L is defined by the distance between different ends of the string on the AdS_5 boundary ($U \to \infty$), it is easy to compute L from (5); one finds (with quark and antiquark at $x = -L/2$ and $L/2$ respectively)

$$L = \frac{R^2}{\sqrt{2}U_T} \frac{\sqrt{a^2 - 1}}{a} [K(f_1(a)) - K(f_2(a))] \tag{6}$$

where $a = U_0/U_T$, $K(k)$ is the complete elliptic integral of the first kind [11], and $f_1(a)$ and $f_2(a)$ are given by

$$f_1(a) = \frac{a + 1}{\sqrt{2(a^2 + 1)}}, \tag{7}$$
$$f_2(a) = \frac{a - 1}{\sqrt{2(a^2 + 1)}}.$$

It is important to note that $f_1(a)$ and $f_2(a)$ are respectively the modulus and complementary modulus of the elliptic functions in the sense that $f_1(a)^2 + f_2(a)^2 = 1$. Since U_T is the position of the horizon we consider only $U_0 > U_T$, i.e. $a \geq 1$.

The energy of solution (5) is interpreted as the interquark potential in the context of AdS/CFT correspondence, and is readily derived using the constant of motion (4), i.e.,

$$E_{q\bar{q}} = \lim_{\tilde{\Lambda} \to \infty} \frac{1}{2\pi} \int_{U_0}^{\tilde{\Lambda}} dx \sqrt{U'^2 + \frac{U^4 - U_T^4}{R^4}} = \frac{U_T}{\pi} \lim_{\Lambda \to \infty} \int_a^\Lambda dy \sqrt{\frac{y^4 - 1}{y^4 - a^4}} \tag{8}$$

where Λ is a cutoff parameter. To regularize $E_{q\bar{q}}$, we have to subtract the quark masses [1]. The finite form of the interquark potential is then

$$E_{q\bar{q}}^{(Reg)} = \frac{U_T}{\pi} \lim_{\Lambda \to \infty} \left[\int_a^\Lambda dy \sqrt{\frac{y^4 - 1}{y^4 - a^4}} - (\Lambda - 1) \right]. \tag{9}$$

This expression can also be evaluated in terms of standard elliptic functions [13]. We obtain

$$E_{q\bar{q}}^{(Reg)} = \frac{U_T}{\pi} \left[1 + \sqrt{\frac{a^2 + 1}{2}} \left[\frac{a - 1}{2a} K(f_1(a)) + \frac{a + 1}{2a} K(f_2(a)) - E(f_1(a)) - E(f_2(a)) \right] \right], \tag{10}$$

where E is the complete elliptic integral of the second kind [11].

The L–dependence of $E_{q\bar{q}}^{(Reg)}$ for various temperatures is plotted in Fig. 1. As mentioned above, the appearance of cusps at finite temperatures strongly suggests that there exists a functional relation between $E_{q\bar{q}}^{(Reg)}$ and L as we know from our experience with the thermo-dynamical treatment of a van der Waals gas or recent investigation of systems allowing the tunneling of large spins [14]. To find this relation it is more convenient to introduce the modulus $\kappa = f_2(a) = (a - 1)/\sqrt{2(a^2 + 1)}$ explicitly. Since $a \geq 1^2$, κ is defined in the region $0 \leq \kappa \leq 1/\sqrt{2}$. Inverting the relation we have

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[2]This is a consequence of Ref. [15] that D–branes are located at the horizon. Although we have a somewhat different opinion, we follow this condition here. Our own opinion will be discussed elsewhere.

$$a = \frac{1 + 2\kappa\kappa'}{1 - 2\kappa^2} \tag{11}$$

where $\kappa' \equiv \sqrt{1 - \kappa^2}$. Then L and $E_{q\bar{q}}^{(Reg)}$ expressed in terms of κ and κ' become:

$$L = \frac{R^2}{\sqrt{2}U_T} \frac{2\sqrt{\kappa\kappa' + 2\kappa^2\kappa'^2}}{1 + 2\kappa\kappa'} \left[K(\kappa') - K(\kappa) \right], \tag{12}$$

$$E_{q\bar{q}}^{(Reg)} = \frac{U_T}{\pi} \left[1 + \frac{\sqrt{1 + 2\kappa\kappa'}}{1 - 2\kappa^2} \left[\frac{\kappa(\kappa + \kappa')}{1 + 2\kappa\kappa'} K(\kappa') + \frac{1 + \kappa(\kappa' - \kappa)}{1 + 2\kappa\kappa'} K(\kappa) - E(\kappa') - E(\kappa) \right] \right].$$

Since the thermodynamical relations are usually realized on the level of first derivatives, we compute $dL/d\kappa$ and $dE_{q\bar{q}}^{(Reg)}/d\kappa$ whose explicit forms are

$$\frac{dL}{d\kappa} = \frac{R^2}{\sqrt{2}U_T} \frac{1}{\kappa'(1 + 2\kappa\kappa')\sqrt{\kappa\kappa' + 2\kappa^2\kappa'^2}} \tag{13}$$
$$\times \left[(1 + 4\kappa^3\kappa')K(\kappa') + (1 + 4\kappa\kappa'^3)K(\kappa) - 2(1 + 2\kappa\kappa')(E(\kappa) + E(\kappa')) \right],$$

$$\frac{dE_{q\bar{q}}^{(Reg)}}{d\kappa} = \frac{U_T}{\pi} \frac{1}{\kappa'(\kappa'^2 - \kappa^2)^2\sqrt{1 + 2\kappa\kappa'}}$$
$$\times \left[(1 + 4\kappa^3\kappa')K(\kappa') + (1 + 4\kappa\kappa'^3)K(\kappa) - 2(1 + 2\kappa\kappa')(E(\kappa) + E(\kappa')) \right].$$

We observe that the coefficients of the complete elliptic integrals in the brackets of $dL/d\kappa$ and $dE_{q\bar{q}}^{(Reg)}/d\kappa$ are identical. This means that $dE_{q\bar{q}}^{(Reg)}/dL$ is independent of these complete elliptic integrals, i.e.,

$$\frac{dE_{q\bar{q}}^{(Reg)}}{dL} = \frac{\sqrt{U_0^4 - U_T^4}}{2\pi R^2}. \tag{14}$$

The right–hand side of Eq.(14) is the regularized energy of the constant solution $U = U_0$, i.e. $E^{(Reg)}(U_0)$. Thus we obtain finally the functional relation

$$\frac{dE_{q\bar{q}}^{(Reg)}}{dL} = E^{(Reg)}(U_0). \tag{15}$$

This relation is seen to be very similar to the relation between dynamical quantities of a classical Euclidean point particle in quantum mechanical tunneling: $dS_E/dP = \mathcal{E}$ where S_E, P, and $\mathcal{E}$ are Euclidean action, period and energy of the classical particle (cf. e.g. [16]).

It is interesting to compare Eq. (15) with the case of a van der Waals gas [17]. Our plot of $E_{q\bar{q}}^{(Reg)}$–vs–L is completely analogous to the plot of enthalpy–vs–pressure of the gas whose

equation of state plotted as pressure–vs–volume corresponds to the plot of L–vs–U_0. Hence, the plot of $E_{q\bar{q}}^{(Reg)}$-vs-L is completely determined from the plot of L-vs-U_0 up to the constant. If the plot of L–vs–U_0 has n extremum points, the plot of $E_{q\bar{q}}^{(Reg)}$–vs–L has n bifurcation points.

In order to understand our result in greater generality we consider the Nambu–Goto action (1) in an arbitrary space–time background. Then the static action will generally reduce to

$$S_{NG} = \frac{\tilde{\tau}}{2\pi} \int dx \sqrt{G_1(U)U'^2 + G_2(U)} \tag{16}$$

where $G_1(U)$ and $G_2(U)$ are metric–dependent functions. The constant of motion in this case is

$$\frac{G_2(U)}{\sqrt{G_1(U)U'^2 + G_2(U)}} = const \equiv \sqrt{G_2(U_0)}. \tag{17}$$

Hence, we choose $U = U_0$ as an extremum point in the string configuration. We assume $G_1(U_0) \neq 0$ and $G_2(U_0) \neq 0$. Although this assumption excludes particular space-time geometries, we do not think that this assumption restricts our general argument crucially. From the constant of motion it is easy to show that the solution of the static Nambu–Goto action (16) obeys the integral equation,i.e.,

$$x = \int_{U_0}^{U} dU \sqrt{\frac{G_1(U)G_2(U_0)}{G_2(U)[G_2(U) - G_2(U_0)]}} \tag{18}$$

¿From this integral equation it is straightforward to show that the distance L between the string ends at $U \to \infty$ and the string energy $\mathcal{E}$ become

$$L = 2\sqrt{G_2(U_0)} \int_{U_0}^{\infty} dU \sqrt{\frac{G_1(U)}{G_2(U)[G_2(U) - G_2(U_0)]}}, \tag{19}$$

$$\mathcal{E} = \frac{1}{\pi} \int_{U_0}^{\infty} dU \sqrt{\frac{G_1(U)G_2(U)}{G_2(U) - G_2(U_0)}}.$$

Using the Leibnitz chain rule one can prove directly

$$\frac{d\mathcal{E}}{dU_0} \approx -\frac{\sqrt{G_1(U_0)G_2(U_0)}}{\pi} \lim_{U \to U_0} \frac{1}{\sqrt{G_2(U) - G_2(U_0)}},$$

$$\frac{dL}{dU_0} \approx -2\sqrt{G_1(U_0)} \lim_{U \to U_0} \frac{1}{\sqrt{G_2(U) - G_2(U_0)}} \tag{20}$$

which recovers our relation

$$\frac{d\mathcal{E}}{dL} = \frac{1}{2\pi}\sqrt{G_2(U_0)} \equiv \mathcal{E}^{(Reg)}(U_0). \tag{21}$$

In our opinion, this kind of relation is not restricted to Nambu–Goto actions. We expect that a similar relation can be established for a Born–Infeld action, which is under study.

Finally we comment briefly on confinement, using our relation (15). Since confinement and deconfinement depend on the L–dependence of $E_{q\bar{q}}^{(Reg)}$, say $E_{q\bar{q}}^{(Reg)} \propto L^\alpha$ where $\alpha \geq 1$ and $\alpha \leq -1$ represent confinement and deconfinement phases respectively, it is important to see the behavior of $dE_{q\bar{q}}^{(Reg)}/dL$ with respect to L; this is shown in Fig. 2. Each line in Fig. 2 has confining and deconfining parts. Fig. 2 also indicates that as the temperature increases, the confining force becomes strong (large α) while available L is reduced, which is physically predictable. At low temperature α approaches 1 and hence the area law of the Wilson loop is recovered in the supergravity picture [18] (i.e. the $T = 0$ curve starts at the origin with confining part along the abscissa, that is, with constant slope implying $dE_{q\bar{q}}^{(Reg)}/dL$ independent of L, and as the temperature increases the confining part has a larger slope, i.e. larger α). It should be noted that the potential energy of the confining part is always larger than that of the deconfining part. This implies that our understanding of confinement is still incomplete. A full understanding of quark confinement seems to require a deeper understanding of the relation between gauge theories and statistical physics like that of black holes.

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FIGURES

FIG. 1.   The L–dependence of $E_{q\bar{q}}^{(Reg)}$. The appearance of the cusps in this figure strongly suggests that there exists a hidden relation between $E_{q\bar{q}}^{(Reg)}$ and L.

FIG. 2.   The L–dependence of $dE_{q\bar{q}}^{(Reg)}/dL$. Each line consists of a confining (lower) and a deconfining (upper) part. This figure indicates that a stronger confining force is required at high temperatures. At low temperatures the area law of the Wilson loop is recovered.
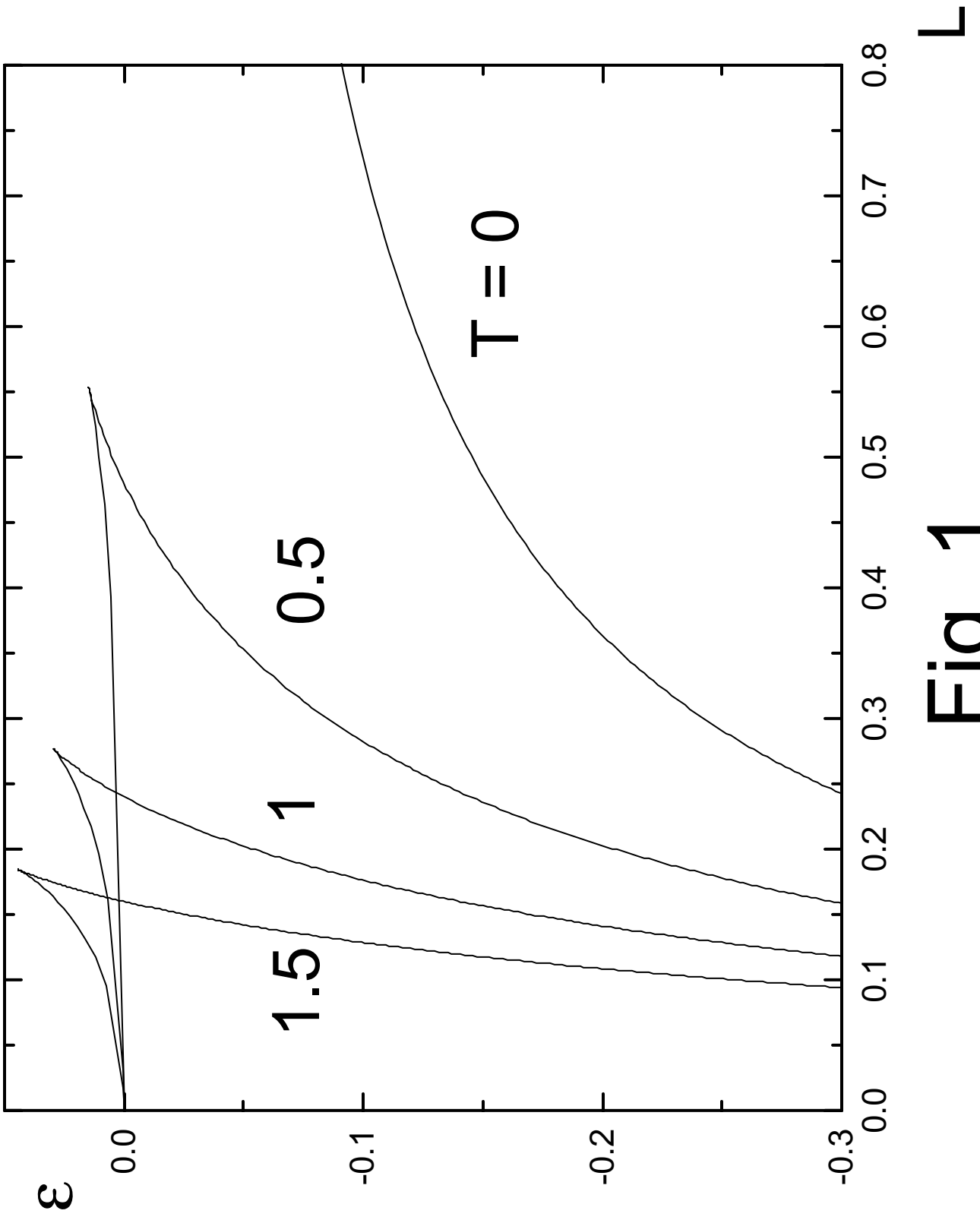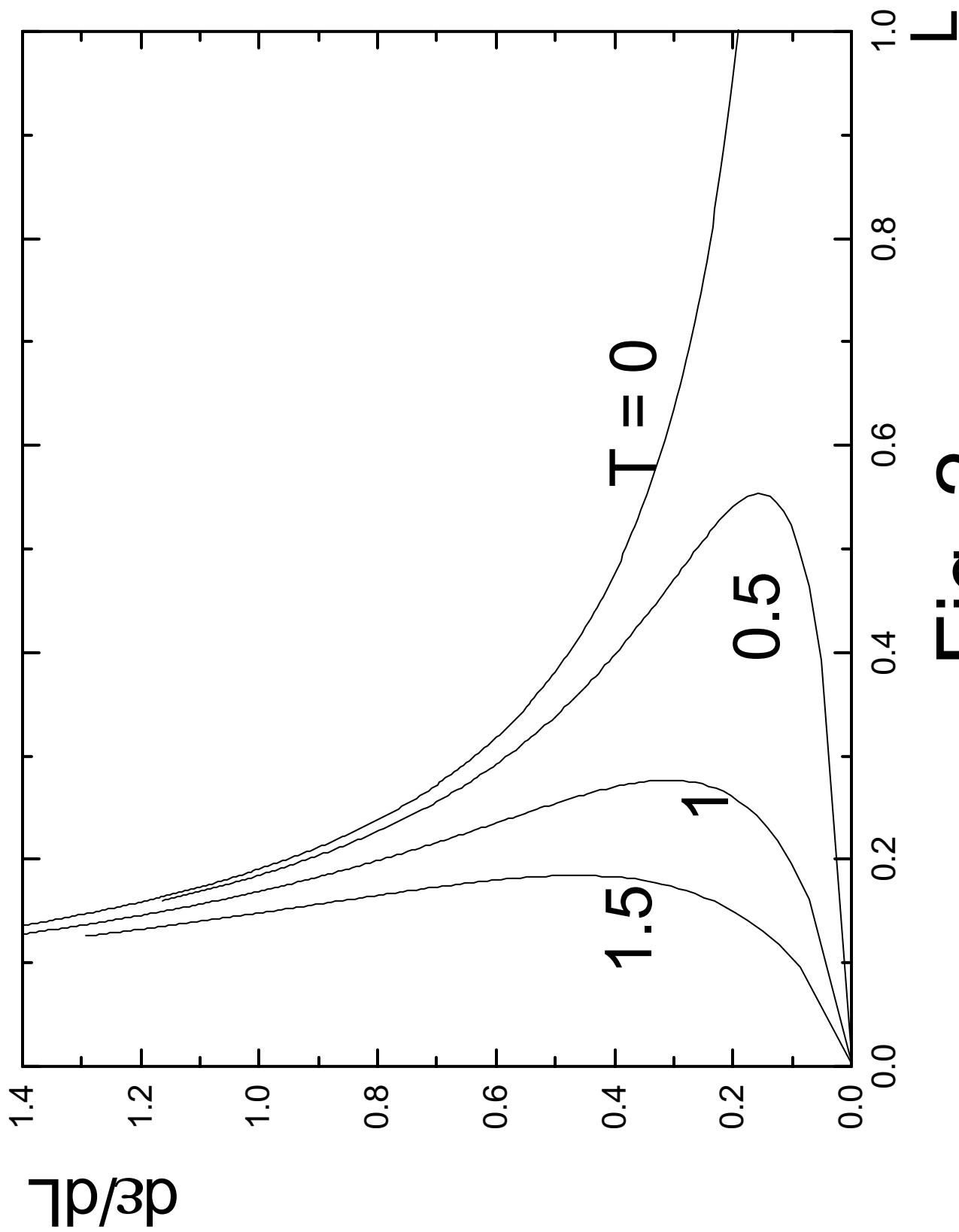
Fig. 1

Fig. 2