# The role of infrared divergence for decoherence

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**Abstract**

Continuous and discrete superselection rules induced by the interaction with the environment are investigated for a class of exactly soluble Hamiltonian models. The environment is given by a Boson field. Stable superselection sectors emerge if and only if the low frequences dominate and the ground state of the Boson field disappears due to infrared divergence. The models allow uniform estimates of all transition matrix elements between different superselection sectors.

## 1 Introduction

Superselection rules are the basis for the emergence of classical physics within quantum theory. But despite of the great progress in understanding superselection rules, see e.g. [20], quantum mechanics and quantum field theory do not provide enough superselection rules to infer the classical probability of "facts" from quantum probability. This problem is most often discussed in the context of measurement of quantum mechanical objects. In an important paper about the process of measurement Hepp [10] has presented a class of models for which the dynamics induces superselection sectors. Hepp starts with a very large algebra of observables – essentially all observables with the exception of the "observables at infinity" which constitute an a priory set of superselection rules – and the superselection sectors emerge in the weak operator convergence. But it has soon been realized that the algebra of observables, which is relevant for the understanding of the process of measurement [8] [2] and, more generally for the understanding of the classical appearance of the world [21] [12] [9] can be severely restricted. Then strong or even uniform operator convergence is possible.

In this paper results of Chap.7 of the book [9] and of the article [13] are extended. After a short introduction to superselection rules and the dynamics of subsystems we prove in Sect.3 that uniform operator estimates are possible also for continuous superselection rules induced by the environment. In Sect.4 we investigate a class of Hamiltonian models with an environment given by a Boson field. The restriction to the Boson sector corresponds to a van Hove model [11]. As the main result of the paper we prove for this class of models:
– The superselection sectors are induced by the infrared contributions of the Boson field.
– The superselection sectors are stable for $t \to \infty$ if and only if the Boson field is infrared divergent.

This type of infrared divergence has been studied by Schroer [19] more than thirty years ago. The Boson field is still defined on the Fock space but the ground state of the Boson field disappears in the continuum.

## 2 Induced superselection rules

We start with a few mathematical notations. Let $\mathcal{H}$ be a separable Hilbert space, then the following spaces of linear operators are used.
$\mathcal{B}(\mathcal{H})$: The linear space of all bounded operators $A$ with the operator norm $\|A\|$.

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$\mathcal{T}(\mathcal{H})$: The linear space of all nuclear operators $A$ with the trace norm $\|A\|_1 = \mathrm{tr}\sqrt{A^+A}$.
$\mathcal{D}(\mathcal{H})$: The set of all positive nuclear operators $W$ with a normalized trace, $\mathrm{tr}\,W = 1$.

We consider standard quantum mechanics and quantum field theory where any state of a quantum system is represented by a statistical operator $W \in \mathcal{D}(\mathcal{H})$ - the rank one projection operators thereby correspond to the pure states - and any bounded observable is represented by an operator $A \in \mathcal{B}(\mathcal{H})$. Without additional knowledge about the structure of the system we have to assume that the set of all states corresponds to $\mathcal{D}(\mathcal{H})$, and the operator algebra of all (bounded) observables coincides with $\mathcal{B}(\mathcal{H})$. In quantum field theory the superposition principle is partially restricted to superselection sectors, see e.g. [20]. The projection operators onto the superselection sectors commute with all observables of the theory: they are classical observables. But there remains an essential problem for the understanding of the classical appearance of the world: Only very few superselection rules can be found in quantum mechanics and quantum field theory. A possible solution is the emergence of superselection rules due to decoherence caused by the dynamics.

Let $A = A(0) \to A(t) = \mathcal{T}_t(A) \in \mathcal{B}(\mathcal{H})$ denote the dynamics in the Heisenberg picture. If there exists a family of projection operators $\{P_m, m \in \mathbf{M}\}$ with the properties $P_m P_n = 0$ for $m \neq n$ and $\sum_n P_n = I$, such that transition matrix elements $(f \mid A(t)g)$ between different sectors $f \in \mathcal{H}_m = P_m\mathcal{H}$, $g \in \mathcal{H}_n = P_n\mathcal{H}$, $m \neq n$, vanish for all observables $A \in \mathcal{B}(\mathcal{H})$ for $t \to \infty$, the subspaces $\mathcal{H}_m = P_m\mathcal{H}$, $m \in \mathbf{M}$, are denoted as superselection sectors induced by the dynamics $\mathcal{T}_t$.

This definition can be applied to the Hamiltonian dynamics $A = A(0) \to A(t) = \mathcal{T}_t(A) := U^+(t)AU(t)$ where $U(t) = \exp(-iHt)$ is the unitary group generated by the Hamiltonian $H$. As a simple example we consider a Hamiltonian $H$ on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with one bound state at energy $E_0$ in the 1-dim. subspace $\mathcal{H}_0$ and with an absolutely continuous spectrum in the subspace $\mathcal{H}_1$. Then for $f_0 \in \mathcal{H}_0$ and $f_1 \in \mathcal{H}_1$ with $\|f_{0,1}\| = 1$ we calculate $(f_0 \mid A(t)f_1) = e^{iE_0 t}(A^+(0)f_0 \mid U(t)f_1) \to 0$, since $U(t)f_1$ converges weakly to zero. The subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are therefore induced superselection sectors of the Hamiltonian dynamics. If $P_{0,1}$ denote the projection operators onto the subspaces $\mathcal{H}_{0,1}$ then the off-diagonal part $P_0A(t)P_1$ converges in the weak operator norm to zero. But neither strong nor, a fortiori, uniform convergence holds for $P_0A(t)P_1$ unless $P_0A(t)P_1 \equiv 0$. More refined examples have been given by Hepp [10]. Thereby an essential consequence of the Hamiltonian time evolution or any other automorphic time evolution is the restriction to a weak operator convergence. Moreover, as has been emphasized by Bell [3], the time scale can be arbitrarily long, such that the practical use of such models is questionable.

A strong or even uniform suppression of the off-diagonal matrix elements of all observables can be obtained by the restriction to a subsystem [8] [2] [21]. In the following we consider an open system, i.e. a system $\mathcal{S}$ which interacts with an environment $\mathcal{E}$, such that the total system $\mathcal{S} \times \mathcal{E}$ satisfies the usual Hamiltonian dynamics. The Hilbert space $\mathcal{H}_{S\times E}$ of the total system $\mathcal{S} \times \mathcal{E}$ is the tensor space $\mathcal{H}_S \otimes \mathcal{H}_E$ of the Hilbert spaces for $\mathcal{S}$ and for $\mathcal{E}$. If the state of the total system is $W \in \mathcal{D}(\mathcal{H}_{S+E})$, then the state of the subsystem is given by the reduced statistical operator $\rho = \mathrm{tr}_E W \in \mathcal{D}(\mathcal{H}_S)$. The dynamics of the states of the total system $W \in \mathcal{D}(\mathcal{H}_{S\times E}) \to W(t) = U(t)W(0)U^+(t) \in \mathcal{D}(\mathcal{H}_{S\times E})$ with the unitary group $U(t) = \exp(-iHt)$, generated by the total Hamiltonian $H$, yields the dynamics of the statistical operator $\rho(t) = \mathrm{tr}_E\, U(t)W(0)U^+(t) \in \mathcal{D}(\mathcal{H}_S)$ of the subsystem $\mathcal{S}$. In the following we assume that the initial state factorizes $W = \rho \otimes \omega$ with $\rho \in \mathcal{D}(\mathcal{H}_S)$ and a fixed reference state $\omega \in \mathcal{D}(\mathcal{H}_E)$ of the environment. Then the dynamics in the Heisenberg picture of the system $\mathcal{S}$ is easily calculated as

$$A \in \mathcal{B}(\mathcal{H}_S) \to A(t) = \mathcal{T}_t(A) := \mathrm{tr}_E\, U^+(t)(A \otimes I_E)U(t)\omega \in \mathcal{B}(\mathcal{H}_S). \qquad (1)$$

Before we investigate induced superselection sectors we generalize the definition given above to the case of continuous superselection sectors. The finite or countable set of projection operators $\{P_m, m \in \mathbf{M}\}$ is substituted by a strongly continuous family of projection operators $P(\Delta)$ indexed by measurable subsets $\Delta \subset \mathbb{R}$, see e.g. [16] or [2]. These projection operators have to satisfy

$$\begin{cases} P(\Delta_1 \cup \Delta_2) = P(\Delta_1) + P(\Delta_2) \text{ and } P(\Delta_1)P(\Delta_2) = 0 \text{ if } \Delta_1 \cap \Delta_2 = \emptyset \\ P(\emptyset) = 0,\ P(\mathbb{R}) = 1. \end{cases} \tag{2}$$

If we chose for $\{P(\Delta), \Delta \subset \mathbb{R}\}$ a general (right continuous) spectral family, the case of discrete superselection rules is included in (2).

The dynamics of the total system induces superselection rules in the system $\mathcal{S}$ if there exists a right continuous family of projection operators (2) $\{P_S(\Delta) \mid \Delta \subset \mathbb{R}\}$ defined on the Hilbert space $\mathcal{H}_S$, such that the off-diagonal contributions of all statistical operators of the system $\mathcal{S}$ vanish for $t \to \infty$, i.e. $P(\Delta_1)\rho(t)P(\Delta_2) \to 0$ if $t \to \infty$ and $\Delta_1 \cap \Delta_2 = \emptyset$, or in the Heisenberg picture, $P_S(\Delta_1)A(t)P_S(\Delta_2) \to 0$ if $t \to \infty$ and $\Delta_1 \cap \Delta_2 = \emptyset$ for all observables $A \in \mathcal{B}(\mathcal{H}_S)$.

## 3 Soluble models

In the following we present models for which the Hamiltonian of the total system provides a family of projection operators $\{P_S(\Delta), \Delta \subset \mathbb{R}\}$ on $\mathcal{H}_S$ such that the off-diagonal elements of any bounded observable of the system $\mathcal{S}$ can be estimated with the operator norm. We derive a uniform decrease

$$\|P_S(\Delta_1)A(t)P_S(\Delta_2)\| \to 0 \text{ if } t \to \infty \tag{3}$$

for arbitrary bounded observables $A \in \mathcal{B}(\mathcal{H}_S)$ and arbitrary disjoint closed intervals $\Delta_1 \cap \Delta_2 = \emptyset$.

The models have the following structure. The total Hamiltonian is defined on the tensor space $\mathcal{H}_{S\times E} = \mathcal{H}_S \otimes \mathcal{H}_E$ as

$$\begin{aligned} H_{S\times E} &= H_S \otimes I_E + I_S \otimes H_E + F \otimes G \\ &= \left(H_S - \frac{1}{2}F^2\right) \otimes I_E + \frac{1}{2}\left(F \otimes I_E + I_S \otimes G\right)^2 + I_S \otimes \left(H_E - \frac{1}{2}G^2\right) \end{aligned} \tag{4}$$

where $H_S$ is the positive Hamiltonian of $\mathcal{S}$, $H_E$ is the positive Hamiltonian of $\mathcal{E}$, and $F \otimes G$ is the interaction potential between $\mathcal{S}$ and $\mathcal{E}$ with operators $F$ on $\mathcal{H}_S$ and $G$ on $\mathcal{H}_E$. To guarantee that $H_{S\times E}$ is self-adjoint and semibounded we assume

1) The operators $F$ and $F^2$ ($G$ and $G^2$) are essentially self-adjoint on the domain of $H_S$ ($H_E$). The operators $H_S - \frac{1}{2}F^2$ and $H_E - \frac{1}{2}G^2$ are semibounded.

Since $F^2 \otimes I_E \pm 2F \otimes G + I_S \otimes G^2$ are positive operators, the operator $F \otimes G$ is $(H_S \otimes I_E + I_S \otimes H_E)$-bounded with relative bound one, and Wüst's theorem, see e.g. Theorem X.14 in [18], implies that $H_{S\times E}$ is essentially self-adjoint on the domain of $H_S \otimes I_E + I_S \otimes H_E$. Moreover $H_{S\times E}$ is obviously semibounded.

To derive induced superselection rules we need the rather severe restriction

2) The operators $H_S$ and $F$ commute strongly, i.e. their spectral projections commute.

So far no model with Hamiltonian dynamics has been presented which violates this assumption and allows the uniform estimate (3) of induced superselection sectors. If the Hamiltonian includes a scattering potential it is possible to abandon this assumption. But then the off-diagonal terms $P(\Delta_1)A(t)P(\Delta_2)$ decrease only in the strong operator topology, see [14].

The operator $F$ has a spectral decomposition $F = \int_{\mathbb{R}} \lambda P_S(d\lambda)$ with a right continuous family of projection operators $P_S(\Delta)$ indexed by measurable subsets $\Delta \subset \mathbb{R}$. We shall see below that exactly the projection operators of this spectral decomposition determine the induced superselection sectors.

As a consequence of assumption 2) we have $[H_S, P_S(\Delta)] = 0$ for all intervals $\Delta \subset \mathbb{R}$. The Hamiltonian (4) has therefore the form $H_{S\times E} = H_S \otimes I_E + \int_{\mathbb{R}} P_S(d\lambda) \otimes (H_E + \lambda G)$. The operator $|G| = \sqrt{G^2}$ has the upper bound $|G| \leq aG^2 + (4a)^{-1} I$ with an arbitrarily small constant $a > 0$. Since $G^2$ is $H_E$-bounded with relative bound 2, the operator $G$ is $H_E$-bounded with an arbitrarily small bound. The Kato-Rellich theorem, see e.g. [18], implies that the operators $H_E + \lambda G$ are self-adjoint on the domain of $H_E$ for all $\lambda \in \mathbb{R}$. The unitary evolution $U(t) := \exp(-iH_{S\times E} t)$ of the total system can therefore be written as $U(t) = \left(\mathrm{e}^{-iH_S t} \otimes I_E\right) \int dP_S(\lambda) \otimes \mathrm{e}^{-i(H_E + \lambda G)t}$. The dynamics of the observables (1) follows as

$$A(t) = \mathrm{e}^{iH_S t} \left( \int \int \chi\left(\alpha, \beta; t\right) P_S(d\alpha)\, A\, P_S(d\beta) \right) \mathrm{e}^{-iH_S t} \tag{5}$$

with the trace

$$\chi(\alpha, \beta; t) = \mathrm{tr}_E \left( \mathrm{e}^{i(H_E + \alpha G)t} \mathrm{e}^{-i(H_E + \beta G)t} \omega \right). \tag{6}$$

The emergence of dynamically induced superselection rules depends on an estimate of this trace. For the models investigated below, we obtain for a large class of reference states $\omega$ (actually a dense set within $\mathcal{D}(\mathcal{H}_E)$) the bounds

$$\left| \frac{\partial^n}{\partial \alpha^n} \chi(\alpha, \beta; t) \right| \leq c \left(1 + (\alpha - \beta)^2 \psi(t)\right)^{-\gamma},\ n = 0, 1, \tag{7}$$

with a function $\psi(t) \geq 0$ which diverges for $t \to \infty$ like a power $t^\delta$, $0 < \delta < 1$, and an exponent $\gamma > 0$ which can be a large number. If $\Delta_1$ and $\Delta_2$ are intervals with a distance $\delta > 0$ then the operator norm of $P_S(\Delta_1)A(t)P_S(\Delta_2)$ is estimated in the Appendix A as

$$\|P_S(\Delta_1)A(t)P_S(\Delta_2)\| \leq const\, \|A\| \left(1 + \delta^2 \psi(t)\right)^{-\gamma}. \tag{8}$$

For operators $F$ with a discrete spectrum $F = \sum \lambda_n P_n^S$ uniform norm estimates have already been derived in Sect. 7.6 of [9]. In this case the bound with $n = 1$ in (7) is obsolete.

A simple class of explicitly soluble models which yield the estimates (7) can be obtained under the additional assumption

3) The Hamiltonian $H_E$ and the potential $G$ commute strongly. The operator $G$ has an absolutely continuous spectrum.

Such models have been investigated (for operators $F$ with a discrete spectrum) by Araki [2] and by Zurek [21], see also Sect. 7.6 of [9] and [14]. Under the assumption 3) the trace (6) simplifies to $\chi(\alpha, \beta; t) = \mathrm{tr}_E \left(\mathrm{e}^{i(\alpha-\beta)Gt}\omega\right)$. Let $G = \int_{\mathbb{R}} \lambda P_E(d\lambda)$ be the spectral representation of the operator $G$. Then the measure $d\mu(\lambda) := \mathrm{tr}_E \left(P_E(d\lambda)\, \omega\right)$ is absolutely continuous with respect to the Lebesgue measure for any $\omega \in \mathcal{D}(\mathcal{H}_E)$, and the function $\chi(t) := \mathrm{tr}_E \left(\mathrm{e}^{iGt}\omega\right) = \int_{\mathbb{R}} \mathrm{e}^{i\lambda t}\, d\mu(\lambda)$ vanishes for $t \to \infty$. But to obtain a decrease which is effective in sufficiently short time, we need an additional smoothness condition on $\omega$. This condition does not impose restrictions on the statistical operator $\rho \in \mathcal{D}(\mathcal{H}_S)$ of the system $\mathcal{S}$. We assume that $G\omega \in \mathcal{T}(\mathcal{H}_E)$ and, moreover, that the integral operator, which represents $\omega$ in the spectral representation of

$G$, is a sufficiently differentiable function vanishing at the boundary points of the spectrum. Then the measure $d\mu(\lambda) = \mathrm{tr}_E\,(P_E(d\lambda)\,\omega)$ has a smooth density, and we can derive a strong decrease of its Fourier transform $\chi(t)$ and its derivative, $\left|\frac{d^n}{dt^n}\chi(t)\right| \leq C_\gamma(1+t^2)^{-\gamma}, n = 0, 1,$ with arbitrarily large values of $\gamma$. That implies bounds (7) with $\psi(t) = t^2$.

# 4 The interaction with a Boson field

In this section we present a model without the restriction 3). Preliminary results have already been reported in [14]. We choose a system $\mathcal{S}$ which satisfies the constraints 1) and 2). The environment given by a Boson field is investigated in details below. As essential result we derive the uniform estimates (7). Consequently the off-diagonal elements of the operator $F$ are suppressed as given in (8). As specific example we may consider a particle on the real line with velocity coupling. The Hilbert space of the particle is $\mathcal{H}_S = \mathcal{L}^2(\mathbb{R})$. The Hamiltonian and the interaction potential of the particle are

$$H_S = \frac{1}{2}P^2 \text{ and } F = P \tag{9}$$

where $P = -i\frac{d}{dx}$ is the momentum operator of the particle. The identity $H_S - \frac{1}{2}F^2 = 0$ guarantees the positivity of the first term in (4). Decoherence then yields superselection rules for the momentum of the particle.

As Hilbert space $\mathcal{H}_E$ we choose the Fock space of symmetric tensors $\mathcal{F}(\mathcal{H}_1)$ based on the one particle Hilbert space $\mathcal{H}_1$. The inner product of $\mathcal{F}(\mathcal{H}_1)$ is denoted by $(.\mid.)$. The Hamiltonian is generated by a one-particle Hamilton operator $M$ on $\mathcal{H}_1$ with the following properties

(i) $M$ is a positive operator with an absolutely continuous spectrum,

(ii) $M$ has an unbounded inverse $M^{-1}$.

The spectrum of $M$ is (a subset of) $\mathbb{R}_+$, which – as a consequence of the second assumption – includes zero. The Hamiltonian of the free field is then the derivation $H_E = d\Gamma(M)$ generated by $M$, see Appendix B. Let $a^+(f)$ denote the creation operator of the one-particle state $f \in \mathcal{H}_1$ and $a(f) = (a^+(f))^+$ the corresponding annihilation operator, normalized to $[a(f), a^+(g)] = (f \mid g)$. The interaction potential $G$ is then chosen as the self-adjoint field operator $G = \Phi(h) := a^+(h) + a(h)$, where $h \in \mathcal{H}_1$ satisfies the additional constraint

$$2\left\|M^{-\frac{1}{2}}h\right\| \leq 1. \tag{10}$$

This constraint guarantees that $H_E - \frac{1}{2}\Phi^2(h)$ is bounded from below, and the Hamiltonian (4) is a well defined semibounded operator on $\mathcal{F}(\mathcal{H}_{S\times E})$, see Appendix B.

To derive induced superselection sectors for the observable $P$ we have to estimate the time dependence of the traces (6) $\chi_{\alpha\beta}(t) := \mathrm{tr}_E U_{\alpha\beta}(t)\omega$, $\alpha \neq \beta$, where $\omega$ is the reference state of the Boson field, and the unitary operators $U_{\alpha\beta}(t)$ are given by

$$U_{\alpha\beta}(t) := \exp(iH_\alpha t)\exp(-iH_\beta t), \text{ with } H_\alpha = H_E + \alpha\Phi(h),\ \alpha, \beta \in \mathbb{R}. \tag{11}$$

The Hamiltonians $H_\alpha$ are Hamiltonians of the van Hove model [11]. In the Appendix B we prove the following results for reference states $\omega$ which are finite superpositions or mixtures of coherent states.

1. If the vector $h$ also satisfies $M^{-1}h \in \mathcal{H}_1$ one can use the standard methods of the van Hove model to evaluate the traces $\chi_{\alpha\beta}(t) = \mathrm{tr}_E U_{\alpha\beta}(t)\omega$. These traces do not vanish for $t \to \infty$. But one can achieve a strong decrease which persists for some finite time interval. This period can be arbitrarily large; but inevitably, recurrences exist.

2. If $M^{-1}h \notin \mathcal{H}_1$ the low energy contribution of the interaction potential dominates, and $\chi_{\alpha\beta}(t)$ vanishes for $t \to \infty$ if $\alpha \neq \beta$. If the vector $h$ satisfies some additional regularity condition at small energies, there exists a uniform limit $\lim_{t\to\infty} \chi_{\alpha\beta}(t) = 0$ for all $\alpha, \beta$ with $|\alpha - \beta| \geq \delta > 0$, and zero can be approached within a short time.

The assumption $M^{-1}h \notin \mathcal{H}_1$ is therefore necessary and sufficient for the emergence of superselection rules, which persist for $t \to \infty$. In this case the Boson field is infrared divergent. It is still defined on the Fock space, but its ground state disappears in the continuum, see [19].

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# A Norm estimates of observables

In the following $P_S(\Delta)$ with intervals $\Delta \subset \mathbb{R}$ denotes the spectral family of the potential $F$. Let $\Delta_1$ and $\Delta_2$ be closed intervals of the real axes, and let $(\alpha,\beta) \in \Delta_1 \times \Delta_2 \subset \mathbb{R}^2 \to \chi(\alpha,\beta) \in \mathbb{C}$ be a differentiable function with the uniform bounds $|\chi(\alpha,\beta)| \leq c_1$ and $\left|\frac{\partial}{\partial\beta}\chi(\alpha,\beta)\right| \leq c_2$. Then $\beta \in \Delta_2 \to T_2(\beta) = \int_{\Delta_1} \chi(\alpha,\beta) P_S(d\alpha) \in \mathcal{B}(\mathcal{H}_S)$ is a differentiable family of operators with the norm estimates $\|T_2(\beta)\| \leq c_1$ and $\|T_2'(\beta)\| \leq c_2$. If $A \in \mathcal{B}(\mathcal{H}_S)$ is a bounded operator, the function $\beta \in \Delta_2 \to T(\beta) = T_2(\beta)A \in \mathcal{B}(\mathcal{H}_S)$ is again differentiable with the uniform estimates

$$\|T(\beta)\| \leq c_1 \|A\| \text{ and } \|T'(\beta)\| \leq c_2 \|A\| \tag{12}$$

For all intervals $\Delta_2$ the Stieltjes integrals $\int_{\Delta_2} T(\beta) P_S(d\beta)$ are well defined. Let $\Delta_2 = [a,b]$ be an interval of finite length. Then partial integration yields the operator identity $\int_{\Delta_2} T(\beta)P_S(d\beta) = T(b)E(b) - T(a)E(a) - \int_{\Delta_2} T'(\beta)E(\beta)d\beta$ with the projection operators $E(\beta) := P_S((-\infty,\beta]))$, and the inequalities (12) imply the bound

$$\left\| \int_{\Delta_2} T(\beta) P_S(d\beta) \right\| \leq (2c_1 + |\Delta_2|\, c_2)\, \|A\|\,. \tag{13}$$

The norm of $P_S(\Delta_1)A(t)P_S(\Delta_2)$, where $A(t)$ is the Heisenberg operator (5), can now be estimated using (13). If $\Delta_1$ and $\Delta_2$ are disjoint intervals with a distance $\delta$, the constants $c_1$ and $c_2$ have to be substituted by the upper bounds in (7), i.e. $c_1 = c_2 = c\left(1 + \delta^2\psi(t)\right)^{-\gamma}$.

# B The van Hove model

Let $F \circ G$ denote the symmetric tensor product of the Fock space $\mathcal{F}(\mathcal{H}_1)$ with vacuum $1_{vac}$. For all $f \in \mathcal{H}_1$ the exponential vectors $\exp f = 1_{vac} + f + \frac{1}{2} f \circ f + ...$ converge within $\mathcal{F}(\mathcal{H}_1)$, the inner product being $(\exp f \mid \exp g) = \exp(f \mid g)$. The linear span of all exponential vectors $\{\exp f \mid f \in \mathcal{H}_1\}$ is dense in $\mathcal{F}(\mathcal{H}_1)$. The creation operators $a^+(f)$ are uniquely determined by $a^+(f)\exp g = f \circ \exp g = \frac{\partial}{\partial\lambda}\exp(f + \lambda g) \mid_{\lambda=0}$, $f,g \in \mathcal{H}_1$ and the annihilation operators are given by $a(g)\exp f = (g \mid f)\exp f$. These operators satisfy the standard commutation relations $[a(f), a^+(g)] = (f \mid g)$. If $M$ is a operator on $\mathcal{H}_1$ then $\Gamma(M)$ is uniquely defined as operator on $\mathcal{F}(\mathcal{H}_1)$ by $\Gamma(M)\exp f := \exp(Mf)$, and the derivation $d\Gamma(M)$ is defined by $d\Gamma(M)\exp f := (Mf) \circ \exp f$.

As explicit example we may take $\mathcal{H}_1 = \mathcal{L}^2(\mathbb{R}^n)$ with inner product
$(f \mid g) = \int_{\mathbb{R}^n} \overline{f(k)} g(k) d^n k$. The one-particle Hamilton operator can be chosen as

$(Mf)(k) := \varepsilon(k)f(k)$ with the positive energy function $\varepsilon(k) = c|k|$, $c > 0$, $k \in \mathbb{R}^n$. Let $a_k^{\#}$, $k \in \mathbb{R}^n$, denote the distributional creation/annihilation operators, such that $a^+(f) = \int a_k^+ f(k)d^nk$ and $a(f) = \int a_k \overline{f(k)}d^nk$, then the Hamiltonian $H_E = d\Gamma(M)$ coincides with $H_E = \int \varepsilon(k)a_k^+ a_k d^nk$.

For arbitrary elements $g \in \mathcal{H}_1$ the unitary Weyl operators are defined on the set of exponential vectors by $T(g)\exp f = \mathrm{e}^{-(g|f)-\frac{1}{2}\|g\|^2}\exp(f+g)$. This definition is equivalent to $T(g) = \exp(a^+(g) - a(g))$. The Weyl operators are characterized by the properties

$$\begin{gathered} T(g_1)T(g_2) = T(g_1+g_2)\exp(-i\mathrm{Im}\,(g_1 \mid g_2)) \\ (1_{vac} \mid T(g)1_{vac}) = \exp\left(-\tfrac{1}{2}\|g\|^2\right). \end{gathered} \tag{14}$$

The time evolution on the Fock space is given by $U(t) = \exp(-iH_Et) = \Gamma(V(t))$ with $V(t) := \exp(-iMt)$. For exponential vectors we obtain $U(t)\exp f = \exp(V(t)f)$. From these equations the dynamics of the Weyl operators follows as

$$U^+(t)T(g)U(t) = T\left(V^+(t)\,g\right). \tag{15}$$

For fixed $h \in \mathcal{H}_1$ the unitary operators $T^+(h)U(t)T(h)$, $t \in \mathbb{R}$, form a one parameter group which acts on exponential vectors as
$T^+(h)U(t)T(h)\exp f = \exp\left((h \mid V(t)(f+h)-f) - \|h\|^2\right)\exp(V(t)(f+h)-h)$. For $h \in \mathcal{H}_1$ with $Mh \in \mathcal{H}_1$ the generator of this group is easily identified with $T^+(h)H_ET(h) = H_E + \Phi(Mh) + (h \mid Mh)$, where $\Phi(.)$ is the field operator. This identity was first derived by Cook [5] by quite different methods. If $h$ satisfies $M^{-1}h \in \mathcal{H}_1$ we obtain

$$T^+(M^{-1}h)H_ET(M^{-1}h) - \left\|M^{-\frac{1}{2}}h\right\|^2 = H_E + \Phi(h) \tag{16}$$

which is the Hamiltonian of the van Hove model [11], see also, [4] p.166ff, and [7].

For all $h \in \mathcal{H}_E$ with $M^{-\frac{1}{2}}h \in \mathcal{H}_E$ the field operator $\Phi(h)$ satisfies the estimate

$$\|\Phi(h)\psi\| \le 2\left\|M^{-\frac{1}{2}}h\right\|\left\|\sqrt{H_E}\psi\right\| + \|h\|\,\|\psi\|, \tag{17}$$

where $\psi \in \mathcal{F}(\mathcal{H}_1)$ is an arbitrary vector in the domain of $H_E$, see e.g. eq. (2.3) of [1]. As consequences we obtain

**Lemma 1** *The operators $H_E + \lambda\Phi(h)$, $\lambda \in \mathbb{R}$, are self-adjoint on the domain of $H_E$ if $h \in \mathcal{H}_1$ and $M^{-\frac{1}{2}}h \in \mathcal{H}_1$. The operator $H_E - \frac{1}{2}\Phi^2(h)$ has the lower bound $H_E - \frac{1}{2}\Phi^2(h) \ge -\|h\|^2$, if $h \in \mathcal{H}_1$ and $\left\|M^{-\frac{1}{2}}h\right\| \le 2^{-1}$.*

**Proof.** From (17) and the numerical inequality $\sqrt{x} \le ax + (4a)^{-1}$, valid for $x \ge 0$ and $a > 0$, we obtain a bound $\|\Phi(h)\psi\| \le c_1\|H_E\psi\| + c_2\|\psi\|$ with positive numbers $c_1$, $c_2 > 0$ where $c_1$ can be chosen arbitrarily small. Then the Kato-Rellich Theorem yields the first statement.
From (17) we obtain
$\|\Phi(h)\psi\|^2 \le 4\left\|M^{-\frac{1}{2}}h\right\|^2(\psi \mid H_E\psi) + 4\left\|M^{-\frac{1}{2}}h\right\|\|h\|\left\|\sqrt{H_E}\psi\right\|\|\psi\| + \|h\|^2\|\psi\|^2$
$\le 8\left\|M^{-\frac{1}{2}}h\right\|^2(\psi \mid H_E\psi) + 2\|h\|^2\|\psi\|^2$. Hence the operator inequalities
$0 \le \frac{1}{2}\Phi^2(h) \le 4\left\|M^{-\frac{1}{2}}h\right\|^2 H_E + \|h\|^2 I_E$ hold, and we have derived the second statement. ■

Therefore the total Hamiltonian (4) is semibounded, and the unitary operators $U_\lambda(t) = \exp(-i(H_E + \lambda\Phi(h))t)$ are well defined if (10) is satisfied.

In a first step we evaluate the expectation value of (11) $U_{\alpha\beta}(t) = U_\alpha(-t)U_\beta(t)$ for a coherent state (= normalized exponential vector) $\exp\left(f - \frac{1}{2}\|f\|^2\right)$ under the additional constraint $M^{-1}h \in \mathcal{H}_1$. This assumption allows to use the identity (16) which reduces all calculations to the Weyl relations and the vacuum expectation (14). The extension to the general case, which violates $M^{-1}h \in \mathcal{H}_1$, can then be performed by a continuity argument.

If $M^{-1}h \in \mathcal{H}_1$ the identity (16) implies
$U_\lambda(t) = T(-\lambda M^{-1}h)U_0(t)T(\lambda M^{-1}h)\exp\left(i\lambda^2\left(h \mid M^{-1}h\right)t\right)$. Then $U_{\alpha\beta}(t) = U_\alpha(-t)U_\beta(t)$ can be evaluated with the help of (14) and (15) with the result

$$\begin{aligned} U_{\alpha\beta}(t) &= T\left((\alpha-\beta)\left(V^+(t)-I\right)M^{-1}h\right)\exp\left(-i\varphi_1(t)\right), \\ \varphi_1(t) &= (\alpha^2-\beta^2)\left\{\left(h \mid M^{-1}h\right)t + \left(M^{-1}h \mid M^{-1}\sin(Mt)h\right)\right\}. \end{aligned} \tag{18}$$

Let $\omega(f)$ denote the projection operator onto the normalized coherent state
$\exp\left(f - \frac{1}{2}\|f\|^2\right)$, $f \in \mathcal{H}_1$, then $\mathrm{tr}_E U_{\alpha\beta}(t)\omega(f)$ is evaluated as
$(1_{vac} \mid T^+(f)U_{\alpha\beta}(t)T(f)1_{vac}) = \left(1_{vac} \mid T\left((\alpha-\beta)\left(V^+(t)-I\right)M^{-1}h\right)1_{vac}\right)\exp\left(-i\varphi(t)\right)$
with the phase
$\varphi(t) = 2(\alpha-\beta)\,\mathrm{Im}\left(f \mid \left(I - V^+(t)\right)M^{-1}h\right) + (\alpha^2-\beta^2)\left(\left(M^{-1}h \mid ht + M^{-1}\sin(Mt)h\right)\right)$. Using the second identity of (14) we finally obtain

$$\mathrm{tr}_E U_{\alpha\beta}(t)\omega(f) = \exp\left(-\frac{(\alpha-\beta)^2}{2}\left\|\left(V^+(t)-I\right)M^{-1}h\right\|^2\right)\exp\left(-i\varphi\right). \tag{19}$$

Under the assumption $M^{-1}h \in \mathcal{H}_1$ the norm $\left\|\left(V^+(t)-I\right)M^{-1}h\right\|$ is uniformly bounded in $t$ and the trace (19) does not vanish for $t \to \infty$. But nevertheless one can achieve a strong decrease which persists for some finite time interval. This period can be chosen arbitrarily large if the low energy contributions are strong; but inevitably, recurrences exist [14].

For vectors $h \in \mathcal{H}_1$ with $M^{-\frac{1}{2}}h \in \mathcal{H}_1$ but $M^{-1}h \in \mathcal{H}_1$ we first prove that $\mathrm{tr}_E U_{\alpha\beta}(t)\omega(f)$ is again given by the identity (19). Then we derive the essential statement that the norm $\left\|\left(V^+(t)-I\right)M^{-1}h\right\|$ diverges for $t \to \infty$, and consequently superselection sectors are induced for all $\alpha \neq \beta$.

The operators $H_E + \lambda\Phi(h)$ are self-adjoint on the domain of $H_E$ if $h \in \mathcal{H}_1$ and $M^{-\frac{1}{2}}h \in \mathcal{H}_1$. Therefore it is possible to extend the result (19) to Hamilton operators which satisfy these constraints but violate $M^{-1}h \in \mathcal{H}_1$. To make this statement more explicit we introduce the norm

$$|||h||| := \|h\| + \left\|M^{-\frac{1}{2}}h\right\|. \tag{20}$$

Let $h_n \in \mathcal{H}_1$, $n = 1, 2, ...$, be a sequence of real vectors which converges in this topology to a vector $h$, then we know from (17) and the proof of Lemma 1 that there exist two null sequences of positive numbers $c_{1n}$ and $c_{2n}$ such that

$$\|\left(\Phi(h_n) - \Phi(h)\right)\psi\| \leq c_{1n}\|\left(H_E + \Phi(h)\right)\psi\| + c_{2n}\|\psi\|.$$

Hence the operators $H_E + \Phi(h_n)$ converge strongly to $H_E + \Phi(h)$. Then Theorem 4.4 of [15] or Theorem 3.17 of [6] imply the strong convergence of $U(h_n;t) = \exp\left(-i\left(H_E + \Phi(h_n)\right)t\right)$ to $U(h;t) = \exp\left(-i\left(H_E + \Phi(h)\right)t\right)$, uniformly in all intervals $0 \leq t \leq s < \infty$. The operators $U_{\alpha\beta,n}(t) := \exp\left(i\left(H_E + \alpha\Phi(h_n)\right)t\right)\exp\left(-i\left(H_E + \beta\Phi(h_n)\right)t\right)$ converge therefore in the weak operator topology to $U_{\alpha\beta}(t)$. For $n = 1, 2, ..$ we can calculate the corresponding traces $\mathrm{tr}_E U_{\alpha\beta,n}(t)\omega(f)$ with the result (19) where $h$ has to be substituted by $h_n$. Since (19) is continuous in the variable $h$ in the topology (20) the limit for $n \to \infty$ is again given by (19).

To derive the divergence of $\left\|(V^+(t) - I)\, M^{-1}h\right\|$ for $t \to \infty$ we introduce the spectral resolution $P_M(d\lambda)$ of the one-particle Hamilton operator $M$. The energy distribution of the vector $h \in \mathcal{H}_1$ is given by the measure $d\sigma_h(\lambda) = (h \mid P_M(d\lambda)h)$. The norm of $(V^+(t) - I)\, M^{-1}h$ is the square root of

$$\psi(t) := \left\|(V^+(t) - I)\, M^{-1}h\right\|^2 = 4 \int_{\mathbb{R}_+} \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda). \tag{21}$$

This integral is well defined for all $h \in \mathcal{H}_1$, and $\psi(t)$ is differentiable for $t \in \mathbb{R}$.

**Lemma 2** *If $M^{-1}h \notin \mathcal{H}_1$, i.e.*

$$\int_\varepsilon^\infty \lambda^{-2}\, d\sigma_h(\lambda) \nearrow \infty \quad \text{if } \varepsilon \to +0, \tag{22}$$

*then the integral (21) diverges for $t \to \infty$.*

**Proof.** Since the operator $M$ has an absolutely continuous spectrum, the measure $d\sigma_h(\lambda)$ is absolutely continuous with respect to the Lebesgue measure $d\lambda$ on $\mathbb{R}_+$. Consequently, the measure $\lambda^{-2}\, d\sigma_h(\lambda)$ is absolutely continuous with respect to the Lebesgue measure on any interval $(\varepsilon, \infty)$ with $\varepsilon > 0$. The identity $\sin^2 \frac{\lambda t}{2} = \frac{1}{2}(1 - \cos \lambda t)$ and the Lebesgue Lemma therefore imply $\lim_{t\to\infty} \int_\varepsilon^\infty \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) = \frac{1}{2} \int_\varepsilon^\infty \lambda^{-2}\, d\sigma_h(\lambda)$. Given a number $N > 0$ the assumption (22) yields the existence of an $\varepsilon > 0$ such that

$$\lim_{t\to\infty} \int_\varepsilon^\infty \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) = \frac{1}{2} \int_\varepsilon^\infty \lambda^{-2}\, d\sigma_h(\lambda) > N. \tag{23}$$

From the inequality $\int_{\mathbb{R}_+} \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) \geq \int_\varepsilon^\infty \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda)$ we then obtain $\int_0^\infty \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) > N$ for sufficiently large $t$. Since the number $N$ can be arbitrarily large the integral (21) diverges for $t \to \infty$. ■

If $d\sigma_h(\lambda)$ satisfies additional regularity conditions, we can obtain uniform estimates of the divergence. E. g. $d\sigma_h(\lambda) \cong c \cdot \lambda^{2\mu} d\lambda$ with $0 < \mu < \frac{1}{2}$ and $c > 0$ in a neighbourhood of $\lambda = +0$ implies a powerlike divergence $\psi(t) \sim t^{1-2\mu}$.

So far the reference state $\omega$ has been a coherent state. But the results remain obviously true if the reference state is a finite linear combination of coherent states or a finite mixture of coherent states.

As a final remark we indicate a modification of the model, which does not use the absolute continuity of the spectrum of $M$. But we still need a dominating low energy contribution in the interaction. More precisely, we assume that $\sigma_h(\lambda) \equiv \int_0^\lambda d\sigma_h(\alpha)$ behaves at low energies like

$$\lambda^{-2} \sigma_h(\lambda) \nearrow \infty \text{ if } \lambda \to +0. \tag{24}$$

Then we can derive the divergence of (21) by the inequalities $\psi(t) \geq 4 \int_0^{\frac{\pi}{t}} \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) \geq \frac{4}{\pi^2} t^2 \int_0^{\frac{\pi}{t}} d\sigma_h(\lambda) = \frac{4}{\pi^2} t^2\, \sigma_h(\frac{\pi}{t})$ using $\sin x \geq \frac{2}{\pi} x$ if $0 \leq x \leq \frac{\pi}{2}$. For measures $d\sigma_h(\lambda) \sim \lambda^{2\mu} d\lambda$ the assumption (24) is more restrictive than (22) – it excludes $d\sigma_h(\lambda) \sim \lambda d\lambda$ which satisfies the conditions of Lemma 2. But (24) is also meaningful for point measures $d\sigma_h(\lambda)$, and $M$ may be an operator with a pure point spectrum. The Boson field can therefore be substituted by an infinite family of harmonic oscillators, which have zero as accumulation point of their frequencies. Such an example has been discussed – also for KMS states – by Primas [17].

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