Exactly soluble models of decoherence[1]

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Abstract

Superselection rules induced by the interaction with the environment are investigated with the help of exactly soluble Hamiltonian models. Starting from the examples of Araki and of Zurek more general models with scattering are presented for which the projection operators onto the induced superselection sectors do no longer commute with the Hamiltonian. The example of an environment given by a free quantum field indicates that infrared divergence plays an essential role for the emergence of induced superselection sectors. For all models the induced superselection sectors are uniquely determined by the Hamiltonian, whereas the time scale of the decoherence depends crucially on the initial state of the total system.

1 Introduction

One of the puzzles of quantum mechanics is the question, how classical objects can arise in quantum theory. Quantum mechanics is a statistical theory, but its statistics differs on a fundamental level from the statistics of classical objects. The violation of Bell's inequalities and the context dependence of quantum mechanics (Kochen-Specker theorem) illustrate this fact, see e.g. [13].

It is known since a long time that the statistical results of quantum mechanics become consistent with a classical statistics of "facts", if the superposition principle is reduced to "superselection sectors", i.e. coherent orthogonal subspaces of the full Hilbert space. The mathematical structure of quantum mechanics and of quantum field theory provides us with only a few "superselection rules", the most important being the charge superselection rule related to gauge invariance, see e.g. [3] [17] and the references given therein. But there are definitively not enough of these superselection rules to understand classical properties in quantum theory. A possible solution of this problem is the emergence of effective superselection rules due to decoherence caused by the interaction with the environment. These investigations – often related to a discussion of the process of measurement – have developed in the eighties; some references are [1][20][9], but see also the earlier publications [18][19] and [6].

In this article decoherence and the emergence of environment induced superselection rules are investigated on the basis of exactly soluble models. After a short introduction to superselection rules and to the dynamics of subsystems in Sects. 2 and 3, several models are presented in Sect. 4. For a class of simple models, which essentially go back to Araki [1] and Zurek [20], the transition between the induced superselection sectors is suppressed uniformly in trace norm. In a more realistic example with a quantum field as environment, presented in Sect.

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4.2, the infrared behaviour of the environment is of essential importance for the emergence of induced superselection rules. Here uniform estimates, which persist for arbitrary times, are only possible in the limit of infrared divergence. In Sect. 4.3 it is shown that additional scattering processes (by sufficiently smooth potentials) do not alter the induced superselection sectors, but the decoherence is no longer uniform with respect to the initial state of the system.

2 Superselection rules

We start with a few mathematical notations. Let $\mathcal{H}$ be a separable Hilbert space, then the following spaces of linear operators are used.

$\mathcal{B}(\mathcal{H})$: The $\mathbf{R}$-linear space of all bounded self-adjoint operators A. The norm of this space is the operator norm $\|A\|$.

$\mathcal{T}(\mathcal{H})$: The $\mathbf{R}$-linear space of all self-adjoint nuclear operators A. These operators have a pure point spectrum $\alpha_i \in \mathbf{R}$, $i = 1, 2, ...$, with $\sum_i |\alpha_i| < \infty$. The natural norm of this space is the trace norm $\|A\|_1 = \operatorname{tr}\sqrt{A^+A} = \sum_i |\alpha_i|$. Another norm, used in the following sections, is the Hilbert-Schmidt norm $\|A\|_2 = \sqrt{\operatorname{tr} A^+A}$. These norms satisfy the inequalities $\|A\| \leq \|A\|_2 \leq \|A\|_1$.

$\mathcal{D}(\mathcal{H})$: The set of all statistical operators, i.e. positive nuclear operators W with a normalized trace, $\operatorname{tr} W = 1$.

$\mathcal{P}(\mathcal{H})$: The set of all rank one projection operators P^1.
These sets satisfy the obvious inclusions $\mathcal{P}(\mathcal{H}) \subset \mathcal{D}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$.

Any state of a quantum system is represented by a statistical operator $W \in \mathcal{D}(\mathcal{H})$, the elements of $\mathcal{P}(\mathcal{H})$ thereby correspond to the pure states. Any (bounded) observable is represented by an operator $A \in \mathcal{B}(\mathcal{H})$, and the expectation of the observable A in the state W is the trace $\operatorname{tr} WA$. Without additional knowledge about the structure of the system we have to assume that the set of all states corresponds exactly to $\mathcal{D}(\mathcal{H})$, and the set of all (bounded) observables is $\mathcal{B}(\mathcal{H})$. The state space $\mathcal{D}(\mathcal{H})$ has an essential property: it is a convex set, i.e. $W_1, W_2 \in \mathcal{D}(\mathcal{H})$ implies $\lambda_1 W_1 + \lambda_2 W_2 \in \mathcal{D}(\mathcal{H})$ if $\lambda_{1,2} \geq 0$ and $\lambda_1 + \lambda_2 = 1$. Any statistical operator $W \in \mathcal{D}(\mathcal{H})$ can be decomposed into pure states $W = \sum_n w_n P_n^1$ with $P_n^1 \in \mathcal{P}(\mathcal{H})$ and probabilities $w_n \geq 0$, $\sum_n w_n = 1$. An explicit example is the spectral decomposition of W. But there are many other possibilities. It is exactly this arbitrariness that does not allow a classical interpretation of quantum probability. A more detailed discussion of the state space of quantum mechanics can be found in [11].

The arbitrariness of the decomposition of W originates in the superposition principle. In quantum mechanics, especially in quantum field theory, the superposition principle can be restricted by superselection rules. Here we cannot discuss the arguments to establish such rules, for that purpose see e.g. [3][17] and also Chap.6 of [7], or to refute them, see e.g. [12]. Here we only investigate the consequences for the structure of the state space. In a theory with discrete superselection rules like the charge superselection rule, the Hilbert space $\mathcal{H}$ splits into orthogonal superselection sectors $\mathcal{H}_m$, $m \in \mathbf{M}$, such that $\mathcal{H} = \oplus_m \mathcal{H}_m$. Pure states with charge m (in appropriate normalization) are then represented by vectors in $\mathcal{H}_m$, and superpositions of vectors with different charges have no physical interpretation. The projection operators P_m onto the orthogonal subspaces $\mathcal{H}_m$ satisfy $P_m P_n = \delta_{mn}$ and $\sum_m P_m = I$. The set of states is reduced to those statistical operators which satisfy $P_m W = W P_m$ for all projection operators $P_m, m \in \mathbf{M}$. The state space of the system is then $\mathcal{D}^S = \{W \in \mathcal{D}(\mathcal{H}) | W P_m = P_m W,\, m \in \mathbf{M}\}$, and all

statistical operators satisfy the identity $W = \sum_m P_m W P_m$. An equivalent statement is that all observables of such a system have to commute with the projection operators P_m, $m \in \mathbf{M}$, and the set of observables of the system is given by
$\mathcal{B}^S = \{A \in \mathcal{B}(\mathcal{H}) \mid AP_m = P_m A,\, m \in \mathbf{M}\} = \{A \in \mathcal{B}(\mathcal{H}) \mid A = \sum_m P_m A P_m\}$.

The projection operators $\{P_m \mid m \in \mathbf{M}\}$ are themselves observables, which commute with all observables of the system, and they generate a nontrivial centre of the algebra of observables.

In theories with continuous superselection rules the finite or countable set of projection operators $\{P_m, m \in \mathbf{M}\}$ is substituted by a (weakly continuous) family of projection operators $P(\Delta)$ indexed by measurable subsets $\Delta \subset \mathbf{R}$, see e.g. [15] or [1]. These projection operators have to satisfy

$$\begin{cases} P(\Delta_1 \cup \Delta_2) = P(\Delta_1) + P(\Delta_2) \text{ for all intervalls } \Delta_1, \Delta_2 \\ P(\Delta_1)P(\Delta_2) = O \;\text{ if }\; \Delta_1 \cap \Delta_2 = \emptyset, \;\text{ and }\; P(\emptyset) = O,\; P(\mathbf{R}) = 1. \end{cases} \tag{1}$$

The set of observables is now given by $\mathcal{B}^S = \{A \in \mathcal{B}(\mathcal{H}) \mid AP(\Delta) = P(\Delta)A,\, \Delta \subset \mathbf{R}\}$, but there is no formulation of the corresponding set of states within the class of nuclear statistical operators.

The importance of superselection rules for the transition from quantum probability to classical probability is obvious. But there remains an essential problem: Only very few superselection rules can be found in quantum mechanics that are compatible with the mathematical structure and with experiment. A satisfactory solution to this problem is the emergence of effective superselection rules induced by the interaction with the environment.

3 Dynamics of subsystems and induced superselection sectors

In the following we consider an "open system", i.e. a system S which interacts with an "environment" E, such that the total system $S + E$ satisfies the usual Hamiltonian dynamics. The Hilbert space $\mathcal{H}_{S+E}$ of the total system $S+E$ is the tensor space $\mathcal{H}_S \otimes \mathcal{H}_E$ of the Hilbert spaces for S and for E. We assume that the only observables at our disposal are the operators $A \otimes I_E$ with $A \in \mathcal{B}(\mathcal{H}_S)$. If the state of the total system is $W \in \mathcal{D}(\mathcal{H}_{S+E})$, then all expectation values $\mathrm{tr}_{S+E} W(A \otimes I_E)$ can be calculated from the reduced statistical operator $\rho = \mathrm{tr}_E W$ which is an element of $\mathcal{D}(\mathcal{H}_S)$, such that $\mathrm{tr}_S A\rho = \mathrm{tr}_{S+E}(A \otimes I_E)W$ holds for all $A \in \mathcal{B}(\mathcal{H}_S)$. We shall refer to the statistical operator $\rho = \mathrm{tr}_E W$ as the "state" of the subsystem.

As mentioned above we assume the usual Hamiltonian dynamics for the total system, i.e. $W(t) = U(t)WU^+(t)$ with the unitary group $U(t)$, generated by the total Hamiltonian. Except for the trivial case that S and E do not interact, the dynamics of the reduced statistical operator

$$\rho(t) = \mathrm{tr}_E U(t) W U^+(t) \tag{2}$$

is no longer unitary, and it is exactly this dynamics which can produce effective superselection sectors. More explicitly, the Hamiltonian of the total system can provide a family of projection operators $\{P_m,\, m \in \mathbf{M}\}$ which are independent from the initial state, such that the statistical operator behaves like

$$\rho(t) \cong \sum_m P_m \rho(t) P_m \;\text{ for }\; t \to \infty. \tag{3}$$

An equivalent statement is that the superpositions between vectors of different sectors $P_m\mathcal{H}_S$ are strongly suppressed. Any mechanism, which leads to this effect, will be called *decoherence.*

In the case of induced continuous superselection rule the asymptotics is more appropriately described in the Heisenberg picture, as stated above. But the decoherence effect is also seen in the Schrödinger picture: $P(\Delta_1)\rho(t)P(\Delta_2) \to 0$ for $t \to \infty$ if Δ_1 and Δ_2 have a positive distance.

The statement (3) is so far rather vague since it does not specify the asymptotics. A preliminary definition of a *weak* type of decoherence can be formulated as follows.

Definition 1 *The subspaces $P_m\mathcal{H}_S$, $m \in \mathbf{M}$, are denoted as induced superselection sectors, of the dynamics (2), if for all observables $A \in \mathcal{B}(\mathcal{H}_S)$ which have no diagonal matrix elements, i.e. $P_mAP_m = O$, $m \in \mathbf{M}$, the trace*

$$\mathrm{tr}_{S+E}(A \otimes I_E)U(t)WU^+(t) = \mathrm{tr}_S A\rho(t) \tag{4}$$

vanishes if $t \to \infty$ for all initial states $W \in \mathcal{D}_1$ of a dense subset $\mathcal{D}_1 \subset \mathcal{D}(\mathcal{H}_{S+E})$.

It is possible to give an alternative definition with $\mathcal{D}_1$ substituted by $\mathcal{D}(\mathcal{H}_{S+E})$. These definitions are equivalent, as can be easily seen. Assume the statements of Definition 1 are valid for a family of subspaces $\{P_m\mathcal{H}_S,\ m \in \mathbf{M}\}$, then we can find for any $W \in \mathcal{D}(\mathcal{H}_{S+E})$ and any $\varepsilon > 0$ a statistical operator $W_1 \in \mathcal{D}_1$ such that $\|W - W_1\|_1 < \varepsilon$ and $\mathrm{tr}_{S+E}(A \otimes I_E)U(t)W_1U^+(t) \to 0$ if $t \to \infty$ for the specified class of observables A. Since $|\mathrm{tr}_{S+E}(A \otimes I_E)U(t)(W - W_1)U^+(t)| < \varepsilon \|A\|$ the trace (4) vanishes if $t \to \infty$ for all initial states $W \in \mathcal{D}(\mathcal{H}_{S+E})$.

The independence from the initial state justifies the terminology induced "superselection" rules. The Definition 1 has to be supplemented by statements about the time scale of the convergence. For that purpose the following models are investigated. They indicate the essential role of the initial state – especially of the components affiliated to the environment – to achieve decoherence in sufficiently short time.

4 Soluble models

The first class of the presented models has a discrete superselection structure such that the off-diagonal elements of the statistical operator vanish in trace norm $\|.\|_1$

$$\|P_m\rho(t)P_n\|_1 \to 0 \text{ if } t \to \infty \text{ and } m \neq n \tag{5}$$

for an arbitrary initial state $\rho(0) \in \mathcal{D}(\mathcal{H}_S)$. But the asymptotics is more complicated for the more realistic models investigated in Sects. 4.2 and 4.3.

The models of Sects. 4.1 and 4.2 have the following structure. The Hilbert space is $\mathcal{H}_{S+E} = \mathcal{H}_S \otimes \mathcal{H}_E$. The total Hamiltonian has the form

$$H_{S+E} = H_S \otimes I_E + I_S \otimes H_E + V_S \otimes V_E \tag{6}$$

where H_S is the Hamiltonian of S, H_E is the Hamiltonian of E, $V_S \otimes V_E$ is the interaction term between S and E with self-adjoint operators V_S on $\mathcal{H}_S$ and V_E on $\mathcal{H}_E$. We make the following assumptions

1) The operators H_S and V_S commute, $[H_S, V_S] = O$, hence $[H_S \otimes I_E, V_S \otimes V_E] = O$.

2) The operator V_E has an absolutely continuous spectrum.

The assumption 1) is a rather severe restriction, which will be given up in Sect. 4.3, where we admit an additional scattering potential V, which has not to commute with any of the other operators. The assumption 2) has more technical reasons. It implies that estimates can be derived in the limit $t \to \infty$ in agreement with Definition 1. But one can also allow operators with point spectra (as done in [20]), if the spacing of the eigenvalues is sufficiently small. Then the norm in (5) is an almost periodic function, and the suppression of this norm takes place only during a finite time interval $0 \leq t \leq T$. But T can be large enough for all practical purposes.

The operator V_S has the spectral representation $V_S = \int_{\mathbf{R}} \lambda P(d\lambda)$ with a spectral family $\{P(\Delta), \Delta \subset \mathbf{R}\}$ which satisfies (1). We shall see that exactly this spectral family determines the superselection sectors. If V_S has a pure point spectrum, then $P(\Delta)$ is a step function with values P_m, and we can write

$$V_S = \sum_m \lambda_m P_m. \tag{7}$$

As a consequence of assumption 1) we have $[H_S, P(\Delta)] = O$ or $[H_S, P_m] = O$ for $\Delta \subset \mathbf{R}$ or $m \in \mathbf{M}$, respectively. The Hamiltonian (6) has therefore the form (for simplicity we only write the version with the discrete spectrum (7))

$$H_{S+E} = H_S \otimes I_E + \sum_m P_m \otimes \Gamma_m \text{ with} \tag{8}$$

$$\Gamma_m = H_E + \lambda_m V_E. \tag{9}$$

The unitary evolution $U(t) := \exp(-iH_{S+E}t)$ of the total system can be written as $\left(\mathrm{e}^{-\imath H_S t} \otimes I_E\right) \sum_m P_m \otimes \mathrm{e}^{-i\Gamma \lambda_m t}$. The calculation of the reduced dynamics (2) then leads to

$$P_m \rho(t) P_n = P_m \mathrm{e}^{-\imath H_S t} \left(\mathrm{tr}_E \mathrm{e}^{-i\Gamma_m t} W \mathrm{e}^{i\Gamma_n t}\right) \mathrm{e}^{\imath H_S t} P_n, \tag{10}$$

where the operators P_n are the projection operators of the spectral representation (7) of V_S. For a factorizing initial state $W = \rho \otimes \omega$ with $\rho \in \mathcal{D}(\mathcal{H}_S)$ and a reference state $\omega \in \mathcal{D}(\mathcal{H}_E)$ of the environment, the operator (10) simplifies to $P_m \rho(t) P_n = P_m e^{-iH_S t} \rho e^{iH_S t} P_n \, \chi_{m,n}(t)$ with

$$\chi_{m,n}(t) = \mathrm{tr}_E \left(\mathrm{e}^{i\Gamma_n t} \mathrm{e}^{-i\Gamma_m t} \omega\right) \tag{11}$$

and the emergence of dynamically induced superselection rules depends on an estimate of this trace.

4.1 The Araki-Zurek models

The first soluble models for the investigation of the reduced dynamics have been given by Araki [1] and Zurek [20], and the following construction is essentially based on these papers. In addition to the specifications made above, we demand that

3) the Hamiltonian H_E and the potential V_E commute, $[H_E, V_E] = O$.

We first investigate $P_m \rho(t) P_n$ for a factorizing initial state $W = \rho \otimes \omega$. Under the assumption 3) the trace (11) simplifies to $\chi_{m,n}(t) = \mathrm{tr}_E \left(\mathrm{e}^{-i(\lambda_m - \lambda_n) V_E t} \omega \right)$. Let $V_E = \int_{\mathbf{R}} \lambda P_E(d\lambda)$ be the spectral representation of the operator V_E. Then, as a consequence of assumption 2), for any $\omega \in \mathcal{D}(\mathcal{H}_E)$ the measure $d\mu(\lambda) := \mathrm{tr}_E \left(P_E(d\lambda) \omega \right)$ is absolutely continuous with respect to the Lebesgue measure, and the function $\chi(t) := \mathrm{tr} \left(\mathrm{e}^{-\imath V_E t} \omega \right) = \int_{\mathbf{R}} \mathrm{e}^{-i\lambda t} \, d\mu(\lambda)$ vanishes if $t \to \infty$. But to have a decrease which is effective in sufficiently short time, we need an additional smoothness condition on ω (which does not impose restrictions on the statistical operator $\rho \in \mathcal{D}(\mathcal{H}_S)$ of the system S). If the integral operator, which represents ω in the spectral representation of V_E, is a sufficiently differentiable function (vanishing at the boundary points of the spectrum) we can derive estimates like $|\chi(t)| \leq C_\gamma (1 + |t|)^{-\gamma}$ with arbitrarily large values of γ. Such an estimate leads to the upper bound

$$|\chi_{m,n}(t)| \leq C_\gamma (1 + \delta \, |t|)^{-\gamma} \tag{12}$$

if $|\lambda_m - \lambda_n| \geq \delta > 0$, and we obtain an estimate for the norm (5)

$$\| P_m \rho(t) P_n \|_1 \leq C_\gamma (1 + \delta \, |t|)^{-\gamma}. \tag{13}$$

with arbitrary $\rho(0) \equiv \rho \in \mathcal{D}(\mathcal{H}_S)$. The constants $\gamma > 0$, $\delta > 0$ and $C_\gamma > 0$ do not depend on ρ. Moreover one can achieve large values of γ and/or small values of the constant C_γ if the reference state ω is sufficiently smooth.

These results depend on the reference state ω only via the decrease of $\chi(t)$. We could have chosen a more general initial state $W \in \mathcal{D}(\mathcal{H}_{S+E})$

$$W = \sum_\mu c_\mu \, \rho_\mu \otimes \omega_\mu \tag{14}$$

with $\rho_\mu \in \mathcal{D}(\mathcal{H}_S)$, $\omega_\mu \in \mathcal{D}(\mathcal{H}_E)$ and numbers $c_\mu \in \mathbf{R}$ which satisfy $\sum_\mu |c_\mu| < \infty$ and $\sum_\mu c_\mu = \mathrm{tr}\, W = 1$. As a consequence of assumption 2) the space $\mathcal{H}_E$ has infinite dimension. If $\mathcal{H}_S$ is finite dimensional, the set (14) of statistical operators covers the whole space $\mathcal{D}(\mathcal{H}_{S+E})$. If also $\mathcal{H}_S$ is infinite dimensional, this set is dense in $\mathcal{D}(\mathcal{H}_{S+E})$. With the arguments given above for factorizing initial states the statement of Definition 1 can be derived for all initial states (14), and the sectors $P_n \mathcal{H}_S$ are induced superselection sectors in the sense of this definition. Moreover, assuming that the components of the statistical operator W affiliated to the environment are sufficiently smooth functions in the spectral representation of V_E, the sum $\sum_\mu \left| c_\mu \, \mathrm{tr}_E \left(\mathrm{e}^{-\imath(\lambda_m - \lambda_n) V_E t} \omega_\mu \right) \right|$ satisfies a uniform estimate (12), and (13) is still valid. Hence the time scale of the decoherence can be as short as we want without restriction on $\rho(0) = \mathrm{tr}_E W = \sum_\mu c_\mu \, \rho_\mu$.

If the potential V_S has a (partially) continuous spectrum with spectral family $\{P(\Delta),\ \Delta \subset \mathbf{R}\}$, an estimate

$$\| P(\Delta_1) \rho(t) P(\Delta_2) \|_2 \leq C_\gamma (1 + \delta \, |t|)^{-\gamma} \tag{15}$$

can be derived in the weaker Hilbert-Schmidt norm for arbitrary intervals Δ_1 and Δ_2 which have a non-vanishing distance, see Sect. 7.6 of [7].

4.2 The interaction with free fields: the role of infrared divergence for induced superselection sectors

In this section we give up the restriction 3) on the Hamiltonian. Then the estimate of the trace (11) needs more involved calculations. As specific example we consider an environment given by a free Boson field. Such models can be calculated explicitly, and they have often been used as the starting point for Markov approximations.

As Hilbert space $\mathcal{H}_E$ we choose the Fock space based on the one particle space $\mathcal{H}^{(1)} = \mathcal{L}^2(\mathbf{R}_+)$ with inner product $\langle f \mid g \rangle = \int_0^\infty \overline{f(k)} g(k) dk$. The one-particle Hamilton operator, denoted by $\widehat{\varepsilon}$, is the multiplication operator $(\widehat{\varepsilon} f)(k) := \varepsilon(k) f(k)$ with the energy function $\varepsilon(k) = c \cdot k$, $c > 0$, $k \in \mathbf{R}_+$, defined for all functions f with $(1 + \varepsilon(k)) f(k) \in \mathcal{L}^2(\mathbf{R}_+)$. The creation/annihilation operators a_k^+ and a_k are normalized to $\left[a_k, a_{k'}^+\right] = \delta(k - k')$. The Hamiltonian of the environment is then

$$H_E = \int_0^\infty \varepsilon(k) a_k^+ a_k dk. \tag{16}$$

With $a^+(f) = \int_0^\infty f(k) a_k^+ dk$ and $a(f) = \int_0^\infty f(k) a_k dk$ we define field operators by $\Phi(f) := 2^{-\frac{1}{2}} \left(a^+(f) + a(f) \right)$ for real functions $f \in \mathcal{L}^2(\mathbf{R}_+)$. The interaction potential is chosen as $V_E = \Phi(f)$ with

$$f \in \mathcal{L}^2(\mathbf{R}_+) \text{ and } \widehat{\varepsilon}^{-1} f \in \mathcal{L}^2(\mathbf{R}_+), \tag{17}$$

An example for the total Hamiltonian is given by a single particle coupled to the quantum field with velocity coupling

$$\begin{aligned} H_{S+E} &= \tfrac{1}{2} P^2 \otimes I_E + P \otimes \Phi(f) + I_S \otimes H_E \\ &= \tfrac{1}{2} \left(P \otimes I_E + I_S \otimes \Phi(f) \right)^2 + I_S \otimes \left(H_E - \tfrac{1}{2} \Phi^2(f) \right) \end{aligned} \tag{18}$$

If the test function f satisfies $\left\| \widehat{\varepsilon}^{-\frac{1}{2}} f \right\| < 2^{-\frac{1}{2}}$, the Hamiltonian $H_E - \frac{1}{2}\Phi^2(f)$ is bounded from below, and consequently H_{S+E} is bounded from below. Since the particle is coupled to the free field with $V_S = P$, the reduced dynamics yields continuous superselection sectors for the momentum P of the particle.

The operators (9) Γ_m are substituted by $H_\lambda := H_E + \lambda \Phi(f)$, $\lambda \in \mathbf{R}$, which are Hamiltonians of the van Hove model [8]. The restrictions (17) are necessary to guarantee that all operators H_λ, $\lambda \in \mathbf{R}$, are unitarily equivalent and defined on the same domain. To derive induced superselection sectors we have to estimate the time dependence of the traces $\chi_{\alpha\beta}(t) := \mathrm{tr}_E U_{\alpha\beta}(t) \omega$, $\alpha \neq \beta$, where the unitary operators $U_{\alpha\beta}(t)$ are given by

$$U_{\alpha\beta}(t) := \exp(i H_\alpha t) \exp(-i H_\beta t), \tag{19}$$

see (11). In the Appendix we prove the following results for states ω which are mixtures of coherent states.

a) Under the restrictions (17) the traces $\chi_{\alpha\beta}(t)$, $\alpha \neq \beta$, do not vanish for $t \to \infty$.

b) If $\Phi(f)$ has contributions at arbitrarily small energies, $\chi_{\alpha\beta}(t)$ can nevertheless strongly decrease for $\alpha \neq \beta$ within a very long time interval $0 \leq t \leq T$. Estimates like (13) or (15) are substituted by $\|P_m \rho(t) P_n\| \leq f(t)$ or $\|P(\Delta) \rho(t) P(\Delta')\|_2 \leq f(t)$. But in contrast to (13) or (15) the function $f(t)$ increases again for $t > T$.

c) For fixed $\alpha \neq \beta$ a limit $\chi_{\alpha\beta}(t) \to 0$ for $t \to \infty$ is possible if $\widehat{\varepsilon}^{-1} f \in \mathcal{L}^2(\mathbf{R}_+)$ is violated, i.e. in the case of infrared divergence.

A large infrared contribution is therefore essential for the emergence of induced superselection sectors. As in Sect. 4.1 the choice of the initial state W of the total system can be extended to (14) with $\rho_\mu \in \mathcal{D}(\mathcal{H}_S)$ and mixtures of coherent states $\omega_\mu \in \mathcal{D}(\mathcal{H}_E)$. This class of states is again dense in $\mathcal{D}(\mathcal{H}_{S+E})$, and, at least in the infrared divergent case, we obtain induced superselection sectors in the sense of Definition 1.

4.3 Models with scattering

For the models presented in Sects. 4.1 and 4.2 the projection operators onto the effective superselection sectors $P_m \otimes I_S$ (or $P(\Delta) \otimes I_S$) commute with the total Hamiltonian. We now modify the Hamiltonian (6) to

$$H = H_{S+E} + V = H_S \otimes I_E + I_S \otimes H_E + V_S \otimes V_E + V$$

where the operator V is only restricted to be a *scattering* potential. This restriction means that the wave operator $\Omega = \lim_{t\to\infty} e^{iHt} e^{-iH_{S+E}t}$ exists as strong limit. To simplify the arguments we assume that there are no bound states such that the convergence is guaranteed on $\mathcal{H}_{S+E}$ with $\Omega^+ = \Omega^{-1}$. Then the time evolution $U(t) = \exp(-iHt)$ behaves asymptotically as $U_0(t)\Omega^+$ with $U_0(t) = \exp(-iH_{S+E}t)$. More precisely, we have for all $W \in \mathcal{D}(\mathcal{H}_{S+E})$

$$\lim_{t\to\infty} \left\| U(t)WU^+(t) - U_0(t)\Omega^+ W \Omega U_0^+(t) \right\|_1 = 0 \tag{20}$$

in trace norm. Following Sect. 4.1 the reduced trace $\mathrm{tr}_E U_0(t)\Omega^+ W \Omega U_0^+(t)$ produces the superselection sectors $P_m \mathcal{H}_S$ which are determined by the spectrum (7) of V_S. The asymptotics (20) then yields (in the sense of Definition 1) the same superselection sectors for $\rho(t) = \mathrm{tr}_E U(t)WU^+(t)$. Moreover we can derive fast decoherence by additional assumptions on the initial state and on the potential. For that purpose we start with a factorizing initial state $W = \rho(0) \otimes \omega$ with smooth ω. To apply the arguments of Sect. 4.1 to the dynamics $U_0(t)\Omega^+ W \Omega U_0^+(t)$ the statistical operator $\Omega^+ (\rho \otimes \omega) \Omega$ has to be a sufficiently smooth operator on the tensor factor $\mathcal{H}_E$ for all $\rho \in \mathcal{D}(\mathcal{H}_S)$. That is guaranteed if we choose as scattering potential a smooth potential in the sense of Kato [10]. Then both the limits, (20) and
$\lim_{t\to\infty} \left\| P_m \left(\mathrm{tr}_E U_0(t)\Omega^+ W \Omega U_0^+(t) \right) P_n \right\|_1 = 0$, $m \neq n$, are reached in sufficiently short time. Hence $\rho(t)$ can decohere fast into the subspaces $P_m \mathcal{H}_S$ which are determined by the spectrum (7) of V_S. But in contrast to (13) one does not obtain a uniform bound with respect to the initial state $\rho(0)$, since the limit (20) is not uniform in $W \in \mathcal{D}(\mathcal{H}_{S+E})$.
Remark. The restriction that V is a scattering potential is essential. The dominating part $V_S \otimes V_E$ of the interaction $V_S \otimes V_E + V$ still satisfies the assumption 1). In [11] a spin model with an interaction which violates both the constraints, the assumption 1) and the scattering condition, has been investigated. That model can produce superselection sectors only in an approximative sense, where the lower bounds on $\|P_m \rho(t) P_n\|_1$ depend on the magnitude of the non-vanishing commutator.

4.4 Concluding remarks

The investigation of the models proves that the uniform emergence (13) or (15) of effective superselection sectors is consistent with the mathematical rules of quantum mechanics. But this result depends on rather restrictive assumptions on the Hamiltonian. For the more realistic model of a quantum field presented in Sect. 4.2 the suppression persists only for a finite period of time. If the low frequency spectrum dominates, this period of time can be sufficiently large for all practical purposes. Only in the limit of infrared divergence the induced superselection sectors persist for $t \to \infty$. If there is additional scattering as considered in Sect. 4.3 the superselection sectors still exist. But the estimates are no longer uniform in the initial state $\rho(0)$ of the system.

For all these models the induced superselection sectors are fully determined by the Hamiltonian in the sense of Definition 1. The initial state of the total system, especially the smoothness properties of the components related to the environment, determine the time scale in which these sectors emerge.

A The van Hove model

As Hilbert space $\mathcal{H}_E$ we take the Fock space $\mathcal{F}(\mathcal{H}^{(1)})$ based on the one-particle space $\mathcal{H}^{(1)} = \mathcal{L}^2(\mathbf{R}_+)$. For test functions $f, g \in \mathcal{S}(\mathbf{R}_+) \subset \mathcal{L}^2(\mathbf{R}_+)$ the creation and annihilation operators $a^+(f) = \int_0^\infty f(k) a^+(k) dk$ and $a(g) = \int_0^\infty g(k) a(k) dk$ are normalized to $[a(f), a^+(g)] = \langle f \mid g \rangle = \int_0^\infty \overline{f(k)} g(k) dk$. The test functions $f \in \mathcal{S}(\mathbf{R}_+)$ are rapidly decreasing C^∞-functions with a support restricted to $\mathbf{R}_+ = [0, \infty)$. The one-particle Hamiltonian of the free field is $(\widehat{\varepsilon} f)(k) := \varepsilon(k) f(k)$ with the energy function $\varepsilon(k) = c \cdot k$, $c > 0$, for $k \geq 0$. Actually we can choose any positive monotonically increasing and polynomially bounded energy function $\varepsilon(k)$, which has excitations of arbitrarily small energy, $\varepsilon(k)/|k| \to c > 0$ if $k \to 0$. The Hamiltonian of the free field is then (16), and as canonical field and momentum operators we choose $\Phi(f) := \frac{1}{\sqrt{2}}(a^+(f) + a(f))$ and $\Pi(f) := \frac{i}{\sqrt{2}}(a^+(f) - a(f))$. For real test functions we define the Weyl operators

$$T(f,g) := \exp(-i\Pi(f) - i\Phi(g)) = \exp(-i\Pi(f)) \exp(-i\Phi(g)) \, \mathrm{e}^{-i(f|g)/2} \tag{21}$$

These operators satisfy the Weyl relations

$$T(f_1, g_1) T(f_2, g_2) = T(f_1 + f_2, g_1 + g_2) \mathrm{e}^{i((f_1|g_2) - (f_2|g_1))/2} \tag{22}$$

and their expectation value in the vacuum state Ω is

$$\langle \Omega \mid T(f,g)\Omega \rangle = \exp\left(-\frac{1}{4}\|f\|^2 - \frac{1}{4}\|g\|^2\right). \tag{23}$$

With $U(t) = \exp(-iH_E t)$ the time evolution of the Weyl operators is

$$U(-t)T(f,g)U(t) = T\left(\cos(\widehat{\varepsilon} t) f + \sin(\widehat{\varepsilon} t) g, \, \cos(\widehat{\varepsilon} t) g - \sin(\widehat{\varepsilon} t) f\right). \tag{24}$$

If the one-particle Hilbert space $\mathcal{L}^2(\mathbf{R}_+)$ is restricted to the one dimensional space $\mathbf{C}$, all these formulas become formulas of the one dimensional harmonic oscillator of frequency $\widehat{\varepsilon} = \varepsilon > 0$.

The Weyl operator $T(f,0)$ is a translation operator

$$\begin{aligned} T(f,0)H_E T(-f,0) &= \int_0^\infty \varepsilon(k)\left(a^+(k)+\frac{1}{\sqrt{2}}f(k)\right)\left(a(k)+\frac{1}{\sqrt{2}}f(k)\right)dk \\ &= H_E+\Phi(\widehat{\varepsilon}f)+\frac{1}{2}\int\varepsilon(k)\left|f(k)\right|^2 dk. \end{aligned} \tag{25}$$

Hence $T(\widehat{\varepsilon}^{-1}f,0)H_E T(-\widehat{\varepsilon}^{-1}f,0)-\frac{1}{2}\left\|\widehat{\varepsilon}^{-\frac{1}{2}}f\right\|^2 = H_E+\Phi(f)$ is the Hamiltonian of the van Hove model [8], see also [4], [2] p. 166ff, and [5]. The operator $T(\widehat{\varepsilon}^{-1}f,0)$ is well defined if $\widehat{\varepsilon}^{-1}f\in\mathcal{L}^2(\mathbf{R}_+)$. The operator $\Phi(f)$, $f\in\mathcal{L}^2(\mathbf{R}_+)$, is H_E-bounded with relative bound smaller than one (in the sense of the Kato-Rellich theorem, see e.g. [16]) if $\widehat{\varepsilon}^{-\frac{1}{2}}f\in\mathcal{L}^2(\mathbf{R}_+)$, and $\Phi^2(f)$ is H_E-bounded with relative bound smaller than one, if in addition $\left\|\widehat{\varepsilon}^{-\frac{1}{2}}f\right\|<2^{-1}$ holds. Hence the operators H_E and $H_E+\lambda\Phi(f)$, $\lambda\in\mathbf{R}$, are self-adjoint on the same domain of the Fock space if $f\in\mathcal{L}^2(\mathbf{R}_+)$ and $\widehat{\varepsilon}^{-\frac{1}{2}}f\in\mathcal{L}^2(\mathbf{R}_+)$, and moreover, the operator $H_E-\frac{1}{2}\Phi^2(f)$ is bounded from below, if $\left\|\widehat{\varepsilon}^{-\frac{1}{2}}f\right\|<2^{-\frac{1}{2}}$.

The trace (11) is now calculated for the model of Sect. 4.2 with the one parameter family of Hamiltonians

$$H_\lambda := T(\lambda\widehat{\varepsilon}^{-1}f,0)H_E T(-\lambda\widehat{\varepsilon}^{-1}f,0)-\frac{\lambda^2}{2}\left\|\widehat{\varepsilon}^{-\frac{1}{2}}f\right\|^2 = H_E+\lambda\Phi(f),\ \lambda\in\mathbf{R}. \tag{26}$$

As stated above these operators are well defined if (17) holds. In the following $\simeq$ indicates an identity up to a phase factor. The unitary operators (18) can be evaluated with the help of the Weyl relations (22) and the time evolution (24)

$$\begin{aligned} U_{\alpha\beta}(t) &\simeq T(\alpha\widehat{\varepsilon}^{-1}f,0)U(-t)T((\beta-\alpha)\widehat{\varepsilon}^{-1}f,0)U(t)T(-\beta\widehat{\varepsilon}^{-1}f,0) \\ &\simeq T\left((\alpha-\beta)\widehat{\varepsilon}^{-1}(1-\cos\widehat{\varepsilon}t)\,f,\,(\alpha-\beta)(\widehat{\varepsilon}^{-1}\sin\widehat{\varepsilon}t)\,f\right). \end{aligned}$$

We only investigate the trace $\mathrm{tr}_E U_{\alpha\beta}(t)\omega$ for states ω which are mixtures of coherent states. Then the traces decompose into sums of matrix elements of (18) between coherent states $T(f_n,g_n)\Omega$ with f_n, $g_n\in\mathcal{L}^2(\mathbf{R}_+)$, $n=1,2,\ldots$. These matrix elements can be calculated with the help of (24) and the Weyl relations (22). Following (23) the modulus of a matrix element is an exponential of the type

$$\exp\left(-\frac{1}{4}\left\|a+(\alpha-\beta)(1-\cos\widehat{\varepsilon}t)\left(\widehat{\varepsilon}^{-1}f\right)\right\|^2-\frac{1}{4}\left\|b+(\alpha-\beta)\left(\sin\widehat{\varepsilon}t\right)\left(\widehat{\varepsilon}^{-1}f\right)\right\|^2\right) \tag{27}$$

where $a=f_m-f_n\in\mathcal{L}^2(\mathbf{R}_+)$ and $b=g_m-g_n\in\mathcal{L}^2(\mathbf{R}_+)$ are fixed. Under the restrictions (17) the norms of $(1-\cos\widehat{\varepsilon}t)\,(\widehat{\varepsilon}^{-1}f)$ and $(\sin\widehat{\varepsilon}t)\,(\widehat{\varepsilon}^{-1}f)$ are uniformly bounded, and (27) cannot vanish for $t\to\infty$.

But nevertheless, since $\varepsilon^{-2}(1-\cos\varepsilon t)\to\frac{1}{2}t^2$ and $\varepsilon^{-1}\sin\varepsilon t\to t$ if $\varepsilon\to 0$, the norms $\|(1-\cos\widehat{\varepsilon}t)\,(\widehat{\varepsilon}^{-1}f)\|$ and $\|(\sin\widehat{\varepsilon}t)\,(\widehat{\varepsilon}^{-1}f)\|$ may become as large as we want at intermediate times, if $f(k)$ has large contributions at small values of k. The estimate (27) for the matrix elements is then negligible for a long period of time (for arbitrary vectors a and b within some bounded domain).

Only if we give up the second constraint in (17), we can find test functions f such that the norms of $(1-\cos\widehat{\varepsilon}t)\,(\widehat{\varepsilon}^{-1}f)$ and $(\sin\widehat{\varepsilon}t)\,(\widehat{\varepsilon}^{-1}f)$ increase indefinitely for $t\to\infty$ and (27) vanishes in this limit.

That behaviour can be illustrated by the coupling to a free particle. As already mentioned we can restrict the one-particle space $\mathcal{H}^{(1)}$ to the one dimensional space $\mathbf{C}$, and the free field becomes a harmonic oscillator of frequency $\widehat{\varepsilon} = \varepsilon > 0$. In that case (27) is a periodic function of $t \in \mathbf{R}$. In the (singular) limit $\varepsilon \to 0$ we obtain functions $\varepsilon^{-2}(1 - \cos \varepsilon t) \to \frac{1}{2}t^2$ and $(\varepsilon^{-1} \sin \varepsilon t) \to t$ which increase beyond any bound for $t \to \infty$. This limit case corresponds to the Hamiltonian of a free particle

$$H_E = \frac{1}{2}P^2 \text{ with coupling } V_E = Q, \tag{28}$$

and $\mathrm{tr}_E U_{\alpha\beta}(t)\omega$ can be calculated by standard methods, see the article [14] of Pfeifer, who has used this model to discuss the measurement process of a spin. With (28) the Hamiltonian (6) of the total system is unbounded from below (corresponding to infrared divergence in the field theoretic model) and we have $\mathrm{tr}_E U_{\alpha\beta}(t)\omega \to 0$ if $t \to \infty$ for $\alpha \neq \beta$ and for all statistical operators ω of the free particle.

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