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Functional Representations for Fock Superalgebras

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Abstract

The Fock space of bosons and fermions and its underlying superalgebra are represented by algebras of functions on a superspace. We define Gaussian integration on infinite dimensional superspaces, and construct superanalogs of the classical function spaces with a reproducing kernel – including the Bargmann-Fock representation – and of the Wiener-Segal representation. The latter representation requires the investigation of Wick ordering on Z_2-graded algebras. As application we derive a Mehler formula for the Ornstein-Uhlenbeck semigroup on the Fock space.

1 Introduction

The representations which we consider in this paper are superanalogs of the Bargmann-Fock [2][37] and of the Wiener-Segal [45][15][36] representations. In the literature the former representation is also called Bargmann-Segal, or holomorphic, or complex wave representation, and the latter is denoted as Itô-Segal-Wiener or real wave representation. Moreover we include reproducing kernel spaces [1] which can be considered as an abstract version of the Bargmann-Fock representation. These representations realize the Fock space of a theory that includes bosons and fermions as some spaces of functions defined on an infinite dimensional superspace, and taking values in an auxiliary infinite dimensional Grassmann algebra Λ with a Hilbert norm [40][41]. The physical Fock space is the completion of a superalgebra with the $\mathbb{Z}_2$-graded tensor product which combines the symmetric tensor product of its bosonic subspace and the antisymmetric tensor product of its fermionic subspace. The superspaces are subspaces of Grassmann or Λ-modules generated by the one-particle subspaces of the Fock space. The inner product of the spaces corresponding to the Bargmann-Fock and to the Wiener-Segal representations can be defined by an integration with respect to a so called Gaussian supermeasure, which is defined on the whole complex Hilbert space (Bargmann-Fock representation) or on its real subspace (Wiener-Segal representation). Actually in both cases one has to use some extensions of these spaces to get σ-additive measures. The version of Gaussian supermeasure presented in this paper slightly deviates from the supermeasure which was first introduced in [40] and also used in [28].

A great part of the constructions of our paper is devoted to the methods of reproducing kernel spaces. This more abstract approach has been used for the study of stochastic processes and of the bosonic Fock space for a long time, see e.g. [18][14]; but it is also an effective tool for the investigation of the fermionic Fock space [27]. The superanalog of the Bargmann-Fock representation will therefore be developed in the more general context of reproducing kernel spaces. As in the classical case the inner product is only defined in an abstract way (without

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reference to an integral representation). But there is a great flexibility in choosing the domain of the functions, including the full analyticity domain used for the Bargmann-Fock representation and also the real subdomain used for the Wiener-Segal representation.

Already within superalgebra of the the Fock space Wick (or normal) ordering and a Wiener-Segal picture of the Fock space can be defined [23][24][26]. These algebraic constructions are the starting point for the Wiener-Segal representation by functions on superspaces. It is exactly the Wiener-Segal representation which allows to transfer formulas of usual analysis on Gaussian measure spaces to superanalysis. Our paper gives a systematic treatment of this representation, which has scarcely been used in superanalysis.

For all these constructions one needs to consider not only the original Fock space but the Λ-supermodule generated by it; just this supermodule contains coherent states defined on the superspace of the one-particle states. The representations of the Fock space by superfunctions (whose values are elements of Λ) allows to represent operators in Fock space by operators acting on the functional spaces and having a form similar to operators acting on the classical function spaces, but now the fermionic part is included.

The paper is organized as follows. In the short Sect.2 we introduce some terminology and notations of infinite dimensional superanalysis. The adopted version of superanalysis that is often called functional superanalysis, takes its origin (essentially for finite dimensional spaces) in the papers of Rogers [32][34][35], of Jadczyk and Pilch [16], of Vladimirov and Volovich [42][43] and in the book of DeWitt [9]; our approach to the infinite dimensional case is most close to the papers [40][41], see also [19]. We should mention as well the books [3][6] of Berezin whose work had a great influence on superanalysis.

The next two Sects. 3 and 4 are devoted to a description of the Fock space $\mathcal{S}(\mathcal{H})$, generated by a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, of its underlying superalgebra, and of Wick (normal) ordering within this superalgebra. These sections contain some results of the papers [23][24] and [26] in a form suited to an extension to superanalysis. In the next Sect.5 we define a Grassmann module extension of the Fock space, simply called Λ-extension, and denoted by $\mathcal{S}^\Lambda(\mathcal{H})$. This extension plays an essential role in all our further investigations. Within $\mathcal{S}^\Lambda(\mathcal{H})$ we can define a generalization of the usual (bosonic) coherent states. These states are used to represent the space $\mathcal{S}^\Lambda(\mathcal{H})$ – and hence its subspace $\mathcal{S}(\mathcal{H})$ – by Λ-valued functions on the superspace $\mathcal{H}_\Lambda$, which, in the sense of [40], is generated by the Hilbert space $\mathcal{H}$. In Sect.6 we introduce the notion of Gaussian supermeasures, which take over the role of Gaussian measures for superanalysis. Actually we describe a connection between an algebraic picture of integration based on Gaussian functionals (as done without superanalysis in Sect.4 and in [26]), and a more analytic version of integration presented in the language of infinite dimensional distributions. The latter type of integration can be considered also as an infinite dimensional version of the Berezin integration [3][9][33][43][44].

The main results of the paper are contained in Sects.7 and 8. Applying techniques, used for the fermionic Fock space in [27], we construct a reproducing kernel space for the space $\mathcal{S}^\Lambda(\mathcal{H})$, develop a superanalog of the Bargmann-Fock representation, and discuss kernels for some linear operators in the Fock space $\mathcal{S}(\mathcal{H})$ (or also in $\mathcal{S}^\Lambda(\mathcal{H})$). Moreover, with help of a generalized Wick ordering we derive a superanalog of the Wiener-Segal representation, and we prove a superanalog of the Mehler formula for the Ornstein-Uhlenbeck semigroup operating on the extended Fock space $\mathcal{S}^\Lambda(\mathcal{H})$.

The paper has several appendices. In App.A we investigate some norm estimates of tensor algebras. We prove in particular that there does not exist any Hilbert norm on a tensor algebra (including the Grassmann algebra) that satisfies the estimate $\|xy\| \leq c\,\|x\|\,\|y\|$ for all elements x, y with a constant $c = 1$, but we construct such a norm for $c = \sqrt{3}$. The further appendices include some additional properties of Grassmann algebras and of coherent states. In App.D we indicate a method of fermionic integration that uses an ordering prescription on the basic Hilbert space without any Λ-extension.

A remark about supersymmetry should be added. Superanalysis as presented in this paper deals with a theory which combines bosons and fermions, but it is not necessarily related to a supersymmetric theory. Superanalysis just provides an appropriate kinematical basis for such a theory. Supersymmetry is an additional constraint for functionals or operators. We give an example of such a constraint at the end of Sect.4.

We would like to mention that the paper is self-contained, and the reader need not to have any prior knowledge of superanalysis.

2 Terminology and notations

A vector space $\mathcal{E}$ over the complex or the real field is called $\mathbb{Z}_2$-graded if it is a direct sum of two its vector subspaces, which are called – and their elements also – even and odd, $\mathcal{E} = \mathcal{E}_{\bar{0}} \oplus \mathcal{E}_{\bar{1}}$. In this case one denotes by π the function $\pi : x \in (\mathcal{E}_{\bar{0}} \cup \mathcal{E}_{\bar{1}})\backslash\{0\} \to \{0,1\}$, which is the indicator of $\Lambda_1\backslash\{0\}$, i.e. $\pi(x) = k$ if $x \in \mathcal{E}_{\bar{k}}\backslash\{0\}, k = 0,1$, and the value $\pi(x)$ is called the parity of x. A superalgebra is a $\mathbb{Z}_2$-graded vector space $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$ equipped with an associative multiplication having the following properties: if $a, b \in \mathcal{E}_{\bar{0}} \cup \mathcal{E}_{\bar{1}}$ with $ab \neq 0$, then $\pi(ab) = |\pi(a) - \pi(b)|$. A superalgebra is called (super)commutative if $ab = (-1)^{\pi(a)\pi(b)} ba$ for any $a, b \in (\Lambda_{\bar{0}} \cup \Lambda_{\bar{1}})\backslash\{0\}$. We always assume that any superalgebra has a unit that we denote by e_0 (or also by κ_0 if Greek letters are used for the elements of the superalgebra). A $\mathbb{Z}_2$-graded locally convex space (LCS), respectively normed, Banach, Hilbert etc. space is a $\mathbb{Z}_2$-graded vector space $\mathcal{E}_{\bar{0}} \oplus \mathcal{E}_{\bar{1}}$ equipped with a structure of a LCS, respectively, of normed, Banach, Hilbert etc. space, such that $\mathcal{E}$ is a topological sum (Hilbert sum in the Hilbert space case) of its topological vector subspaces (Banach, Hilbert etc. subspaces) $\mathcal{E}_{\bar{0}}$ and $\mathcal{E}_{\bar{1}}$. In this case one also says that the structure of a LCS (respectively, normed, Hilbert space) is compatible with the structure of the $\mathbb{Z}_2$-graded vector space.

A locally convex superalgebra is a superalgebra $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$, equipped with a structure of a LCS that is compatible with the structure of the $\mathbb{Z}_2$ graded vector space Λ and is such that there exists a family $\mathcal{P}$ of seminorms on Λ defining the topology of Λ, the multiplication $\Lambda \times \Lambda \to \Lambda$, $(ab) \to ab$ being continuous with respect to each $p \in \mathcal{P}$. A superalgebra with a Hilbert structure is a locally convex superalgebra Λ whose topology can be defined by a Hilbert norm $p(a) = \|a\|$ (such that $p(e_0) = 1$). If this superalgebra is complete it is called a Hilbert superalgebra. As a general rule such a Hilbert superalgebra is not a Banach algebra in the usual sense, because a norm estimate $\| ab \| \leq \| a \| \| b \|$ for all $a, b \in \Lambda$ is not admitted, see App.A. If not specified otherwise, we assume that the annihilator $\Lambda_{\bar{1}}^{\perp} = \{x \in \Lambda;\ \forall z \in \Lambda_{\bar{1}}, xz = 0\}$ of the odd part of any superalgebra Λ is equal to zero, and hence this superalgebra is infinite dimensional with $\dim \Lambda_{\bar{1}} = \infty$.[3]

A Grassmann algebra is an algebra Λ with a unit e_0 that contains a non-empty finite or countable set of elements $\{e_j \mid j \in \mathbf{B} \subset \mathbb{N}\}$, which are called generators, having the following properties: the set $\{e_0\} \cup \{e_j\}$ is linearly independent; $\forall i, j \in \mathbb{N}, e_i, e_j = -e_j e_i$ and $e_i e_j \neq 0$ if $i \neq j$, the minimal subalgebra of Λ containing the set $\{e_0\} \cup \{e_j\}$ coincides with Λ. Each Grassmann algebra can be equipped with the following structure of superalgebra. Let $\Lambda_{\bar{1}}$ be a vector subspace of Λ generated by all products $e_{j_1} \ldots e_{j_n}, j_n \in \mathbf{B}$ where the number n of multipliers is odd and let $\Lambda_{\bar{0}}$ be a vector subspace of Λ generated by e_0 and by all products $e_{j_1} \ldots e_{j_n}$, where the

[3] If $\dim \Lambda < \infty$ let $\{e_j\}$ be a basis of the vector space $\Lambda_{\bar{1}}$ and let n be the maximal natural number for which there exist n elements, $e_{j_1}, \ldots, e_{j_n}$ of the basis, such that $e_{j_1} \cdots e_{j_n} \neq 0$. Then for any element e_k of the basis $e_k e_{j_1} \cdots e_{j_n} = 0$ and hence $e_{j_1} \cdots e_{j_n} \in \Lambda_{\bar{1}}^{\perp}$.

number of multipliers is even. Then the decomposition $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$ together with the function $\pi : (\Lambda_{\bar{0}} \cup \Lambda_{\bar{1}}) \backslash \{0\} \to \{0, 1\}$ which is the indicator of $\Lambda_{\bar{1}} \backslash \{0\}$, defines the structure of a superalgebra. In App.A we prove that any Grassmann algebra can be equipped with a Hilbert structure, and in the following we always assume that a Grassmann algebra has such a structure. (This structure does not depend on the choice of the set of generators.) The completion of an infinite dimensional Grassmann algebra with a Hilbert structure with respect to the corresponding Hilbert norm is called a Hilbert-Grassmann algebra; hence a Hilbert-Grassmann algebra is a Hilbert superalgebra with vanishing annihilator of the odd part. Everywhere below we take as superalgebra $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$ a Hilbert-Grassmann algebra over the complex numbers. But many definitions and statements are valid for any infinite dimensional superalgebra.

If $\mathcal{E}$ and $\mathcal{G}$ are vector spaces then the symbol $\mathcal{E} \otimes \mathcal{G}$ denotes their algebraic tensor product. If $\mathcal{E}$ and $\mathcal{G}$ are equipped with Hilbert norms then $\mathcal{E} \otimes \mathcal{G}$ is assumed to be equipped with the (uniquely defined) corresponding Hilbert cross-norm, and by $\mathcal{E} \widehat{\otimes} \mathcal{G}$ one denotes the completion of $\mathcal{E} \otimes \mathcal{G}$. For any vector space $\mathcal{E}$ the vector space $\Lambda \otimes \mathcal{E}$ can be provided with different structures of a Λ-module; we will specify them later. If $\mathcal{E}$ is a $\mathbb{Z}_2$-graded vector space, then $\Lambda \otimes \mathcal{E}$ is also equipped with the natural structures of $\mathbb{Z}_2$-graded Λ module; in particular, the $\mathbb{Z}_2$-graded vector space $\Lambda \otimes \mathcal{E}$ with the structure of a Λ-module will be denoted by $\mathcal{E}^{\Lambda}$ and it will be called Λ-supermodule over $\mathcal{E}$. The even $\Lambda_{\bar{0}}$–submodule $\Lambda_{\bar{0}} \otimes \mathcal{E}_{\bar{0}} \oplus \Lambda_{\bar{1}} \otimes \mathcal{E}_{\bar{1}}$ of $\mathcal{E}^{\Lambda}$ is called the Λ-*superspace* over $\mathcal{E}$ and is denoted by $\mathcal{E}_{\Lambda}$. In the usual superanalysis of finite dimensional spaces, $\dim \mathcal{E}_{\bar{0}} = m, \dim \mathcal{E}_{\bar{1}} = n$, the space $\mathcal{E}_{\Lambda}$ can be represented in another way. Choosing a basis $\{e_a\}_{a=1,\ldots,m}$ of $\mathcal{E}_{\bar{0}}$ and a basis $\{f_b\}_{b=1,\ldots,n}$ of $\mathcal{E}_{\bar{1}}$ the variable $\zeta = \zeta_{\bar{0}} + \zeta_{\bar{1}} \in \Lambda_{\bar{0}} \otimes \mathcal{E}_{\bar{0}} \oplus \Lambda_{\bar{1}} \otimes \mathcal{E}_{\bar{1}}$ can be decomposed as $\zeta = \sum_{a=1}^{m} \lambda_a \otimes e_a + \sum_{b=1}^{n} \mu_b \otimes f_b$ with $\lambda_a \in \Lambda_{\bar{0}}$ and $\mu_b \in \Lambda_{\bar{1}}$, and ζ is mapped onto an element $(\lambda_1, ..., \lambda_m; \mu_1, ..., \mu_n) \in \Lambda^{m,n} = (\Lambda_{\bar{0}})^m \times (\Lambda_{\bar{1}})^n$ The space $\Lambda^{m,n} \cong \Lambda_{\bar{0}} \otimes \mathbb{C}^m \oplus \Lambda_{\bar{1}} \otimes \mathbb{C}^n$ is then denoted as superspace.[32][16][9][42]

If $\mathcal{E}$ is equipped with a Hilbert norm then $\mathcal{E}_{\Lambda}$ and $\mathcal{E}^{\Lambda}$ are assumed to be equipped with corresponding Hilbert norms. Moreover, if $\mathcal{E}$ is a Hilbert space, we always assume that the spaces $\mathcal{E}_{\Lambda}$ and $\mathcal{E}^{\Lambda}$ are completed.

If $\mathcal{E}$ and $\mathcal{G}$ are normed spaces then by $\mathcal{L}(\mathcal{E}, \mathcal{G})$ one denotes the space of all continuous mappings of $\mathcal{E}$ into $\mathcal{G}$. If $\mathcal{E}$ and $\mathcal{G}$ are Hilbert spaces, by $\mathcal{L}(\mathcal{E}_{\Lambda}, \mathcal{G}^{\Lambda})$ one denotes the topological Λ-module space of $\Lambda_{\bar{0}}$-linear mappings of $\mathcal{E}_{\Lambda}$ into $\mathcal{G}^{\Lambda}$ equipped with the topology of bounded convergence. A mapping $f : \mathcal{E}_{\Lambda} \to \mathcal{G}^{\Lambda}$ is called (Fréchet) differentiable at $x_0 \in \mathcal{E}_{\Lambda}$ if there exists an element of $\mathcal{L}(\mathcal{E}_{\Lambda}, \mathcal{G}^{\Lambda})$, called the (Fréchet) derivative of f at x_0 and denoted by $f'(x_0)$ such that for any bounded sequence $\eta_n \in \mathcal{E}_{\Lambda}$ the limit $t^{-1} (f(x_0 + t_n\eta_n) - f(x_0)) - f'(x_0)\eta_n \to 0$ if $t_n \to 0$ with $t_n \in R^1, t_n \neq 0$ exists. The derivatives of higher order $f^{(n)}(x), n \in \mathbb{N}$, are defined by induction, such that $f^{(n)}(x) \in \mathcal{L}_n(\mathcal{E}_{\Lambda}, \mathcal{G}^{\Lambda})$ where $\mathcal{L}_n(\mathcal{E}_{\Lambda}, \mathcal{G}^{\Lambda})$ is the vector space of all n-fold $\Lambda_{\bar{0}}$–linear continuous mappings of $\mathcal{E}_{\Lambda} \times \ldots \times \mathcal{E}_{\Lambda}$ into G^{Λ} equipped with the topology of bounded convergence.

3 Fock Spaces and Superalgebras

3.1 $\mathbb{Z}_2$-graded Hilbert spaces

Let $\mathcal{E}$ be a complex separable Hilbert space and $\mathcal{E}^*$ its dual space. Then the direct sum $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}^*$ is again a Hilbert space, the inner product denoted by (f, g), with the structure given by

1. An antiunitary involution $f \in \mathcal{F} \to f^* \in \mathcal{F}$, $f^{**} = f$, which maps $\mathcal{E}$ into $\mathcal{E}^*$ and vice versa.

2. The bilinear symmetric form $\omega_+(f, g) = \langle f|g \rangle$ on $\mathcal{F} \times \mathcal{F}$ that is defined by $\omega_+(f, g) \equiv \langle f|g \rangle := (f^*, g)$. The spaces $\mathcal{E}$ and $\mathcal{E}^*$ are isotropic subspaces of this form.

3. The canonical antisymmetric bilinear form on $\mathcal{F}\times\mathcal{F}:\omega_{-}(f,g)\equiv\langle f|j_{-}g\rangle$,where j_{-} is the mapping of $\mathcal{F}$ into $\mathcal{F}$ defined by $j_{-}(u+v):=u-v$ if $u\in\mathcal{E}$ and $v\in\mathcal{E}^{*}$, i.e. j_{-} interchanges the relative phases between $\mathcal{E}$ and $\mathcal{E}^{*}$.

To have shorter notations we define by j_{+} the identical transformation of $\mathcal{F}(\ j_{+}f=f,f\in\mathcal{F})$ then $\omega_{\pm}(f,g):=\langle f|j_{\pm}g\rangle$. The diagonal subspace

$$\mathcal{F}^{D}:=\{f|f=f^{*},f\in\mathcal{F}\}. \tag{1}$$

is a real Hilbert space, which is isomorphic to $\mathcal{E}_{\mathbb{R}}$, the underlying real space of the complex space $\mathcal{E},\quad u\in\mathcal{E}_{\mathbb{R}}\leftrightarrow f=\frac{1}{\sqrt{2}}(u+u^{*})\in\mathcal{F}^{D}$. The space $\mathcal{F}$ is the complexification of both the spaces $\mathcal{F}^{D}$ and $\mathcal{E}_{\mathbb{R}}$. If $u,v\in\mathcal{E}_{\mathbb{R}}$ and $f=\frac{1}{\sqrt{2}}(u+u^{*}),g=\frac{1}{\sqrt{2}}(v+v^{*})\in\mathcal{F}^{D}$, then $\omega_{+}(f,g)=\langle f\mid g\rangle=\mathrm{Re}\,(u,v)$ and $\omega_{-}(f,g)=i\mathrm{Im}\,(u,v)$.

In the following we denote a Hilbert space $\mathcal{F}$ provided with the symmetric bilinear form $\omega_{+}(f,g)$ as $\mathcal{H}_{\overline{0}}$, and a Hilbert space $\mathcal{F}$ provided with the antisymmetric form $\omega_{-}(f,g)$ as $\mathcal{H}_{\overline{1}}$. The spaces $\mathcal{H}_{\overline{0}}$ and $\mathcal{H}_{\overline{1}}$ can be interpreted as spaces of bosons and of fermions. We do not assume an isomorphism between the spaces $\mathcal{H}_{\overline{0}}$ and $\mathcal{H}_{\overline{1}}$; but see the end of Sect.4. The direct sum $\mathcal{H}=\mathcal{H}_{\overline{0}}\oplus\mathcal{H}_{\overline{1}}$ of these spaces is of crucial importance in the following. With the parity $\pi(f)=k$ if $f\in\mathcal{H}_{\overline{k}}\backslash\{0\}\,,k=0,1$, the space $\mathcal{H}=\mathcal{H}_{\overline{0}}\oplus\mathcal{H}_{\overline{1}}$ is a $\mathbb{Z}_{2}$-graded Hilbert space. The corresponding isotropic subspaces will be denoted as $\mathcal{E}_{\overline{k}}$ or $\mathcal{E}_{\overline{k}}^{*}$, $k=0,1$, and the involution $f\rightarrow f^{*}$ maps $\mathcal{E}_{\overline{k}}$ into $\mathcal{E}_{\overline{k}}^{*}$. One can interpret the spaces $\mathcal{E}_{\overline{k}}$ and $\mathcal{E}_{\overline{k}}^{*},k=0,1$, as the spaces of particles and of antiparticles. The space $\mathcal{H}$ is then provided with a bilinear form ω, defined by

$$\omega(f,g)=\omega_{+}(f_{0},g_{0})+\omega_{-}(f_{1},g_{1}) \tag{2}$$

for $f=f_{0}+f_{1}$ and $g=g_{0}+g_{1}$, with $f_{k},g_{k}\in\mathcal{H}_{\overline{k}},k=0,1$. This form has the symmetry property $\omega(f,g)=(-1)^{\pi(f)\pi(g)}\omega(g,f)$ for all $f,g\in\mathcal{H}_{\overline{0}}\cup\mathcal{H}_{\overline{1}}\setminus\{0\}$. On the other hand we define the bilinear symmetric form

$$\langle f\mid g\rangle=(f^{*}\mid g) \tag{3}$$

and the forms (2) and (3) are related by $\omega(f,g)=\langle f\mid jg\rangle$ where j is defined by

$$j(u+b)=u-v\ \text{ if }\ u\in\mathcal{H}_{\overline{0}}\oplus\mathcal{E}_{\overline{1}}\ \text{ and }\ v\in\mathcal{E}_{\overline{1}}^{*}. \tag{4}$$

3.2 Tensor algebras and superalgebra

The incomplete (algebraic) tensor product of n copies of a Hilbert space $\mathcal{H}$ is denoted as $\mathcal{H}^{\otimes n}$. The algebraic sum of these spaces is the tensor algebra $\mathcal{T}_{fin}(\mathcal{H})=\underset{n\geq0}{\oplus}\mathcal{H}^{\otimes n}$. We provide the spaces $\mathcal{H}^{\otimes n}$ with the standard Hilbert norm $\parallel f_{1}\otimes...\otimes f_{n}\parallel_{n}^{2}:=\prod_{a=1}^{n}\parallel f_{a}\parallel^{2}$. The completed tensor spaces $\mathcal{H}^{\otimes n}$are denoted by $\mathcal{T}_{n}(\mathcal{H})$. The space $\mathcal{T}_{fin}(\mathcal{H})$ is equipped with the Hilbert space norm

$$\parallel F\parallel^{2}=\sum_{n=0}^{\infty}\parallel F_{n}\parallel_{n}^{2}\quad\text{if}\ \ F=\sum_{n=0}^{\infty}F_{n}\ \ \text{with}\ \ F_{n}\in\mathcal{H}^{\otimes n}. \tag{5}$$

Let $\mathcal{H}=\mathcal{H}_{\overline{0}}\oplus\mathcal{H}_{\overline{1}}$ be the graded Hilbert space of the preceding section. The subspace $\mathcal{H}_{\overline{0}}$ $(\mathcal{H}_{\overline{1}})$ carries the bosonic (fermionic) degrees of freedom. We define a bounded linear operator $\mathbf{P}^{(n)}$ on the set of decomposable tensors $f_{1}\otimes...\otimes f_{n}$ of vectors f_{a} with a defined parity, $f_{a}\in\mathcal{H}_{\overline{0}}\cup\mathcal{H}_{\overline{1}}$, by

$$\mathbf{P}^{(n)}f_{1}\otimes...\otimes f_{n}=\frac{1}{n!}\sum_{\sigma}\chi_{\sigma}(f_{1},...,f_{n})f_{\sigma(1)}\otimes...\otimes f_{\sigma(n)} \tag{6}$$

with a sign function $\chi_\sigma(f_1,...,f_n) = (-1)^{N_\sigma}$ where
$N_\sigma = \#\left\{(a_i,a_j) \mid \pi(f_{a_i}) = \pi(f_{a_j}) = 1, \text{ with } i<j \text{ and } a_i>a_j\right\}$ counts the inversions of the fermionic arguments. The sum extends over all permutations σ of the numbers $\{1,...,n\}$. This mapping has a unique extension to a projection operator on the space $\mathcal{T}_n(\mathcal{H})$. The image of the space $\mathcal{H}^{\otimes n}$ is denoted as $\mathcal{H}^{\odot n} = \mathbf{P}^{(n)}\mathcal{H}^{\otimes n}$. The algebraic (finite) sum of the spaces $\mathcal{H}^{\odot n}$

$$\mathcal{S}_{fin}(\mathcal{H}) := \bigoplus_n \mathcal{H}^{\odot n}. \tag{7}$$

is a commutative superalgebra with the graded product $F \in \mathcal{H}^{\odot p}, G \in \mathcal{H}^{\odot q} \to F \odot G \in \mathcal{H}^{\odot(p+q)}$

$$F \odot G := \sqrt{\frac{(p+q)!}{p!q!}}\mathbf{P}^{(p+q)}F \otimes G \tag{8}$$

This product reduces to the symmetric tensor product if only tensors of the space $\mathcal{H}_{\overline{0}}$ are used, and it coincides with the antisymmetric tensor product, if all factors are tensors of the space $\mathcal{H}_{\overline{1}}$.[4] The definitions (5) and (8) yield the usual norm for the symmetric/antisymmetric tensor product $\| f_1 \odot ... \odot f_n \|_n^2 = \| f_1 \vee ... \vee f_n \|_n^2 = \text{per}\langle f_a \mid f_b \rangle$ if $f_a \in \mathcal{H}_{\overline{0}}$, $a=1,...,n$, and $\| f_1 \odot ... \odot f_n \|_n^2 = \| f_1 \wedge ... \wedge f_n \|_n^2 = \det\langle f_a \mid f_b \rangle$ if $f_a \in \mathcal{H}_{\overline{1}}$, $a=1,...,n$. If $\mathcal{G} \subset \mathcal{H}$ is a subset of $\mathcal{H}$ we use the notation $\mathcal{S}_{fin}(\mathcal{G})$ for the subalgebra generated by vectors from $\mathcal{G}$ with the product (8). Hence $\mathcal{S}_{fin}(\mathcal{H}_{\overline{0}})$ is the symmetric tensor algebra of the Hilbert space $\mathcal{H}_{\overline{0}}$, and $\mathcal{S}_{fin}(\mathcal{H}_{\overline{1}})$ is the antisymmetric tensor algebra of the Hilbert space $\mathcal{H}_{\overline{1}}$. These symmetric/antisymmetric tensor algebras will also be denoted as $\mathcal{T}^+_{fin}(\mathcal{H}_{\overline{0}})$ or $\mathcal{T}^-_{fin}(\mathcal{H}_{\overline{1}})$, respectively.

With the product (8) the space $\mathcal{S}_{fin}(\mathcal{H}) = \mathcal{S}_{fin}(\mathcal{H}_{\overline{0}}) \odot \mathcal{S}_{fin}(\mathcal{H}_{\overline{1}})$ is a $\mathbb{Z}_2$-graded algebra with even subspace $\mathcal{S}_{fin}(\mathcal{H}_{\overline{0}}) \odot \left(\oplus_{q\geq 0,even}\mathcal{H}_{\overline{1}}^{\odot q}\right)$ and with the odd subspace $\mathcal{S}_{fin}(\mathcal{H}_{\overline{0}}) \odot \left(\oplus_{q\geq 0,odd}\mathcal{H}_{\overline{1}}^{\odot q}\right)$. The completion of $\mathcal{H}^{\odot n}$ with the norm $\|.\|_n$ is denoted as $\mathcal{S}_n(\mathcal{H})$. The completion of $\mathcal{S}_{fin}(\mathcal{H})$ with the norm (5) $\|\cdot\|$ is the Fock space

$$\mathcal{S}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n(\mathcal{H}). \tag{9}$$

The product of two homogeneous elements $F \in \mathcal{H}^{\odot p}$ and $G \in \mathcal{H}^{\odot q}$ is continuous and can be extended to the completed spaces $F \in \mathcal{S}_p(\mathcal{H}), G \in \mathcal{S}_q(\mathcal{H}) \to F \odot G \in \mathcal{S}_{p+q}(\mathcal{H})$ with the norm estimate, see (5) and (8),

$$\|F \odot G\|^2 \leq \frac{(p+q)!}{p!q!} \|F\|^2 \|G\|^2 . \tag{10}$$

In agreement with the notations given above we shall also write $\mathcal{S}(\mathcal{H}_{\overline{0}})$ for the closed Fock space of symmetric tensors of the bosonic space $\mathcal{H}_{\overline{0}}$ and $\mathcal{S}(\mathcal{H}_{\overline{1}})$ for the closed Fock space of antisymmetric tensors of the fermionic space $\mathcal{H}_{\overline{1}}$.

The superalgebra $\mathcal{S}_{fin}(\mathcal{H})$ is isomorphic to the tensor product

$$\mathcal{S}_{fin}(\mathcal{H}) \cong \mathcal{S}_{fin}(\mathcal{H}_{\overline{0}}) \otimes \mathcal{S}_{fin}(\mathcal{H}_{\overline{1}}) = \mathcal{T}^+_{fin}(\mathcal{H}_{\overline{0}}) \otimes \mathcal{T}^-_{fin}(\mathcal{H}_{\overline{1}}) \tag{11}$$

of the symmetric tensor algebra $\mathcal{T}^+_{fin}(\mathcal{H}_{\overline{0}})$ and of the antisymmetric tensor algebra $\mathcal{T}^-_{fin}(\mathcal{H}_{\overline{1}})$. In this representation the product (8) of factorizing tensors $F = F_0 \otimes F_1, G = G_0 \otimes G_1$ with $F_0, G_0 \in \mathcal{S}_{fin}(\mathcal{H}_{\overline{0}})$ and $F_1, G_1 \in \mathcal{S}_{fin}(\mathcal{H}_{\overline{1}})$ is given by $F \odot G = (F_0 \vee G_0) \otimes (F_1 \wedge G_1)$.

The involution $f \to f^*$ on the space $\mathcal{H}$ can be uniquely extended to an isometric involution $F \to F^*$ on $\mathcal{S}(\mathcal{H})$, which satisfies $(F \odot G)^* = G^* \odot F^*$ for all $F, G \in \mathcal{S}(\mathcal{H})$ for which the graded tensor product exists. This involution induces a bilinear pairing on $\mathcal{S}(\mathcal{H})$

$$\langle F \mid G \rangle = (F^* \mid G) . \tag{12}$$

[4] We also use the notation $F \vee G$ for the symmetric tensor product, and $F \wedge G$ for the antisymmetric tensor product.

This duality has been called *twisted duality* in [24] because the order of the left argument is inverted

$$\langle f_m \odot ... \odot f_1 \mid x_1 \odot ... \odot x_n \rangle = \delta_{mn} \sum \chi_\sigma(x_1, ..., x_n) \left\langle f_1 \mid x_{\sigma(1)} \right\rangle \cdots \left\langle f_n \mid x_{\sigma(n)} \right\rangle \tag{13}$$

The normalizations given in (5) and (8) lead to the following important formula. Let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{H}$ be orthogonal subspaces, $\mathcal{G}_1 \perp \mathcal{G}_2$, then

$$\langle G \odot F \mid X \odot Y \rangle = \langle F \mid X \rangle \langle G \mid Y \rangle \tag{14}$$

holds if $F \in \mathcal{S}(\mathcal{G}_1)$, $X \in \mathcal{S}(\mathcal{G}_1^*)$ and $G \in \mathcal{S}(\mathcal{G}_2), Y \in \mathcal{S}(\mathcal{G}_2^*)$.

The *interior product* or *contraction* of a tensor $F \in \mathcal{S}_{fin}(\mathcal{H})$ with a tensor $Y \in \mathcal{S}(\mathcal{H})$ is the unique element $Y \lrcorner F$ of $\mathcal{S}_{fin}(\mathcal{H})$ for which the identity

$$\langle X \mid Y \lrcorner F \rangle = \langle X \odot Y \mid F \rangle \tag{15}$$

is valid for all $X \in \mathcal{S}_{fin}(\mathcal{H})$. The identity (15) is equivalent to $(X \mid Y \lrcorner F) = (Y^* \odot X^* \mid F)$.

The norm (5) can be generalized to

$$\| F \|_{(\gamma)}^2 = \sum_{n=0}^{\infty} (n!)^\gamma \| F_n \|_n^2 \quad \text{if } F = \sum_{n=0}^{\infty} F_n \quad \text{with } F_n \in \mathcal{H}^{\otimes n} \tag{16}$$

with a parameter $\gamma \in \mathbb{R}$. The completion of $\mathcal{S}_{fin}(\mathcal{H})$ with the norm $\| \, . \, \|_{(\gamma)}$ is denoted by $\mathcal{S}_{(\gamma)}(\mathcal{H})$. These spaces satisfy the obvious inclusions $\mathcal{S}_{(\alpha)}(\mathcal{H}) \subset \mathcal{S}_{(\beta)}(\mathcal{H})$ if $\alpha \geq \beta$, and the Fock space $\mathcal{S}(\mathcal{H})$ is $\mathcal{S}_{(0)}(\mathcal{H})$. The tensor product (8) and the contraction (15) have no continuous extension to the Fock space $\mathcal{S}(\mathcal{H})$, but in App.A we derive the following statements.

Lemma 1 *If $F, G \in \mathcal{S}_{(\gamma)}(\mathcal{H}), \gamma > 0$, then $F \odot G \in \mathcal{S}(\mathcal{H})$ with $\|F \odot G\| \leq c_\gamma \cdot \|F\|_{(\gamma)} \|G\|_{(\gamma)}$. If $Y \in \mathcal{S}(\mathcal{H})$ and $F \in \mathcal{S}_{(\gamma)}(\mathcal{H}), \gamma > 0$, then $Y \lrcorner F \in \mathcal{S}(\mathcal{H})$ with $\|Y \lrcorner F\| \leq c_\gamma \cdot \|Y\| \|F\|_{(\gamma)}$.*

Finally we remind a standard notation for operators on Fock spaces. If T is a (bounded) linear operator on $\mathcal{H}$, then its so called second quantization $\Gamma(T)$ is a linear (bounded) operator on $\mathcal{S}_{(\gamma)}(\mathcal{H})$, defined by $\Gamma(T)e_0 = e_0$ (unit = vacuum state), and $\Gamma(T)\,(f_1 \odot ... \odot f_n) = (Tf_1) \odot ... \odot (Tf_n)$ for $f_a \in \mathcal{H}, a = 1, ..., n$. The linear operator $d\Gamma(T)$ is defined as the derivation of the algebra $\mathcal{S}_{fin}(\mathcal{H})$ generated by the operator T, i.e. $d\Gamma(T)e_0 = 0, d\Gamma(T)f = Tf$ if $f \in \mathcal{H}$, and $d\Gamma(T)\,(F \odot G) = (d\Gamma(T)F) \odot G + F \odot (d\Gamma(T)G)$ for $F, G \in \mathcal{S}_{fin}(\mathcal{H})$.

4 Gaussian functionals and Wick ordering

In this section we present an algebraic formulation of the integration with respect to a Gaussian measure. These algebraic methods, developed in [24] and [26], have the advantage that bosonic and fermionic integration can be treated with the same methods, such that the generalization to the superalgebra $\mathcal{S}_{fin}(\mathcal{H})$ is straightforward. Wick ordering and the Wiener-Segal representation, well known for the bosonic Fock space, have within this framework a transparent generalization for the Fock space $\mathcal{S}(\mathcal{H})$.

The starting point of a Gaussian integration is the bilinear form (2). This form is continuous on $\mathcal{H} \times \mathcal{H}$, and it can be written as

$$\omega(f, g) = \langle \Omega \mid f \odot g \rangle \tag{17}$$

with a tensor $\Omega = \Omega^*$,which is an element of $(\mathcal{H} \odot \mathcal{H})'$, the algebraic dual of $\mathcal{H} \odot \mathcal{H}$. We shall give another characterization of Ω bellow. The tensor Ω separates into a bosonic and a fermionic part $\Omega = \Omega_0 + \Omega_1$ with $\Omega_k \in (\mathcal{E}_{\overline{k}} \odot \mathcal{E}^*_{\overline{k}})' \subset (\mathcal{H} \odot \mathcal{H})', k = 0, 1$. The exponential $\exp\Omega = I + \Omega + \frac{1}{2!}\Omega \odot \Omega + ...$ yields the Gaussian combinatorics on tensors $f_1 \odot f_2 \odot ... \odot f_{2n}$ with $f_a \in \mathcal{H}_{\overline{0}} \cup \mathcal{H}_{\overline{1}}, a = 1, ..., 2n$,

$$\langle \exp\Omega | f_1 \odot f_2 \odot ... \odot f_{2n} \rangle = \frac{1}{2^n n!} \sum_\sigma \chi_\sigma (f_1, ..., f_{2n}) \omega(f_{\sigma(1)}, f_{\sigma(2)}) ... \omega(f_{\sigma(2n-1)}, f_{\sigma(2n)}) \quad (18)$$

where the sum extends over all permutations σ of the numbers $\{1, ..., 2n\}$ and the sign function χ_σ has been defined for (6). If all $f_a \in \mathcal{H}_{\overline{0}}$, $a = 1, ..., 2n$, the sign function is $\chi_\sigma = 1$, and the sum (18) agrees with the hafnian $\mathrm{haf}(\omega(f_a, f_b))$, and if $f_a \in \mathcal{H}_{\overline{1}}$, $a = 1, ..., 2n$, the sum (18) agrees with the pfaffian $\mathrm{pf}(\omega(f_a, f_b))$, see e.g.[7]. The functional

$$L(F) := \langle \exp\Omega | F \rangle \quad (19)$$

is then linearly extended to $\mathcal{S}_{fin}(\mathcal{H})$. Moreover, if $F, G \in \mathcal{S}_{fin}(\mathcal{E})$ a simple calculation leads to the identity

$$L(F_1^* \odot F_2) = (F_1 | F_2). \quad (20)$$

The functional (19) is therefore also defined on products $F_1^* \odot F_2$ with $F_{1,2} \in \mathcal{S}(\mathcal{E})$, but it cannot be extended to the full Fock space $\mathcal{S}(\mathcal{H})$.

The interior product (15) can be used to define the linear operator $W : \mathcal{S}_{fin}(\mathcal{H}) \to \mathcal{S}_{fin}(\mathcal{H})$

$$WF := \exp(-\Omega) \rfloor F \quad (21)$$

This mapping is denoted as *Wick ordering* or *normal ordering* of the algebra $\mathcal{S}_{fin}(\mathcal{H})$. It satisfies the reality condition $WF^* = (WF)^*$. Since $\langle \exp\Omega \odot F | \exp(-\Omega) \rfloor G \rangle = \langle F | G \rangle$ for all $F, G \in \mathcal{S}_{fin}(\mathcal{H})$, the inverse operator of W is $W^{-1}F = \exp\Omega \rfloor F$. The operators W and W^{-1} cannot be extended to continuous mapping on the Fock space $\mathcal{S}(\mathcal{H})$. If $\mathcal{S}_{fin}(\mathcal{H})$ is restricted to $\mathcal{S}_{fin}(\mathcal{H}_{\overline{0}}) = \mathcal{T}^+_{fin}(\mathcal{H}_{\overline{0}})$ we obtain the bosonic Wick ordering $W_0 F := \exp(-\Omega_0) \rfloor F$, and if $\mathcal{S}_{fin}(\mathcal{H})$ is restricted to $\mathcal{S}_{fin}(\mathcal{H}_{\overline{1}}) = \mathcal{T}^-_{fin}(\mathcal{H}_{\overline{1}})$ we obtain the fermionic Wick ordering $W_1 F := \exp(-\Omega_1) \rfloor F$ in agreement with the use of Wick ordering in quantum field theory, see [26].

The fundamental algebraic identity to derive an $\mathcal{L}^2$-picture (Wiener-Segal picture) for superalgebras is [23][24][26]

$$\langle \exp\Omega | F^* \odot G \rangle = (W^{-1}F | J W^{-1} G) \quad (22)$$

with $F, G \in \mathcal{S}_{fin}(\mathcal{H})$, where the linear operator J is defined on $\mathcal{S}_{fin}(\mathcal{H})$ by

$$JF = (-1)^q F \;\text{ if }\; F \in \mathcal{S}_{fin}(\mathcal{H}_{\overline{0}}) \odot \mathcal{E}_{\overline{1}}^{\odot p} \odot (\mathcal{E}^*_{\overline{1}})^{\odot q}, \quad (23)$$

i.e. $J = \Gamma(j)$ with j given in (4). Since (22) is the basis of the Wiener-Segal representation in Sect.8, we give a proof of (22) in App. C. If we restrict the tensors F and G to the symmetric tensor algebra $\mathcal{S}_{fin}(\mathcal{H}_{\overline{0}})$ then $J_+ = id$, and the sesquilinear form

$$\langle \exp\Omega_0 | F^* \vee G \rangle = (W_0^{-1} F | W_0^{-1} G) \quad (24)$$

is positive definite and can be diagonalized by Wick ordering $\langle \exp\Omega_0 | W_0 F^* \vee W_0 G \rangle = (F|G)$ as well known for Gaussian integrals. If we choose tensors F and G of the exterior algebra $\mathcal{S}_{fin}(\mathcal{H}_{\overline{1}})$ the sesquilinear form $\langle \exp\Omega_1 | F^* \wedge G \rangle = (W^{-1}F | J W^{-1} G)$ is not positive. A positive sesquilinear form on $\mathcal{S}_{fin}(\mathcal{H})$ can be derived with the help of the antilinear invertible mapping [24][26]

$$F \to F^\dagger := (WJW^{-1}F)^* = WJ_* W^{-1} F^*, \quad (25)$$

where J_* is defined by $J_* F := (-1)^p F = (-1)^{p+q} JF$ for $F \in \mathcal{S}_{fin}(\mathcal{H}_{\overline{0}}) \odot \mathcal{E}_{\overline{0}}^{\odot p} \odot (\mathcal{E}^*_{\overline{1}})^{\odot q}$. The mapping (25) reduces to $F \to F^*$ if $F \in \mathcal{S}_{fin}(\mathcal{H}_{\overline{0}} \oplus \mathcal{E}_{\overline{1}})$, but it is not an involution[5] on $\mathcal{S}_{fin}(\mathcal{H}_{\overline{1}})$.

[5]The more complicated structure of $F^\dagger$ is related to the fact that $\mathcal{T}^-_{fin}(\mathcal{H}_{\overline{1}})$ is not a C*-algebra.

For $F \in \mathcal{S}_{fin}(\mathcal{H}_{\overline{0}}) \odot \mathcal{E}_{\overline{0}}^{\odot p} \odot (\mathcal{E}_{\overline{1}}^{*})^{\odot q}$ we calculate $F^{\dagger\dagger} := (F^{\dagger})^{\dagger} = WJ_{*}JW^{-1}F = (-1)^{p+q}F$, hence $F^{\dagger\dagger} = (-1)^{n}F$ if $F \in \mathcal{S}_{fin}(\mathcal{H}_{\overline{0}}) \odot \mathcal{H}_{\overline{1}}^{\odot n}$. Inserted in (22) the conjugation (25) yields the positive definite form $\left\langle \exp\Omega | F^{\dagger} \odot G \right\rangle = (W^{-1}F|W^{-1}G)$ for $F, G \in \mathcal{S}_{fin}(\mathcal{H})$. When this form is diagonalized by Wick ordering, we exactly recover the inner product of the Fock space $\mathcal{S}(\mathcal{H})$

$$\left\langle \exp\Omega | (WF)^{\dagger} \odot (WG) \right\rangle = (F|G). \tag{26}$$

An identity equivalent to (22) is $\langle \exp\Omega | F \odot G \rangle = \left\langle W^{-1}F | JW^{-1}G \right\rangle$, which yields that the bilinear form

$$F, G \in \mathcal{S}_{fin}(\mathcal{H}) \to \langle \exp\Omega | (WF) \odot (WG) \rangle = \langle F \mid JG \rangle \tag{27}$$

is continuous in the norm (5).

So far all arguments started from the algebraic tensor space $\mathcal{S}_{fin}(\mathcal{H})$. There is another approach using an extension of the Hilbert space $\mathcal{H}$ to a triplet of Hilbert spaces $\mathcal{H}^{+} \subset \mathcal{H} \subset \mathcal{H}^{-}$ with Hilbert-Schmidt embeddings. The continuity of the bilinear form (17) implies that the tensor Ω is an element of $\mathcal{S}_2(\mathcal{H}^{-})$, and therefore $\exp\Omega$ converges within $\mathcal{S}_{(-\gamma)}(\mathcal{H}^{-})$ if $\gamma > 0$, see App.A. The subscript (γ) indicates the modified norm (16). The functional (19) is therefore a continuous mapping $F \in \mathcal{S}_{(\gamma)}(\mathcal{H}^{+}) \to L(F) \in \mathbb{C}$.. Moreover, normal ordering is a continuous mapping $F \in \mathcal{S}_{(\gamma)}(\mathcal{H}^{+}) \to WF \in \mathcal{S}(\mathcal{H})$, see Corollary 3 in App.A.

We would like to add a remark about positive functionals on antisymmetric tensor algebras. Let $\mathcal{T}_{fin}^{-}(\mathcal{K})$ be the algebra of antisymmetric tensors of a complex Hilbert space $\mathcal{K}$ with involution as introduced in Sect 3.1. If $L(F)$ is a linear functional on $\mathcal{T}_{fin}^{-}(\mathcal{K})$, which satisfies $L(F^{*} \wedge F) \geq 0$ for all $F \in \mathcal{T}_{fin}^{-}(\mathcal{K})$, then $L(F)$ has the trivial form $L(F) = \alpha \langle e_0 \mid F \rangle$, where e_0 is the unit of the algebra and α is a non-negative real number. More general functionals are possible, if we demand the positivity condition $L(F^{*} \wedge F) \geq 0$ only for a restricted class of tensors $F \in \mathcal{T}_{fin}^{-}(\mathcal{V})$ where $\mathcal{V} \subset \mathcal{H}$ is a closed linear subspace such that $\mathcal{V}^{*} \cap \mathcal{V} = \{0\}$. A non–trivial example of such a functional is the Gaussian functional (19) with $\mathcal{V} = \mathcal{E}_{\overline{1}} \subset \mathcal{H}_{\overline{1}} = \mathcal{K}$, see (20) restricted to $\mathcal{S}_{fin}(\mathcal{E}_{\overline{1}}) = \mathcal{T}_{fin}^{-}(\mathcal{E}_{\overline{1}})$. Another important example of this type of positivity condition is the Osterwalder-Schrader positivity of the Euclidean quantum field theory of fermions, see e.g. [13], Sect.3.2.

A final remark about *supersymmetry* should be added. The Fock space $\mathcal{S}(\mathcal{H})$ is sometimes called supersymmetric Fock space. But such a notation is misleading. Supersymmetry is an additional property of functionals or operators defined on $\mathcal{S}(\mathcal{H})$; it is a relation between the values of such a functional (operator) on $\mathcal{S}(\mathcal{H}_{\overline{0}})$ with those on $\mathcal{S}(\mathcal{H}_{\overline{1}})$. Let us assume that the spaces $\mathcal{H}_{\overline{0}}$ and $\mathcal{H}_{\overline{1}}$ are isomorphic spaces with an isometric isomorphism $\Theta : \mathcal{H}_{\overline{0}} \to \mathcal{H}_{\overline{1}}$. We define an operator $S : \mathcal{H} \to \mathcal{H}$, which maps $\mathcal{H}_{\overline{0}} \to \mathcal{H}_{\overline{1}}$ and $\mathcal{H}_{\overline{1}} \to \mathcal{H}_{\overline{0}}$ (i.e. S is an odd operator) by $Sf = -\Theta f_0 + \Theta^{-1} f_1$ if $f = f_0 + f_1$, $f_k \in \mathcal{H}_{\overline{k}}$, $k = 0, 1$. Then the Gaussian functional $L(F)$ satisfies the supersymmetry relation $L(D_S F) = 0$ for all $F \in \mathcal{S}_{fin}(\mathcal{H})$, where D_S is the antiderivation of the algebra $\mathcal{S}_{fin}(\mathcal{H})$, which is generated by the odd operator S, i.e. $D_S f = Sf$ if $f \in \mathcal{H}$, see [13]. The restriction to tensors of rank 2 yields the identity $\omega(Sf, g) + (-1)^{\pi(f)}\omega(f, Sg) = 0$, if $f \in \mathcal{H}_{\overline{0}} \cup \mathcal{H}_{\overline{1}}$ and $g \in \mathcal{H}$, for the bilinear form (2) or (17). This form is therefore in the strict sense a supersymmetric form (with respect to the operator S).

5 Superspace and analytic functions

5.1 Superspaces

Let $\mathcal{H} = \mathcal{H}_{\overline{0}} \oplus \mathcal{H}_{\overline{1}}$ be a $\mathbb{Z}_2$-graded Hilbert space as introduced in Sect.3.1 and Λ be an infinite dimensional Grassmann algebra considered as superalgebra $\Lambda = \Lambda_{\overline{0}} \oplus \Lambda_{\overline{1}}$, see Sect.2. Then the

Λ-module $\mathcal{H}^\Lambda = \Lambda \otimes \mathcal{H}$ is a subspace of the algebra $\mathcal{S}^\Lambda_{fin}(\mathcal{H}) := \Lambda \otimes \mathcal{S}_{fin}(\mathcal{H})$. The product of $\mathcal{S}^\Lambda_{fin}(\mathcal{H})$ is generated by[6] $\zeta_1 \cdot \zeta_2 = \lambda_1\lambda_2 \otimes (f_1 \odot f_2)$ for elements $\zeta_i = \lambda_i \otimes f_i \in \Lambda \otimes \mathcal{H}$, $i = 1, 2$. If the factors ζ_i are confined to the space $(\Lambda_{\overline{0}} \otimes \mathcal{H}_{\overline{0}}) \oplus (\Lambda_{\overline{1}} \otimes \mathcal{H}_{\overline{1}}) \subset \Lambda \otimes \mathcal{H}$, this product is commutative. The general rule for the product of two decomposable elements of the algebra $\mathcal{S}^\Lambda_{fin}(\mathcal{H})$ is $\Xi_i = \lambda_i \otimes F_i \in \Lambda \otimes \mathcal{S}_{fin}(\mathcal{H}) = \mathcal{S}^\Lambda_{fin}(\mathcal{H})$, $i = 1, 2$,

$$\Xi_1 \cdot \Xi_2 = \lambda_1\lambda_2 \otimes (F_1 \odot F_2), \tag{28}$$

if $\Xi_i = \lambda_i \otimes F_i \in \Lambda \otimes \mathcal{S}_{fin}(\mathcal{H}) = \mathcal{S}^\Lambda_{fin}(\mathcal{H})$, $i = 1, 2$. In the following we shall assume that also Λ has a parity conserving involution $\lambda \to \lambda^*$. Then

$$(\lambda \otimes F)^* = \lambda^* \otimes F^* \tag{29}$$

induces an involution on $\mathcal{S}^\Lambda_{fin}(\mathcal{H})$. The bilinear form (12) has a unique Λ-bilinear extension $\Xi_1, \Xi_2 \in \mathcal{S}^\Lambda_{fin}(\mathcal{H}) \to \langle \Xi_1 \mid \Xi_2 \rangle \in \Lambda$ such that

$$\langle \Xi_1 \mid \Xi_2 \rangle = \lambda_1\lambda_2 \langle F_1 \mid F_2 \rangle \tag{30}$$

if $\Xi_i = \lambda_i \otimes F_i$ with $\lambda_i \in \Lambda, F_i \in \mathcal{S}_{fin}(\mathcal{H}), i = 1, 2$. This form satisfies the symmetry relation $\langle \Xi_1 \mid \Xi_2 \rangle^* = \langle \Xi_2^* \mid \Xi_1^* \rangle$. Correspondingly, the inner product of $\mathcal{S}_{fin}(\mathcal{H})$ has an extension to a Λ-sesquilinear form on $\mathcal{S}^\Lambda_{fin}(\mathcal{H}) \times \mathcal{S}^\Lambda_{fin}(\mathcal{H})$

$$(\Xi_1 \mid \Xi_2) := \langle \Xi_1^* \mid \Xi_2 \rangle \tag{31}$$

if $\Xi_i \in \mathcal{S}^\Lambda_{fin}(\mathcal{H}), i = 1, 2$. With the identification $F \in \mathcal{S}_{fin}(\mathcal{H}) \Longrightarrow \kappa_0 \otimes F \in \mathcal{S}^\Lambda_{fin}(\mathcal{H})$, the algebra $\mathcal{S}_{fin}(\mathcal{H})$ has a natural injection in $\mathcal{S}^\Lambda_{fin}(\mathcal{H})$, which is consistent with the definitions (30) and (31). The factorization (14) has an extension to the bilinear form (30). We only state the following result. If $F_{1,2} \in \mathcal{S}(\mathcal{E})$ and $\Xi_{1,2} \in \mathcal{S}^\Lambda(\mathcal{E})$ then

$$\langle \Xi_1^* \Xi_2 \mid F_1^* \odot F_2 \rangle = \langle \Xi_1^* \mid F_2 \rangle \langle \Xi_2 \mid F_1^* \rangle. \tag{32}$$

For the superalgebra Λ we choose the Grassmann algebra $\Lambda = \oplus_{p \geq 0} \Lambda_p$, with a norm $\|\lambda\|^2_\Lambda = \sum_{n=0}^\infty (p!)^{-2} \|\lambda_p\|_p^2$ if $\lambda = \sum_{p=0}^\infty \lambda_p, \lambda_p \in \Lambda_p$. Here Λ_p is the subspace of Λ generated by the products of p linearly independent generators. Then the norm of the unit κ_0 is $\|\kappa_0\|_\Lambda = 1$, and the antisymmetric tensor product $\lambda_1, \lambda_2 \in \Lambda \to \lambda_1\lambda_2 \in \Lambda$ is continuous with the estimate, see App.A,

$$\|\lambda_1\lambda_2\|_\Lambda \leq c \|\lambda_1\|_\Lambda \|\lambda_2\|_\Lambda \tag{33}$$

where the constant is $c = \sqrt{3}$. A smaller value of the constant c might be possible, but $c \geq \sqrt{\frac{4}{3}}$ necessarily holds for a Hilbert norm.[7] The inner product of Λ is denoted by $(.\mid.)_\Lambda$. We introduce the unique Hilbert cross norms $\|\Xi\|_p$ on the spaces $\mathcal{S}^\Lambda_p(\mathcal{H}) = \Lambda \widehat{\otimes} \mathcal{S}_p(\mathcal{H})$ which satisfy $\|\Xi\|_p = \|\lambda\|_\Lambda \|F\|_p$ if $\Xi = \lambda \otimes F$ with $\lambda \in \Lambda$ and $F \in \mathcal{S}_p(\mathcal{H})$ for $p \in \mathbb{N}$.. The algebraic tensor product $\mathcal{S}^\Lambda_{fin}(\mathcal{H}) := \Lambda \otimes \mathcal{S}_{fin}(\mathcal{H})$ is then provided with the norms $\|\Xi\|_{(\gamma)} \in \mathbb{R}_+, \gamma \in \mathbb{R}$, defined by $\|\Xi\|^2_{(\gamma)} = \sum_{p=0}^\infty (n!)^\gamma \|\Xi_p\|_p^2$ if $\Xi = \sum_{p=0}^\infty \Xi_p$ with $\Xi_p \in \mathcal{S}^\Lambda_p(\mathcal{H})$. The completion of $\mathcal{S}^\Lambda_{fin}(\mathcal{H})$ with this norm is denoted as $\mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}) = \Lambda \widehat{\otimes} \mathcal{S}_{(\gamma)}(\mathcal{H})$. The space $\mathcal{S}^\Lambda_{(0)}(\mathcal{H})$ will simply be denoted

[6] The tensor product is the algebraic tensor product. The definition used here corresponds to the "Grassmann envelope of the first kind" in [6], p.92.

[7] Instead of the Grassman algebra with the Hilbert norm we could have chosen an infinite dimensional Banach-Grassmann algebra as defined in [32]. In that case the product is continuous with a constant $c = 1$ in (33), but we have to use the projective tensor product of Banach spaces to define $\Lambda \widehat{\otimes} \mathcal{S}(\mathcal{H})$. Then the norm estimates become more involved, but the final results remain unchanged.

by $\mathcal{S}^\Lambda(\mathcal{H})$. With the identification $F \in \mathcal{S}_{(\gamma)}(\mathcal{H}) \Longrightarrow \kappa_0 \otimes F \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H})$ the space $\mathcal{S}_{(\gamma)}(\mathcal{H})$ has a natural isometric injection into $\mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H})$.

The completed Λ-module $\Lambda\widehat{\otimes}\mathcal{H}$ will be denoted by $\mathcal{H}^\Lambda$. For the representation of the Fock space $\mathcal{S}(\mathcal{H})$ by functions the even part of the module $\Lambda\widehat{\otimes}\mathcal{H}$ is the fundamental space. We denote as *superspace* the completed space $\mathcal{H}_\Lambda = (\Lambda_{\overline{0}}\widehat{\otimes}\mathcal{H}_{\overline{0}})\oplus(\Lambda_{\overline{1}}\widehat{\otimes}\mathcal{H}_{\overline{1}}) \subset \mathcal{H}^\Lambda$. This space is a $\Lambda_{\overline{0}}$-module, and it can by obtained as the $\Lambda_{\overline{0}}$-extension of the *restricted superspace* $\mathcal{H}^{res}_\Lambda = \kappa_0\widehat{\otimes}\mathcal{H}_{\overline{0}}\oplus(\Lambda_1\widehat{\otimes}\mathcal{H}_{\overline{1}}) \cong \mathcal{H}_{\overline{0}}\oplus(\Lambda_1\widehat{\otimes}\mathcal{H}_{\overline{1}})$, where Λ_1 is the generating Hilbert space of the Grassmann algebra Λ,

$$\mathcal{H}_\Lambda = \Lambda_{\overline{0}}\widehat{\otimes}\mathcal{H}^{res}_\Lambda. \tag{34}$$

The bilinear form (2) or (17) has a unique Λ-bilinear and continuous extensions to $\mathcal{H}^\Lambda \times \mathcal{H}^\Lambda$, which is again denoted by ω. Restricted to the superspace

$$(\xi,\eta) \in \mathcal{H}_\Lambda \times \mathcal{H}_\Lambda \to \omega(\xi,\eta) = \langle \Omega \mid \xi\eta\rangle \in \Lambda \tag{35}$$

is a symmetric form, $\omega(\xi,\eta) = \omega(\eta,\xi)$.

All $\mathbb{C}$-linear (bounded) operators on $\mathcal{H}$ or $\mathcal{S}(\mathcal{H})$ have a unique Λ-linear extension to $\mathcal{H}^\Lambda = \Lambda\widehat{\otimes}\mathcal{H}$ or $\mathcal{S}^\Lambda(\mathcal{H}) = \Lambda\widehat{\otimes}\mathcal{S}(\mathcal{H})$. If $T : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ then the Λ-extension T: $\mathcal{S}^\Lambda(\mathcal{H}) \to \mathcal{S}^\Lambda(\mathcal{H})$ satisfies $\mathrm{T}(\lambda \otimes F) = \lambda \otimes TF$ for $\lambda \in \Lambda$ and $F \in \mathcal{S}(\mathcal{H})$. The operator T' is called the transposed operator of T, if $\langle T'F \mid G\rangle = \langle F \mid TG\rangle$ holds for all $F, G \in \mathcal{S}(\mathcal{H})$. The Λ-extension T' of T' satisfies

$$\left\langle \mathrm{T}'\Xi \mid H\right\rangle = \langle \Xi \mid \mathrm{T}H\rangle \tag{36}$$

for all $\Xi, H \in \mathcal{S}^\Lambda(\mathcal{H})$.

Given an orthonormal basis $\{e_a \in \mathcal{H}, a \in \mathbb{N}\}$ of $\mathcal{H}$, any $\zeta \in \mathcal{H}^\Lambda$ has the representation $\zeta = \sum_{a=1}^\infty \lambda_a \otimes e_a$ with $\lambda_a \in \Lambda, \sum_a \|\lambda_a\|^2_\Lambda < \infty$. The norm of ζ is calculated as $\|\zeta\|^2 = \|\zeta\|_1^2 = \sum_a \|\lambda_a\|^2_\Lambda$. The norm of a product $\zeta_1\zeta_2...\zeta_p$ of elements $\zeta_n \in \mathcal{H}^\Lambda, n = 1, ..., p \geq 2$, is estimated with the help of (33)

$$\|\zeta_1\zeta_2...\zeta_p\|_p \leq c^{p-1}\sqrt{p!}\prod_{n=1}^p \|\zeta_n\|. \tag{37}$$

Given an orthonormal basis $\{E_n \in \mathcal{S}(\mathcal{H}),\ n \in \mathbb{N}\}$ of the Fock space $\mathcal{S}(\mathcal{H})$ any $\Xi \in \mathcal{S}^\Lambda(\mathcal{H})$ can be decomposed as $\Xi = \sum_n \lambda_n \otimes E_n$ with $\lambda_n \in \Lambda$. The norm is $\|\Xi\|^2 = \sum_n \|\lambda_n\|^2_\Lambda$. The Λ-sesquilinear form (31) of $\Xi_1 = \sum_n \lambda_{1,n} \otimes E_n$ and $\Xi_2 = \sum_n \lambda_{2,n} \otimes E_n$ is calculated as $(\Xi_1 \mid \Xi_2) = \sum_n \lambda^*_{1,n}\lambda_{2,n} \in \Lambda$. The estimate (33) yields the bound $\|(\Xi_1 \mid \Xi_2)\|_\Lambda \leq c\|\Xi_1\|\|\Xi_2\|$ and, since the involution is isometric, the Λ-bilinear form (30) is majorized by the same bound

$$\|\langle\Xi_1 \mid \Xi_2\rangle\|_\Lambda \leq c\|\Xi_1\|\|\Xi_2\|. \tag{38}$$

The bilinear form (30) $\mathcal{S}^\Lambda_{fin}(\mathcal{H}) \times \mathcal{S}^\Lambda_{fin}(\mathcal{H}) \to \Lambda$ can also be extended to a bilinear continuous pairing $\mathcal{S}^\Lambda_{(-\gamma)}(\mathcal{H}) \times \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}) \to \Lambda$.

5.2 Coherent states

The use of coherent states for the representation of the bosonic Fock space [21] can be generalized to superalgebras. But it is only the Λ-extended superalgebra $\mathcal{S}^\Lambda(\mathcal{H})$ for which the simple definition by the exponential series is possible

$$\zeta \in \mathcal{H}_\Lambda \to \exp\zeta = \sum_{p=0}^\infty \frac{1}{p!}\zeta^p \in \mathcal{S}^\Lambda(\mathcal{H}) \tag{39}$$

with the product (28). The absolute convergence of this series follows from the estimate (37), $\|\exp\zeta\|^2 \leq \sum_{p=0}^{\infty}(p!)^{-2}\|\zeta^p\|^2 \leq \sum_{p=0}^{\infty}(p!)^{-1}c^{2p}\|\zeta\|^{2p} < \infty$. Actually the series converges in any space $\mathcal{S}^{\Lambda}_{(\gamma)}(\mathcal{H})$ with $\gamma < 1$. The series (39) satisfies the usual functional relation $\exp(\zeta_1+\zeta_2) = (\exp\zeta_1)(\exp\zeta_2)$ for $\zeta_1, \zeta_2 \in \mathcal{H}_\Lambda$. Since $\|\exp\zeta - 1 - \zeta\| \leq const\,\|\zeta\|^2$ if $\|\zeta\| \leq 1$, the mapping (39) is Fréchet differentiable at $\zeta = 0$, and as a consequence of the functional relation it is Fréchet differentiable everywhere. Moreover, as Fréchet differentiable mapping between complex Banach spaces it is analytic. More explicitly, for $\zeta = \sum_{a\in\mathbf{A}}\lambda_a \otimes f_a$ the series expansion (39) is given by $\exp\zeta = \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{(a_1,\ldots,a_p)\in\mathbf{A}^p}\lambda_{a_1}\cdots\lambda_{a_p}\otimes f_{a_1}\odot\ldots\odot f_{a_p}$. If $\zeta = \kappa_0 \otimes f \in \Lambda_{\overline{0}}\otimes\mathcal{H}_{\overline{0}}$ we obtain $\exp\zeta = \kappa_0 \otimes \exp f$ with the usual coherent states $\exp f$ of the bosonic Fock space $\mathcal{T}^+(\mathcal{H}_{\overline{0}})$.

The $\mathbb{C}$-linear span of the coherent states (39) will be denoted by $\mathcal{C}(\mathcal{H}_\Lambda)$ and the Λ-linear span is called $\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$. The $\mathbb{C}$-linear span of the coherent states of the bosonic Fock space is a dense set of this Fock space. But neither $\mathcal{C}(\mathcal{H})$ nor $\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$ are dense in $\mathcal{S}^\Lambda(\mathcal{H})$. E.g. the space $\kappa_0\widehat{\otimes}\mathcal{H}_{\overline{1}}$ is orthogonal to all elements of $\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$. Nevertheless, many calculations of superanalysis can be reduced to calculations on the set $\mathcal{C}(\mathcal{H}_\Lambda)$. Given a bounded linear mapping $T : \mathcal{S}(\mathcal{H}) \to \mathcal{B}$, where $\mathcal{B}$ is a complex Hilbert space, e.g. $\mathbb{C}$ or $\mathcal{S}(\mathcal{H})$, this operator has a unique extension to a bounded Λ-linear mapping $\mathrm{T} = \kappa_0 \otimes T : \mathcal{S}^\Lambda(\mathcal{H}) \to \Lambda\widehat{\otimes}\mathcal{B}$. The image of T on the set of coherent states completely determines T and consequently T.

Lemma 2 *The linear operator $T : \mathcal{S}(\mathcal{H}) \to \mathcal{B}$ is uniquely determined by $\mathrm{T}\Xi$ with $\Xi \in \mathcal{C}(\mathcal{H}_\Lambda)$.*

Proof The restriction $T\mid_{\mathcal{S}(\mathcal{H}_{\overline{0}})}$ can be obtained from $\mathrm{T}\exp(\kappa_0 \otimes f) = \kappa_0 \otimes (T\exp f)$ with $f \in \mathcal{H}_{\overline{0}}$. The mapping $f \in \mathcal{H}_{\overline{0}} \to \mathrm{T}\exp(\kappa_0 \otimes f) \in \Lambda\widehat{\otimes}\mathcal{B}$ is analytic, and we can calculate $T(f_1 \vee \ldots \vee f_n)$, $f_a \in \mathcal{H}_{\overline{0}}$, from the derivative $\frac{\partial^n}{\partial z_1\ldots\partial z_n}\mathrm{T}\left(\exp\kappa_0 \otimes \sum_{a=1}^n z_a f_a\right)$, $z_a \in \mathbb{C}$, at $z_1 = \ldots = z_n = 0$.

For fermionic arguments $\zeta \in \Lambda_{\overline{1}}\widehat{\otimes}\mathcal{H}_{\overline{1}}$ we may choose a finite sum $\zeta = \sum_{a=1,\ldots,n}\kappa_a \otimes g_a$ with orthonormal elements $\kappa_a \in \Lambda_1$ and arbitrary vectors $g_a \in \mathcal{H}_{\overline{1}}$. Then $(\kappa_1\ldots\kappa_n \mid \mathrm{T}\exp\zeta)_\Lambda = (n!)^{-3}T(g_1 \wedge \ldots \wedge g_n)$, and T is determined on $\mathcal{S}_{fin}(\mathcal{H}_{\overline{1}})$.

The image of $T(F)$ with $F = (f_1 \vee \ldots \vee f_m) \odot (g_1 \wedge \ldots \wedge g_n)$, $f_a \in \mathcal{H}_{\overline{0}}, g_b \in \mathcal{H}_{\overline{1}}, a = 1, \ldots, m$, $b = 1, \ldots, n$, can be calculated from $\mathrm{T}\exp\zeta$ with $\zeta = \sum_{a=1}^m \kappa_0 \otimes f_a + \sum_{b=1}^n \kappa_b \otimes g_b$. Then linearity and continuity determine T uniquely. □

Lemma 2 has an obvious generalization to closable unbounded operators, if the Λ-linear span of $\mathcal{S}^\Lambda_{fin}(\mathcal{H}) \cup \mathcal{C}(\mathcal{H}_\Lambda)$ belongs to the domain of T.

5.3 Analytic functions

We define the topological space $\mathcal{A}(\mathcal{H}_\Lambda)$ of analytic functions on the superspace $\mathcal{H}_\Lambda$ as the linear space of all entire analytic functions $\zeta \in \mathcal{H}_\Lambda \to \varphi(\zeta) \in \Lambda$ with the topology of uniform convergence on bounded sets. We define a mapping $\mathcal{S}^\Lambda(\mathcal{H}) \to \mathcal{A}(\mathcal{H}_\Lambda)$ by

$$\Xi \in \mathcal{S}^\Lambda(\mathcal{H}) \to \varphi_\Xi(\zeta) = \langle \exp\zeta \mid \Xi \rangle \in \Lambda, \tag{40}$$

where the pairing $\langle . \mid . \rangle$ refers only to $\mathcal{S}(\mathcal{H})$.

Lemma 3 *For $\Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H})$ with $\gamma > -1$ the function (40) is an entire analytic function $\mathcal{H}_\Lambda \to \Lambda$. If $\gamma = 0$ it satisfies the uniform bound $\|\varphi_\Xi(\zeta)\|_\Lambda \leq \|\Xi\|\exp\left(2^{-1}c^2\|\zeta\|^2\right)$.*

Proof The function (40) has the series representation

$$\varphi_\Xi(\zeta) = \sum_{p=0}^{\infty}\frac{1}{p!}\langle \zeta^p \mid \Xi_p \rangle \tag{41}$$

where Ξ_p is the projection $\Xi_p \in \Lambda\widehat{\otimes}\mathcal{S}_p(\mathcal{H})$ of $\Xi = \sum_{p=0}^{\infty} \Xi_p$. The homogeneous polynomial $\zeta \to \langle \zeta^p \mid \Xi_p \rangle \in \Lambda$ is holomorphic because it is holomorphic on all finite dimensional subspaces and it is continuous with the bound $\|\langle \zeta^p \mid \Xi_p \rangle\|_\Lambda \leq \sqrt{p!}c^p \|\Xi_p\|_p \|\zeta\|^p$, see (37) and (38). The series (41) converges uniformly with the bound

$$\|\varphi_\Xi(\zeta)\|_\Lambda \leq \sum_p \frac{c^p}{\sqrt{p!}} \|\Xi_p\|_p \|\zeta\|^p \leq \sqrt{\left(\sum_p (p!)^\gamma \|\Xi_p\|^2\right)\left(\sum_q \frac{c^{2q}}{(q!)^{1+\gamma}} \|\zeta\|^{2q}\right)} \leq \|\Xi\|_{(\gamma)} f_\gamma(\|\zeta\|) \tag{42}$$

where the function $f_\gamma(t) = \sqrt{\sum_q \frac{c^{2q}}{(q!)^{1+\gamma}} t^{2q}}$ is locally bounded for all $t \geq 0$ if $\gamma > -1$. For $\gamma = 0$ this function is $f_0(t) = \exp \frac{c^2 t^2}{2}$. □

If $\zeta = \sum_{a \in \mathbf{A}} \lambda_a \otimes x_a \in \mathcal{H}_\Lambda$ with $a \in \mathbf{A} \subset \mathbb{N}$ the expansion (41) has the explicit form

$$\begin{aligned} \varphi_\Xi(\zeta) &= \sum_{p=0}^{\infty} \tfrac{1}{p!} \sum_{(a_1,\ldots,a_p) \in \mathbf{A}^p} \lambda_{a_1} \cdots \lambda_{a_p} \left\langle \Xi_p \mid x_{a_1} \odot \cdots \odot x_{a_p} \right\rangle \\ &= \sum_{p=0}^{\infty} \tfrac{1}{\sqrt{p!}} \sum_{(a_1,\ldots,a_p) \in \mathbf{A}^p} \lambda_{a_1} \cdots \lambda_{a_p} \left\langle \Xi_p \mid x_{a_1} \otimes \cdots \otimes x_{a_p} \right\rangle \end{aligned} \tag{43}$$

The correct symmetrization of the series in the second line follows from the correlated parities of the factors λ_a. For $\Xi \in \mathcal{S}^\Lambda_{fin}(\mathcal{H})$ and a finite set $\mathbf{A}$ this series has only a finite number of terms.

The Fréchet derivative of (40) with increment $\zeta_1 \in \mathcal{H}_\Lambda$ is $\varphi'_\Xi(\zeta)(\zeta_1) = \lim_{t \to +0} t^{-1} (\varphi_\Xi(\zeta + t\zeta_1) - \varphi_\Xi(\zeta)) = \sum_{p=0}^{\infty} \frac{1}{p!} \langle \zeta^p \zeta_1 \mid \Xi_{p+1} \rangle = \sum_{p=0}^{\infty} \frac{1}{p!} \langle \zeta_1 \zeta^p \mid \Xi_{p+1} \rangle$, where ζ_1 appears as left factor. More generally the n-th Fréchet derivative of (40) with increments $\zeta_1, \ldots, \zeta_n \in \mathcal{H}_\Lambda$ is $\varphi^{(n)}_\Xi(\zeta)(\zeta_1, \ldots, \zeta_n) = \sum_{p=0}^{\infty} \frac{1}{p!} \langle \zeta_1 \cdots \zeta_n \zeta^p \mid \Xi_{p+n} \rangle$. This derivative has the uniform bound, see (42),

$$\left\| \varphi^{(n)}_\Xi(\zeta)(\zeta_1 .., \zeta_n) \right\|_\Lambda \leq \sqrt{n!} 2^n c^n \|\Xi\|_{(\gamma)} \|\zeta_1\| \cdots \|\zeta_n\| f_\gamma(\|\zeta\|) \tag{44}$$

for all $\zeta \in \mathcal{H}_\Lambda$ and any $\gamma > -1$.

The restriction of (40) to $\Xi = \kappa_{\overline{0}} \otimes F, F \in \mathcal{S}(\mathcal{H})$, will be denoted by $\varphi_F(\zeta)$. The function space $\mathcal{A}(\mathcal{H}_\Lambda)$ is an algebra with respect to multiplication of the functions. An essential point of the theory of these analytic functions is

Theorem 1 *The mapping $F \in \mathcal{S}_{fin}(\mathcal{H}) \to \varphi_F(\zeta) \in \mathcal{A}(\mathcal{H}_\Lambda)$ is an homomorphism of the algebras, i.e. $\varphi_{F \odot G}(\zeta) = \varphi_F(\zeta)\varphi_G(\zeta)$ if $F, G \in \mathcal{S}_{fin}(\mathcal{H})$.*

Proof The definitions (8) and (40) yield for $F \in \mathcal{S}_p(\mathcal{H})$, $G \in \mathcal{S}_q(\mathcal{H})$ and $\zeta = \sum_{a \in \mathbf{A}} \lambda_a \otimes x_a \in \mathcal{H}_\Lambda$

$$\begin{aligned} \varphi_{F \odot G}(\zeta) &= \tfrac{1}{(p+q)!} \sum_{(a_1,\ldots,a_{p+q}) \in \mathbf{A}^{p+q}} \lambda_{a_1} \cdots \lambda_{a_{p+q}} \left\langle F \odot G \mid x_{a_1} \odot \cdots \odot x_{a_{p+q}} \right\rangle \\ &= \tfrac{1}{(p+q)!\sqrt{p!q!}} \sum_{(a_1,\ldots,a_{p+q}) \in \mathbf{A}^{p+q}} \lambda_{a_1} \cdots \lambda_{a_{p+q}} \times \\ &\quad \sum_\sigma \chi_\sigma(x_{a_1}, \ldots, x_{a_{p+q}}) \left\langle F \mid x_{\sigma(a_1)} \otimes \cdot \cdot \otimes x_{\sigma(a_p)} \right\rangle \left\langle G \mid x_{\sigma(a+1)} \otimes \cdots \otimes x_{\sigma(a_{p+q})} \right\rangle \end{aligned}$$

where $\sum_\sigma$ extends over all permutations σ of $\{1, \ldots, p+q\}$. Since λ_a has the same parity as x_a, the last expression leads to

$$\varphi_{F \odot G}(\zeta) = \frac{1}{\sqrt{p!q!}} \sum_{(a_1,\ldots,a_{p+q}) \in \mathbf{A}^{p+q}} \lambda_{a_1} \cdots \lambda_{a_{p+q}} \left\langle F \mid x_{a_1} \otimes \cdot \cdot \otimes x_{a_p} \right\rangle \left\langle G \mid x_{a+1} \otimes \cdots \otimes x_{a_{p+q}} \right\rangle$$

Following the second identity (43) this function is $\varphi_F(\zeta)\varphi_G(\zeta)$, and linearity in F and G completes the proof. □

The analytic functions (40) provide a representation of the Λ-extended Fock space $\mathcal{S}^\Lambda(\mathcal{H})$.

Lemma 4 *The mapping* $\Xi \in \mathcal{S}^\Lambda(\mathcal{H}) \to \varphi(\zeta) \in \mathcal{A}(\mathcal{H}_\Lambda)$ *is continuous and injective.*

Proof The continuity follows from the estimate (42). The injectivity is a consequence of the stronger Lemma 5. □

The image of the mapping (40) is denoted as $\mathcal{F}^\Lambda(\mathcal{H}_\Lambda)$. The analytic functions of this space can be restricted to arguments ζ in the Hilbert space $\mathcal{H}_\Lambda^{res} = \kappa_0 \widehat{\otimes} \mathcal{H}_{\overline{0}} \oplus \Lambda_1 \widehat{\otimes} \mathcal{H}_{\overline{1}}$ and they are entire analytic functions in the variable $\zeta \in \mathcal{H}_\Lambda^{res}$. Knowing the functions (40) on $\mathcal{H}_\Lambda^{res}$ the values on $\mathcal{H}_\Lambda$ can be obtained by analytic continuation.

Corollary 2 *The functions* $\varphi \in \mathcal{F}^\Lambda(\mathcal{H}_\Lambda)$ *are uniquely determined by their restriction to* $\mathcal{H}_\Lambda^{res}$.

The proof is included in the proof of Lemma 5.

In the following sections we also use restrictions of the analytic functions to the diagonal real subspace. Let $\mathcal{H}$ be a Hilbert space with the structure $\mathcal{H} = \mathcal{E} \oplus \mathcal{E}^*$ as introduced in Sect. 3.1. If $\xi = \sum_{a \in \mathbf{A}} \lambda_a \otimes x_a \in \Lambda \widehat{\otimes} \mathcal{E}$ then the involution (29) leads to $\xi^* = \sum_{a \in \mathbf{A}} \lambda_a^* \otimes x_a^* \in \Lambda \otimes \mathcal{E}^*$ and $\xi + \xi^* \in \Lambda \widehat{\otimes} \mathcal{H}$. On the other hand each $\zeta \in \Lambda \widehat{\otimes} \mathcal{H}$ can be uniquely decomposed into $\zeta = \xi_1 + \xi_2^*$ with $\xi_{1,2} \in \Lambda \widehat{\otimes} \mathcal{E}$. Therefore, if $\zeta \in \Lambda \widehat{\otimes} \mathcal{H}$ satisfies the reality condition $\zeta^* = \zeta$, it has the unique decomposition $\zeta = \xi + \xi^*$ with $\xi \in \Lambda \widehat{\otimes} \mathcal{E}$. If $\zeta^* = \zeta \in \mathcal{H}_\Lambda$ then $\xi \in \mathcal{E}_\Lambda = \Lambda_{\overline{0}} \widehat{\otimes} \mathcal{E}_{\overline{0}} \oplus \Lambda_{\overline{1}} \widehat{\otimes} \mathcal{E}_{\overline{1}}$. In agreement with (1) we define the real subspace

$$\mathcal{H}_\Lambda^D = \{\zeta \mid \zeta^* = \zeta, \zeta \in \mathcal{H}_\Lambda\}. \tag{45}$$

The restriction of the function (40) $\varphi_\Xi(\zeta)$ to this real space

$$\zeta \in \mathcal{H}_\Lambda^D \to \varphi_\Xi(\zeta) = \langle \exp \zeta \mid \Xi \rangle = (\exp \zeta \mid \Xi) \in \Lambda \tag{46}$$

is an infinite Fréchet differentiable function for all $\Xi \in \mathcal{S}^\Lambda(\mathcal{H})$. We denote by $\mathcal{A}(\mathcal{H}_\Lambda^D)$ the space of all functions which are restrictions of the analytic functions $\varphi(\zeta) \in \mathcal{A}(\mathcal{H}_\Lambda)$ to arguments $\zeta \in \mathcal{H}_\Lambda^D$. We provide $\mathcal{A}(\mathcal{H}_\Lambda^D)$ with the topology of uniform convergence for each Fréchet derivative on open bounded sets. The estimates (42) and (44) remain valid for $\zeta \in \mathcal{H}_\Lambda^D$. The functions (46) satisfy the simple reality condition $\varphi_{\Xi^*}(\zeta) = (\varphi_\Xi(\zeta))^*$ only if $\Xi \in \Lambda_{\overline{0}} \widehat{\otimes} \mathcal{S}(\mathcal{H})$.

Lemma 5 *For* $\gamma > -1$ *the mapping*

$$\Xi \in \mathcal{S}_{(\gamma)}^\Lambda(\mathcal{H}) \to \varphi_\Xi(\zeta) \in \mathcal{A}(\mathcal{H}_\Lambda^D) \tag{47}$$

is continuous and injective.

Proof The continuity of (47) follows from the estimates (42) and (44). To derive the injectivity it is sufficient to prove that $\varphi_\Xi(\zeta) = 0$ implies $\Xi = 0$, because $\mathcal{S}^\Lambda(\mathcal{H})$ and $\mathcal{A}(\mathcal{H}_\Lambda^D)$ are linear spaces. The function $t \in \mathbb{R} \to \varphi_\Xi(t\zeta) = \langle \exp t\zeta \mid \Xi \rangle = \sum_{n=0}^\infty \frac{t^n}{n!} \langle \zeta^n \mid \Xi_n \rangle$, with $\zeta \in \mathcal{H}_\Lambda^D$ fixed, is infinite differentiable, and the derivative of order n evaluated at $t = 0$ is the polynomial $\xi \in \mathcal{E}_\Lambda \to \langle (\xi + \xi^*)^n \mid \Xi_n \rangle \in \Lambda$. We choose $\xi = \alpha \otimes x + \sum_{b=1,\dots,n} \vartheta_b \otimes y_b$ with $\alpha \in \Lambda_{\overline{0}}$, $x \in \mathcal{E}_{\overline{0}}$, and $\vartheta_a \in \Lambda_{\overline{1}}$, $y_a \in \mathcal{E}_{\overline{1}}$, for $a = 1, \dots, n$. The polynomial $\langle (\xi + \xi^*)^n \mid \Xi_n \rangle$ is then a polynomial in the variables $\lambda = (\alpha, \vartheta_1, \dots, \vartheta_n) \in \Lambda^{1,n}$ as considered in App.B. Assume $\varphi_\Xi(\zeta) = \langle \exp \zeta \mid \Xi \rangle = 0$ for $\zeta \in \mathcal{H}_\Lambda^D$, then the n-th derivative of $\varphi_\Xi(t\zeta)$ satisfies $\langle \zeta^n \mid \Xi_n \rangle = 0$ for all $\zeta \in \mathcal{H}_\Lambda^D$, and consequently all polynomials in the variables $\lambda = (\alpha, \vartheta_1, \dots, \vartheta_n) \in \Lambda^{1,n}$ are identical zero. As a consequence of Lemma 13 in App.B we obtain $\left\langle x^{\odot p} \odot x^{*\odot q} \odot y_{a_1} \odot \dots \odot y_{a_r} \odot y_{b_1}^* \odot \dots \odot y_{b_s}^* \mid \Xi_n \right\rangle = 0$ for all index sets $1 \le a_1 < \dots < a_r \le n$, $1 \le b_1 < \dots < b_s < n$ and all numbers $p, q, r, s \ge 0$ with $p + q + r + s = n$. Since the vectors $x \in \mathcal{E}_{\overline{0}}$ and $y_a \in \mathcal{E}_{\overline{1}}$, $a = 1, \dots, n$, are arbitrary, the tensor

$\Xi_n \in \Lambda\widehat{\otimes}\mathcal{S}_n(\mathcal{H})$ vanishes. This argument applies to all tensors Ξ_n, $n = 0, 1, ...,$ and $\Xi = \sum_{n=0}^{\infty} \Xi_n$ has to vanish.

Following Corollary 4 in App. B we can take $\xi = \alpha \otimes x + \sum_{b=1,...,n} \vartheta_b \otimes y_b$ with $\alpha \in \mathbb{C}$, $x \in \mathcal{E}_{\overline{0}}$, and $\vartheta_a \in \Lambda_1$, $y_a \in \mathcal{E}_{\overline{1}}$, $a = 1, ..., n$, for the proof given above. Hence already the values of $\varphi_\Xi(\zeta)$ for $\zeta \in \mathcal{H}_\Lambda^{res}$ determine Ξ uniquely. □

The image of the mapping (47) is denoted as $\mathcal{F}_{(\gamma)}^{\Lambda}(\mathcal{H}_\Lambda^D)$. If Ξ is restricted to $\Xi = \kappa_0 \otimes F$ with $F \in \mathcal{S}_{(\gamma)}(\mathcal{H})$ the image is called $\mathcal{F}_{(\gamma)}(\mathcal{H}_\Lambda^D)$, such that $\mathcal{F}_{(\gamma)}^{\Lambda}(\mathcal{H}_\Lambda^D) = \Lambda\widehat{\otimes}\mathcal{F}_{(\gamma)}(\mathcal{H}_\Lambda^D)$. For $\gamma = 0$ the subscript (γ) is omitted in agreement with preceding notations.

For the investigations in Sects.6-8 it is convenient to consider the functions (46) as functions on the space $\mathcal{E}_\Lambda$ choosing the argument $\zeta = \xi + \xi^*$ with $\xi \in \mathcal{E}_\Lambda$. We shall therefore write

$$\varphi_\Xi(\xi, \xi^*) = \langle \exp(\xi + \xi^*) \mid \Xi \rangle, \xi \in \mathcal{E}_\Lambda \tag{48}$$

and we denote the corresponding function spaces by $\mathcal{F}_{(\gamma)}^{\Lambda}(\mathcal{E}_{\Lambda\mathbb{R}})$. The reason for this notation comes from the fact that the arguments are actually taken from $\mathcal{E}_{\Lambda\mathbb{R}}$, the underlying real space of $\mathcal{E}_\Lambda$. It is convenient to introduce for the functions (48) derivatives along vectors belonging to the subspaces $\mathcal{E}_\Lambda$ and $\mathcal{E}_\Lambda^*$, similar to the partial derivatives $\frac{\partial}{\partial z_a}$ or $\frac{\partial}{\partial \overline{z}_a}$ for functions defined on $\mathbb{C}^n$, $\frac{\partial}{\partial z_a} = \frac{1}{2}\left(\frac{\partial}{\partial x_a} - i\frac{\partial}{\partial y_a}\right)$ and $\frac{\partial}{\partial \overline{z}_a} = \frac{1}{2}\left(\frac{\partial}{\partial x_a} + i\frac{\partial}{\partial y_a}\right)$ if $z_a = x_a + iy_a$ with $x_a, y_a \in \mathbb{R}$, $a = 1, ..., n$. We define the derivative of a function $\varphi(\xi, \xi^*)$ along $\eta \in \mathcal{E}_\Lambda$ or $\eta^* \in \mathcal{E}_\Lambda^*$ by

$$\begin{aligned} \varphi'(\xi, \xi^*)(\eta) &= \tfrac{1}{2}\lim_{t\to+0} t^{-1}\left(\varphi(\xi + t\eta, \xi^* + t\eta^*) - i\varphi(\xi + it\eta, \xi^* - it\eta^*) - (1-i)\varphi(\xi, \xi^*)\right), \\ \varphi'(\xi, \xi^*)(\eta^*) &= \tfrac{1}{2}\lim_{t\to+0} t^{-1}\left(\varphi(\xi + t\eta, \xi^* + t\eta^*) + i\varphi(\xi + it\eta, \xi^* - it\eta^*) - (1+i)\varphi(\xi, \xi^*)\right). \end{aligned} \tag{49}$$

For this definition only arguments of the real subspace have been used. If the function $\varphi(\xi, \xi^*)$ can be analytically continued to arguments $(\xi, \eta^*) \in \mathcal{E}_\Lambda \times \mathcal{E}_\Lambda^*$, an assumption which will be satisfied in almost all our applications, these definitions agree with
$\varphi'(\xi, \xi^*)(\eta) = \lim_{t\to+0} t^{-1}\left(\varphi(\xi + t\eta, \xi^*) - \varphi(\xi, \xi^*)\right)$ and
$\varphi'(\xi, \xi^*)(\eta^*) = \lim_{t\to+0} t^{-1}\left(\varphi(\xi, \xi^* + t\eta^*) - \varphi(\xi, \xi^*)\right)$.

The spaces $\mathcal{F}_{(\gamma)}^{\Lambda}(\mathcal{H}_\Lambda^D)$ or $\mathcal{F}_{(\gamma)}^{\Lambda}(\mathcal{E}_{\Lambda\mathbb{R}})$ are again Hilbert spaces with the norm induced by the mapping (47). This mapping transfers also the Λ-bilinear form (30) to a Λ-bilinear form $\mathcal{F}_{(-\gamma)}^{\Lambda}(\mathcal{E}_{\Lambda\mathbb{R}}) \times \mathcal{F}_{(\gamma)}^{\Lambda}(\mathcal{E}_{\Lambda\mathbb{R}}) \to \Lambda$

$$\langle \varphi_\Xi \parallel \varphi_H \rangle := \langle \Xi \mid H \rangle. \tag{50}$$

The space $\mathcal{F}^\Lambda(\mathcal{H}_\Lambda^D) \cong \mathcal{F}^\Lambda(\mathcal{E}_{\Lambda\mathbb{R}})$ is a reproducing kernel space and will be investigated in more detail in Sect.7.1.

6 Gaussian functionals and supermeasures

6.1 Superdistributions

In this section we introduce a notion of the supermeasure in a form adapted to the investigation of Gaussian functionals. As in the last part of Sect.5.3 we consider a topological space of $\Lambda-$valued functions $f(\zeta)$ defined on the real diagonal subspace (45) $\mathcal{H}_\Lambda^D$, or, equivalently, on $\mathcal{E}_\Lambda$ (or more precisely on $\mathcal{E}_{\Lambda\mathbb{R}}$). We assume that these functions are restriction of $\Lambda_{\overline{0}}-$analytic functions on $\mathcal{H}_\Lambda$. The space of all these functions is a $\Lambda-$module and will be denoted by $\mathcal{Z}$. (In the concrete case to be considered in Sect.6.2 the space $\mathcal{Z}$ is a subspace of $\mathcal{F}^\Lambda(\mathcal{E}_{\Lambda\mathbb{R}})$.) For each $\eta \in \mathcal{E}_\Lambda$ let $\varphi_{[\eta]}$ be the function on $\mathcal{E}_{\Lambda\mathbb{R}}$ defined as $\varphi_{[\eta]}(\xi, \xi^*) = \exp i \langle \xi + \xi^* \mid \eta + \eta^* \rangle$. Here we adopt the notations of Sect.5.3 for functions defined on $\mathcal{E}_{\Lambda\mathbb{R}}$. We assume that $\varphi_{[\eta]}(\xi, \xi^*) \in \mathcal{Z}$ for all $\eta \in \mathcal{E}_\Lambda$ and that the function $\eta \longmapsto \varphi_{[\eta]} : \mathcal{E}_{\Lambda\mathbb{R}} \to \mathcal{Z}$ is Gâteau differentiable. Finally, let $\mathcal{Z}'$

be the topological $\Lambda-$module of the continuous $\Lambda-$linear mappings of $\mathcal{Z}$ into Λ. The elements of $\mathcal{Z}'$ will be called $\mathcal{Z}$-distributions[8] or also superdistribution on $\mathcal{E}_{\Lambda\mathbb{R}}$. If $v \in \mathcal{Z}'$ then the Fourier transform $\tilde{v}$ of v is the function on $\mathcal{E}_{\Lambda\mathbb{R}}$ taking values in Λ and defined by

$$\tilde{v}(\eta,\eta^*) = \langle v \parallel \varphi_{[\eta]} \rangle = \langle v \parallel \exp i \langle . \mid \eta + \eta^* \rangle \rangle, \tag{51}$$

where the $\Lambda-$bilinear pairing between $\mathcal{Z}'$ and $\mathcal{Z}$ is denoted by $\langle . \parallel . \rangle$. We will assume that the Fourier transform of v defines v uniquely.

A $\mathcal{Z}$-distribution v is called the canonical Gaussian distribution on $\mathcal{E}_{\Lambda\mathbb{R}}$, if $\widetilde{v}(\eta,\eta^*) = \exp(-\frac{1}{2}\omega(\eta+\eta^*,\eta+\eta^*))$, where ω is the restriction of the bilinear form (35) to $\mathcal{H}^D_\Lambda \times \mathcal{H}^D_\Lambda$; in this context the form ω is called correlation functional of v. Since $\frac{1}{2}\omega(\eta+\eta^*,\eta+\eta^*) = \langle \eta^* \mid \eta \rangle = (\eta \mid \eta)$, it is the restriction of the sesquilinear $\Lambda-$extension (31) of the positive definite inner product of $\mathcal{H}$ to $\mathcal{E}_{\Lambda\mathbb{R}} \times \mathcal{E}_{\Lambda\mathbb{R}}$. The canonical Gaussian distribution on $\mathcal{E}_{\Lambda\mathbb{R}}$ has therefore the Fourier transform

$$\tilde{v}(\eta,\eta^*) = \exp\left(-\frac{1}{2}\omega(\eta+\eta^*,\eta+\eta^*)\right) = e^{-\langle \eta^* \mid \eta \rangle}. \tag{52}$$

The general Gaussian distribution is characterized by the Fourier transform $e^{-b(\eta,\eta^*)}$ with a quadratic correlation functional $b(\eta,\eta^*) : \mathcal{E}_{\Lambda\mathbb{R}} \to \Lambda_{\overline{0}}$ which is derived from the $\Lambda-$extension of some positive (semi)definite form on $\mathcal{H} \times \mathcal{H}$.

6.2 The Gaussian functional on the Fock space

The linear functional (19) has a unique Λ-linear extension to $\mathcal{S}^\Lambda_{fin}(\mathcal{H})$

$$\Xi \in \mathcal{S}^\Lambda_{fin}(\mathcal{H}) \to L(\Xi) = \langle \exp\Omega \mid \Xi \rangle \in \Lambda. \tag{53}$$

For the following investigations we choose as representation of $\mathcal{S}^\Lambda(\mathcal{H})$ the function space $\mathcal{F}^\Lambda(\mathcal{E}_{\Lambda\mathbb{R}})$ with the functions (48) $\varphi_\Xi(\xi,\xi^*) = \langle \exp(\xi+\xi^*) \mid \Xi \rangle, \xi \in \mathcal{E}_\Lambda$. The linear functional (53) is well defined for $\Xi \in \mathcal{S}^\Lambda_{fin}(\mathcal{H})$, but for a theory of integration we would like to apply it also to tensors which do not have finite rank. For that purpose it is convenient to use a triplet of Hilbert spaces $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$ with Hilbert-Schmidt embeddings, already introduced in Sect.4. The spaces $\mathcal{E}^+$ and $\mathcal{E}^{-*}$ (or $\mathcal{E}^-$ and $\mathcal{E}^{+*}$) are dual spaces with respect to the bilinear pairing (3). The tensor Ω is an element of $\mathcal{S}_2(\mathcal{H}^-)$ and $\exp\Omega \in \mathcal{S}_{(-\gamma)}(\mathcal{H}^-)$ with $0 < \gamma < 1$. The space $\mathcal{S}_{(-\gamma)}(\mathcal{H}^-)$ can be embedded into $\mathcal{S}^\Lambda_{(-\gamma)}(\mathcal{H}^-) = \Lambda\widehat{\otimes}\mathcal{S}_{(-\gamma)}(\mathcal{H}^-)$, and the functional (53) is a continuous functional on $\mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+) = \Lambda\widehat{\otimes}\mathcal{S}_{(\gamma)}(\mathcal{H}^+)$. Any tensor $\Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+) \subset \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H})$ can be represented by a differentiable functions in the space $\mathcal{F}^\Lambda_{(\gamma)}(\mathcal{E}^-_{\Lambda\mathbb{R}}) \subset \mathcal{F}^\Lambda_{(\gamma)}(\mathcal{E}_{\Lambda\mathbb{R}})$ following the constructions of Sect.5.3. Here the inclusion $\mathcal{F}^\Lambda_{(\gamma)}(\mathcal{E}^-_{\Lambda\mathbb{R}}) \subset \mathcal{F}^\Lambda_{(\gamma)}(\mathcal{E}_{\Lambda\mathbb{R}})$ means that the restriction of a function $\varphi \in \mathcal{F}^\Lambda_{(\gamma)}(\mathcal{E}^-_{\Lambda\mathbb{R}})$ to arguments $\xi \in \mathcal{E}_\Lambda$ is an element of $\mathcal{F}^\Lambda_{(\gamma)}(\mathcal{E}_{\Lambda\mathbb{R}})$. The tensor $\exp\Omega \in \mathcal{S}_{(-\gamma)}(\mathcal{H}^-)$ is represented by the function $v(\xi,\xi^*) = \langle \exp(\xi+\xi^*) \mid \exp\Omega \rangle = e^{\omega(\xi,\xi^*)}$, which is a differentiable function in the variable $\xi \in \mathcal{E}_{\Lambda\mathbb{R}}$, but it is not an element of $\mathcal{F}^\Lambda_{(-\gamma)}(\mathcal{E}_{\Lambda\mathbb{R}})$, we only have $v(\xi,\xi^*) \in \mathcal{F}^\Lambda_{(-\gamma)}(\mathcal{E}^+_{\Lambda\mathbb{R}})$. Using the bilinear pairing (50) the functional (53) is a linear functional on $\mathcal{F}^\Lambda_{fin}(\mathcal{E}_{\Lambda\mathbb{R}}) = \left\{ \varphi_\Xi(\xi,\xi^*) \mid \Xi \in \mathcal{S}^\Lambda_{fin}(\mathcal{H}) \right\}$

$$\langle v \parallel \varphi_\Xi \rangle = \langle \exp\Omega \mid \Xi \rangle \tag{54}$$

and it is can be extended to a continuous functional on the space $\mathcal{F}^\Lambda_{(\gamma)}(\mathcal{E}^-_{\Lambda\mathbb{R}}) = \left\{ \varphi_\Xi(\xi,\xi^*) \mid \Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+) \right\}$. Following (20) we also know that the functional is well defined

[8] For a general theory of distributions on infinite dimensional spaces see [8].

on the linear span of all functions $\varphi_{\Xi^*H}(\xi,\xi^*)$ with $\Xi, H \in \mathcal{S}^\Lambda(\mathcal{E})$.The functional (54) will be written as integral

$$\langle \exp \Omega \mid \Xi \rangle = \int \varphi_\Xi(\xi,\xi^*) dv(\xi,\xi^*) \tag{55}$$

and $dv(\xi,\xi^*)$ is called the Gaussian *supermeasure* related to the quadratic form $\omega(\xi,\xi^*)$. The identity (55) is (formally) equivalent to $\int \exp(\xi+\xi^*) dv(\xi,\xi^*) = \exp\Omega$. The Fourier transform (51) of the distribution v is

$$\int e^{i\langle \xi+\xi^* \mid \eta+\eta^* \rangle} dv(\xi,\xi^*) = \langle \exp\Omega \mid \exp i(\eta+\eta^*)\rangle = e^{-\frac{1}{2}\omega(\eta+\eta^*,\eta+\eta^*)} = e^{-\langle \eta^* \mid \eta\rangle} \tag{56}$$

with $\eta \in \mathcal{E}_\Lambda$. Here the first identity follows from the definition (55) and the calculation of the functional is performed in App.C. The result (57) generalizes the statements for numerical Gaussian (pro)measures. Moreover $\exp\Omega$ or, more precisely, $v(\xi,\xi^*) = \langle \exp(\xi+\xi^*) \mid \exp\Omega\rangle$ is the canonical Gaussian distribution as defined by the Fourier transform (52). The analytic continuation of (56) leads to the Fourier-Laplace transform

$$\int e^{\langle \xi+\xi^* \mid \varsigma+\tau^* \rangle} dv(\xi,\xi^*) = \langle \exp\Omega \mid \exp(\varsigma+\tau^*)\rangle = e^{\frac{1}{2}\omega(\varsigma+\tau^*,\varsigma+\tau^*)} = e^{\langle \tau^* \mid \varsigma\rangle} \tag{57}$$

with $\varsigma, \tau \in \mathcal{E}_\Lambda$.

The functional (53) is originally defined on $\mathcal{S}^\Lambda_{fin}(\mathcal{H})$, and the supermeasure is a promeasure on the space of all finitely based functions $\mathcal{F}^\Lambda_{fin}(\mathcal{E}_{\Lambda\mathbb{R}}) = \left\{ \varphi_\Xi(\xi,\xi^*) \mid \Xi \in \mathcal{S}^\Lambda_{fin}(\mathcal{H}) \right\}$. But since the functional is defined on $\mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+), \gamma > 0$, this promeasure can be extended to distributions on $\mathcal{E}^-_{\Lambda\mathbb{R}}$, the space on which all test functions $\varphi_\Xi(\xi,\xi^*), \Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+)$, are continuous functions. The class of integrable functions includes the spaces $\mathcal{F}^\Lambda_{(\gamma)}(\mathcal{E}^-_{\Lambda\mathbb{R}}) = \left\{ \varphi_\Xi(\xi,\xi^*), \Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+) \right\}$ for any $\gamma > 0$. For these functions we derive the following statement

Lemma 6 *If $\varphi(\xi,\xi^*) \in \mathcal{F}^\Lambda_{(\gamma)}(\mathcal{E}^-_{\Lambda\mathbb{R}})$ for some $\gamma > 0$, then also $\varphi(\xi+\varsigma,\xi^*+v^*)$ with $\varsigma, v \in \mathcal{E}^-_\Lambda$ and $\varphi(z_1\xi, z_2\xi^*)$ with $z_{1,2} \in \mathbb{C}$ are integrable functions.*

Proof We assume $0 < \gamma < 1$. From the definition of the function follows $\varphi(\xi,\xi^*) = \varphi_\Xi(\xi,\xi^*)$ with a tensor $\Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+)$. and therefore $\varphi(\xi+\varsigma,\xi^*+v^*) = \langle \exp(\xi+\xi^*) \mid \exp(\varsigma+\tau^*) \lrcorner\, \Xi\rangle$. For $\varsigma, v \in \mathcal{E}^-_\Lambda$ we have $\exp(\varsigma+\tau^*) \in \mathcal{S}^\Lambda_{(\alpha)}(\mathcal{H}^-), \alpha < 1$, and Lemma 4 implies $\exp(\varsigma+\tau^*) \lrcorner\, \Xi \in \mathcal{S}^\Lambda_{(\beta)}(\mathcal{H}^+)$ with $0 < \beta < \gamma$. Hence $\langle \exp(\xi+\xi^*) \mid \exp(\varsigma+\tau^*) \lrcorner\, \Xi\rangle \in \mathcal{F}^\Lambda_{(\beta)}(\mathcal{E}^-_{\Lambda\mathbb{R}})$ is integrable.

For $z_{1,2} \in \mathbb{C}$ the mapping $\zeta = \xi + \eta^* \to \mathrm{A}[z_1,z_2]\zeta := z_1\xi + z_2\eta^*$ for $\xi,\eta \in \Lambda\widehat{\otimes}\mathcal{E}^\pm$ is a bounded operator on $\Lambda\widehat{\otimes}\mathcal{H}^\pm$, and $\Gamma(\mathrm{A})$ is a bounded extension on $\mathcal{S}^\Lambda_{(\alpha)}(\mathcal{H}^\pm), \alpha \in \mathbb{R}$. From $\langle \mathrm{A}[z_1,z_2]\zeta \mid \varsigma\rangle = \langle \zeta \mid \mathrm{A}[z_2,z_1]\varsigma\rangle$ for $\zeta \in \Lambda\widehat{\otimes}\mathcal{H}^-$ and $\varsigma \in \Lambda\widehat{\otimes}\mathcal{H}^+$ we obtain $\varphi(z_1\xi, z_2\xi^*) = \langle \exp(\xi+\xi^*) \mid \Gamma(\mathrm{A}[z_2,z_1])\Xi\rangle$. Since $\Gamma(\mathrm{A}[z_2,z_1])\Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+)$ if $\Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+)$ the proof is complete. □

The Gaussian functional (53) is the tensor product of the functional $\langle \exp\Omega_0 \mid .\rangle$ which operates non-trivially only on the bosonic tensors of $\mathcal{S}^\Lambda_{fin}(\mathcal{H}_{\overline{0}})$, and of the functional $\langle \exp\Omega_1 \mid .\rangle$ which operates on the fermionic tensors of $\mathcal{S}^\Lambda_{fin}(\mathcal{H}_{\overline{1}})$. The bosonic functional can be calculated with the help of the canonical Gaussian promeasure $\mu_0(dx)$ on $\mathcal{E}_{\overline{0}} \cong \kappa_0 \otimes \mathcal{E}_{\overline{0}} \subset \Lambda_{\overline{0}}\widehat{\otimes}\mathcal{E}_{\overline{0}}$, see App.D.1. Hence the functional (53) can be evaluated with the vector valued Gaussian (pro)measure $\mu_0(dx)\exp\Omega_1$ defined on $\mathcal{E}_{\overline{0}\mathbb{R}}$ (or $\mathcal{E}^-_{\overline{0}\mathbb{R}}$) with values in $\mathcal{S}_{(-\gamma)}(\mathcal{H}^-_{\overline{1}})$. Equivalently, the Gaussian superdistribution v with Fourier transform (52) has the interpretation of the vector valued promeasure $\mu_0(dx)v_1 : \mathcal{E}^-_{\overline{0}\mathbb{R}} \to \mathcal{Z}'_{\overline{1}}$ where $\mathcal{Z}_{\overline{1}} = \mathcal{F}^{\Lambda}_{(\gamma)}(\Lambda_{\overline{1}}\widehat{\otimes}\mathcal{E}^-_{\overline{1}}), 0 < \gamma < 1$. The original definition of the supermeasure in [40] is related to this interpretation.

There is another method to calculate the functional (19) by Gaussian integration. In App.D we refer to a representation of the tensor algebra $\mathcal{T}^-_{fin}(\mathcal{H})$ by an algebra of functions with the help of an ordering prescription without Λ-extension. Then the Gaussian functional (19), including the fermionic part, can be evaluated with a positive Gaussian measure on $\mathcal{E}^-_{\mathbb{R}} = \mathcal{E}^-_{\overline{0}\mathbb{R}} \oplus \mathcal{E}^-_{\overline{1}\mathbb{R}}$. For more details see App.D.

7 Reproducing kernel spaces and Bargmann-Fock representation for superalgebras

The representation of the bosonic Fock space as a Hilbert space of (anti)holomorphic functions (Bargmann-Fock representation or holomorphic representation) is an effective tool in quantum mechanics [2][4] and in quantum field theory [37][38][25]. This Hilbert space of functions is a reproducing kernel Hilbert space, and a great part of the calculations done within this representation only refer to this property. In the reproducing kernel space technique the inner product can be defined in a rather abstract way, and one has a great flexibility in choosing the domain on which the functions are defined (exploiting the fact that an analytic function is already uniquely determined on rather small sets), see e.g. [27], where these techniques have been applied to the fermionic Fock space. In this section we generalize these constructions to analytic functions on superspaces. In Sect.7.1 we restrict the domain of the analytic functions φ_F to the real subspace $\mathcal{H}^D_\Lambda$. Actually a restriction to even smaller but densely embedded subspaces is possible. In Sect.7.2 we use the full analyticity of the functions and investigate the Bargmann-Fock representation. The resulting function spaces are isomorphic reproducing kernel spaces, which exhibit many features known from the analysis of the bosonic Fock space. In the Bargmann-Fock representation one has an integral representation for the inner product, where the "measure" is the supermeasure, defined in Sect.6.

7.1 Reproducing kernel spaces

We define a sesquilinear form on $\mathcal{F}^\Lambda(\mathcal{H}^D_\Lambda)$ by

$$(\varphi_\Xi \parallel \varphi_H) := (\Xi \mid H), \tag{58}$$

if $\Xi, H \in \mathcal{S}^\Lambda(\mathcal{H})$. This form is positive definite if $\Xi = id \otimes F$ and $H = id \otimes G$ with $F, G \in \mathcal{S}(\mathcal{H})$, and it defines an inner product on $\mathcal{F}(\mathcal{H}^D_\Lambda)$. The mapping (47) is therefore an isomorphism between the Hilbert spaces $\mathcal{S}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H}^D_\Lambda)$. If $\Xi = id \otimes F$ we simply write φ_F instead of $\varphi_{id\otimes F}$. We define a kernel function $\mathcal{H}^D_\Lambda \times \mathcal{H}^D_\Lambda \to \Lambda_{\overline{0}}$ by

$$K(\zeta_1, \zeta_2) := (\exp\zeta_1 \mid \exp\zeta_2) = e^{(\zeta_1|\zeta_2)} = e^{\langle\zeta_1|\zeta_2\rangle}. \tag{59}$$

The identity used on the right side of this definition is calculated in App.C. If $\eta \in \mathcal{H}^D_\Lambda$ then $\exp\eta \in \mathcal{S}^\Lambda(\mathcal{H})$, and the function $\zeta \to K(\zeta, \eta)$ is an element of $\mathcal{F}^\Lambda(\mathcal{H}^D_\Lambda)$. The importance of this kernel function follows from

Lemma 7 *The functions $\varphi(\zeta) \in \mathcal{F}^\Lambda(\mathcal{H}^D_\Lambda)$ satisfy the identity*

$$(K(.,\eta) \parallel \varphi) = \varphi(\eta) \tag{60}$$

with the kernel (59).

Proof Assume $\varphi(\zeta) \in \mathcal{F}^\Lambda(\mathcal{H}^D_\Lambda)$. Then Lemma 5 and the definition of $\mathcal{F}^\Lambda(\mathcal{H}^D_\Lambda)$ imply that there exists a unique element $\Xi \in \mathcal{S}^\Lambda(\mathcal{H})$ such that $\varphi(\zeta) = \varphi_\Xi(\zeta)$. On the other hand for $\eta \in \mathcal{H}^D_\Lambda$ we have $G := \exp\eta \in \mathcal{S}^\Lambda(\mathcal{H})$ and $\varphi_G(\zeta) = (\exp\zeta \mid \exp\eta) = K(\zeta,\eta)$. Hence (58) implies $(K(.,\eta) \parallel \varphi_\Xi) = (\exp\eta \mid \Xi) = \varphi_\Xi(\eta)$. □

The space $\mathcal{F}^\Lambda(\mathcal{H}^D_\Lambda)$ has therefore $K(\zeta_1,\zeta_2)$ as reproducing kernel, and $\mathcal{F}(\mathcal{H}^D_\Lambda) \subset \mathcal{F}^\Lambda(\mathcal{H}^D_\Lambda)$ is a Hilbert space with reproducing kernel. But since the functions in $\mathcal{F}(\mathcal{H}^D_\Lambda)$ have values in a non-commutative algebra, $\mathcal{F}(\mathcal{H}^D_\Lambda)$ differs from the usual function spaces, and the kernel function $K(\zeta_1,\zeta_2)$ does not satisfy all of the properties of a reproducing kernel as given in [1]. E.g. the function $\zeta \to K(\zeta,\eta), \eta \in \left(\Lambda_{\overline{1}}\widehat{\otimes}\mathcal{H}_{\overline{1}}\right) \cap \mathcal{H}^D_\Lambda$, is not an element of the Hilbert space $\mathcal{F}(\mathcal{H}^D_\Lambda)$.

7.2 Bargmann-Fock representation

The functions (46) have a unique analytic extension to arguments $\zeta = \xi + \eta^*$,with $\xi \in \mathcal{E}_\Lambda$ and $\eta^* \in \mathcal{E}^*_\Lambda$, see Lemma 3. We shall denote the analytically continued functions as

$$\varphi_\Xi(\xi,\eta^*) = \langle \exp(\xi+\eta^*) \mid \Xi\rangle = (\exp(\xi^*+\eta) \mid \Xi) \tag{61}$$

with independent arguments $\xi \in \mathcal{E}_\Lambda$ and $\eta^* \in \mathcal{E}^*_\Lambda$.[9] The space of these functions with $\Xi \in \mathcal{S}^\Lambda(\mathcal{H})$ is denoted as $\mathcal{F}^\Lambda\left(\mathcal{E}_\Lambda \times \mathcal{E}^*_\Lambda\right)$. If Ξ is restricted to $\Xi = id \otimes F$ with $F \in \mathcal{S}(\mathcal{H})$, the space is called $\mathcal{F}\left(\mathcal{E}_\Lambda \times \mathcal{E}^*_\Lambda\right)$, and these functions satisfy the identities

$$\begin{cases} (\varphi_F(\xi,\eta^*))^* = \varphi_{F^*}(\xi^*,\eta), \\ \varphi_F(\xi,\eta^*)\varphi_G(\xi,\eta^*) = \varphi_{F\odot G}(\xi,\eta^*), \end{cases} \tag{62}$$

for the second identity see Lemma 1. The space $\mathcal{F}^\Lambda\left(\mathcal{E}_\Lambda \times \mathcal{E}^*_\Lambda\right)$ is of course isomorphic to the space $\mathcal{F}^\Lambda(\mathcal{H}^D_\Lambda)$, and $\mathcal{F}\left(\mathcal{E}_\Lambda \times \mathcal{E}^*_\Lambda\right)$ is still a Hilbert space with a reproducing kernel. Using the variables $\xi \in \mathcal{E}_\Lambda$ and $\eta^* \in \mathcal{E}^*_\Lambda$ the kernel (59) has the analytic continuation

$$K(\xi_1,\eta_1^*;\xi_2,\eta_2^*) = e^{\langle \xi_1 | \xi_2^*\rangle + \langle \eta_1^* | \eta_2\rangle}, \tag{63}$$

and all arguments of Sect.7.1 can be repeated.

As an advantage of the analytic continuation of the argument $\zeta \in \mathcal{H}^D_\Lambda$ to $\xi + \eta^* \in \mathcal{E}_\Lambda \oplus \mathcal{E}^*_\Lambda$ we can write the form (58) as integral.

Lemma 8 *For $\Xi, H \in \mathcal{S}^\Lambda_{fin}(\mathcal{H})$ the sesquilinear form (58) has the integral representation*[10]

$$(\Xi \mid H) = \int (\varphi_\Xi(\xi, \mathrm{j}\eta^*))^*\, \varphi_H(\xi,\eta^*) dv(\xi,\xi^*) dv(\eta,\eta^*). \tag{64}$$

Proof We first assume $\Xi = \kappa_0 \otimes F$ and $H = \kappa_0 \otimes G$ with $F, G \in \mathcal{S}_{fin}(\mathcal{H})$. To prove (64) in this case it is sufficient to consider tensors $F = F_1^* \odot F_2$ and $G = G_1^* \odot G_2$ with $F_{1,2} \in \mathcal{S}_{fin}(\mathcal{E})$ and $G_{1,2} \in \mathcal{S}_{fin}(\mathcal{E})$. Then $\varphi_F(\xi,\eta^*) = \langle \exp\eta^* \mid F_2\rangle \langle \exp\xi \mid F_1^*\rangle$ and
$\varphi_G(\xi,\eta^*) = \langle \exp\eta^* \mid G_2\rangle \langle \exp\xi \mid G_1^*\rangle$, see (32). The integrand of (64) is therefore
$\langle \exp\xi^* \mid F_1\rangle \langle \exp\eta \mid JF_2^*\rangle \langle \exp\eta^* \mid G_2\rangle \langle \exp\xi \mid G_1^*\rangle =$
$\langle \exp\xi^* \mid F_1\rangle \langle \exp(\eta+\eta^*) \mid G_2 \odot JF_2^*\rangle \langle \exp\xi \mid G_1^*\rangle =$
$(-1)^{\pi(F_1)(\pi(G_2)+\pi(F_2))} \langle \exp(\eta+\eta^*) \mid G_2 \odot JF_2^*\rangle \langle \exp(\xi+\xi^*) \mid G_1^* \odot F_1\rangle$. Here J is the operator (23). Only terms with $\pi(G_2) = \pi(F_2)$, for which the sign function is $+1$, contribute to the

[9] More precisely, φ depends on variables $(\xi_{\mathbf{R}}, \eta_{\mathbf{R}}) \in \mathcal{E}^{(1)}_{\Lambda\mathbf{R}} \times \mathcal{E}^{(2)}_{\Lambda\mathbf{R}}$, where $\mathcal{E}^{(1,2)}_{\Lambda\mathbf{R}}$ are two copies of $\mathcal{E}_{\Lambda\mathbf{R}}$, the underlying real space of $\mathcal{E}_\Lambda$.

[10] Here the integration is extended over two independent copies of $\mathcal{E}_{\Lambda\mathbf{R}}$, see the foregoing footnote. The operator j is the Λ-linear extension of (4). The measure is considered as promeasure on $\mathcal{E}_{\Lambda\mathbf{R}}$.

integration (55) with $dv(\eta,\eta^*)$. Moreover $\langle \exp\Omega \mid G_2 \odot JF_2^* \rangle = \langle \exp\Omega \mid F_2^* \odot G_2 \rangle = (F_2 \mid G_2)$, and

$$\int (\varphi_F(\xi, \mathrm{j}\eta^*))^* \varphi_G(\xi,\eta^*) dv(\xi,\xi^*) dv(\eta,\eta^*) = (F \mid G)$$

follows for all F and $G \in \mathcal{S}_{fin}(\mathcal{H})$. The Λ-extension of this identity to $\Xi = \lambda \otimes F$ and $H = \mu \otimes G$ with $\lambda, \mu \in \Lambda$ and $F, G \in \mathcal{S}_{fin}(\mathcal{H})$ is calculated for the left side using the definitions (30) (61), and for the right side with (31). It yields for both sides an additional left factor λ^* and an additional right factor μ. The identity (64) then follows by Λ-linearity. $\square$

If Ξ, H are restricted to $\Xi = \kappa_0 \otimes F$ and $H = \kappa_0 \otimes G$ with $F, G \in \mathcal{S}_{fin}(\mathcal{H})$, the integral (64) yields the inner product of the Fock space $\mathcal{S}(\mathcal{H})$. The antilinear mapping

$$\varphi_F(\xi,\eta^*) \to (\varphi_F(\xi, \mathrm{j}\eta^*))^* = \langle JF^* \mid \exp(\xi^* + \eta) \rangle, \tag{65}$$

which is needed to calculate the integrand, is not an involution, see the discussion of the conjugation (25) in Sect.4. The problem comes from the inversion j which contributes to the fermionic terms $F \in \mathcal{S}(\mathcal{E}_{\overline{1}}^*)$. The inner product of the fermionic Fock space is usually written only for the Fock space $\mathcal{T}^-(\mathcal{E}_{\overline{1}})$ of just one species of fermions, see e.g. Sect.2.4 of [10]. If $F, G \in \mathcal{S}(\mathcal{H}_{\overline{0}} \oplus \mathcal{E}_{\overline{1}})$ then (64) simplifies to $\int (\varphi_F(\xi))^* \varphi_G(\xi) dv(\xi,\xi^*) = (F \mid G)$ without the phase inversion j.

As in the case of the classical Bargmann-Fock space [22][46] we can extend Lemma 8 to all functions φ_Ξ with $\Xi \in \mathcal{S}^\Lambda(\mathcal{H})$. But then the functions should be extended to arguments $(\xi,\eta^*) \in \mathcal{E}_\Lambda^- \times \mathcal{E}_\Lambda^{-*}$, and the supermeasure has to be taken as distribution on $\mathcal{E}_\Lambda^-$. The corresponding space of functions is then more appropriately denoted by $\mathcal{F}^\Lambda(\mathcal{E}_\Lambda^- \times \mathcal{E}_\Lambda^{-*})$. The extended functions are not necessarily analytic on the space $\mathcal{E}_\Lambda^- \times \mathcal{E}_\Lambda^{-*}$.

7.3 Kernels and symbols of operators

In this subsection we give a short introduction to integral operators in the Bargmann-Fock representation. The class of linear operators which allows such a representation includes all bounded operators, and also differential operators. Actually it is more convenient to use the techniques of the reproducing kernel spaces of Sect.7.1 for these constructions. The kernels of the Bargmann-Fock representation can then be recovered by analytic continuation in the arguments.

The sesquilinear form (58) is related to the bilinear form (50) by $(\varphi_\Xi \parallel \varphi_H) = \langle \varphi_{\Xi^*} \parallel \varphi_H \rangle$. Since $\eta^* = \eta$ for $\eta \in \mathcal{H}_\Lambda^D$ the identity (60) can also be written as $\langle K(.,\eta) \parallel \varphi \rangle = \varphi(\eta)$. Let Υ be the mapping of $\mathcal{S}^\Lambda(\mathcal{H})$ onto $\mathcal{F}^\Lambda(\mathcal{H}_\Lambda^D) \subset \mathcal{A}(\mathcal{H}_\Lambda^D)$ defined in Lemma 5. Given an operator T on $\mathcal{S}(\mathcal{H})$ with Λ-extension T on $\mathcal{S}^\Lambda(\mathcal{H})$ we denote by $\mathrm{T}_\mathcal{F}$ its representation on $\mathcal{F}^\Lambda(\mathcal{H}_\Lambda^D)$, i.e. $\mathrm{T}_\mathcal{F}$ is defined by $\mathrm{T}_\mathcal{F}\Upsilon = \Upsilon\mathrm{T}$.

Proposition 1 *For any $\varphi \in \mathcal{F}^\Lambda(\mathcal{H}_\Lambda^D)$ and any $\eta \in \mathcal{H}_\Lambda^D$*

$$(\mathrm{T}_\mathcal{F}\varphi)(\eta) = \langle K_T(.,\eta) \parallel \varphi \rangle \tag{66}$$

where the kernel $K_T : \mathcal{H}_\Lambda^D \times \mathcal{H}_\Lambda^D \to \Lambda$ is given by

$$K_T(\zeta_1,\zeta_2) := \langle \mathrm{T}\exp\zeta_1 \mid \exp\zeta_2 \rangle = \big\langle \exp\zeta_1 \mid \mathrm{T}'\exp\zeta_2 \big\rangle = \big(\exp\zeta_1 \mid \mathrm{T}'\exp\zeta_2\big). \tag{67}$$

Proof The proof is quite similar to that of Lemma 7. If $\varphi \in \mathcal{F}^\Lambda(\mathcal{H}_\Lambda^D)$ then there exists a unique element $\Xi \in \mathcal{S}^\Lambda(\mathcal{H})$ for which $\varphi = \varphi_\Xi$. For any $\eta \in \mathcal{H}_\Lambda^D$ we have $\exp\eta \in \mathcal{S}^\Lambda(\mathcal{H})$, and hence, if $\eta_1, \eta_1 \in \mathcal{H}_\Lambda^D$, and if K_T is defined by (67) then
$\langle K_T(.,\eta_2) \parallel \varphi \rangle = \langle \langle \exp . \mid \mathrm{T}'\exp\eta_2 \rangle \parallel \varphi_\Xi \rangle = \langle \mathrm{T}'\exp\eta_2 \mid \Xi \rangle = \langle \exp\eta_2 \mid \mathrm{T}\Xi \rangle = \mathrm{T}_\mathcal{F}\varphi$. Here T' is the transposed operator (36). $\square$

Take as example a (bounded) operator $A : \mathcal{H} \to \mathcal{H}$, and let T be its second quantization $T = \Gamma(A) : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$. This operator is represented on $\mathcal{F}^{\Lambda}(\mathcal{H}^D_{\Lambda})$ as $\varphi(\zeta) \to (\mathrm{T}_{\mathcal{F}}\varphi)(\zeta) = \varphi(A\zeta)$, and the kernel of $\Gamma(A)$ is $K_{\Gamma(A)}(\zeta_1, \zeta_2) = \langle \Gamma(\mathrm{A}) \exp \zeta_1 \mid \exp \zeta_2 \rangle = \langle \exp(\mathrm{A}\zeta_1) \mid \exp \zeta_2 \rangle = e^{\langle \mathrm{A}\zeta_1 | \zeta_2 \rangle}$. If $A = I$ then $\Gamma(I)$ is the identity operator on $\mathcal{S}(\mathcal{H})$, and we obtain the reproducing kernel (59).

As further examples we consider two unbounded mappings, which are Λ-extensions of the creation and the annihilation operators. The mapping $\Xi \in \mathcal{S}^{\Lambda}(\mathcal{H}) \to \mathrm{T}_{\varpi}\Xi := \varpi\Xi \in \mathcal{S}^{\Lambda}(\mathcal{H})$ with an element $\varpi \in \mathcal{H}_{\Lambda}$ is an extension of the creation operator on $\mathcal{S}(\mathcal{H})$. On the space $\mathcal{F}^{\Lambda}(\mathcal{H}^D_{\Lambda})$ it is a multiplication operator: $\langle \exp \zeta \mid \varpi\Xi \rangle = \langle \zeta \mid \varpi \rangle \langle \exp \zeta \mid \Xi \rangle$. The kernel of this operator is $K_T(\zeta_1, \zeta_2) = \langle \varpi \exp \zeta_1 \mid \exp \zeta_2 \rangle = \langle \varpi \mid \zeta_2 \rangle\, e^{\langle \zeta_1 | \zeta_2 \rangle}$.
The mapping $\Xi \in \mathcal{S}^{\Lambda}(\mathcal{H}) \to \mathrm{C}_{\varpi}\Xi := \varpi \lrcorner \Xi \in \mathcal{S}^{\Lambda}(\mathcal{H})$ with an element $\varpi \in \mathcal{H}_{\Lambda}$ is an extension of the annihilation operator on $\mathcal{S}(\mathcal{H})$. The kernel of this operator is
$K_C(\zeta_1, \zeta_2) = \langle \varpi \lrcorner \exp \zeta_1 \mid \exp \zeta_2 \rangle = \langle \exp \zeta_1 \mid \varpi \exp \zeta_2 \rangle = \langle \zeta_1 \mid \varpi \rangle\, e^{\langle \zeta_1 | \zeta_2 \rangle}$.
If $\varpi \in \mathcal{H}^D_{\Lambda}$ the operator C_{ϖ} is mapped by Υ onto a differential operator on the space of functions $\mathcal{F}^{\Lambda}(\mathcal{H}^D_{\Lambda})$: $\langle \exp \zeta \mid \varpi \lrcorner \Xi \rangle = \langle \varpi \exp \zeta \mid \Xi \rangle = \varphi^{(1)}_{\Xi}(\zeta)(\varpi)$, the Fréchet derivative with increment ϖ.

As in the usual analysis one can define the *Wick symbol* of an operator T by, see e.g. [4][25][5],

$$w_T(\zeta_1, \zeta_2) := e^{-\langle \zeta_1 | \zeta_2 \rangle} K_T(\zeta_1, \zeta_2) = K_T(\zeta_1, \zeta_2) e^{-\langle \zeta_1 | \zeta_2 \rangle} \in \Lambda_{\overline{0}} \tag{68}$$

such that the identity operator has the symbol 1, and the symbols of differential operators are polynomials. The operator symbols offer therefore a simple prescription to relate classical differential operators to operators on the Fock space $\mathcal{S}(\mathcal{H})$. Another application is their use in the evaluation of Feynman path integrals, see the (formal) calculations in [5] or [10].

8 Wiener-Segal representation

8.1 The Wiener-Segal representation and Gauss transform

The Wiener-Segal representation [45][15][36] is based on Wick ordering which has a well defined meaning also for superalgebras. For $\Xi \in \mathcal{S}^{\Lambda}_{fin}(\mathcal{H})$ we calculate the Wick ordered tensor $\mathrm{W}\Xi \in \mathcal{S}^{\Lambda}_{fin}(\mathcal{H})$ following the rule (21): $\mathrm{W}(\lambda \otimes F) = \lambda \otimes WF$ for $\lambda \in \Lambda$ and $F \in \mathcal{S}_{fin}(\mathcal{H})$. Wick ordered polynomials are then defined as elements of $\mathcal{F}^{\Lambda}(\mathcal{E}_{\Lambda\mathbb{R}})$ by

$$\Phi_{\Xi}(\xi, \xi^*) := \langle \exp(\xi + \xi^*) \mid \mathrm{W}\Xi \rangle = \langle \exp(\xi + \xi^* - \Omega) \mid \Xi \rangle. \tag{69}$$

For $\Xi = \kappa_0 \otimes F$ with $F \in \mathcal{S}_{fin}(\mathcal{H})$ we simply write $\Phi_F(\xi, \xi^*)$. These polynomials form again an algebra under pointwise multiplication which is isomorphic to the superalgebra $\mathcal{S}_{fin}(\mathcal{H})$. But in contrast to Theorem 1 the rule is now

$$\Phi_F(\xi, \xi^*)\Phi_G(\xi, \xi^*) = \Phi_{F \triangle G}(\xi, \xi^*), \tag{70}$$

where $F \triangle G$ is Le Jan's Wiener-Grassmann product, which is generated as associative product by $f \triangle g = f \odot g + \omega(f, g)$ for $f, g \in \mathcal{H}$, see [17][26][31]. The polynomials (69) are continuous functions of $\xi \in \mathcal{E}_{\Lambda}$. The functions (69) can be "integrated" with respect to the supermeasure

$$\int \Phi_{\Xi}(\xi, \xi^*) dv(\xi, \xi^*) = \langle \exp \Omega \mid \mathrm{W}\Xi \rangle = \langle 1 \mid \Xi \rangle. \tag{71}$$

Motivated by (65), we define the conjugation

$$\Phi^{\dagger}_{\Xi}(\xi, \xi^*) := (\Phi_{\mathrm{J}\Xi}(\xi, \xi^*))^* \tag{72}$$

where J is the Λ-linear extension of (23).

Lemma 9 *The Λ-sesquilinear form (31) can be calculated as the integral*

$$\int \Phi_{\Xi}^{\dagger}(\xi,\xi^*)\Phi_H(\xi,\xi^*)dv(\xi,\xi^*) = (\Xi \mid H) \tag{73}$$

for all $\Xi, H \in \mathcal{S}^{\Lambda}_{fin}(\mathcal{H})$.

Proof We first restrict the tensors to $\Xi = \kappa_0 \otimes F$ and $H = \kappa_0 \otimes G$ with $F, G \in \mathcal{S}_{fin}(\mathcal{H})$. From the definition (69) and Lemma 1 follows that also the product of the functions Φ_F and Φ_G can be integrated

$$\int \Phi_F(\xi,\xi^*)\Phi_G(\xi,\xi^*)dv(\xi,\xi^*) = \langle \exp\Omega \mid WF \odot WG\rangle \overset{(27)}{=} \langle F \mid JG\rangle\,. \tag{74}$$

For $\Xi = \kappa_0 \otimes F$ the antilinear mapping (72) is related to (25) by $(\Phi_{JF}(\xi,\xi^*))^* = \langle WJF \mid \exp(\xi+\xi^*)\rangle^* = \left\langle (WF)^\dagger \mid \exp(\xi+\xi^*)\right\rangle$. The identity (26) implies then that the sesquilinear form $\int \Phi_F^{\dagger}(\xi,\xi^*)\Phi_G(\xi,\xi^*)dv(\xi,\xi^*)$ is the inner product of $\mathcal{S}(\mathcal{H})$

$$\int \Phi_F^{\dagger}(\xi,\xi^*)\Phi_G(\xi,\xi^*)dv(\xi,\xi^*) = (F \mid G)\,. \tag{75}$$

Both sides of the identity (74) have a unique extension to tensors $\Xi = \lambda \otimes F$ and $H = \mu \otimes G$ with $\lambda, \mu \in \Lambda$ and $F, G \in \mathcal{S}_{fin}(\mathcal{H})$. The definitions (30) (31) (69) and (72) lead to an additional left factor λ^* and an additional right factor μ, see the proof for Lemma 8. Then Λ-linearity finally yields (73). □

So far the functions (69) have been introduced only for $\Xi \in \mathcal{S}^{\Lambda}_{fin}(\mathcal{H})$. But the forms (74) and (75) are well defined for the whole Fock space $\mathcal{S}(\mathcal{H})$, and they have their continuous extensions to $\mathcal{S}^{\Lambda}(\mathcal{H})$. We can therefore define the completion of the space of functions (69) to $\left\{\Phi_{\Xi}(\xi,\xi^*) \mid \Xi \in \mathcal{S}^{\Lambda}(\mathcal{H})\right\}$. Since Wick ordering is not continuous within $\mathcal{S}(\mathcal{H})$, the resulting functions have to be considered as generalized functions. We choose a triplet of Hilbert spaces $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$ with Hilbert-Schmidt embeddings. Then the Wick ordered tensor WF is defined as element of $\mathcal{S}(\mathcal{H})$ for all tensors F in a dense subset $\mathcal{S}_{(\gamma)}(\mathcal{H}^+) \subset \mathcal{S}(\mathcal{H}), \gamma > 0$, see Corollary 3 in App.A. Hence, if $\Xi \in \Lambda\widehat{\otimes}\mathcal{S}_{(\gamma)}(\mathcal{H}^+)$ with $\gamma > 0$, the functions (69) are differentiable functions, which can be analytically continued to functions in $\mathcal{F}\left(\mathcal{E}_{\Lambda}^{-} \times \mathcal{E}_{\Lambda}^{-*}\right)$. Since $\mathcal{S}_{(\gamma)}(\mathcal{H}^+)$ has a Hilbert topology, we can use the bilinear form (74) to define generalized functions Φ_H for all H in the (topological) dual space $\Lambda\widehat{\otimes}\mathcal{S}_{(-\gamma)}(\mathcal{H}^-)$. This space of generalized functions includes $\left\{\Phi_{\Xi}(\xi,\xi^*) \mid \Xi \in \mathcal{S}^{\Lambda}(\mathcal{H})\right\}$.

The space of functions $\{\Phi_F(\xi,\xi^*) \mid F \in \mathcal{S}(\mathcal{H})\}$ is by construction a Hilbert space with the (formal $\mathcal{L}^2$-) inner product (75). This space of functions is denoted as $\mathcal{W}(\mathcal{E}^{-}_{\Lambda\mathbb{R}})$, see the discussion of the domain of the supermeasure in Sect.6.2. As images of the Fock space $\mathcal{S}(\mathcal{H})$ the spaces $\mathcal{F}\left(\mathcal{E}_{\Lambda\mathbb{R}}\right)$ or $\mathcal{F}\left(\mathcal{E}_{\Lambda}^{-} \times \mathcal{E}_{\Lambda}^{-*}\right)$ and $\mathcal{W}(\mathcal{E}^{-}_{\Lambda\mathbb{R}})$ are isomorphic. The Λ-extension is again denoted by $\mathcal{W}^{\Lambda}(\mathcal{E}^{-}_{\Lambda\mathbb{R}})$. If $\Xi \in \mathcal{S}^{\Lambda}_{(\gamma)}(\mathcal{H}^+) = \Lambda\widehat{\otimes}\mathcal{S}_{(\gamma)}(\mathcal{H}^+)$ with $\gamma > 0$ both the functions $\Phi_{\Xi}(\xi,\xi^*) \in \mathcal{W}^{\Lambda}(\mathcal{E}^{-}_{\Lambda\mathbb{R}})$ and $\varphi_{\Xi}(\xi,\eta^*) \in \mathcal{F}^{\Lambda}\left(\mathcal{E}_{\Lambda}^{-} \times \mathcal{E}_{\Lambda}^{-*}\right)$ are differentiable functions, moreover φ_{Ξ} and Φ_{Ξ} can be analytically continued to functions on $\mathcal{E}_{\Lambda}^{-} \times \mathcal{E}_{\Lambda}^{-*}$, such that φ_{Ξ} and $\Phi_{\Xi} \in \mathcal{F}^{\Lambda}\left(\mathcal{E}_{\Lambda}^{-} \times \mathcal{E}_{\Lambda}^{-*}\right)$. As in the analysis of the bosonic Fock space [22][46], these functions can be related by the Gauss transform.

Lemma 10 *For* $\Xi \in \mathcal{S}^{\Lambda}_{(\gamma)}(\mathcal{H}^+), 0 < \gamma < 1$, *the functions (61) and (69) are related by the integral transforms*

$$\Phi_{\Xi}(\xi,\xi^*) = \int \varphi_{\Xi}\left(\xi + i\zeta, \xi^* + i\zeta^*\right) dv(\zeta,\zeta^*) \tag{76}$$

and

$$\varphi_\Xi(\xi, \eta^*) = \int \Phi_\Xi(\zeta + \xi, \eta^* + \xi^*) dv(\zeta, \zeta^*). \tag{77}$$

Proof For $\Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+), 0 < \gamma < 1$, the functions φ_Ξ and Φ_Ξ are elements of $\mathcal{F}^\Lambda_{(\alpha)}(\mathcal{E}^-_{\Lambda\mathbb{R}})$ with $0 < \alpha < \gamma$ and Lemma 6 implies that the functions $\varphi_\Xi(\xi + i\zeta, \xi^* + i\zeta^*)$ and $\Phi_\Xi(\zeta + \xi, \eta^* + \xi^*)$ are integrable. Following Lemma 2 it is sufficient to verify the identities (76) and (77) for coherent states. If $\varsigma, \tau \in \mathcal{H}^+$ then $\Xi = \exp(\varsigma + \tau^*) \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+) \subset \mathcal{S}^\Lambda(\mathcal{H})$ has the Bargmann-Fock representation $\varphi_\Xi(\xi, \eta^*) = \langle \exp(\xi + \eta^*) \mid \exp(\varsigma + \tau^*) \rangle = e^{\langle \xi | \tau^* \rangle + \langle \eta^* | \varsigma \rangle}$, see (90) in App.C. The Wick ordered form of Ξ is $\mathrm{W} \exp(\varsigma + \tau^*) = \exp(\varsigma + \tau^*) e^{-\langle \tau^* | \varsigma \rangle} \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+)$, see (93), and it is represented by the function

$$\Phi_\Xi(\zeta, \zeta^*) = \langle \exp(\zeta + \zeta^*) \mid \exp(\varsigma + \tau^*) \rangle \, e^{-\langle \tau^* | \varsigma \rangle} = e^{\langle \zeta | \tau^* \rangle + \langle \zeta^* | \varsigma \rangle - \langle \tau^* | \varsigma \rangle} \tag{78}$$

in the Wiener-Segal representation. The analytic continuation of this function is $\Phi_\Xi(\xi, \eta^*) = e^{\langle \xi | \tau^* \rangle + \langle \eta^* | \varsigma \rangle - \langle \tau^* | \varsigma \rangle} \in \mathcal{F}\left(\mathcal{E}^-_\Lambda \times \mathcal{E}^{-*}_\Lambda\right)$. The identities (76) and (77) can now easily be derived from the Laplace transform (57) of the supermeasure. The integral on the right side of (76) is calculated as $\int e^{\langle \zeta + i\eta | \tau^* \rangle + \langle \zeta^* + i\eta^* | \varsigma \rangle} dv(\eta, \eta^*) = e^{\langle \xi | \tau^* \rangle + \langle \eta^* | \varsigma \rangle} e^{-\langle \tau^* | \varsigma \rangle} = \Phi_\Xi(\zeta, \zeta^*)$, and (77) follows from $\int e^{\langle \zeta + \xi | \tau^* \rangle + \langle \zeta^* + \eta^* | \varsigma \rangle - \langle \tau^* | \varsigma \rangle} dv(\zeta, \zeta^*) = e^{\langle \xi | \tau^* \rangle + \langle \eta^* | \varsigma \rangle} = \varphi_\Xi(\xi, \eta^*)$. □

We can restrict the argument in (77) to the real subspace with $\xi = \eta$. Then we do not need the analytic continuation of Φ_Ξ, and (77) is an integral transform from a dense subspace of $\mathcal{W}^\Lambda(\mathcal{E}^-_{\Lambda\mathbb{R}})$ into the reproducing kernel space $\mathcal{F}^\Lambda(\mathcal{E}_{\Lambda\mathbb{R}})$.

8.2 The Ornstein-Uhlenbeck semigroup

With help of the Wiener-Segal representation many results of classical analysis can be transferred to superanalysis. As example we consider the Ornstein-Uhlenbeck semigroup on the Fock space $\mathcal{S}(\mathcal{H})$, and its representation by the Mehler formula. For the bosonic case this semigroup and the Mehler formula have been investigated in detail by Meyer [30] and by Kusuoka [29]. The role of the Ornstein-Uhlenbeck semigroup in the quantum field theory of bosons can clearly be seen in [39]. So far results for the fermionic Fock space have only been obtained using an ordering prescription, which is unstable against perturbations. [27]

To avoid technical complications we assume that A is a strictly positive-definite selfadjoint operator on $\mathcal{H}$ with a pure point spectrum satisfying the additional properties

i) The operator A is even, i.e. $A\mathcal{H}_{\overline{k}} \subset \mathcal{H}_{\overline{k}}, k = 0, 1$.

ii) The operator A is real, i.e. $Af^* = (Af)^*$ for $f \in \mathcal{H}$.

Then $e^{-At}, t \geq 0$, is a contraction semigroup on $\mathcal{H}$, and $\Gamma(e^{-At}), t \geq 0$, is by definition the Ornstein-Uhlenbeck semigroup with generator $d\Gamma(A)$ on $\mathcal{S}(\mathcal{H})$.

On the dense subset $\left\{ \Phi_\Xi(\xi, \xi^*) \mid \Xi \in \mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H}^+), 0 < \gamma < 1 \right\} \subset \mathcal{W}(\mathcal{E}^-_{\Lambda\mathbb{R}})$ we define for $t \geq 0$ the transformations

$$(P_t\Phi)(\zeta, \zeta^*) = \int \Phi(e^{-\mathrm{A}t}\zeta - \sqrt{I - e^{-2\mathrm{A}t}}\eta, e^{-\mathrm{A}t}\zeta^* - \sqrt{I - e^{-2\mathrm{A}t}}\eta^*) dv(\eta, \eta^*). \tag{79}$$

For the classical Wiener-Segal representation this is the Mehler formula of the Ornstein-Uhlenbeck semigroup, see e.g. [30][29]. But also in superanalysis we have

Lemma 11 *The Ornstein-Uhlenbeck semigroup* $\Gamma(e^{-At}), t \geq 0$, *is represented by the family of transformations (79).*

Proof It is sufficient to give the proof for coherent states. The Ornstein-Uhlenbeck semigroup has a unique extension $\Gamma(e^{-At}), t \geq 0$, to the space $\mathcal{S}^{\Lambda}(\mathcal{H})$. These transformations map the coherent state $\Xi = \exp(\varsigma + \tau^*) \in \mathcal{S}^{\Lambda}(\mathcal{H})$ onto $\Gamma(e^{-At})\Xi = \exp(e^{-At}\varsigma + e^{-At}\tau^*)$. The Wick ordered form is $W\Gamma(e^{-At})\Xi = \exp(e^{-At}\varsigma + e^{-At}\tau^*)e^{-\langle e^{-2At}\tau^* | \varsigma \rangle}$.

The integral (79) can easily be calculated for the function (78). The Fourier-Laplace transform (57) yields

$$\begin{aligned}
(P_t\Phi_\Xi)(\zeta, \zeta^*) &= e^{\langle e^{-At}\zeta | \tau^* \rangle + \langle e^{-At}\zeta^* | \varsigma \rangle - \langle e^{-2At}\tau^* | \varsigma \rangle} = e^{\langle \zeta | e^{-At}\tau^* \rangle + \langle \zeta^* | e^{-At}\varsigma \rangle - \langle e^{-2At}\tau^* | \varsigma \rangle} \\
&= \left\langle \exp(\zeta + \zeta^*) \mid \exp(e^{-At}\varsigma + e^{-At}\tau^*) \right\rangle e^{-\langle e^{-2At}\tau^* | \varsigma \rangle} \\
&= \left\langle \exp(\zeta + \zeta^*) \mid W\Gamma(e^{-At})\Xi \right\rangle .
\end{aligned}$$

Hence (79) is a representation of the Ornstein-Uhlenbeck semigroup. □

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A Norm estimates for tensor algebras

In this section we present Hilbert norm estimates for rather general $\mathbb{Z}$-graded algebras $\mathcal{A}$ over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.. We assume a structure $\mathcal{A} = \oplus_{n=0}^{\infty}\mathcal{A}_n$ as considered for the tensor algebras in Sect.3.2. Thereby the one dimensional space $\mathcal{A}_0 = \mathbb{K}$ is spanned by the unit e, and the product $F \circ G$ maps $\mathcal{A}_p \times \mathcal{A}_q$ into $\mathcal{A}_{p+q}$ for all $p, q \in \{0, 1, ...\}$. The algebra is provided with the Hilbert norm

$$\|F\|^2 = \sum_{n=0}^{\infty} w(n) \|F_n\|_n^2 \quad \text{if} \quad F = \sum_{n=0}^{\infty} F_n \quad \text{with} \quad F_n \in \mathcal{A}_n. \tag{80}$$

Here $\|.\|_n$ is a Hilbert norm of $\mathcal{A}_n$, and the factors $w(n)$ are positive weights with the normalization $w(0) = 1$. We assume a norm estimate of the product of homogeneous elements as given in (10)[11]

$$\|F_p \circ G_q\|_{p+q}^2 \leq \frac{(p+q)!}{p!q!} \|F_p\|_p^2 \|G_q\|_q^2 \tag{81}$$

if $F_p \in \mathcal{A}_p$ and $G_q \in \mathcal{A}_q$.

Proposition 2 *If the norm is defined with the weights* $w(n) = (n!)^\gamma, \gamma \leq -2$, *then the product of the algebra is continuous with the uniform norm estimate*

$$\|F \circ G\| \leq \sqrt{3}\, \|F\| \, \|G\| \, . \tag{82}$$

Proof The norm of $F \circ G$ for $F = \sum_{n=0}^{\infty} F_n$ and $G = \sum_{n=0}^{\infty} G_n$ with $F_n, G_n \in \mathcal{A}_n$ is given by

$$\begin{aligned}
&\|F \circ G\|^2 = \left\| \textstyle\sum_{m,n} F_m \circ G_n \right\|^2 \\
&\leq |F_0 G_0|^2 + 3 \left(|F_0|^2 \left\| \textstyle\sum_{n=1}^{\infty} G_n \right\|^2 + \left\| \textstyle\sum_{m=1}^{\infty} F_m \right\|^2 |G_0|^2 + \left\| \textstyle\sum_{m\geq 1, n \geq 1} F_m \circ G_n \right\|^2 \right) \\
&\leq |F_0 G_0|^2 + 3 \textstyle\sum_{n \geq 1} w(n) \left(|F_0|^2 \|G_n\|_n^2 + \|F_n\|_n^2 |G_0|^2 + \left\| \textstyle\sum'_{p+q=n} F_p \circ G_q \right\|_n^2 \right)
\end{aligned}$$

[11] Here we keep the normalizations (10) of the superalgebra in Sect.3.2. The factor $\frac{(p+q)!}{p!q!}$ can be easily absorbed by a redefinitionn of the norms $\|.\|_p$.

The symbol $\sum'$ means summation with the constraint $p \geq 1, q \geq 1$. The sum $\sum_{p+q=n, p\geq 1, q\geq 1} \ldots = \sum'_{p+q=n} \ldots$ has $n-1$ terms, hence

$$\left\| \sum'_{p+q=n} F_p \circ G_q \right\|_n^2 \leq (n-1) \sum'_{p+q=n} \|F_p \circ G_q\|_n^2 \overset{(81)}{\leq} (n-1) \sum'_{p+q=n} \binom{n}{p} \|F_p\|_p^2 \|G_q\|_q^2.$$

If $w(n)$ is chosen such that

$$(p+q-1)\frac{(p+q)!}{p!q!} w(p+q) \leq w(p)w(q) \quad \text{for all } \; p, q \geq 1, \tag{83}$$

we finally obtain $\sum_{n\geq 1} w(n) \left\| \sum'_{p+q=n} F_p \circ G_q \right\|_n^2 \leq \left(\sum_{p\geq 1} w(p) \|F_p\|_p^2 \right) \left(\sum_{q\geq 1} w(q) \|G_q\|_q^2 \right)$, hence

$$\begin{aligned} &\|F \circ G\|^2 \\ &\leq |F_0 G_0|^2 + 3 \left(|F_0|^2 \left\| \sum_{n=1}^\infty G_n \right\|^2 + \left\| \sum_{m=1}^\infty F_m \right\|^2 |G_0|^2 + \left\| \sum_{m=1}^\infty F_m \right\|^2 \left\| \sum_{n=1}^\infty G_n \right\|^2 \right) \\ &\leq 3 \|F\|^2 \|G\|^2 . \end{aligned}$$

With $w(n) = \frac{1}{n!}\alpha(n)$ the inequalities (83) are equivalent to $(p+q-1)\alpha(p+q) \leq \alpha(p)\alpha(q)$ for all $p, q \geq 1$. A function which satisfies these constraints is $\alpha(n) = (n!)^{-\gamma}, \gamma \geq 1$. □

It might be possible to derive an estimate (82) with a constant $c < \sqrt{3}$, but the value $c = 1$ is definitely excluded due to the following

Lemma 12 *Let $\mathcal{A}$ be an algebra over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with dimension* $\dim \mathcal{A} \geq 2$. *If this algebra satisfies the properties*
i)$\mathcal{A}$ is provided with a Hilbert inner product $(.\mid.)$ normalized at the unit e, $\|e\|^2 = (e \mid e) = 1$,
ii)there exists at least one element $f \in \mathcal{A}, f \neq 0$, such that each two of the elements e, f and $f^2 = f \circ f$ are orthogonal,
then the norm estimate $\|F \circ G\| \leq c \|F\| \|G\|$ is not valid for some $F, G \in \mathcal{A}$, if $c < \sqrt{\frac{4}{3}}$.

Proof Since $f \neq 0$ we can normalize it and assume $\|f\| = 1$. Take $F = e + \alpha f$ with $\alpha \in \mathbb{R}$.. Then $F^2 = e + 2\alpha f + \alpha^2 f^2$ and $\|F^2\|^2 = 1 + 4\alpha^2 + \alpha^4 \|f^2\|^2$. On the other hand $\|F\|^2 = 1 + \alpha^2$, and $\|F^2\|^2 \leq c^2 \|F^2\|^4$ implies $1 + 4\alpha^2 + \alpha^4 \|f^2\| \leq c^2(1+\alpha^2)^2$. But this inequality is true for all $\alpha \geq 0$ only if $c^2 \geq \sup_{\alpha\geq 0} \frac{1+4\alpha^2}{(1+\alpha^2)^2} = \frac{4}{3}$. □

This Lemma applies to $\mathbb{Z}$-graded algebras as considered above with any(!) choice of the weights $w(n) > 0, w(0) = 1$. For these algebras we can simply choose f from the generating space $\mathcal{A}_1$. The Lemma is true for any algebra with unit, which has two linearly independent nilpotent elements f_1 and f_2. In that case there always exists a nilpotent element $\alpha f_1 + \beta f_2 \neq 0$, which is orthogonal to the unit element, such that the second condition is satisfied.

The Grassmann superalgebra Λ of the main text is exactly chosen as the antisymmetric tensor algebra with the product defined according to (8) with projection onto antisymmetric tensors, and a norm (80) with $w(n) = (n!)^{-2}$.

By a slight modification of the estimates given for the proof of Proposition 2 one can obtain another interesting result. Let $w_\gamma(n), \gamma \in \mathbb{R}$, be a one parameter family of weights with $w_\alpha(n) \leq w_\beta(n)$ if $\alpha \leq \beta$ for all $n \in \mathbb{N}$. The corresponding norms (80) are denoted by $\|F\|_{(\gamma)}$. If these weights satisfy[12]

$$(p+q+1)\frac{(p+q)!}{p!q!} w_\gamma(p+q) \leq c \cdot w_\alpha(p) w_\beta(q) \quad \text{for } \; p, q \geq 0, \tag{84}$$

[12]For this estimate the one dimensional space generated by the unit has not to be separated, and the sum $\sum_{p+q=n} \ldots$ contains $n+1$ terms.

for some $\alpha, \beta, \gamma \in \mathbb{R}$, then the product of $\mathcal{A}$ is estimated by $\|F \circ G\|_{(\gamma)} \leq \sqrt{c} \cdot \|F\|_{(\alpha)} \|G\|_{(\beta)}$ with exactly that constant which appears in (84). If we now choose $w_\gamma(n) = (n!)^\gamma$, the estimate (84) is satisfied if $\gamma < \min(\alpha, \beta)$, because $(p+q)! \leq 2^{p+q} p! q!$, and $p \in \{0, 1, ...\} \to (p+1) 2^p (p!)^{-1-\varepsilon}$ is a bounded function for any fixed $\varepsilon > 0$. Hence we have derived

Proposition 3 *If the algebra $\mathcal{A}$ is equipped with the family of norms $\|.\|_{(\gamma)}$ induced by the weight functions $w_\gamma(n) = (n!)^\gamma, \gamma \in \mathbb{R}$, the product satisfies the norm estimates*

$$\|F \circ G\|_{(\gamma)} \leq c \cdot \|F\|_{(\alpha)} \|G\|_{(\beta)}$$

if $\gamma < \min(\alpha, \beta)$ with a constant c, which depends on the parameters α, β and γ.

The norm estimates of this appendix can be applied to the superalgebra $\mathcal{S}_{fin}(\mathcal{H})$ of Sect. 3.2. If we chose a norm (80) with a weight $w(n) = (n!)^\gamma, \gamma \in \mathbb{R}$, the completion of $\mathcal{S}_{fin}(\mathcal{H})$ is denoted as $\mathcal{S}_{(\gamma)}(\mathcal{H})$. These spaces satisfy the inclusions $\mathcal{S}_{(\alpha)}(\mathcal{H}) \subset \mathcal{S}_{(\beta)}(\mathcal{H})$ if $\alpha \leq \beta$. The Fock space of Sect 3.2 is $\mathcal{S}_{(0)}(\mathcal{H}) = \mathcal{S}(\mathcal{H})$, and the spaces $\mathcal{S}_{(-\gamma)}(\mathcal{H})$ and $\mathcal{S}_{(\gamma)}(\mathcal{H})$ are dual with respect to the pairing (12). Following Proposition 3 the graded tensor product (8) can be extended from $\mathcal{S}_{fin}(\mathcal{H})$ to any space $\mathcal{S}_{(\gamma)}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ with $\gamma < 0$, such that $F, G \in \mathcal{S}_{(\gamma)}(\mathcal{H}) \to F \odot G \in \mathcal{S}(\mathcal{H})$.

For the constructions presented in Sects. 4-8 the convergence of the exponential series of vectors (coherent states) and of tensors of rank 2 (Gaussian functionals) are important. The bound (81) (or the equivalent estimate (10)) imply

$$\left\| f^{\odot n} \right\|_n \leq \sqrt{n!} \, \|f\|^n \quad \text{if } f \in \mathcal{H} \quad \text{and} \quad \left\| F^{\odot n} \right\|_{2n}^2 \leq \frac{(2n)!}{2^{2n}} \|F\|_2^{2n} \quad \text{if } F \in \mathcal{H}^{\odot 2}. \tag{85}$$

A simple estimate then leads to

$$\exp f \in \mathcal{S}_{(\gamma)}(\mathcal{H}) \ \text{ if } \gamma < 1, \quad \text{and} \quad \exp F \in \mathcal{S}_{(\gamma)}(\mathcal{H}) \ \text{ if } \gamma < 0. \tag{86}$$

The convergence of $\exp F$ within $\mathcal{S}_{(0)}(\mathcal{H})$ can only be derived from (85), if the norm of F is small.

The tensor Ω related to the bilinear continuous form (17) is not an element of $\mathcal{S}_2(\mathcal{H})$. We have to choose a triplet of Hilbert spaces $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$ with Hilbert-Schmidt embeddings[13], then $\Omega \in \mathcal{S}_2(\mathcal{H}^-)$. Following (86) the tensor $\exp \Omega$ is an element of $\mathcal{S}_{(-\alpha)}(\mathcal{H}^-)$ with arbitrarily small $\alpha > 0$, and the functional (19) is continuous on the dual space $\mathcal{S}_{(\alpha)}(\mathcal{H}^+)$. To obtain a domain on which Wick ordering is defined we first derive

Proposition 4 *If $Y \in \mathcal{S}_{(-\alpha)}(\mathcal{H}^-)$ and $F \in \mathcal{S}_{(\gamma)}(\mathcal{H}^+), 0 < \alpha < \gamma$, then $Y \lrcorner F \in \mathcal{S}_{(\alpha)}(\mathcal{H}^+) \subset \mathcal{S}(\mathcal{H})$.*

Proof Let $H \in \mathcal{S}_{(-\alpha)}(\mathcal{H}^-)$. Following Proposition 3 the tensor $H \odot Y$ is an element of $\mathcal{S}_{(-\gamma)}(\mathcal{H}^-)$, and $F \in \mathcal{S}_{(\gamma)}(\mathcal{H}^+) \to \langle H \odot Y \mid F \rangle$ is a continuous functional, which satisfies $|\langle H \odot Y \mid F \rangle| \leq const \, \|H\|_{(-\alpha)} \|F\|_{(\gamma)}$ with the norms of $\mathcal{S}_{(-\alpha)}(\mathcal{H}^-)$ and $\mathcal{S}_{(\gamma)}(\mathcal{H}^+)$, respectively. Since $\langle H \odot Y \mid F \rangle = \langle H \mid Y \lrcorner F \rangle$ by definition of the contraction, the tensor $Y \lrcorner F$ is an element of $\mathcal{S}_{(\alpha)}(\mathcal{H}^+) \subset \mathcal{S}(\mathcal{H})$. □

Corollary 3 *If $F \in \mathcal{S}_{(\gamma)}(\mathcal{H}^+), \gamma > 0$, then $WF = \exp(-\Omega) \lrcorner F \in \mathcal{S}_{(\alpha)}(\mathcal{H}^+) \subset \mathcal{S}(\mathcal{H})$ with $0 < \alpha < \gamma$.*

[13] These embeddings should induce the corresponding triplets for the subspaces $\mathcal{E}_{\overline{0}}, \mathcal{E}_{\overline{1}}, \mathcal{E}_{\overline{0}}^*$ and $\mathcal{E}_{\overline{1}}^*$, such that also $\mathcal{H}^-$ and $\mathcal{H}^+$ are Hilbert spaces with the structure presented in Chap.3.1.

B Polynomials on Grassmann Algebras

In this appendix we derive statements about polynomials defined on an infinite dimensional Grassmann algebra Λ with involution. The Grassmann algebra Λ is the antisymmetric tensor algebra over a complex Hilbert space $\mathcal{F}$. It has the direct sum structure $\Lambda = \oplus_{p=0}^{\infty}\Lambda_p$ with $\Lambda_p = \mathcal{T}_p^-(\mathcal{F})$, and it is provided with a Hilbert norm (80) such that the product is continuous. For the proofs it is convenient to use a real orthonormal basis $\kappa_a = \kappa_a^* \in \mathcal{F} = \Lambda_1, a \in \mathbf{N}$. Then $\kappa_{\mathbf{A}} = (-i)\kappa_{a_1}...\kappa_{a_n}, \mathbf{A} = \{a_1 < ... < a_n\} \subset \mathbb{N}$ forms a real basis of Λ_n. Any element $\rho \in \Lambda$ has the representation $\rho = \sum_{\mathbf{A}\subset P(\mathbb{N})} h(\mathbf{A})e_{\mathbf{A}}$ with complex coefficients $h(\mathbf{A})$, where $P(\mathbb{N})$ is the power set of $\mathbb{N}$, i.e. the set of all finite subsets of $\mathbb{N}$.

Proposition 5 *If an element ρ of the Grassmann algebra Λ satisfies one of the following conditions*

$$\begin{cases} i)\ \rho\lambda = 0 \quad \textit{for all } \lambda \in \Lambda_1, \quad \textit{or} \\ ii)\ \rho\lambda\lambda^* = 0 \quad \textit{for all } \lambda \in \Lambda_1, \end{cases}$$

then $\rho = 0$.

Proof Any element of the Grassmann algebra Λ has the series representation $\rho = \sum_{\mathbf{A}\in P(\mathbb{N})} h(\mathbf{A})\kappa_{\mathbf{A}}$. To derive the consequences of assumption i) we choose for λ a basis element $\lambda = \kappa_b, b \in \mathbb{N}$. Then $0 = \rho\kappa_b = \sum_{\mathbf{A}\in P(\mathbb{N}),\mathbf{A}\cap\{b\}=\emptyset} h(\mathbf{A})e_{\mathbf{A}\cup\{b\}}$ implies $h(\mathbf{A}) = 0$ for all $\mathbf{A} \in P(\mathbb{N})$ with $\mathbf{A}\cap\{b\} = \emptyset$. Any set $\mathbf{A} \in P(\mathbb{N})$ is a finite set, and for any $\mathbf{A} \in P(\mathbb{N})$ there exist $b \in \mathbb{N}$, such that $\mathbf{A}\cap\{b\} = \emptyset$ is satisfied. Hence assumption i) implies $h(\mathbf{A}) = 0$ for all $\mathbf{A} \in P(\mathbb{N})$.

To derive the consequences of assumption ii) we choose $\lambda = \kappa_b + i\kappa_{b+1}, b \in \mathbb{N}$. Then assumption ii) implies $h(\mathbf{A}) = 0$ for all $\mathbf{A} \in P(\mathbb{N})$ with $\mathbf{A}\cap\{b, b+1\} = \emptyset$. Since $b \in \mathbb{N}$ is arbitrary, we obtain again $h(\mathbf{A}) = 0$ for all $\mathbf{A} \in P(\mathbb{N})$ without restriction. □

In Sect.7.1 we have to evaluate continuous polynomials on the underlying real space of $\Lambda^{1,n} = \Lambda_{\overline{0}} \times (\Lambda_{\overline{1}})^n$. These polynomials have the structure

$$\begin{aligned} &\lambda = (\alpha, \vartheta_1, ..., \vartheta_n) \in \Lambda_{\mathbb{R}}^{1,n} \to \\ &\mathrm{pol}(\lambda) = \textstyle\sum_{(p,q,\mathbf{A},\mathbf{B})} \rho(p, q, \mathbf{A}, \mathbf{B})\alpha^p\alpha^{*q}\vartheta_{\mathbf{A}}\vartheta_{\mathbf{B}}^* \in \Lambda, \end{aligned}$$

where the sum extends over all numbers $p = 0, 1, ..., m$ and $q = 0, 1, ..., m$ and all ordered subsets $\mathbf{A}, \mathbf{B} \subset \{1, ..., n\}$. The factors $\vartheta_{\mathbf{A}}$ and $\vartheta_{\mathbf{B}}^*$ are the fermionic monomials $\vartheta_{\mathbf{A}} = \vartheta_{a_1}...\vartheta_{a_r}$ and $\vartheta_{\mathbf{B}}^* = \vartheta_{b_1}^*...\vartheta_{b_s}^*$ if $\mathbf{A} = \{a_1 < a_2 < ... < a_r\}$ and $\mathbf{B} = \{b_1 < b_2 < ... < b_s\}$. The number of elements of these sets is $|\mathbf{A}| = r \le n$ and $|\mathbf{B}| = s \le n$. The coefficients $\rho(p, q, \mathbf{A}, \mathbf{B})$ are arbitrary elements of Λ. The polynomial is homogeneous of degree N if only terms with $p+q+r+s = N$ contribute.

Lemma 13 *The polynomial* $\mathrm{pol}(\lambda)$ *is identical zero on* $\Lambda_{\mathbb{R}}^{1,n}$ *if and only if all coefficients* ρ *vanish.*

Proof If all $\rho(p, q, \mathbf{A}, \mathbf{B}) = 0$ then $\mathrm{pol}(\lambda) = 0$ follows obviously for $\lambda \in \Lambda_{\mathbb{R}}^{1,n}$. To derive the inverse statement we substitute the arguments $\alpha \in \Lambda_{\overline{0}}$ by $z\alpha \in \Lambda_{\overline{0}}$ and $\vartheta_a \in \Lambda_{\overline{1}}$ by $z_a\vartheta_a \in \Lambda_{\overline{1}}$ with complex numbers $z = x + iy, z_a = x_a + iy_a$ where $x, x_a \in \mathbb{R}$ and $y, y_a \in \mathbb{R}, a = 1, ..., n$. The function $\mathrm{pol}(z\alpha, ...)$ is then a polynomial in the real variables x, x_a and y, y_b, or, equivalently, a polynomial in the variables z, z_a and $\overline{z}, \overline{z}_b$, $a, b = 1, ..., n$. Since the multiplication is continuous within Λ, this polynomial is a differentiable function in z, z_a and $\overline{z}, \overline{z_b}$, and the partial derivative $\left(\frac{\partial}{\partial z}\right)^p \left(\frac{\partial}{\partial z_{a_1}}\right) \cdots \left(\frac{\partial}{\partial z_{a_r}}\right) \left(\frac{\partial}{\partial \overline{z}}\right)^q \left(\frac{\partial}{\partial \overline{z}_{b_1}}\right) \cdots \left(\frac{\partial}{\partial \overline{z}_{b_s}}\right) \mathrm{pol}(z\alpha, ...)$ at $z = z_1 = ... = \overline{z}_n$ is exactly $\rho(p, q, \mathbf{A}, \mathbf{B})p!q!\alpha^p\alpha^{*q}\vartheta_{\mathbf{A}}\vartheta_{\mathbf{B}}^*$. If $\mathrm{pol}(\lambda)$ is identical zero, all derivatives vanish, and we have $\rho(p,q,\mathbf{A},\mathbf{B})\alpha^p\alpha^{*q}\vartheta_{\mathbf{A}}\vartheta^*_{\mathbf{B}}=0$ for all $p,q=0,1,...,m$ and all subsets $\mathbf{A},\mathbf{B}\subset\{1,...,n\}$ with arbitrary $(\alpha,\vartheta_1,...,\vartheta_n)\in\Lambda^{1,n}$. For the bosonic arguments we can choose $\alpha=\alpha^*=id$. The remaining identities with fermionic factors $\vartheta_a\in\Lambda_1, a=1,...,n$, can then be reduced by induction with the help of Proposition 5 to $\rho(p,q,\mathbf{A},\mathbf{B})=0$. □

Since the proof has only used arguments $\alpha\in\mathbb{C}$ and $\vartheta_a\in\Lambda_1\subset\Lambda_{\overline{1}}$ we have even derived

Corollary 4 *The polynomial* $\mathrm{pol}(\lambda)$ *is uniquely determined by arguments* $\lambda=(\alpha,\vartheta_1,...,\vartheta_n)$ *with* $\alpha\in\mathbb{C}\subset\Lambda_{\overline{0}}$ *and* $\vartheta_a\in\Lambda_1\subset\Lambda_{\overline{1}}, a=1,...,n$.

Lemma and Corollary have an obvious generalization to polynomials on the spaces $\Lambda^{m,n}=(\Lambda_{\overline{0}})^m\times(\Lambda_{\overline{1}})^n$, with $m,n\in\mathbb{N}$.

C Calculations for coherent states

C.1 Basic identities

The Λ-bilinear forms $(\xi,\eta)\in\mathcal{H}_\Lambda\times\mathcal{H}_\Lambda\to\langle\xi\mid\eta\rangle\in\Lambda$ and $\omega(\xi,\eta)\in\Lambda$ are uniquely defined by linear extension to the algebraic superspace $\mathcal{H}^{alg}_\Lambda=\Lambda_{\overline{0}}\otimes\mathcal{H}_{\overline{0}}\oplus\Lambda_{\overline{1}}\otimes\mathcal{H}_{\overline{1}}$ and by closure. The form ω is symmetric on $\mathcal{H}_\Lambda\times\mathcal{H}_\Lambda$, and we have

$$\omega(\xi,\eta)=\langle\xi\mid \mathrm{j}\eta\rangle \tag{87}$$

for all $\xi,\eta\in\mathcal{H}_\Lambda$, where j is the Λ-extension of (4).If $\xi\in\mathcal{E}^*_\Lambda$ and/or $\eta\in\mathcal{E}_\Lambda=\Lambda_{\overline{0}}\otimes\mathcal{E}_{\overline{0}}\oplus\Lambda_{\overline{1}}\otimes\mathcal{E}_{\overline{1}}$, this identity implies $\langle\xi\mid\eta\rangle=\omega(\xi,\eta)$. The bilinear form (30) has the factorization

$$\langle\xi_m\cdots\xi_1\mid\eta_1\cdots\eta_n\rangle=\delta_{mn}\sum_\sigma\left\langle\xi_1\mid\eta_{\sigma(1)}\right\rangle...\left\langle\xi_n\mid\eta_{\sigma(n)}\right\rangle \tag{88}$$

for $\xi_a,\eta_b\in\mathcal{H}_\Lambda, a=1,...,m$ and $b=1,...,n$, see (13). The summation extends over all permutations σ of $\{1,...,n\}$. Using the identity (87) we finally obtain with the Λ-extension of (23), denoted by J,

$$\langle\xi_m\cdots\xi_1\mid \mathrm{J}\eta_1\cdots\eta_n\rangle=\delta_{mn}\sum_\sigma\omega(\xi_1,\eta_{\sigma(1)})...\omega(\xi_n,\eta_{\sigma(n)}). \tag{89}$$

C.2 Bilinear forms and contractions of coherent states

The bilinear pairing (88) of two coherent states (39) yields

$$\langle\exp\xi\mid\exp\eta\rangle=\sum_{p=0}^\infty\left(\frac{1}{p!}\right)^2 p!\,\langle\xi\mid\eta\rangle^p=e^{\langle\xi|\eta\rangle}\in\Lambda_{\overline{0}}. \tag{90}$$

Since $(\exp\xi)^*=\exp\xi^*$, this identity implies $(\exp\xi\mid\exp\eta)=e^{(\xi,\eta)}$. The contraction (15) has a Λ-extension which we denote by the same symbol $\lrcorner$. Since $\exp\eta\in\mathcal{S}^\Lambda_{(\gamma)}(\mathcal{H})$ for any $\gamma<1$ the contraction $\exp\eta\lrcorner\,\Xi$ is a well defined element of $\mathcal{S}^\Lambda(\mathcal{H})$ if $\Xi\in\mathcal{S}^\Lambda_{(\alpha)}(\mathcal{H})$ with $\alpha>0$. The proof of this statement follows by $\Lambda-$extension from Proposition 4 in App. A. As simple application we calculate

$$\exp\eta\lrcorner\exp\xi=e^{\langle\xi|\eta\rangle}\exp\xi \tag{91}$$

if $\xi,\eta\in\mathcal{H}_\Lambda$. This identity follows from
$\langle\exp\zeta\mid\exp\eta\lrcorner\exp\xi\rangle=\langle\exp(\eta+\zeta)\mid\exp\xi\rangle\overset{(90)}{=}e^{\langle\eta+\zeta|\xi\rangle}=e^{\langle\xi|\eta\rangle}\langle\exp\zeta\mid\exp\xi\rangle$, and Lemma 5. The result is also true for arguments $\xi\in\mathcal{H}^+_\Lambda,\eta\in\mathcal{H}^-_\Lambda$.

C.3 Gaussian functionals and Wick ordering

For $\xi, \eta \in \mathcal{E}_\Lambda$ we have $\exp(\xi + \eta^*) = (\exp \eta)^* \cdot (\exp \xi) \in \mathcal{S}(\mathcal{E}_\Lambda^*) \cdot \mathcal{S}(\mathcal{E}_\Lambda)$, and $\langle \exp \Omega \mid \exp(\xi + \eta^*) \rangle = (\exp \eta \mid \exp \xi) = e^{(\eta|\xi)} = e^{\frac{1}{2}\omega(\xi+\eta^*,\xi+\eta^*)}$ follows from (20) and (90). Hence

$$\langle \exp \Omega \mid \exp \zeta \rangle = e^{\frac{1}{2}\omega(\zeta,\zeta)} \tag{92}$$

follows for all $\zeta \in \mathcal{H}_\Lambda$.

For Wick ordering we first take a $\xi \in \mathcal{H}_\Lambda^+$ then $\exp \xi \in \mathcal{S}_{(\gamma)}^\Lambda(\mathcal{H}^+)$ with $0 < \gamma < 1$. Following Proposition 4 in App.A the contractions $\exp(\pm\Omega) \lrcorner \exp \xi$ are defined, and we calculate $\left\langle \exp \eta \mid \mathrm{W}^{\pm 1} \exp \xi \right\rangle = \langle \exp \eta \cdot \exp(\mp\Omega) \mid \exp \xi \rangle = \langle \exp(\mp\Omega) \mid \exp \eta \lrcorner \exp \xi \rangle \overset{(91)}{=}$ $e^{(\eta|\xi)} \langle \exp(\mp\Omega) \mid \exp \xi \rangle = e^{(\eta|\xi) \mp \frac{1}{2}\omega(\xi,\xi)}$ for arbitrary $\eta \in \mathcal{H}_\Lambda^-$. Then (90) and Lemma 5 yield

$$\mathrm{W}^{\pm 1} \exp \xi = e^{\mp \frac{1}{2}\omega(\xi,\xi)} \exp \xi. \tag{93}$$

This result has a unique continuous extension to $\xi \in \mathcal{H}_\Lambda$.

For the proof of the fundamental identity (22) we first derive

$$\langle \exp \Omega \mid (\exp \xi)\, (\exp \eta) \rangle = \left\langle \mathrm{W}^{-1} \exp \xi \mid \mathrm{J}\mathrm{W}^{-1} \exp \eta \right\rangle \tag{94}$$

with $\xi, \eta \in \mathcal{H}_\Lambda$. Following (90) the left side of (94) is $e^{\frac{1}{2}\omega(\xi+\eta,\xi+\eta)}$. The right side is calculated with the help of (89) and (93) as $e^{\frac{1}{2}\omega(\xi,\xi)} e^{\frac{1}{2}\omega(\eta,\eta)} \langle \exp \xi \mid \mathrm{J} \exp \eta \rangle = e^{\frac{1}{2}\omega(\xi,\xi) + \frac{1}{2}\omega(\eta,\eta) + \omega(\xi,\eta)}$, and (94) is valid for all $\xi, \eta \in \mathcal{H}_\Lambda$. The bilinear mapping $(F, G) \in \mathcal{S}_{fin}(\mathcal{H}) \times \mathcal{S}_{fin}(\mathcal{H}) \to \langle \exp \Omega \mid F \odot G \rangle$ has a unique Λ-extension. The identity (94) and the arguments used for the derivation of Lemma 2 imply that

$$\langle \exp \Omega \mid F \odot G \rangle = \left\langle W^{-1} F \mid J W^{-1} G \right\rangle \tag{95}$$

is true for all $F, G \in \mathcal{S}_{fin}(\mathcal{H})$. Since $(W^{-1}F)^* = W^{-1}F^*$, this identity is equivalent to (22).

D Gaussian integrals

In this appendix we indicate a method which allows to calculate the Gaussian functional (19) on the superalgebra $\mathcal{S}_{fin}(\mathcal{H})$ by Gaussian integration without use of superanalysis.

D.1 The bosonic functional

The tensors $F \in \mathcal{S}(\mathcal{H}_{\overline{0}}) = \mathcal{T}^+(\mathcal{H}_{\overline{0}})$ of the bosonic Fock space can be represented by the functions

$$x \in \mathcal{E}_{\overline{0}} \to \varphi_F(x, x^*) = \langle F \mid \exp(x + x^*) \rangle\,. \tag{96}$$

These functions are restrictions of the functions (48) to $\Xi = id \otimes F$ and to arguments $\zeta = id \otimes x$, $x \in \mathcal{E}_{\overline{0}}$.

The bosonic part of the Gaussian functional (19) can be calculated by integration with respect to the canonical Gaussian promeasure μ_0 on $\mathcal{E}_{\overline{0}}$ (more precisely on the underlying real space $\mathcal{E}_{\overline{0}\mathbb{R}}$). This promeasure is characterized by the Laplace transform

$$\int_{\mathcal{E}_{\overline{0}\mathbb{R}}} e^{\langle f^* | x \rangle + \langle g | x^* \rangle} \mu_0(dx, dx^*) = e^{\langle f^* | g \rangle} \quad \text{if } f, g \in \mathcal{E}_{\overline{0}}. \tag{97}$$

Lemma 14 *The Gaussian functional (19) restricted to $F \in \mathcal{S}_{alg}(\mathcal{H}_{\overline{0}}) = \mathcal{T}^{+}_{alg}(\mathcal{H}_{\overline{0}})$ has the integral representation*

$$\langle \exp \Omega_0 \mid F \rangle = \int_{\mathcal{E}_{\overline{0}\mathbb{R}}} \varphi_F(x, x^*) \mu_0(dx, dx^*). \tag{98}$$

Proof The proof follows from a simple calculation for coherent states $F = \exp f, f \in \mathcal{H}_{\overline{0}}$. Then (18) implies $\langle \exp \Omega_0 \mid \exp(f^* + g) \rangle = e^{\omega(f^*, g)} = e^{\langle f^* | g \rangle}$. A comparison with (97) yields the result. □

D.2 The fermionic functional

There is no canonical counterpart to the functions (96) in the fermionic case. But using the ordering prescription on the Hilbert space $\mathcal{H}_{\overline{1}}$ one can transfer a great part of the constructions given for the symmetric tensor algebra in Sect.D.1 to the antisymmetric tensor algebra. These techniques have been introduced in quantum field theory by K. O. Friedrichs [11], and they have been further developed by Klauder[20] and by Garbaczewski and Rzewuski[12], for the following calculations see [26][27]. In this appendix we only use the ordering of a basis. The cited literature gives more general ordering prescriptions. The disadvantage of all these methods is their explicit dependence on the chosen ordering prescription, which is not invariant against general unitary transformations. On the other hand, these constructions do not need an extension to a superspace.

Let $\{e_a \mid a \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{E}_{\overline{1}}$. If $\mathbf{A}$ is the finite ordered subset $\mathbf{A} = \{a_1 < a_2 < ... < a_p\} \subset \mathbb{N}$ of the integers, the tensor $e_{\mathbf{A}} = e_{a_1} \wedge ... \wedge e_{a_p}$ is an element of an ON basis of $\mathcal{T}^-_p(\mathcal{E}_{\overline{1}})$ and $e^*_{\mathbf{A}} = e^*_{a_p} \wedge ... \wedge e^*_{a_1}$ is an element of a basis of $\mathcal{T}^-_p(\mathcal{E}^*_{\overline{1}})$. Moreover, $\{e^*_{\mathbf{A}} \wedge e_{\mathbf{B}} \mid \mathbf{A} \in P(\mathbb{N}), \mathbf{B} \in P(\mathbb{N})\}$, where $P(\mathbb{N})$ is the power set of $\mathbb{N}$, is an orthonormal basis of the Fock space $\mathcal{S}(\mathcal{H}_{\overline{1}}) = \mathcal{T}^-(\mathcal{H}_{\overline{1}})$. Any $F \in \mathcal{S}(\mathcal{H}_{\overline{1}}) = \mathcal{T}^-(\mathcal{H}_{\overline{1}})$ has the decomposition $F = \sum_{\mathbf{A},\mathbf{B}} f(\mathbf{A}, \mathbf{B}) e^*_{\mathbf{A}} \wedge e_{\mathbf{B}}$, where the sum extends over $\mathbf{A} \in P(\mathbb{N})$ and $\mathbf{B} \in P(\mathbb{N})$. The fermionic part of the tensor (17) Ω has the representation $\Omega_1 = \sum_{a=1}^{\infty} e^*_a \wedge e_a$ (with respect to any orthonormal basis of $\mathcal{E}_{\overline{1}}$), and the exponential is calculated as $\exp \Omega_1 = \sum_{\mathbf{A} \in P(\mathbb{N})} e^*_{\mathbf{A}} \wedge e_{\mathbf{A}}$. The Gaussian functional has therefore the representation

$$\langle \exp \Omega_1 \mid F \rangle = \sum_{\mathbf{A} \in P(\mathbb{N})} f(\mathbf{A}, \mathbf{A}) \tag{99}$$

for all $F \in \mathcal{T}^-_{fin}(\mathcal{H}_{\overline{1}})$. To define functions like (96) also for tensors $F \in \mathcal{T}^-_{fin}(\mathcal{H}_{\overline{1}})$ we need a nontrivial substitute for the coherent states. We first introduce the numerical monomials $x \in \mathcal{H}_{\overline{1}} \to x(\mathbf{A}) := \prod_{a \in \mathbf{A}} (e_a, x) \in \mathbb{C}$.. The substitutes for powers "$\frac{1}{n!} x^n$" are then the tensors $\sum_{\mathbf{A}, |\mathbf{A}|=n} x(\mathbf{A}) e_{\mathbf{A}} \in \mathcal{T}^-_n(\mathcal{E}_{\overline{1}})$, and the substitute for a coherent state is the absolutely converging series $E(x, x^*) = \sum_{\mathbf{A},\mathbf{B}} x(\mathbf{A}) x(\mathbf{B}) e^*_{\mathbf{A}} \wedge e_{\mathbf{B}} \in \mathcal{T}^-(\mathcal{H}_{\overline{1}})$, where the summation extends over $\mathbf{A} \in P(\mathbb{N})$ and $\mathbf{B} \in P(\mathbb{N})$. It is exactly this definition where the ordering prescription enters. Fermionic functions can then be defined in the same way as the bosonic functions (96)

$$\psi_F(x, x^*) := \langle E(x, x^*) \mid F \rangle = \sum_{\mathbf{A},\mathbf{B}} f(\mathbf{B}, \mathbf{A}) \overline{x(\mathbf{A})} x(\mathbf{B}) \tag{100}$$

We integrate the numerical function (100) with the canonical Gaussian promeasure μ_1 of the space $\mathcal{E}_{\overline{1}\mathbb{R}}$, see (97) for the corresponding measure on $\mathcal{E}_{\overline{0}\mathbb{R}}$, and we obtain exactly the Gaussian functional (99)

$$\int_{\mathcal{E}_{\overline{1}\mathbb{R}}} \psi_F(x, x^*) \mu_1(dx, dx^*) = \sum_{\mathbf{A} \in P(\mathbb{N})} f(\mathbf{A}, \mathbf{A}). \tag{101}$$

The promeasures μ_0 and μ_1 have unique extensions to $\sigma-$additive measures on $\mathcal{E}^-_{\overline{0}\mathbb{R}}$, or $\mathcal{E}^-_{\overline{1}\mathbb{R}}$, respectively. The functions $\{\varphi_F(x, x^*) \mid F \in \mathcal{S}(\mathcal{H}_{\overline{0}})\}$ (or $\{\psi_G(x, x^*) \mid G \in \mathcal{S}(\mathcal{H}_{\overline{1}})\}$) are then square integrable functions of the variables $x \in \mathcal{E}^-_{\overline{0}\mathbb{R}}$ (or $x \in \mathcal{E}^-_{\overline{1}\mathbb{R}}$).

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