The Stability of $D2$–Branes in the Presence of an RR field

D.K. Park[1,2*], S. Tamaryan[1,3†] and H.J.W. Müller–Kirsten[1§]

1. Department of Physics, University of Kaiserslautern, D-67653 Kaiserslautern, Germany

2. Department of Physics, Kyungnam University, Masan, 631-701, Korea

3. Theory Department, Yerevan Physics Institute, Yerevan–36, 375036, Armenia

Abstract

A Born–Infeld theory describing a $D2$–brane coupled to a 3–form RR potential is reconsidered in order to investigate the stability of its nonsingular solutions with finite energy. The condition of stability of the solutions is established and the stable solutions and their shape are determined.

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1 Introduction

Recently in ref.[1] the question was considered whether a string can tunnel to a $D2$–brane in the presence of a uniform background RR field, and it was shown that the string can indeed nucleate the spheroidal bulge of a $D2$–brane and can tunnel to a toroidal $D2$–brane. The tunneling was described by bounces in Euclidean time and the rate of decay of the string into the toroidal $D2$–brane was deduced. The transition process was investigted in more detail in ref.[2], and the order of the quantum–classical transitions was determined depending on the magnitude of the applied RR field. All the solutions considered in these cases are unstable as stated in ref.[1] and therefore they must decay to the stable configurations [3, 4]. The source of this instability will be exposed here.

One can argue that stable solutions must also exist because in the given background of the RR field the configuration which has minimal energy and hence is a true minimum of the energy functional will be stable. Our intention here is to find this solution and to show that this solution is stable. This then allows us to clarify to which stable configuration the string can tunnel by application

*Email:dkpark@rose.kyungnam.ac.kr

†Email:sayat@physik.uni-kl.de, sayat@moon.yerphi.am

§Email:mueller1@physik.uni-kl.de

of the external force. Moreover, this tunneling is dominant since the transition rate exponent is proportional to the opposite value of the configuration energy, so that the tunneling to the branes with higher energy is exponentially suppressed.

The stability of Born–Infeld particles has been studied in detail in refs.[5, 6] in the case of the $D3$–brane (without an applied RR field). In particular it was shown there that the combined brane–antibrane configuration is unstable, whereas the Born–Infeld string is stable. The latter was also shown earlier in ref.[7], this being a consequence of the preservation of supersymmetry.

In ref.[1] it was shown that in the case of the $D2$–brane in the background of a uniform RR field the Born–Infeld string is unstable and can tunnel to an unstable $D2$–brane. In the present work we show that in the background of the uniform RR field one still has stable $D2$–branes with minimal energy. Tunneling to these stable $D2$–branes dominates other tunnelings. We also show that strings can survive in special cases, and can have negative tension. If there are strings, then those with negative tension are stable and those with positive tension are unstable.

In Section 2 we consider other brane configurations and show that these are physically acceptable, i.e. are nonsingular and have finite energy. We also show that for small values of the RR field there are two types of solutions, one type with higher energy (used in [1]) and a second type with lower energy; for the RR field strength h larger than some critical value only the lower energy solution survives. In Section 3 we consider special solutions. The equation for $D2$–brane configurations into which strings can tunnel admits 3 types of solutions: Periodic solutions in terms of elliptic functions, two constant solutions of cylindrical shape, and solutions which are either finite or vanish exponentially at infinity. We are looking for branes with finite energy, otherwise the tunneling rate would be zero. In particular we are considering the nucleation of the unwrapped string so that for $z\epsilon\mathbf{R}^1$ space only the third type of solutions is physically acceptable. Periodic solutions can be used in the case of a compactified space, when the wrapped string tunnels into the toroidal $D2$–brane [2]. In Section 4 we investigate the stability of such branes using the semiclassical method, i.e. we consider the fluctuation operator describing small deviations of the action in the vicinity of that solution, and demonstrate that this has no negative eigenvalues. Vice versa it may possess a negative eigenvalue for the upper energy solution and imply that this solution is unstable. In Section 5 we conclude with some remarks.

2 Formulation of the problem and determination of solutions with minimal energy

With the convention of $\alpha' = 1$ the action of a $D2$–brane coupled with the 3–form gauge potential A in Born–Infeld approximation is given by [1, 8]

$$I = -\frac{1}{4\pi^2 g}\int d^3\xi\Big\{\sqrt{-det\Big(g^{ind}_{\alpha\beta} + 2\pi\mathcal{F}_{\alpha\beta}\Big)} + \frac{1}{3!}\epsilon^{\alpha\beta\gamma}A_{\mu\nu\rho}\partial_\alpha X^\mu\partial_\beta X^\nu\partial_\gamma X^\rho\Big\} \quad (1)$$

where $\mu,\nu,\rho = 0,\cdots,9$ are spacetime indices, and $\alpha,\beta,\gamma = 0,1,2$ worldvolume indices and g is the string coupling. The dilaton field is taken to be constant and the background field strength $H = dA$ is taken to be uniform and aligned with the brane, i.e. $H_{0123} = h$=const. As in ref.[1] we choose the world volume to be cylindrical and hence define

$$X^0 = t, \;\; X^1 = z, \;\; X^2 = R(t,z)\cos\sigma, \;\; X^3 = R(t,z)\sin\sigma, \;\; \mathcal{E} = 2\pi F_{tz}, \tag{2}$$

and all other X^i = const. After integration over σ the action takes the form

$$I = \int dt \int dz \mathcal{L}, \;\; \mathcal{L} = -\frac{1}{2\pi g}\Big(R\sqrt{1 - \dot{R}^2 - \mathcal{E}^2 + R'^2} - \frac{h}{2}R^2\Big) \tag{3}$$

where dots and primes denote derivatives with respect to t and z respectively. The canonical momentum $D = 2\pi g \delta I/\delta E$ must be constant (cf. refs.[1, 2]) and is given by gn, where n is the number of fundamental strings (cf. [1, 2]), each of tension $1/2\pi$. For static solutions the energy E is given by

$$E = \frac{1}{2\pi g}\int dz \Big\{\sqrt{(1+R'^2)(D^2+R^2)} - \frac{h}{2}R^2\Big\} \tag{4}$$

Variation of E with respect to R yields

$$\frac{\delta E}{\delta R} - \frac{d}{dz}\frac{\delta E}{\delta R'} = 0 \tag{5}$$

which can be reduced to a first order differential equation because it does not contain the variable z explicitly, resulting in

$$\sqrt{\frac{R^2+D^2}{1+R'^2}} - \frac{h}{2}R^2 = C \tag{6}$$

where C is a constant. We rewrite this equation

$$R' = \overset{-}{(+)}\, \frac{h}{hR^2+2C}\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}, \;\; R' \neq 0 \tag{7}$$

with

$$R_\pm{}^2 = \frac{2}{h^2}\Big[(1-Ch) \pm \sqrt{1 - 2Ch + h^2 D^2}\Big] \tag{8}$$

Now we compare the expression (6) for the integration constant C with expression (4) for the energy density. We observe that the second (i.e. "potential") terms are the same and the first ("kinetic") terms differ by a multiplicative factor, i.e. $(1+R'^2)$, and consequently the restriction to positivity of C leads to the positivity of the energy E, while we are interested in the configuration with the lowest lying energy. Hence we abandon that restriction and consider negative values of C. At first sight this may seem dangerous since this suggests a pole in eq.(7) but in spite of this the solution is nonsingular and the energy finite. In this case eq.(7) permits solutions for all values of h and for R_+^2 and R_-^2 real and positive. From eq.(8) we deduce two conditions

$$C^2 \geq D^2, \quad 1 - Ch \geq 0,$$

which for positive values of C imply

$$hD \leq hC \leq 1$$

meaning that h must be less than the critical value $h_C = 1/D$. On the contrary, for negative values of C, keeping in mind that h is positive, we have only the one condition that

$$|C| \geq D$$

so that h can take any value, but $|C|$ is restricted to non–small values. At the end of this Section we present a solution for large values of h, i.e. when $hD >> 1$.

We now demonstrate that solutions for negative values of C are, in principle, also acceptable, i.e. are associated with a finite energy. It is convenient in this case to set

$$C = -\frac{h}{2}a^2 \tag{9}$$

Then for $|C| \geq D$ we have

$$0 \leq R_-^2 \leq a^2 \leq R_+^2$$

and

$$R' = \mp \frac{\sqrt{(R_+^2 - R^2)(R^2 - R_-^2)}}{R^2 - a^2}$$

We consider the approach $R \to a$. In this domain

$$R' \simeq \mp \frac{\lambda}{2}\frac{1}{R-a}, \quad \lambda \simeq \frac{2\sqrt{a^2 + D^2}}{ha}$$

and

$$(R-a)^2 = \lambda |z - z_o|, \quad |R'| = \frac{1}{2}\sqrt{\frac{\lambda}{|z - z_0|}}$$

so that the crucial part of the energy (4) around $z = z_0$ becomes

$$\begin{aligned} & \frac{1}{2\pi g}\int_{z_0-\delta}^{z_0+\delta} dz \sqrt{(1 + R'^2)(D^2 + R^2)} \\ \simeq \quad & \frac{\sqrt{a^2 + D^2}}{2\pi g}\int_{z_0-\delta}^{z_0+\delta} |R'| dz = \frac{\sqrt{\lambda(a^2 + D^2)}}{4\pi g}\int_{z_0-\delta}^{z_0+\delta} \frac{dz}{|z - z_0|^{1/2}} < \infty \end{aligned} \tag{10}$$

We conclude therefore that static solutions for negative values of C also have finite energy.

3 Solutions with negative tension

We now consider some special solutions and their energy. We consider first the special case $C = -D$. In this case $D = ha^2/2, R_- = 0, R_+^2 = 4(1 + hD)/h^2$. Integrating the equation

$$R' = \mp \frac{R}{R^2 - a^2}\sqrt{R_+^2 - R^2}$$

one obtains

$$\mp(z-z_0) = -\sqrt{R_+^2 - R^2} + \frac{a^2}{R_+}\ln\frac{R_+ + \sqrt{R_+^2 - R^2}}{R} \tag{11}$$

In Fig. 1 we show this solution. One can clearly see that the effect of negative values of C is opposite to that of positive C: Whereas positive C yield an elongated spheroidal bulge (cf. refs.[1, 2]), negative C push this bulging in the opposite direction to eventually form a wheel–like structure. We can also calculate the energy of this configuration using eq.(6) with C there replaced by $-D$. With $D = +ng$ one obtains

$$E = \frac{h}{8\pi^2 g}\left(\frac{4\pi}{3}R_+^3\right) - \frac{n}{2\pi}\int dz. \tag{12}$$

We observe that the energy of the bulge (the first term) is the same as in the case considered in ref.[1], but in comparison the tension has changed its sign. Thus strings in the background may have negative tension.

Next we consider the case $C = -ha^2/2$ with $ha >> 1, hD >> 1$. In this case

$$R_\pm^2 \simeq a^2 \pm 2R_0, \quad R_0 \equiv \sqrt{a^2 + D^2}/h, \quad R_0 << a^2, D^2$$

We set (with $\tau' \equiv d\tau/dz$)

$$R^2 \equiv a^2 + \tau.$$

Then eq.(7) becomes

$$\frac{\tau'}{2\sqrt{a^2+\tau}} = \mp\frac{\sqrt{4R_0^2 - \tau^2}}{\tau}. \tag{13}$$

Integration (for $\tau << a^2$) yields the equation ($R^2 = x^2 + y^2$)

$$z^2 + \left(\frac{R^2 - a^2}{2a}\right)^2 = \frac{a^2 + D^2}{h^2 a^2} \tag{14}$$

(with integration constant $z_0 = 0$). This equation describes a deformed circular structure with radius a (neglecting $2\sqrt{a^2 + D^2}/h$) which is obvious if we look at its intersection with the plane $z = 0$. The structure has a thickness $\sqrt{a^2 + D^2}/ha$. Thus in the limit $ha >> 1$ with parameter a fixed, the circular structure becomes a shell of finite radius and small thickness as shown in Fig. 2.

4 The stability of the solutions

We now return to the question of whether the solutions considered above are stable, i.e. are global minima of the energy, or not. For this reason we consider the second variation $\delta^2 E$ of the energy in the vicinity of the classical solution. Straightforward calculation yields

$$\delta^2 E = \frac{1}{4\pi g}\int \delta R \hat{M} \delta R dz \tag{15}$$

where

$$\hat{M} = -\frac{d}{dz}Q\frac{d}{dz} + 2P\frac{d}{dz} + V \tag{16}$$

with

$$Q = \frac{\sqrt{R^2 + D^2}}{(1 + R'^2)^{3/2}}, \quad P = \frac{RR'}{\sqrt{(1 + R'^2)(R^2 + D^2)}}, \quad V = D^2\frac{\sqrt{1 + R'^2}}{(R^2 + D^2)^{3/2}} - h \tag{17}$$

A negative eigenvalue of the fluctuation operator $\hat{M}$ implies instability of the respective solution, since the variation of the solution in the direction of the corresponding eigenfunction decreases the energy. Therefore it suffices to investigate for which solutions $\hat{M}$ has only positive eigenvalues and for which not.

One can derive another expression for $\hat{M}$ which is equivalent to the one above, i.e.

$$\hat{M} = -\frac{1}{R'}\frac{d}{dz}R'^2 Q\frac{d}{dz}\frac{1}{R'} + 2P\frac{d}{dz} - \frac{1}{R'}(QR'')' \tag{18}$$

but allows to present $\delta^2 E$ as a sum of positively defined terms plus a term proportional to C. On the basis of this one can easily distinguish the stable solutions from the unstable ones.

Here we are looking for branes with finite energy. With $z\epsilon\mathbf{R}^1$ this implies the square integrability of the eigenfunctions of $\hat{M}$. The charge D remains fixed by quantisation (as stated earlier). The positivity of all eigenvalues of $\hat{M}$ means that the mean value of $\hat{M}$ over any function with finite norm is positive and vice versa. We assume that $\delta R(z)$ is a square integrable function, i.e.

$$\int \delta R(z)^2 dz < +\infty, \tag{19}$$

and consider the mean value of $\hat{M}$ on that class of functions. This is the same as $\delta^2 E$. The function $R(z)$ is bounded, $R_- \leq R \leq R_+$, and $\delta R(z)$ must vanish at infinity. Consequently we can integrate the term $2\delta RP\frac{d}{dz}\delta R = P\frac{d}{dz}\delta R^2$ by parts and the total derivative must vanish. Also in the first term we can return to the antihermitian operator d/dz to act on the term to the left yielding a minus sign. As a result we have

$$\delta^2 E = \frac{1}{4\pi g}\int \Big[R'^2 Q\Big(\frac{d}{dz}\frac{\delta R}{R'}\Big)^2 + U\delta R^2\Big]dz \tag{20}$$

where

$$U = V - \frac{dP}{dz} - \frac{1}{R'}\Big(QR''\Big) \tag{21}$$

It is worth noting that one can make analogous manipulations with the differential equation for eigenvalues of the operator $\hat{M}$. After appropriate substitutions and eliminating the first order derivative, one obtains the same expression (21) for the effective potential. We prefer the above way which allows us to connect the second variation of the energy directly with the integration constant C. With some algebra one can deduce the explicit expression for U:

$$U = h\frac{R^2}{R^2 + D^2} + hD^2\frac{R^2}{(R^2 + D^2)^2} - 2C\frac{R^2}{(R^2 + D^2)^2} \tag{22}$$

The last term demonstrates in a transparent way the stability of solutions with negative values of C. In contrast, in the case when $C > 0$ and $hD^2 \leq C$, the negative term becomes consequently a source for the instability which is due to negative eigenvalue.

5 Conclusions

In the above we derived the nonsingular, finite–energy solutions of the Born–Infeld theory of a $D2$–brane in the presence of a three–form RR–potential. By considering small fluctuations about the configuration we also demonstrated its stability. The theory, originally proposed in ref.[1], turned out to enable all these considerations to be performed in a relatively simple way; it could therefore serve as a prototype model for more complicated cases.

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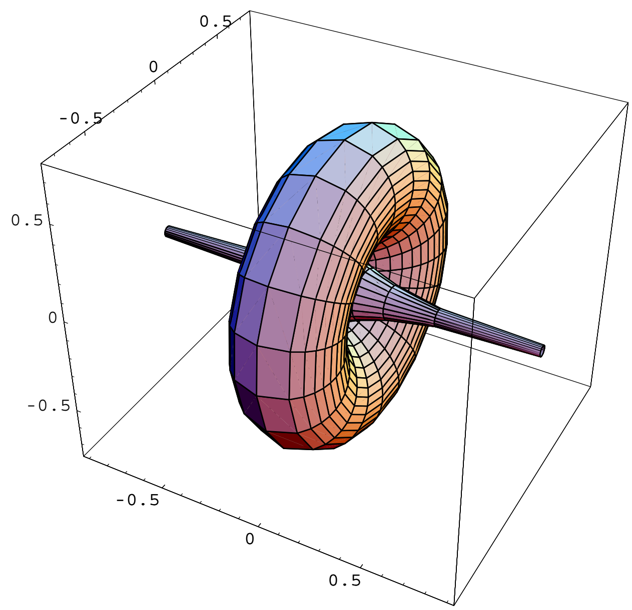

Fig.1

The solution for a negative value of C

$C = -D; \quad D = 0.4\ ; \quad h = 4$

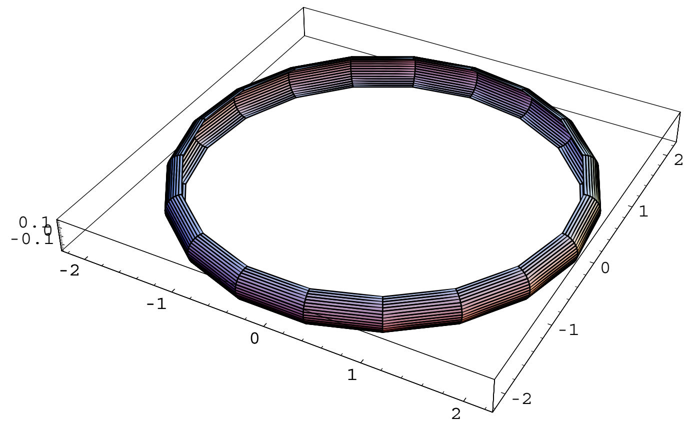

Fig.2

The limit of the string becoming an annular shell

a=2; D=3; h=10