D–Branes and their Absorptivity in Born–Infeld Theory

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Abstract

Standard methods of nonlinear dynamics are used to investigate the stability of particles, branes and D-branes of abelian Born–Infeld theory. In particular the equation of small fluctuations about the D–brane is derived and converted into a modified Mathieu equation and – complementing earlier low–energy investigations in the case of the dilaton–axion system – studied in the high–energy domain. Explicit expressions are derived for the S–matrix and absorption and reflection amplitudes of the scalar fluctuation in the presence of the D-brane. The results confirm physical expectations and numerical studies of others. With the derivation and use of the (hitherto practically unknown) high energy expansion of the Floquet exponent our considerations also close a gap in earlier treatments of the Mathieu equation.

1 Introduction

Recently Born–Infeld gauge theory has attracted considerable interest as the bosonic light–brane approximation or limit of superstring theory[1], and has turned out to be a simple and transparent model in this context [2]. Branes, defined as extended objects in spacetime, can be fundamental or solitonic. The connection of these branes with a $U(1)$ gauge field was motivated by the presence of this field in the massless part of the spectrum of open strings, and by realising that branes with open strings attached to them which satisfy Dirichlet

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boundary conditions, or more generally one brane attached to another, can become classically stable, solitonic objects. It is for this reason that the dynamics of D-branes [3] in Born–Infeld theory is being studied in detail and generalised [4],[5],[6],[7],[8]. Since, in general, a brane may or may not be a solitonic configuration or BPS state, the exploration of this question deserves particular attention. It is often stated that a brane is BPS in view of the vanishing of a fraction of the supersymmetry variation of the associated gaugino field. However, since BPS states (as classically and topologically stable states) and Bogomol'nyi bounds have been studied in great detail in a host of other theories, and the approach in these is practically standard, one would like to understand aspects of Born–Infeld particles in a similar way, also because it is not absolutely clear that D–branes are solitons of string theory in precisely the same way as more familiar topological solitons in field theory. Therefore our first intention in the following is to study Born–Infeld particles with standard methods of nonlinear dynamics in the simplest case of a flat spacetime. We begin with the free Born–Infeld particles, i.e. BIon and catenoid [4]. Using a scale transformation argument [9] we show that these static configurations – which differ from ordinary solitons of nonlinear theories in requiring a special consideration of source terms or boundary conditions (cf. also [10],[11]) – require the number of space dimensions p to be larger than 2. We assume spherical symmetry and study the local stability of these configurations by considering the second variational derivatives of their respective actions. Our conditions for stability are a) that the eigenfunctions of the corresponding operator be square integrable, and b) that the charge e be fixed, with angular fluctuations ignored. We then consider the case of the scalar field corresponding to a single transverse coordinate coupled to the gauge field (here only the electric component), i.e. the catenoid or brane with associated open fundamental string. We distinguish between two types of arguments in deriving the linearised fluctuation equation, and infer the stability of this stringy D-brane. In ref.[12] an explicit and detailed consideration of the Bogomol'nyi bound in a special model of Born–Infeld theory has been given where the central charge of the supersymmetry algebra plays the role of the topological or winding number of ordinary solitons.

Our second intention in the following is the explicit study of the small fluctuation equation about the $D3$–brane in the high energy domain. This equation with singular potential has the remarkable property of being convertible into a modified Mathieu equation which depends only on one coupling parameter which is a product of energy and electric charge. The S–matrix for scattering of the fluctuation off the brane can be obtained in explicit form. The $D3$–brane is therefore one of the very rare examples allowing a detailed study of its properties with explicit expressions for all relevant physical quantities in both low and high energy domains. We therefore expect that also S–duality can be uncovered and studied in this case (although we do not attempt this here). Various other Dp–brane models have been discovered recently whose small fluctuation equations can be reduced to modified Mathieu equations [13, 14, 15] which have then been investigated mainly by computational methods. For the AdS/CFT correspondence the logarithmic corrections to the low energy absorption probability are of particular interest, since these permit a direct relation to the discontinuity of

the cut in the correlation function of the dual two–dimensional quantum field theory. The first such logarithmic correction to the absorption probability was originally obtained in refs.[16, 17, 18] without resorting to the use of Mathieu functions. Subsequently the authors of ref. [13] considered the modified Mathieu equation and used computational methods to generate explicit series expansions up to several orders for the low energy absorption probability. In [19] a different choice of expansions was considered to obtain leading expressions more easily. It is natural to supplement such investigations by exploring also the high energy case, the first such consideration being that of ref.[15]. The analytical high energy results obtained in the following and the complementary low energy results of ref. [19] (we also demonstrate how the S–matrices are related) are therefore directly applicable to these. Singular potentials have been studied from time to time, and have mostly been discarded as pathological. It seems, however, that their real significance lies in the context of curved spaces with black–hole type of absorption [20].

Sections 2 and 3 deal with the BIon and the catenoid, sections 4 and 5 with the Bogomol'nyi limit of the $D3$–brane and the derivation of the linearised fluctuation equation about it. In section 6 we consider this equation in detail in the high energy domain and calculate the rate of absorption of partial waves of the fluctuation field by the brane. That this absorption occurs is attributed to the singularity of the potential. The absorptivity part of the paper may be looked at as the high energy complement to the low energy case of ref.[19] with the same expression of the S–matrix. All these calculations require a matching of wave functions. In the low energy S–wave case simple considerations of Bessel and Hankel functions suffice as was shown in refs. [21]. The low energy limit is, in fact, independent of the choice of matching point, as was shown recently [22]. Our considerations here, however, are general.

2 The BIon

We consider first purely static cases and write the Lagrangian of the static BIon in $p+1$ spacetime dimensions (cf.[4])

$$L = \int d^p x \mathcal{L}, \quad \mathcal{L} = 1 - \sqrt{1-(\partial_i \phi)^2} - \Sigma_p e \phi \delta(\mathbf{r}), \quad \Sigma_p = \frac{p\pi^{\frac{p}{2}}}{(\frac{p}{2})!} \tag{1}$$

$(i = 1, \cdots, p)$ with the charge e held fixed by the constraint

$$e + \frac{1}{\Sigma_p} \int d\sigma_i \frac{\partial_i \phi}{\sqrt{1-(\partial_j \phi)^2}} = 0 \tag{2}$$

Eq. (1) is the Lagrangian one obtains from the world brane action of the pure Born–Infeld $U(1)$ electromagnetic action reduced to the purely electric case with field $E_i = \partial_0 A_i - \partial_i A_0$ and no transverse coordinate. The field A_μ is assumed to

depend on the world brane coordinates $x_\mu, \mu = 0, \cdots, p$. The static BIon equation of motion is

$$\partial_i \left(\frac{\partial_i \phi}{\sqrt{1 - (\partial_i \phi)^2}} \right) = -\Sigma_p e \delta(\mathbf{r}) \tag{3}$$

In the special case $p = 3$ the classical $SO(3)$ symmetric solution, called a BIon, is given by

$$\phi_c(r) = \int_r^\infty \frac{dx}{\sqrt{1 + \frac{x^4}{e^2}}} = \phi_c(0) - \int_0^r \frac{dx}{\sqrt{1 + \frac{x^4}{e^2}}} \stackrel{r \to 0}{\simeq} \left[\phi_c(0) - r + \frac{r^5}{10 e^2} \right] \tag{4}$$

and $\phi_c(0) = \frac{1}{4} B(\frac{1}{4}, \frac{1}{4}) . e^{\frac{1}{2}} = 1.854074677 . e^{\frac{1}{2}}$, B being the Bernoulli function. It is easily verified that this solution satisfies the constraint (2) for any value of r. Defining $\mathbf{E} = -\nabla \phi_c$ (so that $\phi_c = A_0$ with $\partial A_0(x_i, t)/\partial t = 0$ in the static case), and defining $\mathbf{D} = \frac{\partial \mathcal{L}}{\partial \mathbf{E}} = \frac{\mathbf{E}}{\sqrt{1 - \mathbf{E}^2}}$ we have (with $F_{0i} = E_i$)

$$T_{00} = F_{0i} \frac{\partial \mathcal{L}}{\partial F_{0i}} - \mathcal{L} = \mathbf{E} \cdot \mathbf{D} - \mathcal{L} = \frac{1}{\sqrt{1 - \mathbf{E}^2}} - 1 + 4\pi e \phi \delta(\mathbf{r}) \tag{5}$$

The energy H_c of the BIon (obtained by integration over $\mathbf{R}^3$) is then found to be finite, i.e.

$$H_c = \int d\mathbf{x} T_{00} = 4\pi (3.09112) . e^{\frac{3}{2}} \tag{6}$$

and in p dimensions the total energy of the BIon scales correspondingly as $e^{\frac{p}{p-1}}$. The finiteness of the energy depends on the minus sign in (1) and so with (3) on the relation

$$\sqrt{1 - (\phi_c')^2} = -\frac{r^2}{e} \phi_c' = \frac{r^2}{e \sqrt{1 + \frac{r^4}{e^2}}} \tag{7}$$

for $0 \leq r \leq \infty$. It may be noted that by defining $\mathbf{D}$ such that the left hand side of eq.(3) is $\partial_i D_i$, the singularity of the right hand side is associated with $\mathbf{D}$ rather than with $\mathbf{E}$ which is the decisive difference between Maxwell and Born–Infeld electrodynamics. A similar observation applies to the catenoid equation below. The energy of the BIon is seen to be independent of its position which hints at the existence of some kind of collective coordinate. However, exploring this point further is expected to be difficult since a moving charge generates a magnetic field, and hence the electric field alone would not suffice.

We can use a scaling argument [9] to show that here finite energy configurations require p to be larger than or equal to 3. Under a scale transformation $x \to x' = \lambda x, \phi(x) \to \phi_\lambda(x) = \phi(\lambda x), \partial_i \phi(x) \to [\partial_i \phi(x)]_\lambda = \lambda \partial_i \phi(\lambda x)$. The charge e defined by the constraint (2) also changes under the scale transformation, i.e.

$$e \to e_\lambda = -\frac{1}{\lambda^{p-2} \Sigma_p} \int \frac{d\sigma_i \partial_i \phi}{\sqrt{1 - \lambda^2 (\partial_j \phi)^2}} \tag{8}$$

In particular for $p = 3$ and radial symmetry

$$e_\lambda^{(p=3)} = \frac{r^2}{\lambda} \frac{1}{\sqrt{1 - \lambda^2 + \frac{r^4}{e^2}}} \tag{9}$$

and for arbitrary values of λ the r–dependence drops out only if the limit $r \to \infty$ is taken in the evaluation of the integral. Then

$$\frac{e_\lambda^{(p=3)}}{e} \stackrel{r\to\infty}{\longrightarrow} \frac{1}{\lambda} \tag{10}$$

But also $e_{\lambda=1} = e$ for any r. If ϕ_c is stable and $\neq 0$, the energy must be stationary for $\lambda = 1$, i.e. $(\partial H_c/\partial\lambda)_{\lambda=1} = 0$. From this one finds that $p \geq 3$. Also $(\partial^2 H_c/\partial\lambda^2)_{\lambda=1} > 0$ for $p \geq 3$. Eqs.(6) and (10) show that changing the scale changes both the charge and the energy, i.e. if the charge were variable, one could lower the energy and hence the configuration could be unstable. But fixing the charge (e.g. by a quantisation condition) no instability is implied by the scaling condition.

We investigate the stability of the BIon further in the special and exemplary case of $p = 3$ by considering the second functional variation of the static Lagrangian evaluated at $\phi_c(r)$. This can be written and simplified in the following form (ignoring total divergences on the way)

$$\delta^2 L = \frac{1}{2}\int d^3x \delta\phi \hat{A} \delta\phi, \tag{11}$$

where

$$\begin{aligned} \hat{A} &= -\partial_i \frac{1}{[1-(\partial_j\phi_c)^2]^{1/2}}\partial_i - \partial_i \frac{\partial_i\phi_c\partial_j\phi_c}{[1-(\partial_k\phi_c)^2]^{3/2}}\partial_j \\ &= -\frac{1}{r^2}\frac{d}{dr}\frac{r^2}{(1-\phi_c'^2)^{3/2}}\frac{d}{dr} \end{aligned} \tag{12}$$

The operator $\hat{A}$ can also be written

$$\hat{A} = -\frac{1}{(1-\phi_c'^2)^{3/2}}\left\{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{6}{r}\phi_c'^2\frac{d}{dr}\right\} \tag{13}$$

The classical stability of ϕ_c is therefore decided by the spectrum $\{\omega_n\}$ of the small fluctuation equation

$$-\frac{1}{r^2}\frac{d}{dr}\frac{r^2}{(1-\phi_c'^2)^{3/2}}\frac{d}{dr}\psi_n = \omega_n\psi_n \tag{14}$$

We explore first the existence of a zero mode ψ_0, i.e. the case $\omega = 0$. In this case

$$\frac{r^2}{(1-\phi_c'^2)^{3/2}}\frac{d}{dr}\psi_0 = C \tag{15}$$

and so with $\psi_0(\infty) = 0$

$$\psi_0 = -\frac{C}{e^3}\int_r^\infty \frac{x^4}{(1+\frac{x^4}{e^2})^{3/2}}dx = -C\frac{\partial}{\partial e}\int_r^\infty \frac{dx}{(1+\frac{x^4}{e^2})^{1/2}} = -C\frac{\partial\phi_c}{\partial e} \tag{16}$$

The derivative of the classical configuration ϕ_c with respect to the charge e indicates that a perturbation along $\partial\phi_c/\partial e$ around $\phi_c$leaves the static action invariant, i.e. $\phi_c(e,r)$ and $\phi_c(e+\delta e,r)$ have the same action since

$$\left.\frac{\partial\phi_c(e+\delta e,r)}{\partial(\delta e)}\right|_{\delta e=0} = \frac{\partial\phi_c(e,r)}{\partial e}.$$

We now show that the operator $\hat{A}$ does not possess negative eigenvalues, and that therefore the BIon is a classically stable configuration. We let ψ_n be an eigenfunction of the operator $\hat{A}$. Then

$$\begin{aligned}
\int d^3x\psi_n\hat{A}\psi_n &= -4\pi\int_0^\infty dr\psi_n\frac{d}{dr}\frac{r^2}{(1-\phi_c'^2)^{3/2}}\frac{d\psi_n}{dr} \\
&= -4\pi\int_0^\infty dr\left\{\frac{d}{dr}\psi_n\frac{r^2}{(1-\phi_c'^2)^{3/2}}\frac{d\psi_n}{dr} - \frac{r^2}{(1-\phi_c'^2)^{3/2}}\left(\frac{d\psi_n}{dr}\right)^2\right\} \\
&= F + 4\pi\int_0^\infty dr\frac{r^2}{(1-\phi_c'^2)^{3/2}}\left(\frac{d\psi_n}{dr}\right)^2 \qquad (17)
\end{aligned}$$

where $F := F(r)|_0^\infty$ and

$$F(r) = -4\pi\psi_n\frac{r^2}{(1-\phi_c'^2)^{3/2}}\frac{d\psi_n}{dr} = -4\pi e^3\frac{\psi_n}{r^4}\left(1+\frac{r^4}{e^2}\right)^{3/2}\frac{d\psi_n}{dr}. \qquad (18)$$

The second term on the right hand side of eq.(17) is strictly positive. Hence nonpositive eigenvalues imply a nonvanishing negative value of F. ¿From the condition $\int_0^\infty\psi_n^2r^2dr<\infty$, (i.e. $\psi_n \overset{r\to\infty}{\simeq} 1/(r^{1+\epsilon}), \epsilon>0$) it follows that $r^2\psi_nd\psi_n/dr\to 0$ with $r\to\infty$, so that

$$F(r) \overset{r\to\infty}{\simeq} -4\pi r^2\psi_n\frac{d\psi_n}{dr}\to 0$$

and $F(\infty)=0$. Hence

$$F = -F(0) \simeq 4\pi e^3\frac{\psi_n}{r^4}\frac{d\psi_n}{dr}\Big|_{r\to 0} \qquad (19)$$

As $r\to 0$ eq.(14) becomes

$$-\frac{1}{r^2}\frac{d}{dr}\frac{1}{r^4}\frac{d}{dr}\psi_n = \frac{\omega_n}{e^3}\psi_n \qquad (20)$$

In the case of the zero mode

$$\psi_0 \simeq C_1 + C_2r^5, \quad r\to 0 \qquad (21)$$

In this case $F=20\pi e^3C_1C_2$. For $C_1C_2<0$ this is in full compliance with (16) and (4) from which we obtain

$$\psi_0 \simeq -C\left(\frac{B(\frac{1}{4},\frac{1}{4})}{8e^{1/2}} - \frac{r^5}{5e^3}\right).$$

For $\omega_n \neq 0$ the small–r behaviour of ψ_n is

$$\psi_n \simeq C_n \left(1 - \frac{1}{24}\frac{\omega_n}{e^3} r^8 + O(r^{16})\right) \tag{22}$$

so that

$$F = -\frac{4}{3}\pi {C_n}^2 \frac{1}{r^4} r^7 \Big|_{r=0} = 0$$

Thus the conclusion is that for all eigenfunctions ψ_n

$$< \psi_n | \hat{A} | \psi_n > \quad \geq 0 \tag{23}$$

This inequality excludes the possibility of the existence of negative eigenvalues. Hence the BIon is in this sense classically stable.

3 The catenoid

The Lagrangian of the static catenoid in $p+1$ spacetime dimensions and with a source term is given by (cf.[4])

$$L = \int d^p x \mathcal{L}, \quad \mathcal{L} = 1 - \sqrt{1 + (\partial_i y)^2} - \Sigma_p r_0^{p-1} y \delta(\mathbf{r}) \tag{24}$$

where the signs have been chosen such that the energy is positive. Here the scalar field $y(x_i, t)$ originates from gauge field components A_a for $a = p+1, \cdots, (d-1), d$ =dimension, which represent transverse displacements of the brane; here we consider the case of only one such transverse coordinate, i.e. y, all $d-p-1$ of which are essentially Kaluza–Klein remnants of the $d = 10$ dimensional $N = 1$ electrodynamics after dimensional reduction to $p+1$ dimensions. The Euler–Lagrange equation of the static catenoid y_c (static meaning $\partial y(x_i, t)/\partial t = 0$) is given by

$$\partial_i \left(\frac{\partial_i y_c}{\sqrt{1 + (y_c')^2}} \right) = \Sigma_p r_0^{p-1} \delta(\mathbf{r}) \tag{25}$$

so that after integration

$$\frac{\nabla y_c}{\sqrt{1 + (y_c')^2}} = r_0^{p-1} \frac{\mathbf{r}}{r^p} \tag{26}$$

or for $r \geq r_0$

$$y_c' = \bar{(+)} \frac{{r_0}^{p-1}}{\sqrt{r^{2p-2} - {r_0}^{2p-2}}}, \quad \sqrt{1 + {y_c'}^2} = \bar{(+)} \frac{r^{p-1}}{\sqrt{r^{2p-2} - r_0^{2p-2}}} \tag{27}$$

In the case of the catenoid without source term the right hand side of eq.(26) can be taken to originate from a boundary condition such as $\nabla \cdot \left(\frac{\mathbf{r}}{r^p}\right) = 0$. The domain $r \leq r_0$ is the nonsingular throat region (i.e. $y_c(r_0)$) is finite). One may observe that the singularity on the right hand side of eq.(25) is associated with

the entire expression on the left whereas, like $\partial_i \phi_c$ in the BIon case, so now here ∇y_c is finite, i.e. the p-brane or single throat solution is given by

$$y_c(r) = \bar{(+)} \int_r^\infty dr \frac{r_0^{p-1}}{\sqrt{r^{2p-2} - r_0^{2p-2}}} \tag{28}$$

Thus y is double valued. The two possible signs can be taken to define a brane and its antibrane. We show at the end of this section that the solution with the minus sign is the minimum of the action and the solution with the plus sign the maximum of the action. This function is finite at $r = r_0$ and can be expressed in terms of elliptic integrals. For $r_0 = 1$ it is even simpler and has the value $y_c(1) = \bar{(+)} \frac{1}{\sqrt{2}} \mathcal{K}(\frac{1}{\sqrt{2}})$ where $\mathcal{K}$ is the complete elliptic integral of the first kind. Plotted as a function of r, $y_c(r)$ is a monotonically decreasing function starting from r_0; pictured on a 2–dimensional space it looks like an inverted funnel (i.e. the surface swept out by a catenary with boundaries at the openings), thus suggesting the name catenoid. As pointed out in ref.[2], the two possible signs of the square root allow a smooth joining of one such funnel–shaped branch to an inverted one connected by a throat of finite thickness, the resulting structure then representing a brane–antibrane pair. This brane–antibrane pair is joined by the throat of finite thickness r_0 and finite length. In fact, we can rewrite eq.(28) in terms of $\tilde{y}_c(x) = y_c(r_0 x), x = \frac{r}{r_0}$, and for the special case of $p = 3$ as

$$\begin{aligned} \tilde{y}_c(x) &= \bar{(+)} \int_x^\infty \frac{dx}{\sqrt{x^4 - 1}} = \bar{(+)} \int_1^\infty \frac{dx}{\sqrt{x^4 - 1}} \overset{+}{(-)} \int_1^x \frac{dx}{\sqrt{x^4 - 1}} \\ &= \bar{(+)} \frac{1}{\sqrt{2}} \left[\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - cn^{-1}\left(\frac{1}{x}, \frac{\sqrt{2}}{2}\right) \right] \end{aligned} \tag{29}$$

where $x > 1$ and we used formulae of ref.[23]. Inverting this expression we obtain the periodic function

$$x(y) = \left[cn\left(\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) \overset{+}{(-)} \sqrt{2} y, \frac{\sqrt{2}}{2} \right) \right]^{-1} \tag{30}$$

Plotting this expression with x as ordinate, one obtains the picture of a cross section through a chain of periodically recurring funnel–shaped structures to the one side of the throat, i.e. the series $\cup \cup \cup \cup \cdots$ representing a series of brane–antibrane pairs along the abscissa.

Proceeding as in the above case of the static BIon and calculating the second variational derivative we obtain

$$\delta^2 L = \frac{1}{2} \int d^p x \delta y \hat{B} \delta y \tag{31}$$

where for $r \geq r_0$

$$\begin{aligned} \hat{B} &= \partial_i \frac{i}{[1 + (\partial_i y_c)^2]^{1/2}} \partial_i - \partial_i \frac{\partial_i y_c \partial_j y_c}{[1 + (\partial_i y_c)^2]^{3/2}} \partial_j \\ &= \frac{1}{r^2} \frac{d}{dr} \frac{r^2}{\left(1 + y_c'^2\right)^{3/2}} \frac{d}{dr} = \bar{(+)} \frac{1}{r^2} \frac{d}{dr} \frac{(r^4 - r_0^4)^{3/2}}{r^4} \frac{d}{dr} \end{aligned} \tag{32}$$

The operator $\hat{B}$ can also be written

$$\hat{B} = \frac{1}{(1+y_c'^2)^{3/2}} \left\{ \frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} + \frac{6}{r} y_c'^2 \frac{d}{dr} \right\} \tag{33}$$

Since the gauge field components $A_a, a = p+1, \cdots, d-1$ (of which we retain only one), are dynamical, the Lagrangian in the nonstatic case is

$$\mathcal{L} = 1 - \sqrt{1 - (\partial_\mu y)(\partial^\mu y)} - \Sigma_p r_0^{p-1} \delta(\mathbf{r}) \tag{34}$$

and we can obtain the same condition of stability by considering the dynamical fluctuation η, i.e.

$$y(t, \mathbf{x}) = y_c(r) + \eta(t, \mathbf{x}), \quad \eta = \xi(r) e^{i\sqrt{\omega} t}$$

and linearising the time–dependent Euler–Lagrange equation. The square integrable perturbations $\xi(r)$ are the socalled "L^2 deformations" of ref.[4]. The classical stability of y_c is therefore decided by the spectrum $\{\omega\}$ of the small fluctuation equation

$$\frac{1}{r^2}\frac{d}{dr}\frac{r^2}{(1+y_c'^2)^{3/2}}\frac{d}{dr}\psi = \frac{1}{r^2}\frac{d}{dr}\left\{ \overset{-}{(+)} \frac{(r^4 - r_0^4)^{3/2}}{r^4} \right\} \frac{d}{dr}\psi = \omega\psi \tag{35}$$

We explore first the existence of a zero mode ψ_0, i.e. the case $\omega = 0$. In this case

$$\frac{r^2}{(1+y_c'^2)^{3/2}} \frac{d}{dr}\psi_0 = C \tag{36}$$

and so in the case $p = 3$ and $r \geq r_0$

$$\psi_0 = C \int_r^\infty dx \frac{x^4}{(x^4 - r_0^4)^{3/2}} = \frac{C}{2r_0}\frac{\partial y_c}{\partial r_0} \tag{37}$$

so that

$$\psi_0 = C \frac{\partial y_c}{\partial {r_0}^2}$$

Here again the derivative of the classical configuration y_c with respect to the parameter ${r_0}^2$ is indicative of stationarity of the action in a shift of r_0^2.

We now demonstrate that the operator $\hat{B}$ with the minus sign has no negative eigenvalues, and that therefore the free catenoid is a classically stable configuration like the BIon for fixed throat radius r_0. Then

$$\begin{aligned}
\int d^3x \psi \hat{B} \psi &= 4\pi \int_{r_0}^\infty dr \psi \frac{d}{dr} \frac{r^2}{(1+y_c'^2)^{3/2}} \frac{d\psi}{dr} \\
&= 4\pi \frac{r^2}{(1+y_c'^2)^{3/2}} \psi \frac{d\psi}{dr}\Big|_{r_0}^\infty - 4\pi \int_{r_0}^\infty dr \frac{r^2}{(1+y_c'^2)^{3/2}} \left(\frac{d\psi}{dr}\right)^2 \\
&= \overset{-}{(+)}\, 4\pi \frac{(r^4 - r_0^4)^{3/2}}{r^4} \psi \frac{d\psi}{dr}\Big|_{r_0}^\infty \overset{+}{(-)}\, 4\pi \int_{r_0}^\infty dr \frac{(r^4 - {r_0}^2)^{3/2}}{r^4} \left(\frac{d\psi}{dr}\right)^2
\end{aligned} \tag{38}$$

where we used eq.(27). The second term is always positive if the upper sign is chosen. The first term vanishes at infinity with $\int dr r^2\psi^2 < \infty$, since

$$-4\pi\frac{(r^4-r_0^4)^{3/2}}{r^4}\psi\frac{d\psi}{dr} \stackrel{r\to\infty}{\simeq} -4\pi r^2\psi\frac{d\psi}{dr} \to 0.$$

On the other hand, in the case $r \to r_0$, we have

$$-4\pi\frac{(r^4-r_0^4)^{3/2}}{r^4}\psi\frac{d\psi}{dr} \simeq -32\pi\sqrt{r_0}(r-r_0)^{3/2}\psi\frac{d\psi}{dr}$$

As $r \to r_0$ eq.(35) becomes

$$-\frac{8}{r_0^{3/2}}\frac{d}{dr}(r-r_0)^{3/2}\frac{d\psi}{dr} = \omega\psi \tag{39}$$

In the case of the zero mode ψ_0 with $\omega = 0$ the considerations are analogous to those of the BIon case and the sum of the two terms in eq.(38) vanishes. In the case of $\omega \neq 0$ we therefore have

$$\psi \simeq C\left(1-\frac{\omega}{4}{r_0}^{3/2}\sqrt{r-r_0}\right)$$

and

$$\lim_{r\to r_0}\ (r-r_0)^{3/2}\psi\frac{d\psi}{dr} = -\frac{\omega}{8}C^2.\lim_{r\to r_0}{r_0}^{3/2}(r-r_0) = 0$$

This proves that for all eigenfunctions ψ

$$<\psi|\hat{B}|\psi> \ \geq 0.$$

Thus $\hat{B}$ has no negative eigenvalues, and the free throat is classically stable with fixed r_0 for the sign chosen as in eq.(38). Obviously the operator $\hat{B}$ with the plus sign has no positive eigenvalues, which means that we have the maximum of the action. Of course, if we change r_0 (and so consider a different theory), the expectation value of $\hat{B}$ also changes. One should note that the free throat we discuss here is that with vanishing gauge field. The double valuedness of the solution of eq.(25) implies that if one solution is classically stable, the other one is not. Thus a multi–throat solution constructed from these by matching both solutions, if it exists, like the brane–antibrane solution of ref.[2], is expected to be unstable in view of negative as well as positive eigenvalues, and is therefore neither a maximum nor a minimum of the action. In fact, as argued in ref.[4] (after eq.(132)) equilibrium between these should not be possible. The reason for this is that a symmetrical configuration, symmetrical about the plane $x_3 = 0$ for instance, implies $\partial_3 y = 0$ there. Evaluating the stress tensor element T_{33} (even for vanishing gauge field), one obtains a negative quantity which is interpreted as implying an attractive force between the brane and its antibrane in this symmetrically constructed configuration. This is, in fact, the general instability of this configuration discussed in ref.[2].

4 Coupled fields: The D–brane in the Bogomol'nyi limit

In the case of coupled fields ϕ and y (the former with source, the latter without), the Lagrangian of the static case is (cf.[4])

$$\begin{aligned} L &= \int d^p x \mathcal{L},\ \mathcal{L} = 1 - Q - \Sigma_p e\phi\delta(\mathbf{r}), \\ Q &= \left[1 - (\partial_i\phi)^2 + (\partial_i y)^2 + (\partial_i\phi.\partial_i y)^2 - (\partial_i\phi)^2(\partial_j y)^2\right]^{\frac{1}{2}} \end{aligned} \tag{40}$$

¿From the first variation of L we obtain the coupled equations of the fields ϕ and y, i.e. from

$$\begin{aligned} \delta L = &- \int \delta\phi\partial_i \frac{1}{Q}[\partial_i\phi - (\nabla\phi\cdot\nabla y)\partial_i y + (\nabla y)^2\partial_i\phi]d^p x \\ &+ \int \delta y\partial_i \frac{1}{Q}[\partial_i y + (\nabla\phi\cdot\nabla y)\partial_i\phi - (\nabla\phi)^2\partial_i y]d^p x \\ &- \Sigma_p e\int \delta\phi\delta(\mathbf{r})d^p x \end{aligned} \tag{41}$$

(ignoring total divergences).

The source term of the electric field again suggests spherical symmetry. In deriving the two coupled Euler–Lagrange equations one new constant c (apart from e) arises in the integration of the catenoid equation, i.e.

$$\partial_r\left(r^{p-1}\frac{\partial\mathcal{L}}{\partial(\partial_r y)}\right) = 0,\quad r^{p-1}\frac{\partial\mathcal{L}}{\partial(\partial_r y)} = c$$

We have no source term of the y field because, as before, the appropriate effect is provided by the boundary condition defining the width of the throat. The two equations with spherical symmetry are found to be

$$\frac{\phi'}{[1-(\phi')^2+(y')^2]^{\frac{1}{2}}} = -\frac{e}{r^{p-1}},\quad \frac{-y'}{[1-(\phi')^2+(y')^2]^{\frac{1}{2}}} = \frac{c}{r^{p-1}} \tag{42}$$

so that

$$\frac{\phi'}{y'} = \frac{e}{c} \equiv \frac{1}{a} \tag{43}$$

Then

$$(\phi')^2 = \frac{1}{\frac{r^{2(p-1)}}{e^2}+1-a^2},\quad (y')^2 = \frac{a^2}{\frac{r^{2(p-1)}}{e^2}+1-a^2} \tag{44}$$

Thus the family of solutions can be parametrised in terms of the single parameter a as already pointed out in ref. [5]. This parameter is seen to interpolate between the two types of static solutions. The solution y of (35) for various values of a^2 is now the p-brane, i.e.

$$y(r) = \overset{+}{(-)}\, ae\int_r^\infty dr\frac{1}{\sqrt{r^{2(p-1)} - r_0^{2(p-1)}}} \tag{45}$$

where $r_0^{2(p-1)} = e^2(a^2-1)$ and for the solution to make sense we must have $a^2 \geq 1$. If ae in eq.(36) is replaced by $-ae$, the expression represents the corresponding antibrane. Taking $e^2 \to 0, a^2e^2 \to$ const. the electric field is eliminated and we regain the free catenoid solution. In approaching the limit $a^2 \to 1$ the width of the throat becomes infinitesimal with nonvanishing electric field and the configuration can then be considered to be a fundamental string, as argued in ref.[2]. We distinguish between three cases:

$$\begin{aligned}
|a| < 1 \quad &: \\
&\phi = \int_r^\infty \frac{dx}{\sqrt{1-a^2+x^4/e^2}}, y = a\int_r^\infty \frac{dx}{\sqrt{1-a^2+x^4/e^2}}, \\
|a| > 1 \quad &: \\
&\phi = e\int_r^\infty \frac{dx}{\sqrt{x^4-{r_0}^4}}, y = ae\int_r^\infty \frac{dx}{\sqrt{x^4-{r_0}^4}} \\
a = \pm 1 \quad &: \quad \phi = \frac{e}{r}, y = \pm\frac{e}{r}
\end{aligned} \tag{46}$$

We see that for $a^2 = 1$ eq. (34) becomes the first order Bogomol'nyi equation or linearised field equation for y (as in ref.[2])

$$F_{0r} \pm \frac{\partial y}{\partial r} = 0 \tag{47}$$

where $F_{0r} = E_c$ is the static electric field. This is the same equation as that obtained from the vanishing of the supersymmetry variation of the gaugino field Σ for half the number of 16 supersymmetries (for $d = 10$ and $p = 3$) ϵ_+, ϵ_- of ϵ for which $\delta\Sigma = 0$, i.e.

$$\delta_+\Sigma = 0, \quad \delta_-\Sigma \neq 0$$

where – as discussed in the literature [24] – ϵ is the constant spinor of the supersymmetry variation and $\epsilon_\pm$ are its chiral components. Thus $a^2 = 1$ implies BPS configurations, wheras those with $a^2 \neq 1$ are non–BPS. Taking $a^2 = 0$ in eq.(36) we regain the BIon configuration as a local minimum of the energy whereas for vanishing electric field one expects a local maximum, i.e. a sphaleron configuration (as pointed out in [2]).

Next we investigate the second variation of the static L with spherical symmetry. We set

$$\delta^2 L = \frac{1}{2}\int \left\{\delta\phi\hat{M}\delta\phi + \delta y\hat{N}\delta y + \delta\phi\hat{L}\delta y + \delta y\hat{L}^\dagger\delta\phi\right\}d^px \tag{48}$$

Again ignoring total divergences one finds

$$\begin{aligned}
\hat{M} &= -\frac{1}{r^2}\frac{d}{dr}r^2\frac{1+y'^2}{Q^3}\frac{d}{dr}, \\
\hat{N} &= \frac{1}{r^2}\frac{d}{dr}r^2\frac{1-\phi'^2}{Q^3}\frac{d}{dr}, \\
\hat{L} &= \frac{1}{r^2}\frac{d}{dr}r^2\frac{\phi'y'}{Q^3}\frac{d}{dr}
\end{aligned} \tag{49}$$

with $\hat{L} = \hat{L}^\dagger$. We can now rewrite $\delta^2 L$ as

$$\delta^2 L = \frac{1}{2}\int d^3x(\delta\phi, \delta y)\hat{H}\begin{pmatrix} \delta\phi \\ \delta y \end{pmatrix} \tag{50}$$

where

$$\hat{H} = \begin{pmatrix} M & L \\ L^\dagger & N \end{pmatrix} = \frac{1}{r^2}\frac{d}{dr}r^2 h\frac{d}{dr} \tag{51}$$

and

$$h = \frac{1}{Q^3}\begin{pmatrix} -1-y'^2 & y'\phi' \\ y'\phi' & 1-\phi'^2 \end{pmatrix}, \quad \hat{H}^\dagger = \hat{H}, \tag{52}$$

with

$$h^{-1} = Q\begin{pmatrix} -1+\phi'^2 & y'\phi' \\ y'\phi' & 1+y'^2 \end{pmatrix}, \quad \det h = \frac{1}{Q^4}, \tag{53}$$

The small fluctuation equation therefore becomes

$$\hat{H}\psi = \frac{1}{r^2}\frac{d}{dr}r^2 h\frac{d}{dr}\psi = \omega\psi \tag{54}$$

Again we first explore the existence of a zero mode ψ_0 with

$$r^2 h\frac{d}{dr}\psi_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{55}$$

where α and β are constants. Setting

$$\psi_0 = \begin{pmatrix} \phi_0 \\ y_0 \end{pmatrix}$$

and evaluating ψ_0 for the solutions of eq.(46) we obtain with

$$\varphi = -\int_r^\infty \phi'^3(x)dx$$

the relation

$$\psi_0 = \frac{\phi}{e}\left\{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} - \frac{\varphi}{e}(\alpha + a\beta)\begin{pmatrix} 1 \\ a \end{pmatrix}\right\} \tag{56}$$

In the BPS limit with $y' = \phi' = E_c, Q = 1$, the operator $\hat{H}$ of eq.(42) becomes

$$\hat{H} = \frac{1}{r^2}\frac{d}{dr}r^2\begin{pmatrix} -1-E_c^2 & E_c^2 \\ E_c^2 & 1-E_c^2 \end{pmatrix}\frac{d}{dr} \tag{57}$$

Setting

$$\psi_s = \begin{pmatrix} \delta\phi \\ \delta y \end{pmatrix} = \rho(x)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for an arbitrary function $\rho(x)$ we have

$$\hat{H}\psi_s = \frac{1}{r^2}\frac{d}{dr}r^2\rho'\begin{pmatrix} -1-E_c^2 & E_c^2 \\ E_c^2 & 1-E_c^2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{r^2}\frac{d}{dr}r^2\rho'\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{58}$$

Thus for arbitrary $\rho(x)$, we have $\psi_s \hat{H} \psi_s = 0$ implying $\delta^2 L = 0$ or L constant in a specific direction about the BPS configuration. This behaviour may be interpreted as indicative of a local symmetry, in this case of supersymmetry, and so of the cancellation of fermionic and bosonic contributions in the one loop approximation. Here, of course, we have no fermionic contributions and consequently those of the two bosonic fields have opposite signs.

5 Fluctuations about the D–brane

In the following we distinguish clearly between two different types of fluctuations. We consider the above BPS solution for the string attached to the 3–brane as background and consider first a scalar field propagating in a direction along the string and perpendicular to the brane and its anti–brane. The linearised equation of small fluctuations about this background is obtained from the second variational derivative of the action which is the standard procedure and we therefore consider this first (cf. also [25]). Our treatment here is somewhat different (see below) from that in refs.[25]. The resulting fluctuation equation has also been given in ref.[2]. It is necessary to return to the fully time-dependent version, i.e.

$$S = \frac{1}{(2\pi)^p g_s} \int d^{p+1}x \left[1 - \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})} - \Sigma_p e\phi\delta(\mathbf{r}) \right] \tag{59}$$

where in $3+1$ dimensions $F_{\mu\nu} = F_{\mu\nu}(x_0, x_1, x_2, x_3)$. In the electrostatic case with only one scalar field y we have $A_\mu = (A_0, A_1, A_2, A_3, y, 0, 0, 0, 0, 0)$, $F_{0i} = E_i$ and $F_{\mu 4} = \partial_\mu y$ for $i = 1, 2, 3$ and $\mu = 0, 1, 2, 3$. Then

$$\det(\eta_{\mu\nu} + F_{\mu\nu}) = \begin{vmatrix} -1 & E_1 & E_2 & E_3 & \partial_0 y \\ -E_1 & 1 & 0 & 0 & \partial_1 y \\ -E_2 & 0 & 1 & 0 & \partial_2 y \\ -E_3 & 0 & 0 & 1 & \partial_3 y \\ -\partial_0 y & -\partial_1 y & -\partial_2 y & -\partial_3 y & 1 \end{vmatrix} \tag{60}$$

and so

$$\det(\eta_{\mu\nu} + F_{\mu\nu}) = -(1 - \mathbf{E}^2)(1 + \nabla y^2) - (\mathbf{E} \cdot \nabla y)^2 + (\partial_0 y)^2 \tag{61}$$

We consider first the Lagrangian density (remembering that the relevant fields are A_0, A_i and y)

$$\mathcal{L} = 1 - Q, \quad Q = \left[(1 - \mathbf{E}^2)(1 + \nabla y^2) + (\mathbf{E} \cdot \nabla y)^2 - \dot{y}^2 \right]^{\frac{1}{2}} \tag{62}$$

The equations of the static BIon and the static catenoid discussed above follow again from the first variations

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial E_i} &= \frac{1}{Q} \left[E_i(1 + \nabla y^2) - \partial_i y (\mathbf{E} \cdot \nabla y) \right], \\ \frac{\partial \mathcal{L}}{\partial \partial_i y} &= -\frac{1}{Q} \left[\partial_i y (1 - \mathbf{E}^2) + E_i (\mathbf{E} \cdot \nabla y) \right], \\ \frac{\partial \mathcal{L}}{\partial \partial_0 y} &= -\frac{1}{Q} \end{aligned} \tag{63}$$

In the BPS background given by

$$\partial_i y = E_i, \quad \mathbf{E}^2 = (\nabla y)^2 = \mathbf{E}\cdot\nabla y = \frac{e^2}{r^4} \equiv E_c^2 \equiv y_c^2, \quad Q = 1 \tag{64}$$

one finds

$$\frac{\partial^2\mathcal{L}}{\partial E_i \partial E_j} = (1+E_c^2)\delta_{ij}, \frac{\partial^2\mathcal{L}}{\partial\partial_i y \partial\partial_j y} = -(1-E_c^2)\delta_{ij}, \frac{\partial^2\mathcal{L}}{\partial E_i \partial\partial_j y} = -E_c^2\delta_{ij}, \frac{\partial^2\mathcal{L}}{\partial\dot{y}^2} = 1 \tag{65}$$

This enables us to write (ignoring again total divergences in shifting derivatives)

$$\delta^2\mathcal{L} = (\delta A_0, \delta A_i, \delta y)\cdot \cdot \begin{pmatrix} -\partial_i(1+E_c^2)\partial_i & \partial_i(1+E_c^2)\partial_0 & -\partial_i E_c^2\partial_i \\ \partial_0(1+E_c^2)\partial_i & -\partial_0(1+E_c^2)\partial_0 & \partial_0 E_c^2\partial_i \\ -\partial_i E_c^2\partial_i & +\partial_i E_c^2\partial_0 & \partial_i(1-E_c^2)\partial_i - \partial_0\partial_0 \end{pmatrix} \begin{pmatrix} \delta A_0 \\ \delta A_i \\ \delta y \end{pmatrix} \tag{66}$$

In the linear approximation the Euler–Lagrange equations of the fluctuations $\delta y \equiv \eta, \delta E_i = \partial_0\delta A_i - \partial_i\delta A_0$ are therefore given by the following set of three equations

$$-\frac{d^2}{dt^2}\eta + \partial_i(1-E_c^2)\partial_i\eta + \partial_i E_c^2(\partial_0\delta A_i - \partial_i\delta A_0) = 0, \tag{67}$$

$$\frac{d}{dt}(1+E_c^2)(\partial_0\delta A_i - \partial_i\delta A_0) - \frac{d}{dt}E_c^2\partial_i\eta = 0, \tag{68}$$

$$\partial_i(1+E_c^2)(\partial_0\delta A_i - \partial_i\delta A_0) - \partial_i E_c^2\partial_i\eta = 0 \tag{69}$$

The last of these three equations can be seen to be a constraint by appying $\partial/\partial t$ and using the second equation. Substituting from the last

$$\partial_i E_c^2(\partial_0\delta A_i - \partial_i\delta A_0) = \partial_i E_c^2\partial_i\eta - \partial_i(\partial_0\delta A_i - \partial_i\delta A_0)$$

into the first equation we obtain

$$-\frac{d^2}{dt^2}\eta + \triangle\eta - \partial_i(\partial_0\delta A_i - \partial_i\delta A_0) = 0 \tag{70}$$

The second of the three equations can be written in the form

$$(1+E_c^2)(\partial_0\delta A_i - \partial_i\delta A_0) - E_c^2\partial_i\eta = (1+E_c^2)C_i(r) \tag{71}$$

where $\mathbf{C}(r)$ is an arbitrary function. Dividing eq.(60) by $(1+E_c^2)$ and taking the derivative ∂_i, we obtain

$$\begin{aligned} \partial_i(\partial_0\delta A_i - \partial_i\delta A_0) &= \partial_i\frac{E_c^2}{1+E_c^2}\partial_i\eta + \partial_i C_i \\ &= \frac{E^2}{1+E_c^2}\triangle\eta + \frac{2E_c E_c'}{(1+E_c^2)^2}\frac{x_i}{r}\partial_i\eta + \partial_i C_i \end{aligned} \tag{72}$$

Replacing on the right hand side $E_c^2\partial_i\eta$ by the expression in eq.(60)this becomes

$$\partial_i(\partial_0\delta A_i - \partial_i\delta A_0) = \frac{E_c^2}{1+E_c^2}\triangle\eta + \frac{2E_c'}{E_c(1+E_c^2)}\frac{x_i}{r}\left[(\partial_0\delta A_i - \partial_i\delta A_0) - C_i\right] + \partial_i C_i \tag{73}$$

Choosing as gauge fixing condition the relation

$$\frac{2E_c'}{E_c(1+E_c^2)}\frac{x_i}{r}\left[(\partial_0\delta A_i-\partial_i\delta A_0)-C_i\right]+\partial_i C_i=0$$

one obtains the following fluctuation equation for η

$$-(1+E_c^2)\frac{d^2\eta}{dt^2}+\triangle\eta=0 \tag{74}$$

All the relations from (60) to (74) describe perturbations along the string and perpendicular to the brane. Eq. (74) cannot be considered independently of the others as is apparent from the linkage of the fields in the above equations. Thus if one wants to determine the radiation of the string between the brane and the antibrane, one must connect the asymptotic behaviour of the field η with that of the vector field δA_μ.

However, an equation like (74)is also obtained if one evaluates the determinant in the Born–Infeld Lagrangian at the BPS background and with an additional time–dependent scalar η, representing the fluctuation field along a new spatial direction (cf. also ref.[2]). In this case this new scalar field in the D–brane background has no relevance to the string radiation, and we have

$$\det(\eta_{\mu\nu}+F_{\mu\nu})|_{BPS,\eta}=\begin{vmatrix} -1 & E_1 & E_2 & E_3 & 0 & \partial_0\eta \\ -E_1 & 1 & 0 & 0 & E_1 & \partial_1\eta \\ -E_2 & 0 & 1 & 0 & E_2 & \partial_2\eta \\ -E_3 & 0 & 0 & 1 & E_3 & \partial_3 y \\ 0 & -E_1 & -E_2 & -E_3 & 1 & 0 \\ -\partial_0\eta & -\partial_1\eta & -\partial_2\eta & -\partial_3\eta & 0 & 1 \end{vmatrix} \tag{75}$$

and so

$$\det(\eta_{\mu\nu}+F_{\mu\nu})|_{BPS,\eta}=-(1+\mathbf{E}_c{}^2)(\partial_0\eta)^2-(\partial_i\eta)^2-1 \tag{76}$$

Thus the Lagrangian density becomes

$$\mathcal{L}=1-\sqrt{1+(\nabla\eta)^2-(1+\mathbf{E}_c{}^2)(\partial_0\eta)^2} \tag{77}$$

By expanding the square root and retaining only the lowest order terms, we again obtain a fluctuation equation like (65), but this time for η with no relevance to radiation of the string. This is equivalent to studying the scattering of the scalar η off a corresponding supergravity background.

6 Absorption of scalar in background of $D3$ brane

We now consider the equation of small fluctuations, i.e. eq.(74), in more detail. The fluctuation $\eta(t,\mathbf{x})$ represents a scalar field that impinges on the brane which reflects part of it and absorbs part of it depending on the energy ω of the field. The absorption results from and takes place into the singularity of the real potential which corresponds to the black hole with zero event horizon in the analogous

case of the dilaton–axion system of e.g. ref.[19]. This absorption is a classical phenomenon. We therefore consider the equation

$$\triangle_r \xi + \omega^2 \left[1 + \frac{e^2}{r^4}\right] \xi = 0 \tag{78}$$

One can argue that the absorption is a consequence of the nonhermiticity of the potential.

The radial part of this equation is with $\xi = r^{-1}\Psi Y_{lm}$ and angular momentum l

$$\frac{d^2\Psi}{dr^2} + \left[-\frac{l(l+1)}{r^2} + \omega^2\left(1 + \frac{e^2}{r^4}\right)\right]\Psi = 0 \tag{79}$$

This equation is a radial Schrödinger equation for an attractive singular potential $\propto r^{-4}$ but depends only on the single coupling parameter $\kappa = e\omega^2$ for constant positive Schrödinger energy, i.e. for S-waves the equation is with $x = \omega r$ simply

$$\left(\frac{d^2}{dx^2} + 1 + \frac{\kappa^2}{x^4}\right)\Psi = 0 \tag{80}$$

In the following we consider the general case, i.e. $l \neq 0$. The simplified case of the singular potential replaced by an effective delta–function potential has been considered in refs.[2] and [25]. The solutions and properties of such equations have been studied in detail in the literature, in both the small– and large–κ domains and with inclusion of the centrifugal term $-l(l+1)/r^2$ in eq.(79) for the calculation of Regge trajectories $l \to \alpha_n(\omega^2)$ [26], [27],[28],[29]. A recent investigation which attempts to treat arbitrary power singular potentials is ref.[30]. Eq.(79) describes waves above the singular potential well. With the substitutions

$$\Psi(r) = r^{\frac{1}{2}}\psi(r), \quad r = \sqrt{e}e^z, \quad h^2 = e\omega^2, \quad a = l + \frac{1}{2}, \tag{81}$$

the equation becomes the modified Mathieu equation

$$\frac{d^2\psi}{dz^2} + \left[2h^2\cosh 2z - a^2\right]\psi = 0 \tag{82}$$

which has been studied in detail in the literature [31] (though some properties, such as large–h asymptotic expansions of Fourier coefficients, have even now not yet been published). Here we study the S–matrix in the domain of finite values of angular momentum l and $h^2 \neq 0$, i.e. in the domain of h^2 large. Relevant solutions and matching conditions for this case have been developed in [32]and [33]. We follow the latter of these references here since this makes full use of the symmetries of the solutions. Moreover we can determine also the Floquet exponent ν which ref.[32] leaves undetermined and only remarks that the notion that this is a known function of (our) a^2 and h^2 is "partly a convenient fiction".

For convenience we set in eq. (82) as in ref.[33, 34]

$$a^2 = -2h^2 + 2hq + \frac{\triangle(q,h)}{8} \tag{83}$$

where q is a parameter to be determined as the solution of this equation and $\triangle/8$ is the remainder of the large–h asymptotic expansion (83), the various terms of which are determined concurrently with corresponding iteration contributions of the solutions ψ of the equation and are known explicitly to many orders [34]. Then setting in eq. (82)

$$\psi(q,h;z) = A(q,h;z)exp[\pm 2hi\sinh z] \tag{84}$$

we obtain an equation for A which can be written

$$\cosh z\frac{dA}{dz} + \frac{1}{2}(\sinh z \pm iq)A = \pm\frac{1}{4hi}\left[\frac{\triangle}{8}A - \frac{d^2A}{dz^2}\right] \tag{85}$$

We let $A_q(z)$ be the solution of this equation when the right hand side is replaced by zero (i.e. in the limit $h \to \infty$). Then one finds easily

$$A_q(z) = \frac{1}{\sqrt{\cosh z}}\left(\frac{1+i\sinh z}{1-i\sinh z}\right)^{\mp q/4} \overset{z\to\infty}{\simeq} \sqrt{2}e^{-z/2}e^{\mp i\pi q/4} \tag{86}$$

Correspondingly the various solutions ψ are

$$\begin{aligned}
\psi(q,h;z) &= A_q(z)exp[\pm 2hi\sinh z] \overset{z\to\infty}{\simeq} \frac{exp(\pm ihe^z)}{\sqrt{\cosh z}}e^{\mp i\pi q/4}, \\
\psi(q,h;z) &= A_q(z)exp[\pm 2hi\sinh z] \overset{z\to-\infty}{\simeq} \frac{exp(\mp ihe^{|z|})}{\sqrt{\cosh z}}e^{\mp i\pi q/4}
\end{aligned} \tag{87}$$

We make the important observation that given one solution $\psi(q,h;z)$ we can obtain the linearly independent one either as $\psi(-q,-h;z)$ or as $\psi(q,h;-z)$, the expression (83) remaining unchanged. With the solutions as they stand, of course $\psi(q,h;z) = \psi(-q,-h;-z)$. Below we require solutions $He^{(i)}(z), i = 1,2,3,4$, with some specific asymptotic behaviour. We define these in terms of the function

$$Ke(q,h;z) := \frac{exp[i\pi q/4]}{\sqrt{-2ih}}A_q(z)exp[2hi\sinh z] \equiv k(q,h)\psi(q,h;z) \tag{88}$$

Since this function differs from a solution ψ by a factor $k(q,h)$, it is still a solution but not with the symmetry property $\psi(q,h;z) = \psi(-q,-h;-z)$. Instead, after performing this cycle of replacements the function picks up a factor, i.e.

$$Ke(q,h;z) = \frac{k(q,h)}{k(-q,-h)}Ke(-q,-h;-z), \quad \frac{k(q,h)}{k(-q,-h)} = e^{i\frac{\pi}{2}(q+1)} \tag{89}$$

in leading order. One can easily show that the quantity Φ_0 of ref. [32] is related to q by $\Phi_0 = iq\pi/2 + O(1/h)$. In Fig. 1 we show the behaviour of q as a function of h. In order to be able to obtain the S–matrix, we have to match a solution valid at $z = -\infty$ to a combination of solutions valid at $z = \infty$. This is achieved with the help of Floquet solutions $Me_{\pm\nu}(z,h^2)$. As such, these satisfy the same circuit relation as a solution $M^{(1)}_{\pm\nu}(z,h^2)$ of eq.(82) expanded in a series of Bessel functions, i.e. we have the proportionality

$$Me_\nu(z,h^2) = \alpha_\nu M^{(1)}_\nu(z,h^2), \quad \alpha_\nu(h^2) = Me_\nu(0,h^2)/M^{(1)}_\nu(0,h^2) \tag{90}$$

The functions $Me_{\pm\nu}(z,h^2)$ are expansions of the modified (hence 'M' instead of 'm') Mathieu equation in terms of exponentials (hence 'e') which are uniformly convergent in any finite domain of z. For large values of the argument $2h\cosh z$ of the Bessel functions of the modified Mathieu function $M_\nu^{(1)}(z,h^2)$ can be reexpressed in terms of Hankel functions. With the dominant terms of these we can obtain the large $2h\cosh z$ asymptotic behaviour of the Floquet function $Me_{\pm\nu}(z,h^2)$, i.e. for $|z|\to\infty$

$$Me_{\pm\nu}(z,h^2)\simeq exp[\pm i\pi\gamma/2]\frac{\cos(2h\cosh z\mp\nu\pi/2-\pi/4)}{\sqrt{2h\cosh z}} \tag{91}$$

where (with $Me_\nu(-z,h^2)=Me_{-\nu}(z,h^2)$)

$$exp[i\pi\gamma]=\frac{\alpha_\nu(h^2)}{\alpha_{-\nu}(h^2)}=M_{-\nu}^{(1)}(0,h^2)/M_\nu^{(1)}(0,h^2) \tag{92}$$

We now define the following set of solutions of eq.(82) by setting

$$\begin{aligned} He^{(2)}(z,q,h) &= Ke(q,h,z), He^{(1)}(z,q,h)=He^{(2)}(z,-q,-h),\\ He^{(3)}(z,q,h) &= He^{(1)}(-z,q,h), He^{(4)}(z,q,h)=He^{(2)}(-z,q,h) \end{aligned} \tag{93}$$

The solutions so defined have the following asymptotic behaviour (where $\epsilon(z)=(2h\cosh z)^{-1/2}$):

$$\begin{aligned} He^{(1)}(z,q,h) &= \epsilon(z)\cdot exp[-ihe^z-i\frac{\pi}{4}],\quad \Re z>>0,\\ &\overset{r\to\infty}{\sim} \frac{exp[-i\omega r-i\pi/4]}{\sqrt{\omega r}},\\ He^{(2)}(z,q,h) &= \epsilon(z)\cdot exp[ihe^z+i\frac{\pi}{4}],\quad \Re z>>0,\\ &\overset{r\to\infty}{\sim} \frac{exp[i\omega r+i\pi/4]}{\sqrt{\omega r}},\\ He^{(3)}(z,q,h) &= \epsilon(z)\cdot exp[-ihe^{|}z|-i\frac{\pi}{4}],\quad \Re z<<0,\\ He^{(4)}(z,q,h) &= \epsilon(z)\cdot exp[ihe^{|}z|+i\frac{\pi}{4}],\quad \Re z<<0,\\ &\overset{r\to 0}{\sim} \frac{r^{1/2}exp[ie\omega/r+i\frac{\pi}{4}]}{(e\omega)^{1/2}}, \end{aligned} \tag{94}$$

For the following reasons we choose the latter, i.e. the solution $He^{(4)}(z,q,h)$, as our solution at $r=0$. The time–dependent wave function with this asymptotic behaviour is proportional to

$$e^{-i\omega t+ie\omega/r+i\pi/4}$$

Fixing the wave front by setting $\varphi=-\omega t+e\omega/r+\pi/4=const.$ and considering the propagation of this wave front, we have

$$r=\frac{e\omega}{\varphi+\omega t-\pi/4}$$

so that when $t \to \infty : r \to 0$. This means that the origin of coordinates acts as a sink.

With eq. (91) we therefore equate in the domain $\Re z >> 0$:

$$\begin{aligned} Me_\nu(z,h^2) &= \frac{i}{2} exp[i\pi\gamma/2]\Big\{exp[i\nu\frac{\pi}{2}]He^{(1)}(z,q,h) - exp[-i\nu\frac{\pi}{2}]He^{(2)}(z,q,h)\Big\}, \\ Me_{-\nu}(z,h^2) &= \frac{i}{2} exp[-i\pi\gamma/2]\Big\{exp[-i\nu\frac{\pi}{2}]He^{(1)}(z,q,h) \\ &- exp[i\nu\frac{\pi}{2}]He^{(2)}(z,q,h)\Big\}, \end{aligned} \tag{95}$$

where the second relation was obtained by changing the sign on ν in the first. Changing the sign of z we obtain in the domain $\Re z << 0$:

$$\begin{aligned} Me_\nu(-z,h^2) &= Me_{-\nu}(z,h^2) \\ &= \frac{i}{2} exp[i\pi\gamma/2]\Big\{exp[i\nu\frac{\pi}{2}]He^{(3)}(z,q,h) - exp[-i\nu\frac{\pi}{2}]He^{(4)}(z,q,h)\Big\}, \\ Me_\nu(z,h^2) &= \frac{i}{2} exp[-i\pi\gamma/2]\Big\{exp[-i\nu\frac{\pi}{2}]He^{(3)}(z,q,h) \\ &- exp[i\nu\frac{\pi}{2}]He^{(4)}(z,q,h)\Big\}, \end{aligned} \tag{96}$$

These relations are now valid over the entire range of z. Substituting eqs.(96)into eqs.(95) and eliminating $He^{(3)}$ we obtain

$$-\sin\pi\nu.He^{(4)}(z,q,h) = \sin\pi(\gamma+\nu).He^{(1)}(z,q,h) - \sin\pi\gamma.He^{(2)}(z,q,h) \tag{97}$$

In a similar way one obtains the relations

$$\begin{aligned} -\sin\pi\nu.He^{(2)}(z,q,h) &= \sin\pi(\gamma+\nu).He^{(3)}(z,q,h) - \sin\pi\gamma.He^{(4)}(z,q,h) \\ \sin\pi\nu.He^{(1)}(z,q,h) &= -\sin\pi\gamma.He^{(3)}(z,q,h) + \sin\pi(\gamma-\nu).He^{(4)}(z,q,h) \end{aligned} \tag{98}$$

¿From eqs.(89) and (93) we see that $He^{(2)}(z,q,h)$ is proportional to $He^{(3)}(z,q,h)$. ¿From (89) and (98) we see that the proportionality factor is given by

$$exp[i\frac{\pi}{2}(q+1)] = -\frac{\sin\pi(\gamma+\nu)}{\sin\pi\nu} \tag{99}$$

¿From eq.(97) we can now deduce the S–matrix $S_l \equiv e^{2i\delta_l}$, where δ_l is the phase shift. The latter is defined by the following large r behaviour of the solution chosen at $r = 0$, which in our case is the solution $He^{(4)}$. Thus here the S–matrix is defined by (using (97))

$$\begin{aligned} - &\quad \sin\pi\nu\frac{r^{1/2}e^{ie\omega/r+i\pi/4}}{(e\omega)^{1/2}} \\ \stackrel{r\to\infty}{=} &\quad -(-1)^l\frac{\sin\pi(\gamma+\nu)e^{-i\pi/4}}{\sqrt{\omega r}}\Big[\frac{\sin\pi\gamma(-1)^l}{\sin\pi(\gamma+\nu)}e^{i\pi/2}e^{i\omega r} - (-1)^l e^{-i\omega r}\Big] \\ \equiv &\quad \frac{e^{-i\delta}e^{-il\pi/2}}{2i\sqrt{r}}\Big[S_l e^{i\omega r} - (-1)^l e^{-i\omega r}\Big] \end{aligned} \tag{100}$$

¿From this we deduce that

$$S_l = \frac{\sin\pi\gamma}{\sin\pi(\gamma+\nu)} e^{i\pi(l+1/2)} = -\frac{\sin\pi\gamma}{\sin\pi\nu} e^{i\pi(l-\frac{1}{2}q)} \tag{101}$$

We can see the relation of this high–energy (i.e. large $|h|$) expression of the S–matrix to the low–energy expression of ref.[31] by recalling that R of the latter is here $exp(i\pi\gamma)$. With this identification we can write S_l

$$S_l = \frac{R - \frac{1}{R}}{(Re^{i\pi\nu} - \frac{e^{-i\pi\nu}}{R})} e^{i\pi(l+1/2)}, \quad R \equiv e^{i\pi\gamma},$$

which agrees with the S–matrix of ref.[19], i.e. we thus obtained the same exact expression of the S–matrix here with our large–h considerations. In fact, comparison with the considerations given there allows one to write down the reflection and transmission amplitudes A_r and A_t as $A_r = 2i\sin\pi\gamma$ and $A_t = \sin\pi\nu$ respectively. We thus have one and the same expression for the S–matrix for the two asymptotic regions, i.e. in the low energy and high energy domains. One should therefore be able to proceed directly to the large–h case from the exact S–matrix derived in the small–h domain. This is an interesting calculation which we do not attempt to go into here. We only indicate in Appendix A the first necessary step in that direction, i.e. the derivation of large–h asymptotic expansions for the Fourier coefficients of Mathieu functions. In this connection we make the following two observations. 1) Eq.(80) is invariant under interchanges $x \leftrightarrow \kappa/x, \Psi \leftrightarrow \Psi x$ which means that the inner or string region is equivalent or dual to the outer or brane region. 2) Due to the $SL(2,R)$ invariance of the $D3$–brane its action is mapped into that of an equivalent $D3$–brane by S–duality transformations[35] or, in other words, weak–strong duality takes the $D3$–brane into itself[3]. It would be interesting to find some connection between these properties, or equivalently the symmetry which the $SL(2,R)$ invariance of the $D3$–brane action imposes on the S–matrix.

The quantity γ is now to be determined from eq. (99). One finds

$$\sin\pi\gamma = \sin\pi\nu\Big\{ -ie^{i\frac{\pi}{2}q}\cos\pi\nu \pm \sqrt{1+e^{i\pi q}\sin^2\pi\nu}\Big\} \tag{102}$$

It remains to determine the Floquet exponent ν in terms of q and h. In Appendix B we derive the appropriate large–h behaviour of ν for the case of the periodic Mathieu equation. Replacing there the eigenvalue λ by $a = (l+\frac{1}{2})^2$ and observing that h^2 remains h^2, the appropriate relation for our considerations is

$$\begin{aligned} \cos\pi\nu + 1 &= \frac{\pi e^{4h}}{(8h)^{q/2}}\Big[\frac{1+\frac{3(q^2+1)}{64h}}{\Gamma[\frac{3}{4}-\frac{q}{4}]\Gamma[\frac{1}{4}-\frac{q}{4}]} + O(\frac{1}{h^2})\Big] \\ &= \frac{e^{4h}}{(8h)^{q/2}}\Big[\frac{\Big(1+\frac{3(q^2+1)}{64h}\Big)\,\Gamma(\frac{q+1}{2})\cos(\frac{q\pi}{2})}{\sqrt{2\pi}2^{q/2}} + O(\frac{1}{h})\Big] \end{aligned} \tag{103}$$

Since the right hand side grows exponentially with increasing h the Floquet exponent ν must have a large imaginary part. Since the right hand side is real, the

real part of ν must be an integer. Using Stirling's formula we can approximate the equation for $q \simeq h$ (i.e. irrespective of what the value of l is) as

$$\cos \pi\nu + 1 = \sqrt{\frac{h}{2}} \cos(\frac{h\pi}{2})(e^7/32)^{h/2} \simeq \sqrt{\frac{h}{2}} e^{1.8h} \cos(\frac{h\pi}{2}) \tag{104}$$

¿From eq.(101) and eq.(102) we obtain

$$S_l = ie^{il\pi}\Big(\cos \pi\nu - \sqrt{\cos^2 \pi\nu - 1 - e^{iq\pi}}\Big) \tag{105}$$

¿From this we obtain the absorptivity $A(l,h)$ of the l–th partial wave, i.e.

$$A(l,h) := 1 - |S_l|^2 \tag{106}$$

with near asymptotic behaviour

$$A(l,h) \simeq 1 - \frac{2\pi(16h)^q}{e^{8h}\left\{\Gamma(\frac{q+1}{2})\right\}^2} \tag{107}$$

In Figs. 2, 3 and 4 we plot $A(l,h)$ as a function of h. One can clearly see the expected asymptotic approach to unity and in Fig. 2 some sign of rapidly damped oscillations. This behaviour agrees with that obtained on general grounds in ref.[15]. We also observe that in the high energy limit logarithmic contributions as in the low energy expansions, discovered originally in [16, 17, 18], and typical of the low energy expansions of [13] and [19], do not arise. Of course, these plots do not extend down to $h = 0$, since our asymptotic solutions become meaningless in that domain. The continuation to $h = 0$ can be obtained, however, from small–h expansions such as those derived in refs.[13] and [19]. Thus the absorptivity $A(l,h)$ is known over the entire range of h. We observe that $S_l = 0$ for $q = 1, 3, 5, \cdots$, with $[(l + 1/2)^2 + 2h^2]/2h \simeq 1, 3, 5, \cdots$. Only in the plot for $l = 2$ is h sufficiently large to hint at these zeros.

7 Concluding remarks

Branes, whether fundamental or solitonic, play an important role in all aspects of string theory. In particular D–branes have been looked at as string–theory analogues of solitons of simple field theories, and some of their important properties such as charges are well understood. Our first objective in the above was to investigate properties of solitonic objects of Born–Infeld theory in ways familiar from field theory, in particular their classical stability. It was shown that the BIon and the catenoid as distinct, i.e. free objects, are stable configurations whereas the brane–antibrane system is unstable; we also recognised the zero modes associated with these and their significance. We then considered the $D3$–brane of Born–Infeld theory and recognised this as a BPS state that preserves half of the number of supersymmetries as discussed in detail already in [2]. The equation of

small fluctuations about this $D3$–brane was derived and shown to be convertible into a modified Mathieu equation. The low energy solutions of this equation, the S–matrix for scattering of a massless scalar off the brane and the corresponding absorption and reflection amplitudes are similar to those for the dilaton–axion system investigated first in refs.[16, 17, 18], where the important logarithmic contributions were discovered, and then investigated in extensive detail in [13] and [19]. Here we performed the high energy calculations which complement in particular those of [19], thus completing the investigation of the modified Mathieu equation for the purpose of obtaining absorption cross sections for all such cases. In particular the behaviour of the important Floquet exponent involved in these calculations (in general a complex quantity) is now fully understood, the Floquet exponent being vital in the evaluation of the S–matrix which we derive and the calculation of the corresponding absorption amplitudes and cross sections. According to our findings the high energy limit of the absorption cross section does not involve logarithmic contributions, quite contrary to the low energy limit.

The high energy case considered here is not only of interest in the immediate context of the Born–Infeld model considered here, but together with the low–energy case also of considerable interest in connection with the concept of duality which links weak coupling with strong coupling. The $D3$–brane with Schrödinger potential coupling $e\omega^2$, which links the gauge field charge e with energyω of the incoming scalar field is presumably the ideal example for the investigation of this property. Investigations elucidating this aspect are of considerable interest. We also envisage interest in the study of non–BPS configurations, including sphalerons and bounces, as a matter of principle, i.e. even if the effect of these is not of dominant importance. Finally we remark that it should be possible to proceed directly from the S–matrix derived in ref.[19] to the high–energy case here by using appropriate asymptotic expansions for the cylindrical functions and expansion coefficients involved (for the latter such expansions do not seem to have been given in the published literature so far, but we comment on these in Appendix A).

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Appendix A

In ref. [19] on the absorptivity of the $D3$–brane of the dilaton–axion system it was shown that the S–matrix for scattering of a massless scalar field off the brane is given by

$$S = \frac{(R - \frac{1}{R})e^{-i\pi l}}{Re^{i\nu\pi} - \frac{e^{-i\nu\pi}}{R}} \tag{A.1}$$

where

$$R = \frac{M^{(1)}_{-\nu}(0,h)}{M^{(1)}_{\nu}(0,h)},$$

$M^{(1)}_{\nu}(z,h)$ being the modified Mathieu function expanded in terms of Bessel functions, i.e.

$$Me_{\nu}(0,h)M^{(1)}_{\nu}(z,h) = \sum_{r=-\infty}^{\infty} c^{\nu}_{2r}(h^2)J_{\nu+2r}(2h\cosh z)$$

(an expansion with better convergence to use in practice is one in terms of products of Bessel functions as shown in ref.[19]) where $Me_{\nu}(z,h)$ is the Fourier or Floquet solution of the Mathieu equation. In the published literature the coefficients $c^{\nu}_{2r}(h^2)$ have only been considered as power series in rising powers of h^2, and consequently were used in ref. [19] in the small h^2 or low energy domain. It would be very interesting to make the transition to the large–h^2 or high energy case directly from this expression by developing large–h^2 asymptotic expansions of the Mathieu function Fourier coefficients $c^{\nu}_{2r}(h^2)$ (for the Bessel functions the corresponding expansions are known). We know of no publication where such expansions have been given, but one of us (M.–K.) remembers from private communication with the author of ref.[37] that these Stokes–type asymptotic expansions can indeed be obtained. One writes the recurrence relation of the coefficients (cf. [31], p.106)

$$c_{2\rho+2} + c_{2\rho-2} = \frac{[\lambda - (\nu+2\rho)^2]}{h^2}c_{2\rho} \tag{A.2}$$

(the Mathieu equation being $y'' + (\lambda - 2h^2\cos 2x)y = 0$). For $|h^2| \to \infty$ this implies

$$c_{2\rho+2} \propto i^{(2\rho+2)/2}$$

Setting

$$c_{2\rho+2} \equiv b_{\rho+1}, \quad b_{\rho} = i^{\rho}\beta_{\rho}$$

we have

$$\begin{aligned} b_{\rho+1} + b_{\rho-1} &= \frac{[\lambda - (\nu+\rho)^2]}{h^2}b_{\rho}, \\ \beta_{\rho+1} + \beta_{\rho-1} &= \frac{-i[\lambda - (\nu+\rho)^2]}{h^2}\beta_{\rho} \end{aligned} \tag{A.3}$$

¿From this we deduce that the next approximation to $c_{2\rho+2}$ is obtained from

$$\beta_r = 1 + \frac{i}{h^2}\sum_{\rho=0}^{r}\Big[(\rho+\nu)^2 - \lambda\Big] \tag{A.4}$$

The sums on the right hand side can be evaluated. E.g.

$$\sum_{\rho=0}^{r} \rho^2 = 1^2 + 2^2 + 3^2 + \cdots + r^2 = \frac{1}{6} r(r+1)(2r+1)$$

so that one obtains

$$\beta_r = 1 + \frac{i}{h^2}\Big[\frac{r(r+1)(2r+1)}{6} + 2\nu\frac{r(r-1)}{2} + \nu^2 - \lambda\Big] \qquad \text{(A.5)}$$

Proceeding in this way one can indeed obtain the desired asymptotic expansion of the coefficients $c_{2\rho}$. (In fact the asymptotic expansion of the Bessel function – similar to that of a linear combination of Hankel functions – can be obtained from its recurrence relation in a very similar way).

Appendix B

For the determination of the large–h behaviour of the Floquet exponent ν we make use of results of ref.[34]. A fundamental pair y_I, y_{II} of respectively even and odd solutions of the original periodic Mathieu equation with eigenvalue λ defined by

$$y_I(-z) = y_I(z), \quad y_{II}(z) = -y_{II}(-z)$$

can be chosen to satisfy the following boundary conditions (cf. e.g. [31], pp.99,100)

$$y_I(0) = 1, \quad y_{II}(0) = 0, \quad y_I{}'(0) = 0, \quad y_{II}{}'(0) = 1$$

¿From its original defining property the Floquet exponent ν can then be shown to be given by (cf. [31], p.101)

$$\cos \pi\nu = y_I(\pi; \lambda, h^2) \qquad \text{(B.1)}$$

so that (cf.[31], p. 100)

$$\cos \pi\nu + 1 = 2 y_I(\pi/2; \lambda, h^2) y_{II}{}'(\pi/2; \lambda, h^2) \qquad \text{(B.2)}$$

The solutions $y_I(z), y_{II}(z)$ can be identified with the large–h solutions ce, se of ref.[34](there eqs.(64))in terms of functions $A(z), \bar{A}(z)$ as in eq.(84) above with normalization constants N_0, N_0', i.e. in leading order

$$ce(o) = 2N_0 A(0), \quad se'(0) = 4h N_0{}' A(0)$$

from which we deduce in leading order for large $|h|$ that

$$N_0 = 2^{-3/2}, \quad N_0{}' = 2^{-5/2}/h$$

Eqs.(65) of ref.[34] give the large–h expansions of $y_I(\pi/2; \lambda, h^2)$ and $y_{II}{}'(\pi/2; \lambda, h^2)$. Inserting these multiplied by the appropriate normalization constants into eq.(B.2) and retaining the dominant terms for large $|h|$ we obtain

$$\cos \pi\nu + 1 = \frac{\pi e^{4h}}{(8h)^{q/2}}\Big[\frac{1 + \frac{3(q^2+1)}{64h}}{\Gamma[\frac{3}{4} - \frac{q}{4}]\Gamma[\frac{1}{4} - \frac{q}{4}]} + O(\frac{1}{h^2})\Big] \qquad \text{(B.3)}$$

in agreement with a result cited in ref.[31](p.210) from [36] with logarithmic corrections. We, however, see no such logarithmic terms in the simpler formulation of ref.[34]. The relation (B.3) we rediscovered here has practically been unknown, largely in view of the difficulty to extract it from the complicated considerations of ref.[36]. Our derivation above is simple and closes a difficult gap which the author of ref.[32] commented upon with the words: "It is not likely at this stage that an analytic relation will ever be found connecting (our) ν and γ to (our) a^2 and h^2". Our search of later literature did not uncover other derivations. The main source summarizing more recent developments in the field of the Mathieu equation is ref.[38].

Figure Captions

Figure 1

The function $q(h)$ plotted versus h, which, of course, is valid only away from $h = 0$. The plot should be compared with graphs in ref.[32] where a similar but less convenient quantity is used.

Figure 2

The absorptivity $A(l, h)$ for $l = 0$.

Figure 3

The absorptivity $A(l, h)$ for $l = 1$.

Figure 4

The absorptivity $A(l, h)$ for $l = 2$.

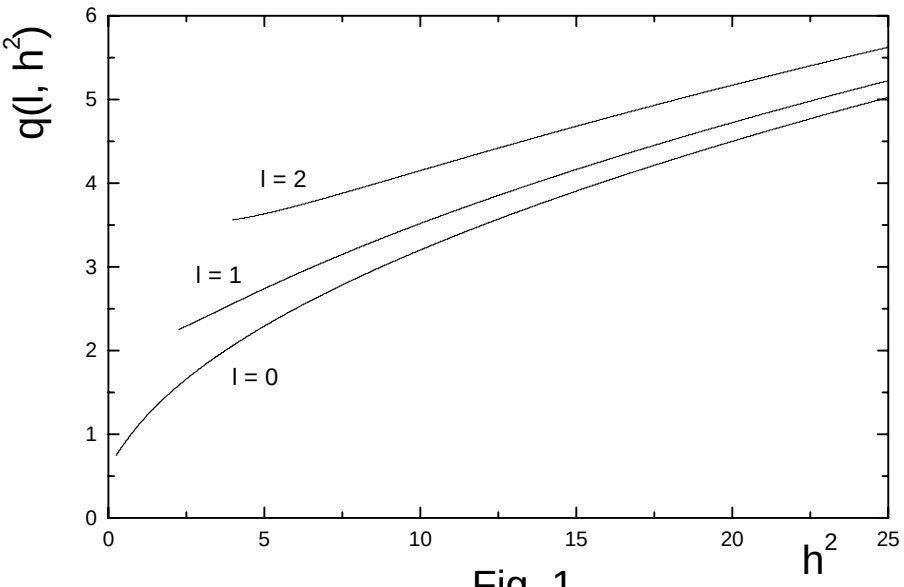


Fig. 1

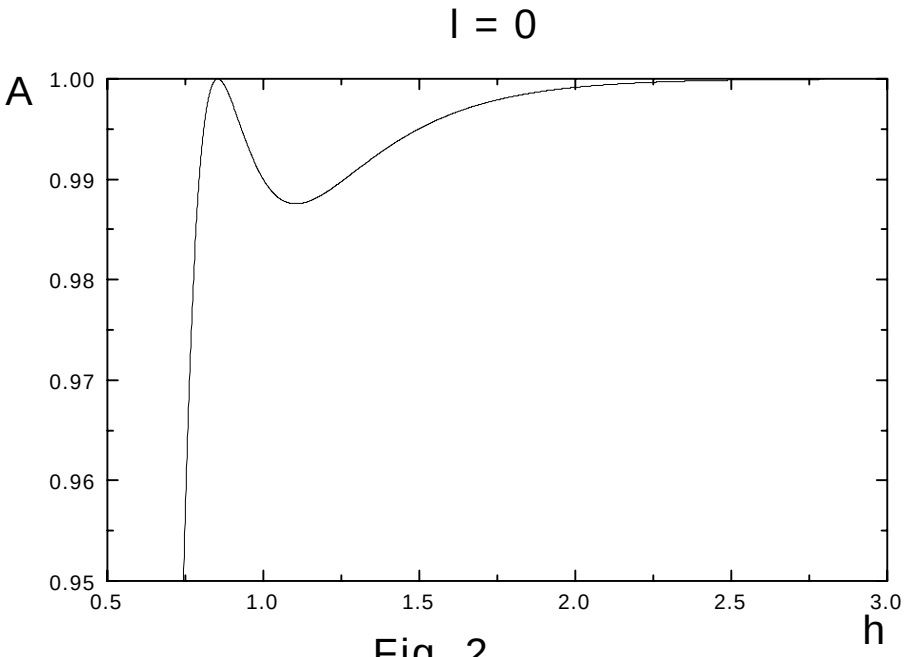
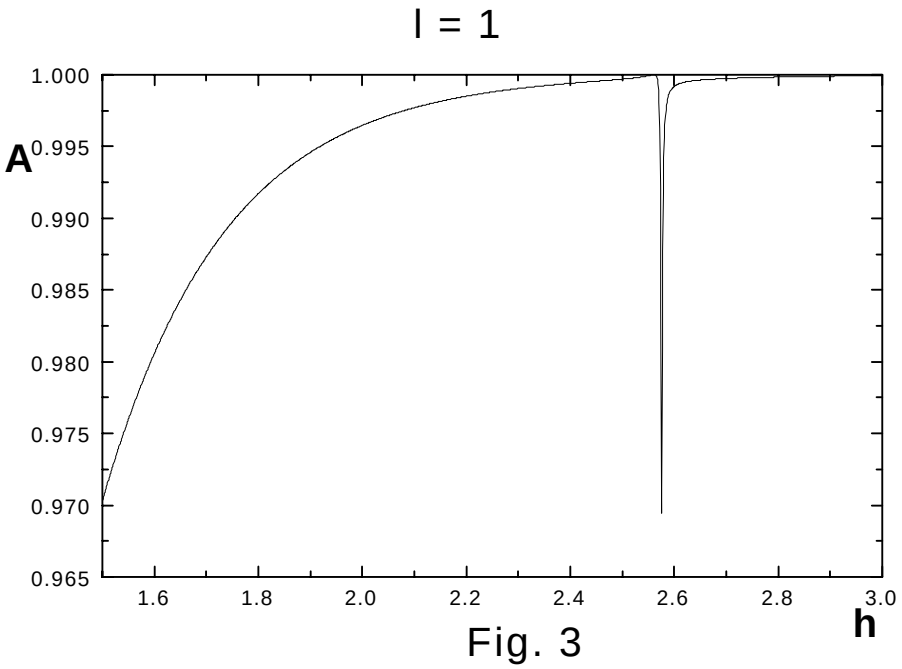


Fig. 2


l = 1
A
h
1.000
0.995
0.990
0.985
0.980
0.975
0.970
0.965
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
Fig. 3

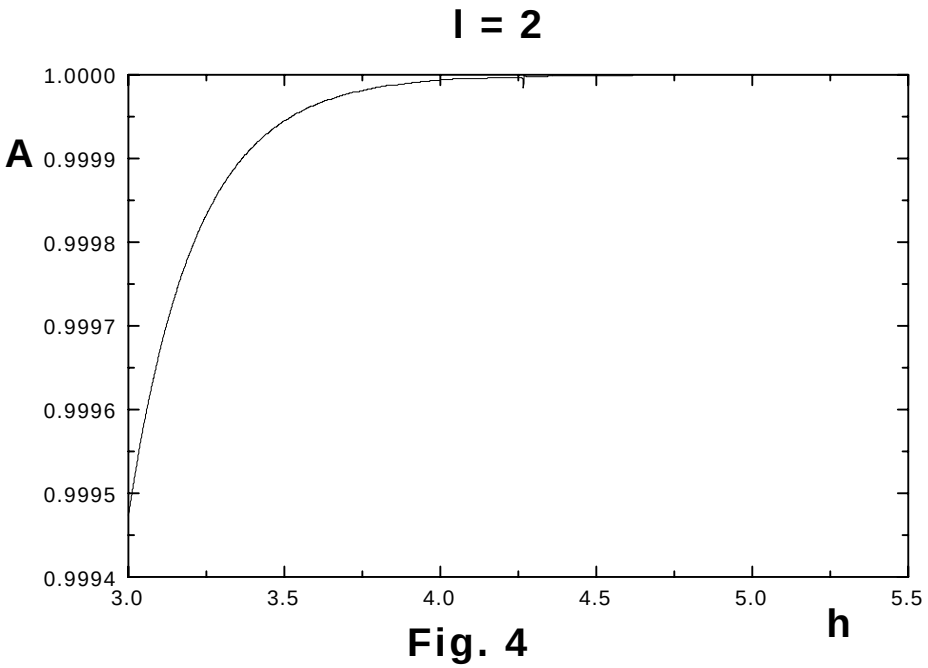

l = 2
A
1.0000
0.9999
0.9998
0.9997
0.9996
0.9995
0.9994
3.0
3.5
4.0
4.5
5.0
5.5
h

Fig. 4