Winding number transitions at finite temperature in the d = 2 Abelian-Higgs model

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Abstract

Following our earlier investigations we examine the quantum–classical winding number transition in the Abelian-Higgs system. It is demonstrated that the winding number transition in this system is of the smooth second order type in the full range of parameter space. Comparison of the action of classical vortices with that of the sphaleron supports our finding.

Recently much attention has been paid to the decay-rate transition between the low-temperature instanton-dominated quantum tunneling regime and the high-temperature sphaleron-dominated thermal activity regime in quantum mechanics [1,2], in field theoretical and gauge models [3–6], and in cosmology [7–9]. In particular the winding number

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transitions in gauge theories are too complicated to handle analytically, and hence most calculations of this type rely on numerical simulation with the help of computers. It is, however, usually difficult to obtain a good physical insight from numerical calculations alone. Hence, it is important to develop alternative methods which enable one to extend the analytical approach as far as possible. Investigations along these directions were developed recently by using nonlinear perturbation theory [10] or by counting the number of negative modes of the full Hessian around the sphaleron configuration [11]. Although these two methods start from completely different points of view, they both yield the same criterion for a sharp first-order transition in the scalar field theories. Since the explicit form of the criterion is model-dependent, it is better to explain briefly how the criterion is derived at this stage. Let u_0 and ϵ_0 be eigenfunction and eigenvalue of the negative mode of the fluctuation operator $\hat{h}$ around sphaleron. Therefore, the sphaleron frequency Ω_{sph} is defined as $\Omega_{sph} \equiv \sqrt{-\epsilon_0}$. Then the type of the transition is determined by computing the nonlinear corrections to the frequency. Let, for example, Ω be a frequency involving the nonlinear corrections. If $\Omega_{sph} - \Omega < 0$, the energy dependence of the period of the periodic instanton becomes a nonmonotonic function. This is easily conjectured from the fact that the energy-dependence of the period exhibits the increasing and decreasing behaviours near the sphaleron and vacuum instanton. From this conjecture and the relation $dS/d\tau = E$ where S, τ, and E are classical action, period, and energy respectively, one can imagine that the temperature dependence of instanton action consists of monotonically decreasing and increasing parts when $\Omega_{sph} - \Omega < 0$ [12], which results in the discontinuity in the derivative of action with respect to temperature and hence, generates the sharp first-order transition. This is the main idea of Ref. [4,10].

Some applications of this criterion to condensed matter physics [10,13,14], field theoretical but non-gauge models [4,15], and cosmology [9,16] verify that this is physically reasonable. However, the usefulness of such a criterion in the case of gauge theories is not clear without an application in a specific model. This is clearly desirable since the winding number transitions in gauge theories imply additional complications such as gauge fixing procedure and it is important to understand the implication of those in physical phenomena such as baryon- and lepton-number violating processes.

In order to obtain some insight into such transitions at higher temperatures, the criterion is here applied to the Abelian–Higgs model, which may be the simplest model among the gauge theories which support both vacuum instanton and sphaleron configurations.

We start with the Euclidean action of the d = 2 Abelian-Higgs model:

$$S_E = \int d\tau dx \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^* D_\mu \phi + \lambda [\mid \phi \mid^2 - \frac{v^2}{2}]^2 \right] \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - igA_\mu$.

It is well-known that action (1) is mathematically equivalent to Ginzburg-Landau theory [17] and supports a vortex solution [18] as a zero temperature solution. The temperature dependence of the classical action for the periodic solution in this model is calculated in Ref. [19] using some special numerical techniques. The final numerical result of Ref. [19] shows that the winding number phase transition in this model is of the smooth second-order type in the range of $1/4 < M_H/M_W < 4$, where $M_H = \sqrt{2\lambda} v$ and $M_W = gv$. In this paper we will follow the method developed in Ref. [4] and show that the type of the transition does not change over the full range of parameter space, *i.e.* it is always of the smooth second order type.

The static solutions for the action (1) whose field equations are

$$\begin{aligned} \partial_\mu F_{\mu\nu} &= ig \left[\phi^* (D_\nu \phi) - (D_\nu \phi)^* \phi \right] \\ D_\mu D_\mu \phi &= 2\lambda \phi (\mid \phi \mid^2 - \frac{v^2}{2}), \end{aligned} \tag{2}$$

can be easily obtained:

$$\begin{aligned} A_0^{sph} &= A_1^{sph} = 0 \\ \phi_{sph} &= \frac{v}{\sqrt{2}} \tanh \sqrt{\frac{\lambda}{2}} vx. \end{aligned} \tag{3}$$

In order to prove that $(A_0^{sph}, A_1^{sph}, \phi_{sph})$ are genuine sphaleron configurations in this model, we introduce a non-contractible loop [20–22]

$$\bar{A}_0 = \bar{A}_1 = 0 \tag{4}$$
$$\bar{\phi} = e^{is}\left[\frac{v}{\sqrt{2}}\cos s + ih(x)\sin s\right]$$

where s is a loop parameter defined in the region $0 \leq s \leq \pi$. Note that $\bar{\phi}$ becomes a trivial vacuum at the end points of s. In addition, the minimizing condition of energy $\mathcal{E}(\phi, A) = \int dx \mathcal{L}_E$, where $\mathcal{L}_E$ is Euclidean Lagrangian density in Eq.(1), makes $h(x)$ to be

$$h(x) = \frac{v}{\sqrt{2}}\tanh\left(\sin s\sqrt{\frac{\lambda}{2}}vx\right). \tag{5}$$

Hence, $h(x)$ coincides with ϕ_{sph} when $s = \pi/2$. It is easy to show that the energy along the minimal energy loop has a maximum at $s = \pi/2$, which proves that $(A_0^{sph}, A_1^{sph}, \phi_{sph})$ are sphaleron configuration.

Chern-Simons number at $\tau = \tau_0$ in this model is defined as

$$\mathcal{N}_{cs} = \frac{1}{\pi v^2}\int_{-\infty}^{\tau_0} d\tau \int_{-\infty}^{\infty} dx \partial_\mu \Omega_\mu \tag{6}$$

where the generalized Chern-Simons current Ω_μ is

$$\Omega_\mu = \epsilon_{\mu\nu}\left[i\phi^* D_\nu \phi - \frac{g^2 v^2}{2}A_\nu\right]. \tag{7}$$

In fact, $\partial_\mu \Omega_\mu$ is a lower bound of $\mathcal{L}_E$ when $g = \sqrt{2\lambda}$. To compute $\mathcal{N}_{cs}$ along the loop, we treat the loop parameter as an Euclidean time-dependent quantity $s = s(\tau)$ with $s(\tau = -\infty) = 0$, $s(\tau = \infty) = \pi$, and $s(\tau = \tau_0) = s_0$. Then it is straightforward to show that $\mathcal{N}_{cs}$ along the loop is

$$\mathcal{N}_{cs} = \frac{s_0}{\pi} - \frac{\sin 2s_0}{2\pi}. \tag{8}$$

Hence, the sphaleron configuration($s_0 = \pi/2$) has half-integer Chern-Simons number whereas the trivial vacuum($s_0 = 0, \pi$) has integer one, which allows us to interpret the sphaleron as a classical solution sitting at the top of the barrier separating the topologically distinct vacua. The classical action corresponding to that of the sphaleron is easily shown to be

$$S_{sph} = \frac{E_{sph}}{T_{sph}} \tag{9}$$

where T_{sph}, the inverse of the sphaleron period, is interpreted as a temperature and

$$E_{sph} = \frac{2\sqrt{2\lambda}}{3} v^3 \tag{10}$$

which is interpreted as the barrier height. Since the sphaleron is a static solution, one may wonder how to define the sphaleron period or frequency. In fact, the sphaleron frequency is defined by using a periodic instanton solution $\phi_{PI}(\tau, x; E)$ which is a time-dependent solution of the Euclidean field equation (2) in the full range of energy $0 < E < E_{sph}$. Since it is well known that $\phi_{PI}(\tau, x; E = 0)$ and $\phi_{PI}(\tau, x; E = E_{sph})$ coincide with vacuum instanton and sphaleron respectively, we define the sphaleron frequency is frequency of $\lim_{E \to E_{sph}} \phi_{PI}(\tau, x; E)$.

In order to be able to examine the type of quantum-classical transition we have to introduce the fluctuation fields around the sphaleron and expand field equations up to the third order in these fields. If, however, one expands Eq.(2) naively, one will realize that the fluctuation operators are not diagonalized and, hence, the spectra of these operators are not obtainable analytically. To solve this problem we fix a gauge as a R_ξ gauge [23,24] by adding as gauge fixing term

$$S_{gf} = \frac{1}{2\xi} \int d\tau dx \left[\partial_\mu A_\mu + \frac{ig}{2} \xi (\phi^2 - \phi^{*2}) \right]^2 \tag{11}$$

to the original action (1). Then, the field equations are slightly changed to

$$\begin{aligned} \partial_\mu F_{\mu\nu} + \frac{1}{\xi} \left[\partial_\mu \partial_\nu A_\mu + ig\xi(\phi \partial_\nu \phi - \phi^* \partial_\nu \phi^*) \right] &= ig \left[\phi^* (D_\nu \phi) - (D_\nu \phi)^* \phi \right] \\ D_\mu D_\mu \phi + ig\phi^* \left[\partial_\mu A_\mu + \frac{ig\xi}{2} (\phi^2 - \phi^{*2}) \right] &= 2\lambda \phi (\mid \phi \mid^2 - \frac{v^2}{2}). \end{aligned} \tag{12}$$

It is easy to show that the sphaleron solution (3) and the corresponding action (9) are not changed under the R_ξ gauge.

We now introduce the fluctuation fields around the sphaleron as follows:

$$A_0(\tau,x) = a_0(\tau,x)$$

$$A_1(\tau,x) = a_1(\tau,x) \tag{13}$$

$$\phi(\tau,x) = \frac{1}{\sqrt{2}}\left[v\tanh\sqrt{\frac{\lambda}{2}}vx + \eta_1(\tau,x) + i\eta_2(\tau,x)\right]$$

where a_0, a_1, η_1, and η_2 are real fields. After introducing the new space-time variables

$$z_0 \equiv \sqrt{\frac{\lambda}{2}}v\tau \tag{14}$$

$$z_1 \equiv \sqrt{\frac{\lambda}{2}}vx,$$

dimensionless parameters

$$\theta \equiv \frac{2M_W}{M_H} = \sqrt{\frac{2g^2}{\lambda}}, \tag{15}$$

and, for convenience, a function of θ

$$s_1 \equiv \sqrt{\theta^2 + \frac{1}{4}} - \frac{1}{2}, \tag{16}$$

one can show that at $\xi = 1$ the field equation (12) can be expanded as

$$\hat{l}\begin{pmatrix} a_0 \\ \rho_+ \\ \rho_- \\ \eta_1 \end{pmatrix} = \hat{h}\begin{pmatrix} a_0 \\ \rho_+ \\ \rho_- \\ \eta_1 \end{pmatrix} + \begin{pmatrix} G_2^{a_0} \\ G_2^{\rho_+} \\ G_2^{\rho_-} \\ G_2^{\eta_1} \end{pmatrix} + \begin{pmatrix} G_3^{a_0} \\ G_3^{\rho_+} \\ G_3^{\rho_-} \\ G_3^{\eta_1} \end{pmatrix} \tag{17}$$

where

$$\hat{l} = \begin{pmatrix} \frac{\partial^2}{\partial z_0^2} & 0 & 0 & 0 \\ 0 & \frac{\partial^2}{\partial z_0^2} & 0 & 0 \\ 0 & 0 & \frac{\partial^2}{\partial z_0^2} & 0 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial z_0^2} \end{pmatrix}, \qquad \hat{h} = \begin{pmatrix} \hat{h}_{a_0} & 0 & 0 & 0 \\ 0 & \hat{h}_{\rho_+} & 0 & 0 \\ 0 & 0 & \hat{h}_{\rho_-} & 0 \\ 0 & 0 & 0 & \hat{h}_{\eta_1} \end{pmatrix}, \tag{18}$$

and the functions G_2 and G_3 are given in the appendix (A.1). Here, ρ_+ and ρ_- are defined as

$$\rho_+ \equiv \frac{1}{\sqrt{\cosh\alpha}}\left[\cosh\frac{\alpha}{2}a_1 + \sinh\frac{\alpha}{2}\eta_2\right], \tag{19}$$
$$\rho_- \equiv \frac{1}{\sqrt{\cosh\alpha}}\left[-\sinh\frac{\alpha}{2}a_1 + \cosh\frac{\alpha}{2}\eta_2\right]$$

where $\alpha = \sinh^{-1} 2\theta$ and

$$\begin{aligned}
\hat{h}_{a_0} &= -\frac{\partial^2}{\partial z_1^2} - \theta^2 \mathrm{sech}^2 z_1 + \theta^2, \\
\hat{h}_{\rho_+} &= -\frac{\partial^2}{\partial z_1^2} - (s_1 - 1)s_1 \mathrm{sech}^2 z_1 + \theta^2, \\
\hat{h}_{\rho_-} &= -\frac{\partial^2}{\partial z_1^2} - (s_1 + 1)(s_1 + 2) \mathrm{sech}^2 z_1 + \theta^2, \\
\hat{h}_{\eta_1} &= -\frac{\partial^2}{\partial z_1^2} - 6\mathrm{sech}^2 z_1 + 4.
\end{aligned} \tag{20}$$

The spatial parts of the fluctuation operators $\hat{h}_{a_0}$, $\hat{h}_{\rho_+}$, $\hat{h}_{\rho_-}$, and $\hat{h}_{\eta_1}$ are various kinds of Pöschl-Teller type operators whose spectra are summarized in Ref. [25]. It is easy to show that the spectra of $\hat{h}_{a_0}$ and $\hat{h}_{\rho_+}$ consist of only positive modes whose explicit forms are not necessary for further study. What we need are only the negative mode of $\hat{h}_{\rho_-}$ whose eigenfunction $\psi_{-1}^{(\rho_-)}$ and eigenvalue $\lambda_{-1}^{(\rho_-)}$ are

$$\begin{aligned}
\psi_{-1}^{(\rho_-)}(z_1) &= 2^{-(s_1+1)}\sqrt{\frac{\Gamma(2s_1+3)}{\Gamma(s_1+1)\Gamma(s_1+2)}}\frac{1}{\cosh^{s_1+1} z_1}, \\
\lambda_{-1}^{(\rho_-)} &= -s_1 - 1,
\end{aligned} \tag{21}$$

and the full spectrum of $\hat{h}_{\eta_1}$, which is summarized in Table I. It is easy to show that the zero mode $\psi_0^{(\eta_1)}$ in Table I is propotional to $\partial\phi_{sph}/\partial z_1$, which indicates the translational symmetry of the Abelian-Higgs system.

Now, we have to carry out the perturbation to derive the criterion for the sharp first–order transition as suggested in Ref. [10,14]. Since $\hat{l}$ and $\hat{h}$ in Eq. (17) are expressed in a matrix form, it is impossible to use the criterion derived in Ref. [10,14] directly. In this case we have to repeat the perturbation procedure with a spectrum of the full spatial fluctuation operator $\hat{h}$ as suggested in Ref. [4]. Computing the nonlinear corrections of the sphaleron frequency Ω perturbatively, one can derive the final result of the criterion for the sharp first-order transition in this model as a following inequality:

$$I_1(\theta, v) + I_2(\theta, v) + I_3(\theta, v) < 0 \tag{22}$$

where

$$\begin{aligned} I_1(\theta, v) &= < \psi_{-1}^{(\rho-)}(z_1) \mid D_1^{(1)} >, \\ I_2(\theta, v) &= < \psi_{-1}^{(\rho-)}(z_1) \mid D_1^{(2)} >, \\ I_3(\theta, v) &= < \psi_{-1}^{(\rho-)}(z_1) \mid D_1^{(3)} > . \end{aligned} \tag{23}$$

Here $D_1^{(1)}(z_1)$, $D_1^{(2)}(z_1)$, and $D_1^{(3)}(z_1)$ are given in the appendix (A.2). Since T_{sph} in Eq. (9) is the inverse of the sphaleron period, the action of the sphaleron becomes

$$S_{sph} = \frac{8\pi}{3\sqrt{s_1+1}} v^2. \tag{24}$$

In deriving S_{sph} in Eq. (24) one has to use the rescaling definition of space-time variables (14) and $\Omega_{sph} = \sqrt{s_1+1}$ which is given in the appendix.

Now, in order to compute $D_1^{(i)}(z_1) \quad i = 1, 2, 3$ we are in a position to compute $g_{\eta_1,1}(z_1)$ and $g_{\eta_1,2}(z_1)$ explicitly which is given in the appendix (A.3). The function $g_{\eta_1,1}(z_1)$ is explicitly derived as follows. We define

$$q_1(z_1) \equiv \hat{h}_{\eta_1}^{-1} \frac{\sinh z_1}{\cosh^{2s_1+3} z_1}$$

or equivalently

$$\hat{h}_{\eta_1} q_1(z_1) = \frac{\sinh z_1}{\cosh^{2s_1+3} z_1}. \tag{25}$$

Multiplying the zero mode of $\hat{h}_{\eta_1}$ with Eq. (25), integrating over z_1 from $-\infty$ to z_1, and performing partial integration twice, we can obtain the first-order differential equation for $q_1(z_1)$. Solving this differential equation one can obtain $q_1(z_1)$ up to the constant of integration. This constant is determined by the fact that $q_1(z_1)$ does not have a zero mode component. Inserting $q_1(z_1)$ into $g_{\eta_1,1}(z_1)$ one can derive the explicit form of $g_{\eta_1,1}(z_1)$ which is

$$g_{\eta_1,1}(z_1) = \frac{1}{4\sqrt{\pi} v} \frac{(s_1 - \frac{1}{2})(s_1+1)\Gamma(s_1 + \frac{1}{2})}{\Gamma(s_1+1)} \frac{u(z_1)}{\cosh^2 z_1} \tag{26}$$

where

$$u(z_1) = \int_0^{z_1} \frac{dy}{\cosh^{2s_1} y}. \tag{27}$$

Next we define

$$q_2(z_1) \equiv \left(\hat{h}_{\eta_1} + 4\Omega_{sph}^2\right)^{-1} \frac{\sinh z_1}{\cosh^{2s_1+3} z_1}. \tag{28}$$

Using the completeness condition as follows

$$q_2(z_1) = \left(\hat{h}_{\eta_1} + 4\Omega_{sph}^2\right)^{-1} \left[\sum_{n=1}^{2} \mid \psi_n^{(\eta_1)} >< \psi_n^{(\eta_1)} \mid + \int dk \mid \psi_k^{(\eta_1)} >< \psi_k^{(\eta_1)} \mid\right] \frac{\sinh z_1}{\cosh^{2s_1+3} z_1}, \tag{29}$$

one can obtain the integral representation of $q_2(z_1)$. Inserting this into $g_{\eta_1,2}(z_1)$, we can derive the final form of $g_{\eta_1,2}(z_1)$

$$\begin{aligned} g_{\eta_1,2}(z_1) = & \frac{1}{2\sqrt{\pi}v} \frac{(s_1 - \frac{1}{2})(s_1+1)(s_1+2)\Gamma(s_1+\frac{1}{2})}{\Gamma(s_1+1)} \\ & \times \Bigg[\frac{3\sqrt{\pi}}{4(4s_1+7)} \frac{\Gamma(s_1+\frac{3}{2})}{\Gamma(s_1+3)} \frac{\sinh z_1}{\cosh^2 z_1} + \frac{2^{2s_1+2}(2s_1+1)(2s_1+3)}{2\pi\Gamma(2s_1+5)} \\ & \times \left[J_1(\theta,z_1) + 3\tanh z_1 \left(J_2(\theta,z_1) - J_4(\theta,z_1)\right) - 3\tanh^2 z_1 J_3(\theta,z_1)\right] \Bigg] \end{aligned} \tag{30}$$

where

$$\begin{aligned} J_1(\theta,z_1) &\equiv \int_0^\infty dk \frac{k\Gamma(s_1+1+\frac{ik}{2})\Gamma(s_1+1-\frac{ik}{2})}{4(s_1+2)+k^2} \sin kz_1, \\ J_2(\theta,z_1) &\equiv \int_0^\infty dk \frac{\Gamma(s_1+1+\frac{ik}{2})\Gamma(s_1+1-\frac{ik}{2})}{4(s_1+2)+k^2} \cos kz_1, \\ J_3(\theta,z_1) &\equiv \int_0^\infty dk \frac{k\Gamma(s_1+1+\frac{ik}{2})\Gamma(s_1+1-\frac{ik}{2})}{(1+k^2)[4(s_1+2)+k^2]} \sin kz_1, \\ J_4(\theta,z_1) &\equiv \int_0^\infty dk \frac{\Gamma(s_1+1+\frac{ik}{2})\Gamma(s_1+1-\frac{ik}{2})}{(1+k^2)[4(s_1+2)+k^2]} \cos kz_1. \end{aligned} \tag{31}$$

Now the computation of $I_1(\theta,v)$, $I_2(\theta,v)$, and $I_3(\theta,v)$ is straightforward and their final form is given in the appendix (A.4). It is very interesting that $I_1(\theta,v)$ and $I_2(\theta,v)$ vanish at $s_1 = 1/2$ or, in terms of θ, at $\theta = \sqrt{3}/2$.

The θ-dependence of $I_1(\theta,v)$, $I_2(\theta,v)$, $I_3(\theta,v)$, and $I_1(\theta,v)+I_2(\theta,v)+I_3(\theta,v)$ is shown in Fig. 1 which shows that the condition for the sharp first-order transition, i.e. (22), does not hold when $\theta < 4$. Ploting $I_1(\theta, v) + I_2(\theta, v) + I_3(\theta, v)$ in the range of large θ, one can confirm numerically that it is a monotonically increasing function which indicates that the sharp first-order transition does not occur in the full range of parameter space. This means the winding number phase transition of this model is always smooth second-order as shown in Ref. [19], where same conclusion was derived by numerical method in the restricted region of parameter space.

There is another indirect method which confirms our conclusion. If the transition is second order and there is no interaction between vacuum instanton and anti-instanton, the condition

$$2S_1 > S_{sph}, \tag{32}$$

where S_1 is the action of one instanton solution, has to be satisfied. Since there is no interaction between vortices at the Bogomol'nyi limit [26,27] which is $\theta = 2$ in this model, we can use the condition (32) to check the credibility of our conjecture. Since $S_1 = \pi v^2$ in this limit, it is easy to show that

$$\frac{S_{sph}}{2S_1} = 0.833 < 1 \tag{33}$$

which supports our conclusion.

In general, there is an interaction between vortices and hence, the condition (32) has to be modified to

$$S_2 > S_{sph}, \tag{34}$$

where S_2 is action of two interacting vortices, at arbitrary θ. S_1 and S_2 at arbitrary θ can be computed numerically by employing the variation method [28]. S_1, S_2 and S_{sph} at various θ are summarized at Table II, which also confirms our finding at $0 < \theta < 4$.

We hope our method can be applicable to the $SU(2)$-Higgs model which is most important to understand the baryon-number violating process. The approach along this direction is under investigation.

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TABLES

Eigenvalue of $\hat{h}_{\eta_1}$	Eigenfunction of $\hat{h}_{\eta_1}$
$\lambda_0^{(\eta_1)} = 0$	$\psi_0^{(\eta_1)}(z_1) = \frac{\sqrt{3}}{2} \frac{1}{\cosh^2 z_1}$
$\lambda_1^{(\eta_1)} = 3$	$\psi_1^{(\eta_1)}(z_1) = \sqrt{\frac{3}{2}} \frac{\sinh z_1}{\cosh^2 z_1}$
$\lambda_k^{(\eta_1)} = 4 + k^2$	$\psi_k^{(\eta_1)}(z_1) = -\frac{1}{\sqrt{2\pi}} \frac{e^{ikz_1}}{(1+ik)(2+ik)} \left[(1+k^2) + 3ik \tanh z_1 - 3\tanh^2 z_1\right]$

TABLE I. Eigenvalues and eigenfunctions of $\hat{h}_{\eta_1}$

θ	$S_{sph}/\pi v^2$	$S_1/\pi v^2$	$S_2/\pi v^2$
0.5	2.43	1.79	4.31
1.0	2.10	1.34	2.94
1.5	1.85	1.13	2.34
2.0	1.67	1.00	2.00
2.5	1.53	0.91	1.78
3.0	1.42	0.85	1.62
3.5	1.33	0.80	1.50
4.0	1.25	0.76	1.40

TABLE II. S_1, S_2 and S_{sph} at various values of θ

Appendix A

In this appendix we collect the lengthy expressions to make the main text to be simple and compact.

In the expansion of equation of motion (17) the higher order terms G_2 and G_3 are

$$
\begin{aligned}
G_2^{a_0} &= \frac{1}{v}\left[\theta\sqrt{\frac{2s_1}{s_1+\frac{1}{2}}}\rho_+\frac{\partial\eta_1}{\partial z_0} + \theta\sqrt{\frac{2(s_1+1)}{s_1+\frac{1}{2}}}\rho_-\frac{\partial\eta_1}{\partial z_0} + 2\theta^2\tanh z_1 a_0\eta_1\right],\\
G_3^{a_0} &= \frac{1}{v^2}\left[\theta^2 a_0\eta_1^2 + \frac{\theta^2 s_1}{2(s_1+\frac{1}{2})}a_0\rho_+^2 + \frac{\theta^2(s_1+1)}{2(s_1+\frac{1}{2})}a_0\rho_-^2 + \frac{\theta^3}{s_1+\frac{1}{2}}a_0\rho_+\rho_-\right],\\
G_2^{\rho_+} &= \frac{1}{v}\left[\theta\sqrt{\frac{2s_1}{s_1+\frac{1}{2}}}a_0\frac{\partial\eta_1}{\partial z_0} + \frac{2\theta^2}{s_1+\frac{1}{2}}\rho_+\frac{\partial\eta_1}{\partial z_1} + \frac{\theta}{s_1+\frac{1}{2}}\rho_-\frac{\partial\eta_1}{\partial z_1}\right.\\
&\qquad\left. + \tanh z_1\left[2\left(\theta^2+\frac{s_1}{s_1+\frac{1}{2}}\right)\rho_+\eta_1 + \frac{2\theta}{s_1+\frac{1}{2}}\rho_-\eta_1\right]\right],\\
G_3^{\rho_+} &= \frac{1}{v^2}\left[\frac{\theta^2 s_1}{2(s_1+\frac{1}{2})}a_0^2\rho_+ + \frac{\theta^3}{2(s_1+\frac{1}{2})}a_0^2\rho_- + \left(\theta^2+\frac{s_1}{s_1+\frac{1}{2}}\right)\rho_+\eta_1^2 + \frac{\theta}{s_1+\frac{1}{2}}\rho_-\eta_1^2 + \frac{1}{2}\frac{s_1^2+\theta^4}{(s_1+\frac{1}{2})^2}\rho_+^3\right.\\
&\qquad\left. + \frac{3\theta}{2(s_1+\frac{1}{2})^2}\left(s_1+\frac{\theta^2}{2}\right)\rho_+^2\rho_- + \frac{\theta}{2(s_1+\frac{1}{2})^2}\left(s_1+1-\frac{\theta^2}{2}\right)\rho_-^3 + \frac{\theta^2(7-2\theta^2)}{4(s_1+\frac{1}{2})^2}\rho_+\rho_-^2\right],\\
G_2^{\rho_-} &= \frac{1}{v}\left[\theta\sqrt{\frac{2(s_1+1)}{s_1+\frac{1}{2}}}a_0\frac{\partial\eta_1}{\partial z_0} - \frac{2\theta^2}{s_1+\frac{1}{2}}\rho_-\frac{\partial\eta_1}{\partial z_1} + \frac{\theta}{s_1+\frac{1}{2}}\rho_+\frac{\partial\eta_1}{\partial z_1}\right.\\
&\qquad\left. + \tanh z_1\left[2\left(\theta^2+\frac{s_1+1}{s_1+\frac{1}{2}}\right)\rho_-\eta_1 + \frac{2\theta}{s_1+\frac{1}{2}}\rho_+\eta_1\right]\right],\\
G_3^{\rho_-} &= \frac{1}{v^2}\left[\frac{\theta^2(s_1+1)}{2(s_1+\frac{1}{2})}a_0^2\rho_- + \frac{\theta^3}{2(s_1+\frac{1}{2})}a_0^2\rho_+ + \left(\theta^2+\frac{s_1+1}{s_1+\frac{1}{2}}\right)\rho_-\eta_1^2 + \frac{\theta}{s_1+\frac{1}{2}}\rho_+\eta_1^2\right.\\
&\qquad\left. + \frac{\theta^4+(s_1+1)^2}{2(s_1+\frac{1}{2})^2}\rho_-^3 + \frac{\theta(s_1+\frac{\theta^2}{2})}{2(s_1+\frac{1}{2})^2}\rho_+^3 + \frac{3\theta(s_1+1-\frac{\theta^2}{2})}{2(s_1+\frac{1}{2})^2}\rho_+\rho_-^2 + \frac{\theta^2(7-2\theta^2)}{4(s_1+\frac{1}{2})^2}\rho_+^2\rho_-\right],\\
G_2^{\eta_1} &= \frac{1}{v}\left[-\theta\sqrt{\frac{2s_1}{s_1+\frac{1}{2}}}\left(\frac{\partial a_0}{\partial z_0}\rho_+ + a_0\frac{\partial\rho_+}{\partial z_0}\right) - \theta\sqrt{\frac{2(s_1+1)}{s_1+\frac{1}{2}}}\left(\frac{\partial a_0}{\partial z_0}\rho_- + a_0\frac{\partial\rho_-}{\partial z_0}\right)\right.\\
&\qquad - \frac{2\theta^2}{s_1+\frac{1}{2}}\left(\rho_+\frac{\partial\rho_+}{\partial z_1} - \rho_-\frac{\partial\rho_-}{\partial z_1}\right) - \frac{\theta}{s_1+\frac{1}{2}}\left(\frac{\partial\rho_+}{\partial z_1}\rho_- + \rho_+\frac{\partial\rho_-}{\partial z_1}\right)\\
&\qquad\left. + \tanh z_1\left[\theta^2 a_0^2 + 6\eta_1^2 + \left(\theta^2+\frac{s_1}{s_1+\frac{1}{2}}\right)\rho_+^2 + \left(\theta^2+\frac{s_1+1}{s_1+\frac{1}{2}}\right)\rho_-^2 + \frac{2\theta}{s_1+\frac{1}{2}}\rho_+\rho_-\right]\right],\\
G_3^{\eta_1} &= \frac{1}{v^2}\left[\theta^2 a_0^2\eta_1 + \left(\theta^2+\frac{s_1}{s_1+\frac{1}{2}}\right)\rho_+^2\eta_1 + \left(\theta^2+\frac{s_1+1}{s_1+\frac{1}{2}}\right)\rho_-^2\eta_1 + \frac{2\theta}{s_1+\frac{1}{2}}\rho_+\rho_-\eta_1 + 2\eta_1^3\right].
\end{aligned}
\tag{A.1}
$$

In Eq. (23) $D_1^i(z_1)$ $i=1,2,3$ are

$$
D_1^{(1)}(z_1) = \frac{2^{-s_1}}{v}\sqrt{\frac{\Gamma(2s_1+3)}{\Gamma(s_1+1)\Gamma(s_1+2)}}
$$

$$\times\left[\frac{(s_1+1)(2s_1^2+s_1+2)}{2s_1+1}\frac{\sinh z_1}{\cosh^{s_1+2} z_1}g_{\eta_1,1}-\frac{s_1(s_1+1)}{s_1+\frac{1}{2}}\frac{1}{\cosh^{s_1+1} z_1}\frac{dg_{\eta_1,1}}{dz_1}\right],$$

$$D_1^{(2)}(z_1)=\frac{2^{-(s_1+1)}}{v}\sqrt{\frac{\Gamma(2s_1+3)}{\Gamma(s_1+1)\Gamma(s_1+2)}} \tag{A.2}$$

$$\times\left[\frac{(s_1+1)(2s_1^2+s_1+2)}{2s_1+1}\frac{\sinh z_1}{\cosh^{s_1+2} z_1}g_{\eta_1,2}-\frac{s_1(s_1+1)}{s_1+\frac{1}{2}}\frac{1}{\cosh^{s_1+1} z_1}\frac{dg_{\eta_1,2}}{dz_1}\right],$$

$$D_1^{(3)}=\frac{3\cdot 2^{-3s_1-6}}{v^2}(1+s_1^2)\left(\frac{s_1+1}{s_1+\frac{1}{2}}\right)^2\left(\frac{\Gamma(2s_1+3)}{\Gamma(s_1+1)\Gamma(s_1+2)}\right)^{\frac{3}{2}}\frac{1}{\cosh^{3s_1+3} z_1},$$

where

$$g_{\eta_1,1}(z_1)=\frac{1}{2\sqrt{\pi}v}\frac{\Gamma(s_1+\frac{1}{2})}{\Gamma(s_1+1)}(s_1-\frac{1}{2})(s_1+1)(s_1+2)\hat{h}_{\eta_1}^{-1}\frac{\sinh z_1}{\cosh^{2s_1+3} z_1},$$

$$g_{\eta_1,2}(z_1)=\frac{1}{2\sqrt{\pi}v}\frac{\Gamma(s_1+\frac{1}{2})}{\Gamma(s_1+1)}(s_1-\frac{1}{2})(s_1+1)(s_1+2)\left(\hat{h}_{\eta_1}+4\Omega_{sph}^2\right)^{-1}\frac{\sinh z_1}{\cosh^{2s_1+3} z_1}, \tag{A.3}$$

and $\Omega_{sph}=\sqrt{-\lambda_{-1}^{(\rho_-)}}=\sqrt{s_1+1}$ is the zeroth order frequency of the sphaleron.

Using the explicit results of $g_{\eta_1,1}(z_1)$ and $g_{\eta_1,2}(z_1)$ it is straightforward to calculate $I_1(\theta,v)$, $I_2(\theta,v)$, and $I_3(\theta,v)$ given in Eq. (23):

$$I_1(\theta,v)=-\frac{1}{4\sqrt{\pi}v^2}(s_1-\frac{1}{2})^2(s_1+1)^2\frac{\Gamma^2(s_1+\frac{1}{2})\Gamma(2s_1+2)}{\Gamma^2(s_1+1)\Gamma(2s_1+\frac{5}{2})},$$

$$I_3(\theta,v)=\frac{3\cdot 2^{2s_1-2}}{v^2\pi}(1+s_1^2)\left(\frac{s_1+1}{s_1+\frac{1}{2}}\right)^2\frac{\Gamma^3(s_1+\frac{3}{2})}{\Gamma(s_1+1)\Gamma(2s_1+\frac{5}{2})},$$

$$I_2(\theta,v)=-\frac{1}{2\pi v^2}\frac{(s_1-\frac{1}{2})(s_1+1)(s_1+2)}{(s_1+\frac{1}{2})}\frac{\Gamma^2(s_1+\frac{3}{2})}{\Gamma^2(s_1+1)} \tag{A.4}$$

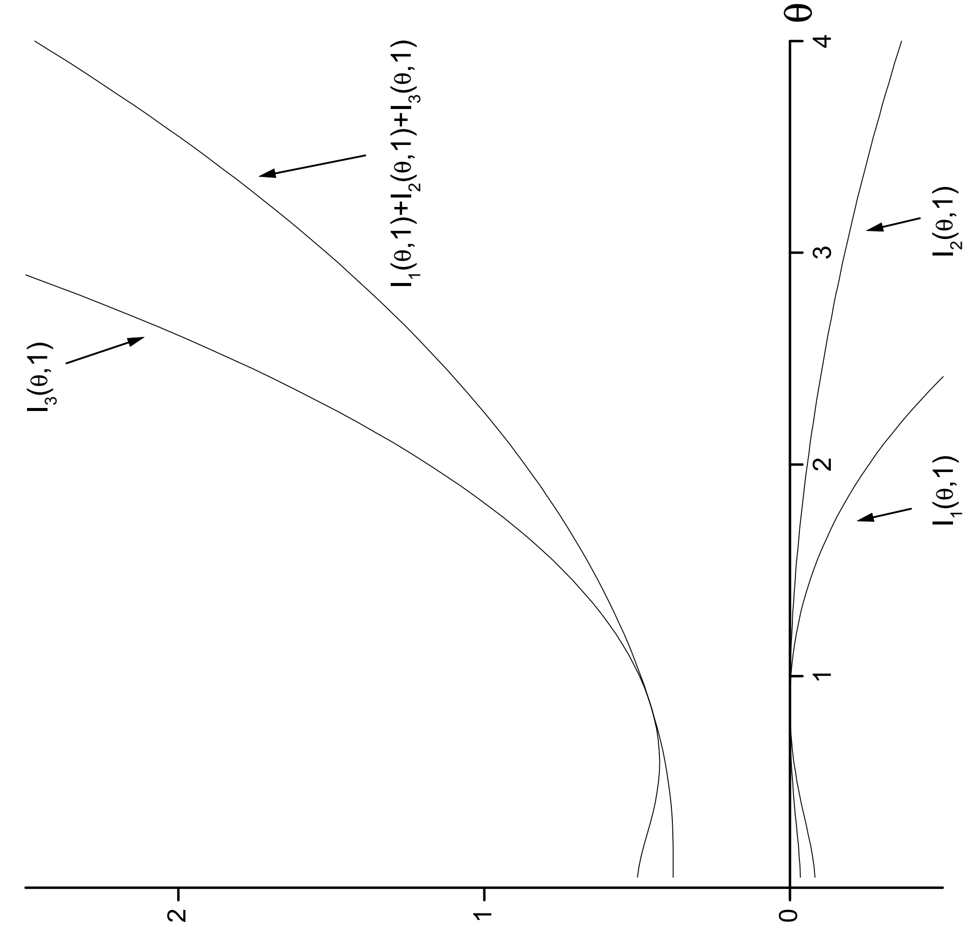
$$\times\left[\frac{3\pi(2s_1-1)(s_1+1)(s_1+2)}{4(2s_1+1)(4s_1+7)}\frac{\Gamma^2(s_1+\frac{3}{2})}{\Gamma^2(s_1+3)}+\right.$$
$$\frac{2^{2s_1+1}(s_1-2)(2s_1+1)(2s_1+3)}{\pi\Gamma(2s_1+5)}\int_0^\infty dz_1\frac{J_5(\theta,z_1)}{\cosh^{2s_1+2} z_1}-$$
$$\frac{3\cdot 2^{2s_1+2}(2s_1+3)(s_1+\frac{1}{2})(2s_1^2+3s_1+2)}{\pi\Gamma(2s_1+5)}\int_0^\infty dz_1\frac{J_2(\theta,z_1)-J_4(\theta,z_1)}{\cosh^{2s_1+2} z_1}+$$
$$\left.\frac{3\cdot 2^{2s_1+2}(s_1+1)(s_1+\frac{3}{2})(2s_1+1)(2s_1+3)}{\pi\Gamma(2s_1+5)}\int_0^\infty dz_1\frac{J_2(\theta,z_1)-J_4(\theta,z_1)}{\cosh^{2s_1+4} z_1}\right],$$

where

$$J_5(\theta,z_1)\equiv\int_0^\infty\frac{k^2\Gamma(s_1+1+\frac{ik}{2})\Gamma(s_1+1-\frac{ik}{2})}{4(s_1+2)+k^2}\cos kz_1. \tag{A.5}$$

FIGURES

FIG. 1. The θ-dependence of I_1, I_2, I_3, and $I_1 + I_2 + I_3$ at $v = 1$. From this figure we can conclude that the winding number transition of the Abelian-Higgs model is smooth second-order in the full parameter range.



$I_3(\theta,1)$
$I_1(\theta,1)+I_2(\theta,1)+I_3(\theta,1)$
$I_2(\theta,1)$
$I_1(\theta,1)$
θ
0
1
2
3
4
0
1
2

Fig. 1