

# Winding Number Transitions in the Mottola–Wipf Model on a Circle

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## Abstract

Winding number transitions from quantum to classical behavior are studied in the case of the 1+1 dimensional Mottola–Wipf model with the space coordinate on a circle for exploring the possibility of obtaining transitions of second order. The model is also studied as a prototype theory which demonstrates the procedure of such investigations. In the model at hand we find that even on a circle the transitions remain those of first order.

## I. INTRODUCTION

The study of transitions from quantum behavior to classical behavior has attracted considerable attention recently as a result of extension of semiclassical vacuum instanton considerations to those of periodic instantons and sphalerons, and the consequent possibility to study such theories at higher temperatures [1–3]. In these earlier investigations the order

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of the transitions was inferred from monotonically decreasing or nondecreasing behavior of the period of the periodic pseudoparticle configurations. But in other contexts [4] it was shown that one can derive criteria for the occurrence of such phase transitions of one type or the other by expanding the field concerned about the sphaleron configuration. This idea was then further exploited [5] to obtain concrete conditions in the form of inequalities for a large class of quantum mechanical models, applications of which can be found in [6–10].

The period  $\beta$  is related to the energy  $E$  in the standard way, i.e.  $E = \frac{\partial S}{\partial \beta}$ , where  $S(\beta)$  is the action of the periodic instanton (bounce) per period. Such periodic instantons (bounces) smoothly interpolate between the zero temperature instantons (bounces) and the static solution named sphaleron at the top of the potential barrier. The sphaleron is responsible for thermal hopping. With increasing temperature thermal hopping becomes more and more important and beyond some critical or crossover temperature  $T_c$  becomes the decisive mechanism. It is more difficult to study such phase transitions in the context of field theory. However, as mentioned, the criterion for a first order transition can be obtained by studying the Euclidean time period in the neighborhood of the sphaleron. If the period  $\beta(E \rightarrow U_0)$  of the periodic instanton (bounce) close to the barrier peak can be found, a sufficient condition for a first order transition is seen to be the inequality  $\beta(E \rightarrow U_0) - \beta_s < 0$  or  $\omega^2 > \omega_s^2$ , where  $U_0$  denotes the barrier height and  $\beta_s$  is the period of small oscillation around the sphaleron. Here  $\omega$  and  $\omega_s$  are the corresponding frequencies. The frequency of the sphaleron  $\omega_s$  is the frequency of small oscillations at the bottom of the inverted potential well. The winding number transition in the  $O(3)\sigma$  model with and without a Skyrme term has been successfully analysed with such a criterion in ref. [6]; other applications have been carried out in refs. [7–11].

In ref. [10] the theory defined by the following euclidean action was considered

$$S_E = \int d\tau dx \left[ \frac{1}{2} \left( \frac{d\phi}{d\tau} \right)^2 + \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + U(\phi) \right],$$

$$U(\phi) = -\frac{\mu^2}{2a^2}(\phi^2 - a^2)^2 + \frac{\mu^2}{2a^2}a^4. \quad (1)$$

It was found that the quantum classical transition is always of the smooth second order

type for a noncompactified spatial coordinate. If one considers the same theory with  $x$  on  $S^1$ , the theory can become first order when the elliptic modulus  $k$  of the periodic solution is subjected to a condition  $k < k^* < 1$ . The limit of decompactification (i.e.  $S^1 \rightarrow \mathbb{R}$ ) is given by  $k = 1$ . If we write our criterion inequality  $f < 0$  and plot  $f$  versus  $k$ , then a first-order transition implies that  $f < 0$  for  $k < k^*$ . Thus one can conclude that either (i) compactification prefers a first-order transition, or (ii) compactification changes the type of transition. In the following we investigate which of these possibilities is right in the case of the compactified Mottola–Wipf model, for which the uncompactified limit is always of first-order, as shown in [6,11]. Since the Mottola–Wipf model serves as an important testing ground of many field theoretical aspects, it is desirable to explore what happens in its case with compactification. If (i) is correct, the compactified case implies a stronger tendency to first-order behavior, if (ii) is correct one expects a sign change in  $f$ , and so a rising behavior below a critical value of  $k$ . In the following we shall see that case (i) applies.

## II. THE SPHALERON AND EXPANSION AROUND IT

The Euclidean time action of the Mottola–Wipf model [12] is given by

$$S = \frac{1}{g^2} \int d\tau dx \left[ \frac{1}{2} \partial_\mu n_a \partial_\mu n_a + (1 + n_3) \right] \quad (2)$$

where  $a = 1, 2, 3, n_a n_a = 1, \mu = \tau, x$ . The equation of motion is

$$(\delta_{ab} - n_a n_b) \square n_b = (\delta_{a3} - n_a n_3). \quad (3)$$

The static and so only  $x$ -dependent sphaleron solution is given by

$$\mathbf{n}_{sph}(x) = (-\sin f_s(x), 0, \cos f_s(x)), \quad f_s(x) = 2 \arcsin[ksn(x)] \quad (4)$$

where  $k$  is the elliptic modulus of the Jacobian elliptic functions involved and (later)  $k'^2 = 1 - k^2$ . For the expansion around the sphaleron configuration we set

$$\mathbf{n}(x, \tau) = \frac{1}{\sqrt{1 + u^2}} \left( -\sin(f_s + v), u, \cos(f_s + v) \right) \quad (5)$$

where  $u(x, \tau)$  and  $v(x, \tau)$  are fluctuation fields. Inserting this ansatz into the equation of motion, we obtain after some tedious calculations the following set of equations

$$\hat{\vartheta} \begin{pmatrix} u \\ v \end{pmatrix} = \hat{h} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} G_2^u(u, v) \\ G_2^v(u, v) \end{pmatrix} + \begin{pmatrix} G_3^u(u, v) \\ G_3^v(u, v) \end{pmatrix} + \dots \quad (6)$$

with operators

$$\hat{\vartheta} = \begin{pmatrix} \frac{\partial^2}{\partial \tau^2} & 0 \\ 0 & \frac{\partial^2}{\partial \tau^2} \end{pmatrix}, \quad \hat{h} = \begin{pmatrix} \hat{h}_u & 0 \\ 0 & \hat{h}_v \end{pmatrix}, \quad (7)$$

In these equations

$$\begin{aligned} G_2^u(u, v) &= 2k \operatorname{sn}(x) \operatorname{dn}(x) uv - 4k \operatorname{cn}(x) uv' \\ G_2^v(u, v) &= -k \operatorname{sn}(x) \operatorname{dn}(x) (u^2 - v^2) + 4k \operatorname{cn}(x) uu' \\ G_3^u(u, v) &= 2u(\dot{u}^2 + u'^2) - u(\dot{v}^2 + v'^2) - \frac{1}{2}(1 - 2k^2 \operatorname{sn}^2(x))u(u^2 - v^2) \\ G_3^v(u, v) &= \frac{1}{6}(1 - 2k^2 \operatorname{sn}^2(x))(v^3 - 3u^2v) + 2u(\dot{u}\dot{v} + u'v') \end{aligned} \quad (8)$$

where the dot and prime imply differentiation with respect to  $\tau$  and  $x$  respectively, and

$$\hat{h}_u = -\frac{\partial^2}{\partial x^2} - (1 + 4k^2 - 6k^2 \operatorname{sn}^2(x)), \quad \hat{h}_v = -\frac{\partial^2}{\partial x^2} - (1 - 2k^2 \operatorname{sn}^2(x)). \quad (9)$$

The equations

$$\hat{h}_u \psi_n(x) = \epsilon_n^u \psi_n(x), \quad \hat{h}_v \phi_n(x) = \epsilon_n^v \phi_n(x) \quad (10)$$

are Lamé equations. The  $2N + 1$ ,  $N = 2, 1$  discrete eigenvalues of these equations and their respective eigenfunctions whose periods are  $4K(k)$  and  $2K(k)$  (cf. [13]) are given in Table 1.

**Table 1**

Discrete eigenfunctions and eigenvalues of  $\hat{h}_u, \hat{h}_v$

$n$	Period	$\psi_n(x)$	$\epsilon_n^u$
0	$2K$	$\psi_0(x) = C_0[sn^2(x) - \frac{1+k^2+\sqrt{1-k^2k'^2}}{3k^2}]$	$\epsilon_0^u = 1 - 2k^2 - 2\sqrt{1 - k^2k'^2}$
1	$4K$	$\psi_1(x) = C_1cn(x)dn(x)$	$\epsilon_1^u = -3k^2$
2	$4K$	$\psi_2(x) = C_2sn(x)dn(x)$	$\epsilon_2^u = 0$
3	$2K$	$\psi_3(x) = C_3sn(x)cn(x)$	$\epsilon_3^u = 3(1 - k^2)$
4	$2K$	$\psi_4(x) = C_4[sn^2(x) - \frac{1+k^2-\sqrt{1-k^2k'^2}}{3k^2}]$	$\epsilon_4^u = 1 - 2k^2 + 2\sqrt{1 - k^2k'^2}$

$n$	Period	$\phi_n(x)$	$\epsilon_n^v$
0	$2K$	$\phi_0(x) = D_0dn(x)$	$\epsilon_0^v = -(1 - k^2)$
1	$4K$	$\phi_1(x) = D_1cn(x)$	$\epsilon_1^v = 0$
2	$4K$	$\phi_2(x) = D_2sn(x)$	$\epsilon_2^v = k^2$

We now consider the spectrum of  $\hat{h}$  and write

$$\hat{h}\xi_n = h_n\xi_n. \quad (11)$$

The eigenfunctions and respective eigenvalues of eq.(11) depend, of course, on the boundary conditions of the theory. We set

$$\mathbf{n}(x, \tau) = (n_1(x, \tau), n_2(x, \tau), n_3(x, \tau))$$

and consider the following boundary conditions:

(i) *Periodic boundary conditions*

$$n_1(x + 4K, \tau) = n_1(x, \tau), n_2(x + 4K, \tau) = n_2(x, \tau), n_3(x + 4K, \tau) = n_3(x, \tau) \quad (12)$$

(ii) *Partially anti-periodic boundary conditions*

$$n_1(x + 2K, \tau) = -n_1(x, \tau), n_2(x + 2K, \tau) = -n_2(x, \tau), n_3(x + 2K, \tau) = n_3(x, \tau) \quad (13)$$

Then up to the first order

$$\mathbf{n} = \mathbf{n}_{sph} + \delta \mathbf{n} \quad (14)$$

where

$$\mathbf{n}_{sph} = (-2ksn(x)dn(x), 0, 2dn^2(x) - 1), \quad \delta \mathbf{n} = (-(2dn^2(x) - 1)v, u, -2ksn(x)dn(x)v). \quad (15)$$

Thus in the case of periodic boundary conditions we need fluctuation fields  $u$  and  $v$  which have the period  $4K$  and all 8 cases of Table 1 are possible. The eigenfunctions and eigenvalues of eq.(11) are then of the following type:

$$\begin{aligned} \xi_0 &= (\psi_0, 0), & \epsilon_0^u &< 0, \\ \xi_1 &= (\psi_1, 0), & \epsilon_1^u &< 0, \\ \xi_2 &= (0, \phi_0), & \epsilon_0^v &< 0, \\ \xi_3 &= (\psi_2, 0), & \epsilon_2^u &= 0, \\ \xi_4 &= (0, \phi_1), & \epsilon_1^v &= 0, \\ \xi_5 &= (\psi_3, 0), & \epsilon_3^u &> 0, \\ \xi_6 &= (0, \phi_2), & \epsilon_2^v &> 0, \\ \xi_7 &= (\psi_4, 0), & \epsilon_4^u &> 0. \end{aligned} \quad (16)$$

Hence  $\hat{h}$  has three negative modes and two zero modes in this case. Thus it is impossible to analyse the phase transition problem properly in the case of periodic boundary conditions.

In the second case of partially antiperiodic boundary conditions our findings of the spectrum of  $\hat{h}$  are as follows. Considering again eq.(15) one can show that with  $2K$ -antiperiodic  $u$  and  $v$  satisfying

$$u(x + 2K) = -u(x), \quad v(x + 2K) = -v(x)$$

the eigenfunctions  $\xi_n$  and eigenvalues  $\epsilon_n$  become those given in Table 2 below.

**Table 2**

Eigenfunctions and eigenvalues of  $\hat{h}$  in case of partially antiperiodic boundary conditions

$n$	$\xi_n$	$h_n$
0	$\xi_0 = (\psi_1, 0)$	$h_0 = \epsilon_1^u = -3k^2$
1, 2	$\xi_1 = (\psi_2, 0), \xi_2 = (0, \phi_1)$	$h_1 = h_2 = 0$
3	$\xi_3 = (0, \phi_2)$	$h_2 = \epsilon_2^v = k^2$

Since this case produces only one negative eigenmode we can consider the winding number transition with our method. Hence, in the next section we will investigate the winding number phase transition in this model with the partially antiperiodic boundary condition. Although the periodic boundary condition in the model is fully examined in Ref. [14], we think the partially antiperiodic boundary condition is physically more reasonable in the sense that it has an uncompactified limit which coincides with the spectrum derived in the *a priori* infinite spatial domain. This fact justifies our choice of this boundary condition.

### III. DERIVATION OF THE CRITERION INEQUALITY

The calculation of the inequality proceeds as in [6]. Without repeating detailed calculational steps we summarize briefly how the criterion for the sharp first-order transition is obtained. Let  $u_0$  and  $\epsilon_0$  be eigenfunction and eigenvalue of the negative mode of the fluctuation operator. Therefore, the sphaleron frequency  $\Omega_{sph}$  is defined as  $\Omega_{sph} \equiv \sqrt{-\epsilon_0}$ . Then the type of the transition is determined by studying the nonlinear corrections to the frequency. Let, for example,  $\Omega$  be a frequency involving the nonlinear corrections. If  $\Omega_{sph} - \Omega > 0$ , the period-vs-energy diagram has monotonically decreasing behavior at least in the vicinity of the sphaleron. This means  $dS/dT$ , where  $S$  and  $T$  are classical instanton action and temperature (inverse of period), is monotonically decreasing around the sphaleron and eventually,  $S$  merges with the sphaleron action smoothly as shown in Ref. [1]. If, on the contrary,  $\Omega_{sph} - \Omega < 0$ , the temperature dependence of instanton action consists of monotonically

decreasing and increasing parts, which results in the discontinuity of  $dS/dT$ . This is the main idea which is used to derive the criterion in Ref. [6].

Recently, similar considerations have been used in the context of the  $SU(2)$ -Higgs model [15,16], where the winding number transition is characterized by the monotonic sign of  $d^2S/d\beta^2$ , where  $\beta$  is the period. From the viewpoint of our criterion it is clear that the non-monotonic behavior of  $dS/d\beta$  exactly corresponds to  $\Omega_{sph} - \Omega < 0$ .

Now, we will compute  $\Omega$  following the procedure in Ref. [6]. We restrict ourselves to a presentation of the main steps. We set (cf. Table 1)

$$u_0(x) = \psi_1(x) = C_1 cn(x) dn(x)$$

Using  $\int_{-K}^K dx \psi_1^*(x) \psi_1(x) = 1$ , the normalization constant  $C_1$  is found to be

$$C_1 = \left[ \frac{2}{3k^2} \left\{ (1+k^2)E - (1-k^2)K \right\} \right]^{-\frac{1}{2}} \quad (17)$$

where  $K(k)$  is the quarter period introduced earlier and  $E(k)$  the complete elliptic integral of the second kind (here and elsewhere we use formulas of ref. [17]). We let  $a$  be a small amplitude around the sphaleron. Then, one can carry out the perturbation with an expansion parameter  $a$ . In the first order perturbation where we have to expand up to the quadratic terms in Eq.(6) it is shown that  $\Omega$  does not have a correction. The next order perturbation where we have to consider terms up to the cubic terms generates a nonlinear correction to  $\Omega$ , which makes the condition for a first order transition to be

$$\Omega_{sph}^2 - \Omega^2 = a^2 \langle u_0 | G_{u,1} \rangle < 0 \quad (18)$$

where  $\langle \psi_1 | \psi_2 \rangle \equiv \int_{-K}^K dx \psi_1^*(x) \psi_2(x)$ ,  $\Omega_{sph} = \sqrt{3}k$ ,

$$\begin{aligned} G_{u,1}(x) = & 2k sn(x) cn(x) dn^2(x) (g_{v,1}(x) + g_{v,2}(x)/2) - 4k C_1 cn^2(x) dn(x) (g'_{v,1}(x) + g'_{v,2}(x)/2) \\ & + \frac{C_1^3}{4} \left[ cn^3(x) dn^3(x) \left\{ 2\Omega_{sph}^2 - \frac{3}{2} + 3k^2 sn^2(x) \right\} \right. \\ & \left. + 6sn^2(x) cn(x) dn(x) \{ dn^2(x) + k^2 cn^2(x) \}^2 \right] \end{aligned} \quad (19)$$



and here

$$g_{v,1}(x) = \frac{k}{2} C_1^2 \hat{h}_v^{-1} \left[ \{ (5 + 4k^2) - 9k^2 sn^2(x) \} sn(x) cn^2(x) dn(x) \right] \quad (20)$$

and

$$g_{v,2}(x) = \frac{k}{2} C_1^2 (\hat{h}_v + 4\Omega_{sph}^2)^{-1} \left[ \{ (5 + 4k^2) - 9k^2 sn^2(x) \} sn(x) cn^2(x) dn(x) \right]. \quad (21)$$

The inequality (18) can be written in the form

$$\frac{\Omega_{sph}^2 - \Omega^2}{a^2} = I_1(k) + I_2(k) + I_3(k) + I_4(k) < 0 \quad (22)$$

where

$$I_i(k) = \langle u_0 | f_i \rangle, \quad i = 1, 2, 3, 4, \quad (23)$$

with

$$\begin{aligned} f_1 &= 2kC_1 sn(x) cn(x) dn^2(x) g_{v,1} - 4kC_1 cn^2(x) dn(x) g'_{v,1} \\ f_2 &= kC_1 sn(x) cn(x) dn^2(x) g_{v,2} - 2kC_1 cn^2(x) dn(x) g'_{v,2} \\ f_3 &= \frac{C_1^3}{4} cn^3(x) dn^3(x) \{ 2\Omega_{sph}^2 - \frac{3}{2} + 3k^2 sn^2(x) \} \\ f_4 &= \frac{3}{2} C_1^3 sn^2(x) cn(x) dn(x) \{ dn^2(x) + k^2 cn^2(x) \}^2. \end{aligned} \quad (24)$$

#### IV. EVALUATION OF THE QUANTITIES $I_i(k)$

It is now necessary to evaluate the quantities entering the criterion inequality. Since this is nontrivial we consider these individually.

##### The quantity $I_1(k)$

We consider first the case of  $k = 1$ . In this case  $g_{v,1}(x)$  becomes

$$g_{v,1}(k = 1, x) = \frac{27}{8} \hat{h}_v \frac{\sinh x}{\cosh^6 x}, \quad \hat{h}_v^{-1}(k = 1) = -\frac{\partial^2}{\partial x^2} + (1 - 2/\cosh^2 x). \quad (25)$$

Since the operator  $\hat{h}_v^{-1}(k = 1)$  is of the usual Pöschl–Teller type, the complete spectrum can be obtained and is summarised in Table 3.

**Table 3**

Eigenfunctions and eigenvalues of  $\hat{h}_v^{-1}(k = 1)$

Eigenvalue of $\hat{h}_v^{-1}(k=1)$	Eigenfunction of $\hat{h}_v^{-1}(k=1)$
Discrete mode: 0	$\langle x 0\rangle = \frac{1}{\sqrt{2}} \frac{1}{\cosh x}$
Continuum mode: $1+k_c^2$	$\langle x k_c\rangle = \frac{e^{ik_c x}}{\sqrt{2\pi(1+ik_c)}}(ik_c - \tanh x)$

Using the completeness relation one can evaluate  $g_{v,1}(k=1, x)$ . We find

$$\begin{aligned}
g_{v,1}(k=1, x) &= \langle x | \frac{27}{8} \hat{h}_v^{-1}(k=1) \left[ |0\rangle\langle 0| + \int dk_c |k_c\rangle\langle k_c| \right] |x\rangle \frac{\sinh x}{\cosh^6 x} \\
&= \frac{3}{16} \frac{\sinh x}{\cosh^4 x} (1 + 2 \cosh^2 x).
\end{aligned} \tag{26}$$

By inserting this into eq.(25) one can verify that this is correct. The  $k=1$  limit of  $I_1(k)$  can now be easily obtained as

$$I_1(k=1) = -\frac{36}{35} = -1.02857. \tag{27}$$

Since this is negative it is indicative of support for a first order contribution.

In order to obtain the  $k$ -dependence of  $I_1(k)$ , we can proceed in two ways: We can derive first an approximate expression, and then the exact one. It is instructive to derive the approximation first. Thus from eq.(20)

$$\begin{aligned}
g_{v,1} &= \frac{k}{2} C_1^2 \hat{h}_v^{-1} \left[ \{ (5 + 4k^2) - 9k^2 \text{sn}^2(x) \} \text{sn}(x) \text{cn}^2(x) \text{dn}(x) \right] \\
&= \frac{k}{2} C_1^2 \sum_n \frac{1}{\epsilon_n^v} \langle \phi_n | \{ (5 + 4k^2) - 9k^2 \text{sn}^2(x) \} \text{sn}(x) \text{cn}^2(x) \text{dn}(x) | \phi_n \rangle.
\end{aligned} \tag{28}$$

Since we know only  $|\phi_1\rangle$  and  $|\phi_2\rangle$  (the former, the zero mode, does not contribute to (28)), we approximate the expression (28) as follows.

$$\begin{aligned}
g_{v,1}(k, x) &= \frac{k}{2} C_1^2 \frac{1}{\epsilon_2^v} \langle \phi_2 | [(5 + 4k^2) - 9k^2 \text{sn}^2(x)] \text{sn}(x) \text{cn}^2(x) \text{dn}(x) | \phi_2 \rangle \\
&= \frac{\pi C_1^2 D_2^2}{32k} (10 - k^2) \text{sn}(x)
\end{aligned} \tag{29}$$

where

$$D_2 = \frac{1}{\sqrt{\frac{2}{k^2}(K - E)}}, \tag{30}$$

$K$  being the quarter period introduced earlier and  $E$  the complete elliptic integral of the second kind. The approximation is valid provided  $|\phi_2 \rangle$  is an isolated discrete mode and the density of higher states is dilute. But if one checks the spectra of  $\hat{h}_v(k=1)$  (see also below) and  $\hat{h}_v$  in general, one can see that  $|\phi_2 \rangle$  is the lowest continuum mode of  $\hat{h}_v(k=1)$ . This is why  $D_2(k=1) = 0$ , and so this approximation is not valid around  $k=1$ , as can also be seen from a plot of  $I_1(k)$  with this approximation in Fig.1.

In Appendix A we derive the exact value of  $g_{v,1}(k, x)$  using the zero mode of  $\hat{h}_v$ . The result is

$$g_{v,1}(k, x) = \frac{k}{2} C_1^2 cn(x) \left[ \frac{1 + k^2(2 - 4K(k)) - 3k^4}{4k^2(1 - k^2)} E(x) + \frac{4k^2 R(k) + k^4 - 1}{4k^2} x + \frac{1}{2} sn(x) cn(x) dn(x) + \frac{R(k)}{1 - k^2} \frac{sn(x) dn(x)}{cn(x)} \right] \quad (31)$$

where

$$R(k) = \frac{1 - k^2(1 + 3k^2)E - (1 - k^2)(1 + k^2)K}{4k^2(E - (1 - k^2)K)}$$

$$E(x) \equiv \int_0^{amx} \sqrt{1 - k^2 \sin^2 \theta} d\theta \equiv \int_0^x dn^2(u) du. \quad (32)$$

In the limit  $k \rightarrow 1$  these quantities are such that

$$\left\{ R(k) \right\}_{k \rightarrow 1}^{lim} = 0, \quad \left\{ \frac{R(k)}{1 - k^2} \right\}_{k \rightarrow 1}^{lim} = 1 \quad (33)$$

and

$$g_{v,1}(k=1, x) = \frac{3}{16} \frac{\sinh x}{\cosh^4 x} \left[ 1 + 2 \cosh^2 x \right] \quad (34)$$

in agreement with eq.(26). The expression  $I_1$  can now be evaluated numerically; its behavior is also shown in Fig.1. One can see agreement with our  $k \rightarrow 1$  limit, and that our initial approximation applies in the small  $k$  region. From the latter we deduce that the Lamé equation has very dilute discrete higher states in that domain.

### The quantity $I_2(k)$

Using the spectrum of  $\hat{h}_v(k=1)$  one can derive an integral representation of  $g_{v,2}(k=1, x)$ , i.e.

$$\begin{aligned}
g_{v,2}(k=1, x) &= \frac{3}{256} \int dk_c \frac{(k_c^2 + 1)(k_c^2 + 9)}{(k_c^2 + 13) \cosh \frac{k_c \pi}{2}} \sin k_c x \\
&+ \tanh x \int dk_c \frac{(k_c^2 + 1)(k_c^2 + 9)}{(k_c^2 + 13) \cosh \frac{k_c \pi}{2}} \cos k_c x.
\end{aligned} \tag{35}$$

With this we obtain for the  $k = 1$  limit of  $I_2(k)$

$$I_2(k=1) = -\frac{9\pi}{8192} K_c \approx -0.09683, \quad K_c \equiv \int_0^\infty \frac{(k_c^2 + 1)^3 (k_c^2 + 9)^2}{(k_c^2 + 13) \cosh^2 \frac{k_c \pi}{2}} \approx 28.0549. \tag{36}$$

The negative sign of  $I_2$  indicates that it also contributes to make the transition of first order. To our knowledge an exact evaluation of  $I_2(k)$  is not possible. We therefore adopt for two reasons the above approximate procedure used for the calculation of  $I_1(k)$ . (i) From the result of the limit  $k = 1$  one can conjecture that  $I_2(k)$  is very small in magnitude compared with  $I_1(k)$ ; hence its contribution would not change the type of transition. (ii) From a knowledge of the type of transition at  $k = 1$ , the interest is shifted to the domain of small  $k$ , and in this region our approximate procedures are valid. In this approximation

$$g_{v,2}(k, x) \approx \frac{\pi C_1^2 D_2^2}{416k} (10 - k^2) \sin x \tag{37}$$

and

$$I_2(k) \approx -\frac{\pi^2 C_1^4 D_2^2}{6656} (10 - k^2)^2. \tag{38}$$

The behavior of  $I_2(k)$  as a function of  $k$  is shown in Fig. 2.

### The quantity $I_3(k)$

In the case of  $I_3(k)$  we obtain

$$\begin{aligned}
I_3(k) &= \frac{C_1^4}{420k^4} \left[ (1 - k^2)(-2 + 111k^2 - 297k^4 - 44k^6)K \right. \\
&\quad \left. + (2 - 112k^2 + 234k^4 + 428k^6 - 88k^8)E \right]
\end{aligned} \tag{39}$$

For  $k = 1$  the value is given in ref. [6]:

$$I_3(k=1) = \frac{87}{140} \approx 0.621429. \tag{40}$$

Since this is positive, it would help towards a second order transition. With these results one can show that  $I_3(k)$  has the correct  $k = 1$  limit. The behavior of  $I_3(k)$  as a function of  $k$  is shown in Fig.3.

### The quantity $I_4(k)$

Proceeding as in the other cases the final result is found to be

$$I_4(k) = \frac{3C_1^4}{315k^4} \left[ - (1 - k^2)(10 - 21k^2 + 48k^4 - 5k^6)K + (10 - 26k^2 + 96k^4 - 26k^6 + 10k^8)E \right]. \quad (41)$$

At  $k = 1$  this is

$$I_4(k = 1) = \frac{12}{35} \approx 0.342857 \quad (42)$$

which also supports a tendency towards a second order transition. One can show that  $I_4(k = 1)$  has the correct  $k = 1$  limit. The behavior of  $I_4(k)$  as a function of  $k$  is shown in Fig.4.

## V. SUMMARY AND CONCLUSIONS

Summarising the above results in the limit  $k \rightarrow 1$ , we obtain

$$\begin{aligned} \frac{\Omega_{sph}^2 - \Omega^2}{a^2} &= \sum_{i=1}^4 I_i(k = 1) = -\frac{36}{35} - \frac{9\pi}{8192}K_c + \frac{87}{140} + \frac{12}{35} \\ &= -1.02857 - 0.09683 + 0.62143 + 0.342857 \\ &= -0.161113. \end{aligned} \quad (43)$$

Thus the transition for  $k = 1$  is of first order. One may note that roughly  $I_2(k = 1) \approx I_1(k = 1)/10$ , which is roughly due to a factor “12” in  $g_{v,2}(k = 1, x)$ . The corresponding plot of action  $S$  versus  $1/T$  is shown in Fig.5. Putting all our results together we see that as we go to smaller values of  $k$  the tendency away from second order behavior is even enhanced, i.e. the transition becomes even more of first order.

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## Appendix A: Exact calculation of $g_{v,1}(k, x)$

We set

$$g_{v,1}(k, x) \equiv \frac{k}{2} C_1^2 q(k, x) \quad (\text{A.1})$$

where

$$q(k, x) = \hat{h}_v^{-1} \left[ \left\{ (5 + 4k^2) - 9k^2 sn^2(x) \right\} sn(x) cn^2(x) dn(x) \right]. \quad (\text{A.2})$$

Hence

$$\hat{h}_v q(k, x) = \left[ (5 + 4k^2) - 9k^2 sn^2(x) \right] sn(x) cn^2(x) dn(x). \quad (\text{A.3})$$

We multiply the expression by  $cn(x)$  and integrate over  $x$  from  $-K$  to  $x$ . Then integrating by parts one obtains

$$\begin{aligned} & \left[ q \frac{d}{dx} cn(x) - cn(x) \frac{dq}{dx} \right]_{x=-K}^x + \int_{-K}^x dx q \hat{h}_v cn(x) \\ &= \int_{-K}^x dx \left[ (5 + 4k^2) - 9k^2 sn^2(x) \right] sn(x) cn^3(x) dn(x). \end{aligned} \quad (\text{A.4})$$

Since  $cn(x)$  is the zero mode of  $\hat{h}_v$ , the second term on the left hand side of this equation is zero. After integrating the right hand side of eq.(A.4) we obtain

$$\frac{dq}{dx} + \frac{sn(x) dn(x)}{cn(x)} q = \frac{5}{4} (1 - k^2) cn^3(x) + \frac{3}{2} k^2 cn^5 x + \frac{R}{cn(x)} \quad (\text{A.5})$$

where

$$R \equiv \left[ cn(x) \frac{dq}{dx} + sn(x) dn(x) q(x) \right]_{x=-K}. \quad (\text{A.6})$$

Eq.(A.5) is a simple linear differential equation and its solution is

$$\begin{aligned} q &= A cn(x) + cn(x) \left[ \frac{1 + k^2(2 - 4R) - 3k^4}{4k^2(1 - k^2)} E(x) + \frac{-1 + 4k^2 R + k^4}{4k^2} x \right. \\ &\quad \left. + \frac{1}{2} sn(x) cn(x) dn(x) + \frac{R}{1 - k^2} \frac{sn(x) dn(x)}{cn(x)} \right] \end{aligned} \quad (\text{A.7})$$

where

$$E(x) \equiv \int_0^{amx} \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^x dn^2(u) du. \quad (\text{A.8})$$

It is now necessary to determine  $A$  and  $R$ . Since we work with partially anti-periodic boundary conditions, we require

$$q(x + 2K) = -q(x). \quad (\text{A.9})$$

Using also  $E(x + 2K) = 2E + E(x)$ , one can show that

$$R = \frac{(1 - k^2) \left[ (1 + 3k^2)E - (1 - k^2)(1 + k^2)K \right]}{4k^2 \left[ E - (1 - k^2)K \right]}. \quad (\text{A.10})$$

The expression can also be derived from the definition of  $R$  in eq.(A.6). In order to determine  $A$  we recall that  $cn(x)$  is simply the zero mode term. Since, obviously,  $q(x)$  does not have a zero mode component, we need the condition

$$\int_{-K}^K dx cn(x)q(x) = 0$$

from which we conclude that  $A = 0$ .

## Figure Captions

Fig. 1:  $I_1(k)$  exact (solid line) and approximate (dashed line)

Fig. 2:  $I_2(k)$

Fig. 3:  $I_3(k)$

Fig. 4:  $I_4(k)$

Fig. 5: Action  $S$  versus  $\tau \equiv 1/T$  demonstrating the first order transition, the bold line determining the transition rate

FIGURES

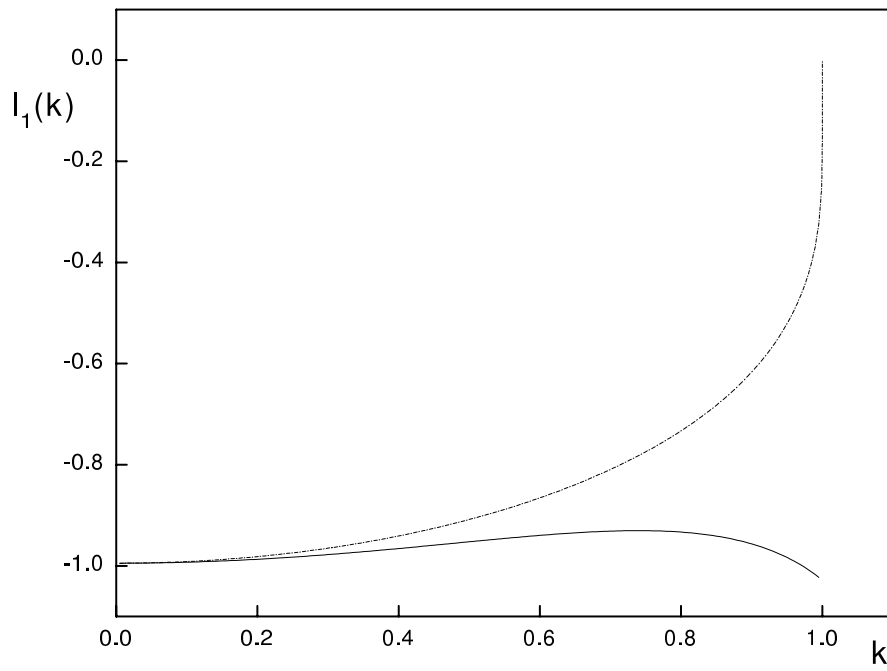


Fig. 1

FIG. 1.  $I_1(k)$  exact (solid line) and approximate (dashed line)

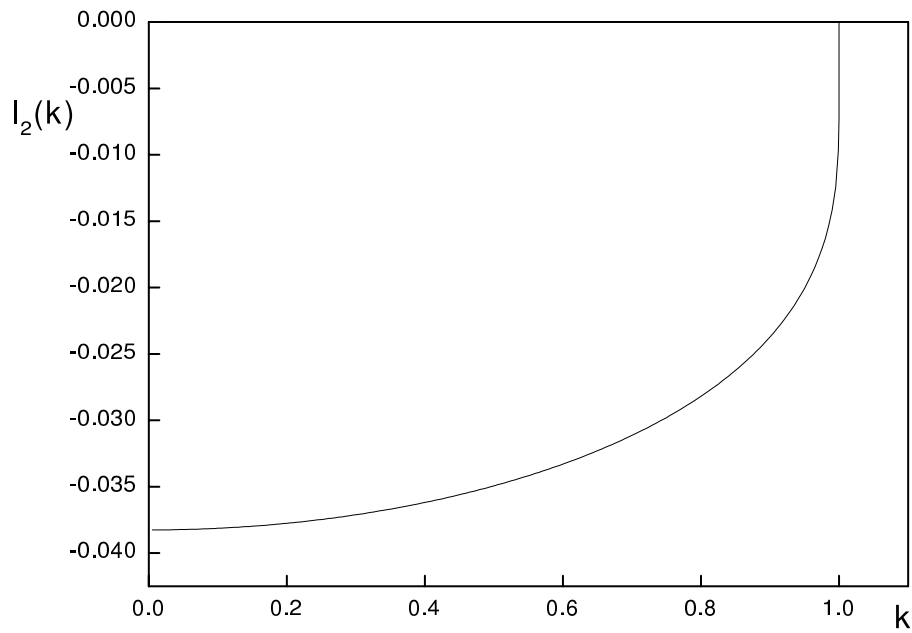


Fig. 2

FIG. 2.  $I_2(k)$

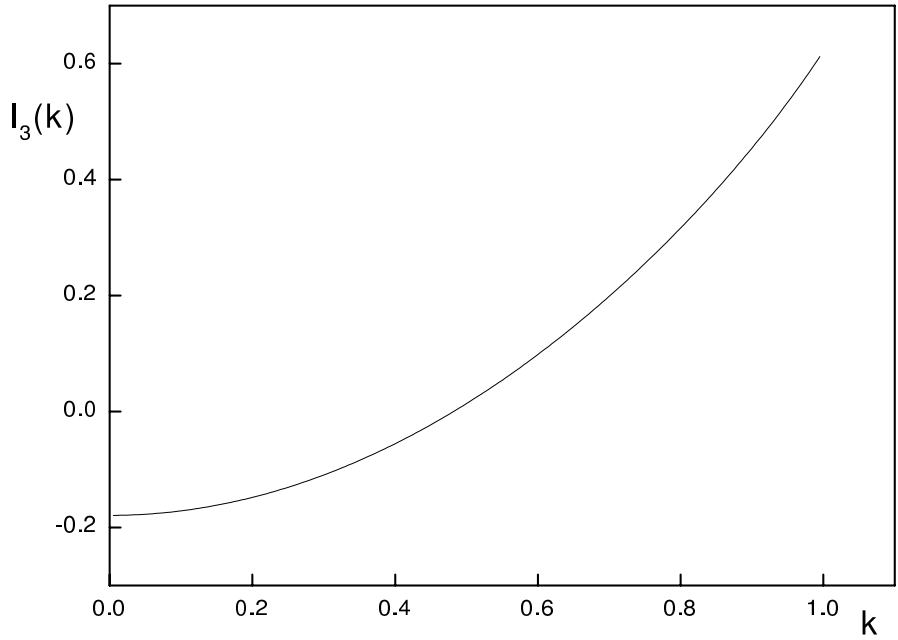


Fig. 3

FIG. 3.  $I_3(k)$

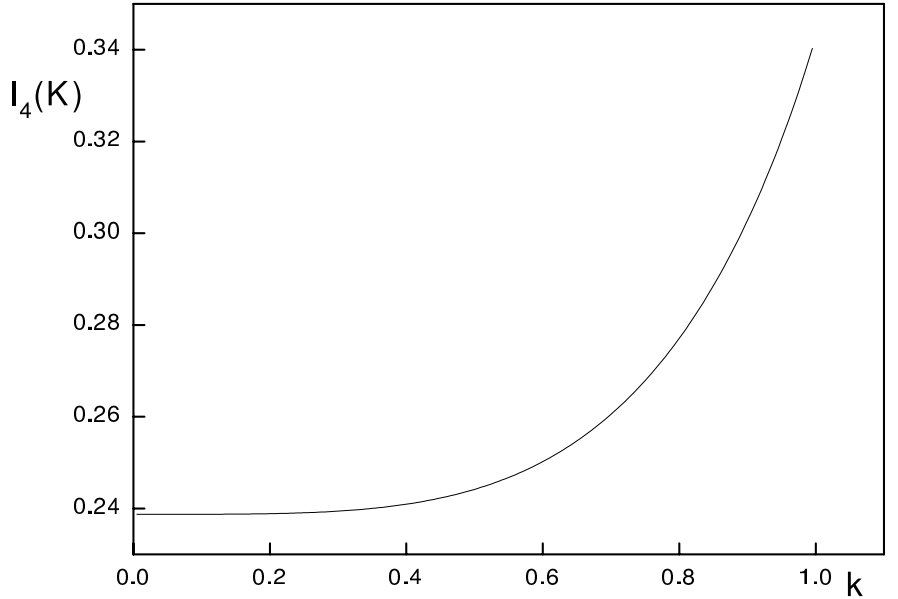


Fig. 4

FIG. 4.  $I_4(k)$

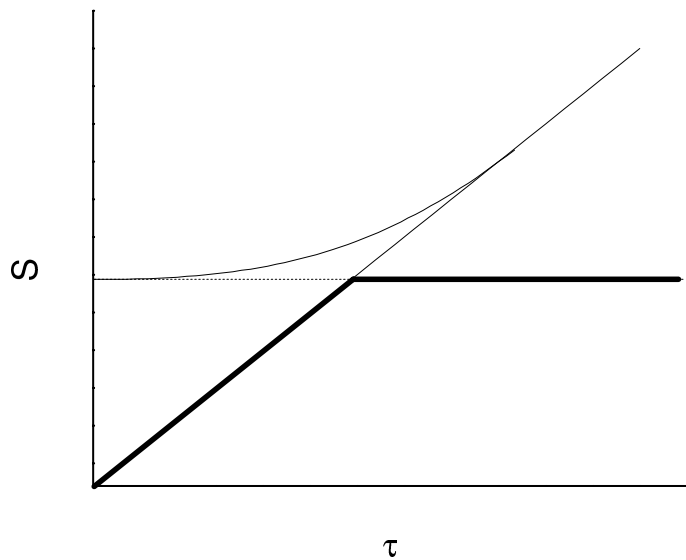


Fig. 5

FIG. 5. Action  $S$  versus  $\tau \equiv 1/T$  demonstrating the first order transition, the bold line determining the transition rate