NONVACUUM PSEUDOPARTICLES, QUANTUM TUNNELING AND METASTABILITY

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ABSTRACT

It is shown that nonvacuum pseudoparticles can account for quantum tunneling and metastability. In particular the saddle-point nature of the pseudoparticles is demonstrated, and the evaluation of path-integrals in their neighbourhood. Finally the relation between instantons and bounces is used to derive a result conjectured by Bogomolny and Fateyev.

1. Introduction

One of the most interesting applications of the Euclidean path-integral approach is the study of semi-classical instabilities or tunneling processes as Hawking and Ross[1] emphasised recently. Instanton transitions related to the possibility of baryon– and lepton–number violation in electroweak theory have attracted widespread attention [2]. It has gradually been realised that vacuum instantons and vacuum bounces which require vacuum boundary conditions may not be appropriate for the description of tunneling at finite, nonzero energy [3]. The investigation of quantum tunneling with a new type of instanton–like configurations which are characterised by nonzero energy and satisfy manifestly nonvacuum boundary conditions is therefore of great interest. In the following we consider the new type of instanton–like and bounce–like configurations called periodic instantons or periodic bounces or sphalerons [3,4,5,6] and investigate their stability for various potentials in (1+1) dimensions. We then calculate their tunneling effects in (1+0) dimensions for various solvable models since their appropriate transitions reduce to quantum mechanical tunneling problems [7,8,9,10]. It is well-known that the latter also play an important role in the investigation of the large order behaviour of perturbation expansions. We close therefore with a discussion of the Bogomolny–Fateyev relation[11] which relates the level splitting of one case to the energy discontinuity of a related case, the main idea behind this being the fact that the instanton is exactly half of the bounce. The latter point has been exploited recently in the discussion of duality in gravity theory[1].

2. Solitons, bounces and sphalerons on S^1 and their stability

We recall first the behaviour of vacuum pseudoparticles in (1+0) dimensional quantum mechanics. For the double–well potential given by

$$V(\phi)=\frac{\mu^2}{2a^2}(\phi^2-a^2)^2 \tag{1}$$

the instanton solution is the well-known expression

$$\phi_c=a\tanh[\mu(\tau+\tau_0)] \tag{2}$$

where $\tau = it$ is the Euclidean time. The configuration ϕ_c which corresponds to a transition between the degenerate vacua has nonzero topological charge and is stable.

The effect of tunneling appears in the level splitting for the potential with two degenerate minima and in the band structure for the sine–Gordon potential with an infinite number of degenerate minima. In the case of the inverted double–well potential

$$V(\phi)=-\frac{\mu^2}{2a^2}(\phi^2-a^2)^2+\frac{\mu^2}{2a^2}a^4 \tag{3}$$

the corresponding classical configuration is the bounce

$$\phi_c=a\sqrt{2}[\cosh(\mu\sqrt{2}\tau)]^{-1} \tag{4}$$

This configuration has zero topological charge and is unstable. The tunneling effect here is the decay of the (sometimes called "false") vacuum state (metastability) . Calculating the imaginary part of the energy [7] one obtains

$$\Im m E_0=\frac{4\mu}{g}[\frac{\sqrt{2}\mu^3}{\pi}]\exp[-\frac{4\sqrt{2}\mu^3}{3g^2}] \tag{5}$$

where $g^2=\frac{\mu^2}{a^2}$. One should note that the vacuum instanton is an odd function of its argument whereas the bounce is even, so that their derivatives (which classically represent velocities) are respectively even and odd. Since these derivatives are also the translational zero modes, i.e. wave functions with eigenvalue zero, the instanton is classically stable, wheras the bounce is not.

We now consider a field $\Phi(x,t)$ in (1+1) dimensions and static configuration $\phi(x)$ with Lagrangian density

$$\mathcal{L}=\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi-U(\Phi) \tag{6}$$

where U is respectively the double–well potential, the inverted double–well potential or the sine–Gordon potential, i.e.

$$\begin{aligned} U_1[\Phi] &= \frac{\mu^2}{2a^2}(\Phi^2-a^2)^2 \\ U_2[\Phi] &= -\frac{\mu^2}{2a^2}(\Phi^2-a^2)^2+\frac{\mu^2}{2a^2}a^4 \\ U_3[\Phi] &= 1+\cos\Phi \end{aligned} \tag{7}$$

The static solution with finite, nonzero energy is given by

$$\frac{1}{2}(\frac{d\phi}{dx})^2-U[\phi]=-\frac{1}{2}c^2 \tag{8}$$

where with $i = 1, 2, 3$

$$\begin{aligned} -U_1[0] \le -\frac{1}{2}{c_1}^2 \le -U_1[a] &= 0 \\ -U_2[0] \le -\frac{1}{2}{c_2}^2 \le -U_2[0] &= 0 \\ -U_3[0] \le -\frac{1}{2}{c_3}^2 \le -U_3[\pm\pi] &= 0 \end{aligned} \tag{9}$$

In these cases

$$\phi = 0, \phi = \pm a, \phi = 0, \pm 2\pi, ... \tag{10}$$

are trivial (constant) solutions and

$$\phi = \pm a, \phi = 0, \phi = \pm \pi \tag{11}$$

are the corresponding vacuum solutions. The modulus k of the elliptic functions with $0 \le k \le 1$ in the three cases is respectively given by

$$\begin{aligned} c_1^2 &= a^2\mu^2(\frac{1-k^2}{1+k^2})^2 \\ c_2^2 &= a^2\mu^2(\frac{1-k^2}{1+k^2})^2 \\ c_3^2 &= 4(1-k^2) \end{aligned} \tag{12}$$

Here and in the following k' is the complementary elliptic modulus defined by $k'^2 = 1 - k^2$. The expressions $\frac{1}{2}c_i^2$ can be regarded as energies of the appropriate pseudoparticles. We now take $x \epsilon S^1$, and we demand periodicity of the solutions, i.e. $\phi(x) = \phi(x + L)$. The nontrivial solutions are in the three cases respectively [5]

$$\begin{aligned} \phi(x) &= \frac{akb(k)}{\mu} sn[b(k)x, k] \\ \phi(x) &= s_+(k) dn[\beta(k)x, \gamma] \\ \phi(x) &= 2 \arcsin[ksn(x), k] \end{aligned} \tag{13}$$

where

$$b(k) = \mu(\frac{2}{1+k^2})^{\frac{1}{2}}, \beta(k) = \frac{\mu}{a} s_+(k), \tag{14}$$

and

$$s_+(k) = a\frac{1+k}{\sqrt{1+k^2}}, \gamma^2 = \frac{4k}{(1+k)^2} \tag{15}$$

In the limit $k^2 \to 1$, i.e. $c^2 \to 0$, we regain the vacuum solutions, i.e.

$$\begin{aligned} \phi(x) &= a \tanh(\mu x) \\ \phi(x) &= a\sqrt{2}[\cosh(\mu\sqrt{2}x)]^{-1} \\ \phi(x) &= 2 \arcsin[\tanh x] \end{aligned} \tag{16}$$

whereas in the limit $k^2 \to 0$ the periodic solutions become the trivial solutions given above. The periodicity requirement implies certain critical values of L, i.e. respectively

$$L = 4nK(k), L = 2nK(k), L = 4nK(k) \tag{17}$$

where $K(k)$ is the complete elliptic integral of the first kind defined by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin\theta^2}} \tag{18}$$

and $n = 1, 2, 3, ...$ (i.e. note that $n = 0$ is excluded). Setting

$$\Phi(x,t) = \phi_c(x) + \sum_m \psi_m(x) e^{i\omega_m t} \tag{19}$$

we obtain the stability or small fluctuation equation

$$(-\frac{d^2}{dx^2} + U''[\phi_c(x)])\psi_m(x) = {\omega_m}^2 \psi_m(x) \tag{20}$$

with

$$\psi_m(x) = \psi_m(x + L) \tag{21}$$

For socalled "classical stability" we must have $\omega_m^2 \geq 0$. For the vacuum solutions $\psi_m \propto \sin$ or $\cos(\frac{2\pi}{L}mx), m = 0, 1, 2, ...$, these conditions are respectively in the three cases

$$\begin{aligned} \omega_m^2 &= 4\mu^2 + \frac{4\pi^2}{L^2}m^2 > 0 \\ \omega_m^2 &= 2\mu^2 + \frac{4\pi^2}{L^2}m^2 > 0 \\ \omega_m^2 &= 1 + \frac{4\pi^2}{L^2}m^2 > 0 \end{aligned} \tag{22}$$

It is seen that these conditions are satisfied. In the case of the trivial solutions

$$\psi_m \propto \sin \text{ or } \cos(\frac{2\pi}{L}mx) \tag{23}$$

at the critical values of L the stability conditions are respectively

$$\begin{aligned} \omega_m^2 &= -2\mu^2 + \frac{4\pi^2}{L^2}m^2 = 2\mu^2(\frac{m^2}{n^2} - 1) \\ \omega_m^2 &= -4\mu^2 + \frac{4\pi^2}{L^2}m^2 = 4\mu^2(\frac{m^2}{n^2} - 1) \\ \omega_m^2 &= -1 + \frac{4\pi^2}{L^2}m^2 = -1 + \frac{m^2}{n^2} \end{aligned} \tag{24}$$

where $m = 0, 1, 2, ...$ and $n = 1, 2, 3, ...$ We see that for $n > m$: $\omega^2 < 0$, i.e. in that case ϕ_c is unstable (a sphaleron).

In the case of the nontrivial solutions the fluctuation equation turns out to be in each case a Lamé equation[5], i.e.

$$\frac{d^2\psi}{dz^2} + [\lambda - N(N+1)\kappa^2 sn^2(z,\kappa)]\psi = 0 \tag{25}$$

The discrete eigenvalues and eigenfunctions in each of the cases of the potentials U_i, with $i = 1,2,3$, are:
U_1:Here $N = 2$, $z = b(k)x$,$\kappa^2 = k^2$ and $\lambda = \frac{\omega^2+2\mu^2}{b^2(k)}$, and the solutions are

$$\begin{aligned}
\psi_1 &= sn(z,k)cn(z,k) \\
\psi_2 &= sn(z,k)dn(z,k) \\
\psi_3 &= cn(z,k)dn(z,k) \\
\psi_{4,5} &= sn^2(z,k) - \frac{[1 + k^2 \pm \sqrt{1-k^2(1-k^2)}]}{3k^2}
\end{aligned} \tag{26}$$

with respectively the following eigenvalues

$$\begin{aligned}
\omega_1^2 &= \frac{6\mu^2}{(1+k^2)} \\
\omega_2^2 &= \frac{6\mu^2 k^2}{(1+k^2)} \\
\omega_3^2 &= 0 \\
\omega_{4,5}^2 &= 2\mu^2(1 \mp \frac{2\sqrt{1-k^2(1-k^2)}}{1+k^2})
\end{aligned} \tag{27}$$

where the second last eigenvalue is seen to be negative and the third is that of the zero mode.
U_2:Here $N = 2$ with $z = \beta(k)x$ and $\kappa^2 = \gamma^2 = \frac{4k}{(1+k)^2}$ and $\lambda = 6 + \frac{\omega^2-2\mu^2}{\beta^2(k)}$. In this case the solutions have respectively the same form as in the first case but with elliptic modulus γ instead of k and the corresponding eigenvalues are

$$\begin{aligned}
\omega_1^2 &= 0 \\
\omega_2^2 &= -\frac{3\mu^2(1-k)^2}{1+k^2} \\
\omega_3^2 &= -\frac{3\mu^2(1+k)^2}{1+k^2} \\
\omega_{4,5}^2 &= -2\mu^2 \mp 2\mu^2\frac{\sqrt{1+14k^2+k^4}}{1+k^2}
\end{aligned} \tag{28}$$

where $\omega_2^2, \omega_3^2, \omega_4^2$are seen to be negative.
U_3:Here $N = 1, \kappa = k, z = x, \lambda = \omega^2 + 1$, and the eigenfunctions are

$$\psi_1 = cn(x,k), \psi_2 = dn(x,k), \psi_3 = sn(x,k) \tag{29}$$

with respectively the following eigenvalues

$$\omega_1^2 = 0,\ \omega_2^2 = k^2 - 1,\ \omega_3^2 = k^2 \tag{30}$$

We see that in this case the second eigenvalue is negative. We also observe that in each case the number of negative eigenvalues is odd and some eigenvalues merge with others in the limit $k^2 \to 1$. A negative eigenvalue implies,of course, that the corresponding configuration is a saddle point. We mention finally that the supersymmetrised versions of the three models discussed here have also been investigated[12].

3. Nonvacuum instantons and tunneling

We now consider the calculation of the level-splitting for the double-well potential by summing contributions originating from nonvacuum instantons and corresponding nonvacuum instanton– anti-instanton pairs[8]. An analogous calculation can be performed for the sine-Gordon potential[10]. Finally we consider the limiting cases of high and low energies, high meaning here energies approaching the top of the tunneling barrier.

We consider a scalar field ϕ in (1+0)-dimensions with mass = 1 and Lagrangian

$$\mathcal{L} = \frac{1}{2}(\frac{d\phi}{dt})^2 - V(\phi) \tag{31}$$

with potential

$$V(\phi) = \frac{\mu^2}{2a^2}(\phi^2 - a^2)^2 \tag{32}$$

Integrating the classical equation we obtain with $\tau = it$

$$\frac{1}{2}(\frac{d\phi_c}{d\tau})^2 - V(\phi_c) = -E_c \tag{33}$$

Integrating we obtain

$$\phi_c = \frac{akb(k)}{\mu} sn[b(k)(\tau + \tau_0)] \tag{34}$$

where we have suppressed the elliptic modulus k. The solution ϕ_c has been dubbed "periodic instanton" [3], "sphaleron" [4] and "bounce" [13]. It is convenient to introduce a new parameter u defined by

$$k^2 = \frac{1-u}{1+u} \tag{35}$$

with $u = \frac{\sqrt{2E_c}}{a\mu}$ and $b(k) = \mu(\frac{2}{1+k^2})^{\frac{1}{2}}$. The Jacobian elliptic function sn has period $\mathcal{T} = 4nK(k)$ for $n = 1, 2, 3, ...$ Setting $b(k)T = K(k)$ we can define as the analogue of the topological charge the quantity

$$Q = \frac{1}{2a}[\phi_c(T) - \phi_c(-T)] = k\sqrt{\frac{2}{1+k^2}} \tag{36}$$

We consider the half period part of the solution from $\tau = -T$ to $\tau = +T$ (and so with $\tau_0 = 0$) as the trajectory of the nonvacuum instanton (as we prefer to call it).We are interested in the transition amplitude $A_{+,-}$ for the transition from one side of the central barrier of the double–well potential to the other. We let $|E>_{\pm}$ be the eigenstates of the same energy E_0 in the two wells (with minima $\phi_{\pm}$) if the presence of the other well is ignored. The finite height of the potential barrier in between splits the $\pm$ degeneracy so that the eigenstates become the odd and even states

$$|E>_{o,e}=\frac{1}{\sqrt{2}}[|E>_{+}\mp|E>_{-}] \tag{37}$$

with eigenvalues $E_o \pm \frac{1}{2}\triangle E$. The desired amplitude then becomes

$$A_{+,-}={}_{+}<E|\exp(-2HT)|\,E>_{-}=-\exp(-2E_0T)\sinh(T\triangle E) \tag{38}$$

The problem is to calculate the shift $\triangle E$.We do this with the help of the path-integral method. In this case the amplitude $A_{+,-}$ can be written

$$A_{+,-}=\int\psi_{E+}(\phi_f)\psi_{E-}(\phi_i)K(\phi_f,\tau_f;\phi_i,\tau_i)d\phi_f d\phi_i \tag{39}$$

where $\tau_f-\tau_i=2T$. The kernel K is given by the Feynman path-integral

$$K(\phi_f,\tau_f;\phi_i,\tau_i)\equiv<\phi_f,\tau_f;\phi_i,\tau_i>=\int\mathcal{D}[\phi]exp[-S] \tag{40}$$

where the Euclidean action S is given by

$$S=\int_{\tau_i}^{\tau_f}[\frac{1}{2}(\frac{d\phi}{d\tau})^2+V(\phi)]d\tau \tag{41}$$

We write the amplitude as a sum over nonvacuum instanton contributions, i.e.

$$A_{+,-}=\sum_{n=0}^{\infty}A_{+,-}^{(2n+1)} \tag{42}$$

where $A_{+,-}^{(2n+1)}$ denotes the contribution of the amplitude for one nonvacuum instanton and n nonvacuum instanton-anti-instanton pairs.

We consider first the one-nonvacuum–instanton contribution. We set

$$\phi(\tau)=\phi_c(\tau)+X(\tau) \tag{43}$$

where $X(\tau)$ is the deviation of $\phi(\tau)$ from the classical trajectory ϕ_c with fixed end-points $X(\tau_i)=X(\tau_f)=0$. We also write $S=S_c+\delta S$ where

$$S_c=W(\phi(\tau_f),\phi(\tau_i),E)+2E_cT \tag{44}$$

Here

$$W\longrightarrow\frac{4\mu a^2}{3}(1+u)^{\frac{1}{2}}[E(k)-uK(k)] \tag{45}$$

in the limits $\phi(\tau_i)=\phi_i\rightarrow-\tilde{a}$ and $\phi(\tau_f)=\phi_f\rightarrow\tilde{a}$, where $-\tilde{a}$ and $\tilde{a}$ are the two middle turning points (where $\tau_i=-T,\tau_f=T$) ($-\tilde{a}'$ and $\tilde{a}'$ are the two other outside turning points) and $E(k)$ is the complete elliptic integral of the second kind.In the one-loop approximation we have

$$\delta S=\int_{\tau_i}^{\tau_f} d\tau[\frac{1}{2}(\frac{dX}{d\tau})^2+X^2(3\frac{\mu^2}{a^2}{\phi_c}^2-\mu^2)]\equiv\int_{\tau_i}^{\tau_f}(X\hat{M}X)d\tau \tag{46}$$

where $\hat{M}$ is the small fluctuation operator evaluated at the classical configuration,i.e.

$$\hat{M}=-\frac{1}{2}\frac{d^2}{d\tau^2}+\mu^2(\frac{3{\phi_c}^2}{a^2}-1) \tag{47}$$

The kernel K defined above is then given by

$$K\equiv exp[-S_c].I \tag{48}$$

where

$$I=\int_{X(\tau_i)=0}^{X(\tau_f)=0}\mathcal{D}\{X\}exp(-\delta S) \tag{49}$$

We now have to evaluate the integral I. The usual analysis starts as follows. One expands the fluctuation X in terms of the complete set of eigenfunctions ψ_n of the small fluctuation operator $\hat{M}$ with $\hat{M}\psi_n=\omega_n^2\psi_n$. Then $X=\sum_n C_n\psi_n$ and

$$\begin{aligned} I &= \int\mathcal{D}\{C_n\}\det(\frac{\partial X}{\partial C_n})\exp(-\sum_n C_n^2\omega_n^2) \\ &= \det(\frac{\partial X}{\partial C_n})\prod_n[\frac{\pi}{\omega_n^2}] \\ &= \det(\frac{\partial X}{\partial C_n})\frac{\pi}{\det\hat{M}} \end{aligned} \tag{50}$$

Here one expects a problem with the negative eigenvalue of the small fluctuation operator obtained above. However, the boundary conditions $X(\tau_i)=X(\tau_f)=0$ remove this negative eigenvalue ω_4^2. This can be seen as follows. The two boundary conditions imply the equations

$$-C_2k'+C_4(1-\frac{\triangle_1+\triangle_2}{3k^2})+C_5(1-\frac{\triangle_1-\triangle_2}{3k^2})=0 \tag{51}$$

and

$$C_2k'+C_4(1-\frac{\triangle_1+\triangle_2}{3k^2})+C_5(1-\frac{\triangle_1-\triangle_2}{3k^2})=0 \tag{52}$$

where $\triangle_1=1+k^2$ and $\triangle_2=\sqrt{1-k^2(1-k^2)}$. These equations imply $C_2=0$ and

$$C_4(1-\frac{\triangle_1+\triangle_2}{3k^2})=-C_5(1-\frac{\triangle_1-\triangle_2}{3k^2}) \tag{53}$$

From the definition of the coefficients C_n we obtain

$$C_4 = \int X\psi_4^\star d\tau = \int sn^2[b(k)\tau]Xd\tau - \int Xd\tau\frac{\triangle_1 + \triangle_2}{3k^2} \tag{54}$$

and

$$C_5 = \int X\psi_5^\star d\tau = \int sn^2[b(k)\tau]Xd\tau - \int Xd\tau\frac{\triangle_1 - \triangle_2}{3k^2} \tag{55}$$

Here $\int_{\tau_i=-T}^{\tau_f=T} X(\tau)d\tau = 0$ if we require the fluctuation to be orthogonal to the zero mode $\frac{d\phi_c}{d\tau}$, i.e.$\int_{-T}^{T} \frac{d\phi_c(\tau)}{d\tau}X(\tau)d\tau = 0$, so that X has to satisfy $X(\tau) = -X(-\tau)$. The above equations therefore imply that $C_4 = C_5$. The previous equation therefore implies that $C_4 = C_5 = 0$. Using the shift–method–transformation[14] we can set

$$X(\tau) = Y(\tau) + N(\tau)\int_{\tau_i}^{\tau} \frac{\dot{N}(\tau')}{N^2(\tau')}Y(\tau')d\tau' \tag{56}$$

with

$$N(\tau) \equiv \frac{d\phi_c}{d\tau} = \frac{kb^2(k)a}{\mu}cn[b(k)\tau]dn[b(k)\tau] \tag{57}$$

and

$$I = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{N(\tau_i)N(\tau_f)\int_{\tau_i}^{\tau_f}\frac{d\tau}{N^2(\tau)}}\right]^{\frac{1}{2}} \tag{58}$$

This expression is singular at the turning point values of τ_f and τ_i, since the "velocities"expressed by the zero modes vanish at the turning points;this is different from the case of vacuum instantons or vacuum bounces in which case the turning points can be reached only asymptotically. Our procedure here is to use the end-point integrations in the expression for the transition amplitude in order to smooth out the singularities in I. One can show that in approaching the limits $\tau_{f,i} \to \pm T$ the following expression holds formally[8]

$$\frac{\partial^2 S_c}{\partial\phi^2(\tau_f)} = \left[N(\tau_f)N(\tau_i)\int_{\tau_i}^{\tau_f}\frac{d\tau}{N^2(\tau)}\right]^{-1} \tag{59}$$

In the integral

$$A_{+,-} = \int \psi_{E+}(\phi_f)\psi_{E-}(\phi_i)\exp\left[-S_c(\phi_f,\phi_i,\tau_f-\tau_i)\right]I(\phi_f,\phi_i,\tau_f,\tau_i)d\phi_f d\phi_i \tag{60}$$

we replace the wave functionals by their respective WKB approximations in the barrier (from $\phi(-T) = -\tilde{a}$ to $\phi(T) = \tilde{a}$) , i.e.we set

$$\begin{aligned}
\psi_{E+}(\phi_f) &= \frac{C_+\exp\left[-\Omega(\phi_f)\right]}{\sqrt{N(\tau_f)}} \equiv \frac{C_+\exp\left[-\int_{\phi_f}^{\tilde{a}}\dot{\phi}d\phi\right]}{\sqrt{N(\tau_f)}} \\
\psi_{E-}(\phi_i) &= \frac{C_-\exp\left[-\Omega(\phi_i)\right]}{\sqrt{N(\tau_i)}} \equiv \frac{C_-\exp\left[-\int_{-\tilde{a}}^{\phi_i}\dot{\phi}d\phi\right]}{\sqrt{N(\tau_i)}}
\end{aligned} \tag{61}$$

where the constants C_+, C_- can be calculated from integrals over the two domains $(-\tilde{a}', -\tilde{a})$ and $(\tilde{a}, \tilde{a}')$ neighbouring the barrier and are given by

$$C_+ = C_- = \left[\frac{\frac{1}{2}}{\int_{\tilde{a}}^{\tilde{a'}} \frac{d\phi}{\sqrt{2(E-V)}}}\right]^{\frac{1}{2}} = \left[\frac{\mu\sqrt{1+u}}{2K(k')}\right]^{\frac{1}{2}} \equiv C \tag{62}$$

We are interested in the limits

$$\phi_i \to \phi(-T) \equiv -\tilde{a}, \phi_f \to \phi(T) \equiv \tilde{a}' \tag{63}$$

We therefore use Taylor expansion in ϕ_f around a nearby point $\phi(\tau_0)$ so that

$$\begin{aligned} \exp\left[-S_c(\phi_f, \phi_i, \tau_f - \tau_i)\right] &= \exp\left[-S_c(\phi(\tau_0), \phi_i, \tau_0 - \tau_i)\right] \cdot \\ &\cdot \exp\left[-\frac{1}{2}(\frac{\partial^2 S_c}{\partial {\phi_f}^2})_{\phi_f=\phi(\tau_0)}(\phi_f - \phi(\tau_0))^2\right] \end{aligned} \tag{64}$$

and

$$\exp\left[-\Omega(\phi_f)\right] = \exp\left[-\frac{1}{2}(\frac{\partial^2 \Omega(\phi_f)}{\partial \phi_f^2})_{\phi_f=\phi(\tau_0)}(\phi_f - \phi(\tau_0))^2\right] \tag{65}$$

in the Gaussian approximation. We also have with $\phi(\tau) = -\phi(-\tau)$

$$\begin{aligned} (\frac{\partial^2 S_c}{\partial \phi_f^2})_{\phi_f=\phi(\tau_0)} &= -\frac{\dot{N}(\phi(\tau_0))}{N(\phi(\tau_0))} + \frac{1}{N^2(\phi(\tau_0)) \int_{-\tau_0}^{\tau_0} \frac{d\tau}{N^2(\tau)}} \\ (\frac{\partial^2 \Omega}{\partial \phi_f^2})_{\phi_f=\phi(\tau_0)} &= +\frac{\dot{N}(\phi(\tau_0))}{N(\phi(\tau_0))} \end{aligned} \tag{66}$$

(here the Ω contribution is seen to be one degree less divergent than that of S_c) and the relations

$$\begin{aligned} S_c(\phi(T), \phi(-T), 2T) &= W(\phi(T), \phi(-T), E_c) + 2E_cT \\ W &= \frac{4\mu a^2}{3}(1+u)^{\frac{1}{2}}[E(k) - uK(k)] \end{aligned} \tag{67}$$

and write $\frac{d\phi_i}{N(\phi_i(\tau))} = d\tau$.The integration with respect to ϕ_f becomes Gaussian and can be carried out first. Then the limit $\tau_0 \to T$ is taken. Finally integrating with respect to τ from $-T$ to T we obtain for the amplitude in the one-loop approximation

$$A_{+,-} = 2TC^2 \exp\left[-W\right] \exp\left[-2E_cT\right] \equiv S_{+,-} \exp\left[-2E_cT\right] \tag{68}$$

(For comparison with the S-matrix calculation to be mentioned below we note here that the factor $\exp[-2E_cT]$ represents the free field evolution part (here in Euclidean time) so that in the limit of vacuum boundary conditions the remaining part $S_{+,-}$ can be looked at as the (here rather unconventional) S-matrix element in the one vacuum instanton approximation between the low-lying nth excited states in the two wells, i.e.$S_{+,-}\equiv 2TC^2\exp[-W]$ for $k^2\rightarrow 1$).

Proceeding similarly in the case of amplitude contributions stemming from one nonvacuum instanton and respectively one or n nonvacuum instanton pairs we obtain

$$\begin{aligned} A^{(3)}_{+,-} &= \int_{-T}^{T} d\tau_1 \int_{-T}^{\tau_1} d\tau_2 \int_{-T}^{\tau_2} d\tau C^3 \exp\left[-3W\right]\exp\left[-2E_cT\right] \\ &= \frac{(2T)^3}{3!}C^3\exp\left[-3W\right]\exp\left[-2E_cT\right] \\ A^{(2n+1)}_{+,-} &= \frac{(2T)^{2n+1}}{(2n+1)!}C^{2n+1}\exp\left[-(2n+1)W\right]\exp\left[-2E_cT\right] \end{aligned} \tag{69}$$

Summing over n and comparing the expression with the expression for $A_{+,-}$ at the beginning, we obtain the WKB level-splitting formula

$$\triangle E = \frac{\mu\sqrt{1+u}}{K(k')}\exp\left[-W\right] \tag{70}$$

We define as weak coupling those values of μa^2 which are such that $g^2\equiv\frac{1}{\mu a^2}<<1$. In this limit the two minima of the potential are widely separated and the central barrier becomes very high. In the following we shall consider high energies as those associated with high quantum states. We therefore replace E_c by the oscillator approximation $E_n=(n+\frac{1}{2})\omega$ where $\omega=2\mu$. In that approximation we have $V(\phi)\simeq 2\mu^2(\phi-\phi_{\pm})^2, \phi_{\pm}=\pm a$ and $\int_{\tilde{a}}^{\tilde{a'}}\sqrt{2(E-V)}d\phi=(n+\frac{1}{2})\pi$. We consider separately the cases of low and high energies.

Low energies. We have $u=\sqrt{2E_c}/a\mu=2g\sqrt{n+\frac{1}{2}}, k^2=\frac{1-u}{1+u}=1-k'^2$. The appropriate expansions (for small u or k'^2) of the elliptic integrals are

$$\begin{aligned} E(k) &= 1+\frac{1}{2}k'^2\{\ln(\frac{4}{k'})-\frac{1}{2}\}+... \\ K(k) &= \ln(\frac{4}{k'})+\frac{1}{4}k'^2\{\ln(\frac{4}{k'})-1\}+... \end{aligned} \tag{71}$$

With these expansions we obtain

$$W=\frac{4}{3g^2}+2(n+\frac{1}{2})\ln(\frac{g}{4})+(n+\frac{1}{2})\ln(n+\frac{1}{2})-(n+\frac{1}{2}) \tag{72}$$

and hence

$$\triangle E_n=\frac{2\mu}{\pi}[\frac{2^4 e}{g^2(n+\frac{1}{2})}]^{n+\frac{1}{2}}e^{-\frac{4}{3g^2}} \tag{73}$$

Using the Stirling relation

$$[\frac{e}{n+\frac{1}{2}}]^{n+\frac{1}{2}}\approx\frac{\sqrt{2\pi}}{n!}\frac{e^{\frac{1}{2}}}{(1+\frac{1}{2n})^{n+\frac{1}{2}}}\approx\frac{\sqrt{2\pi}}{n!} \tag{74}$$

we see that

$$\triangle E_n = \frac{2\sqrt{2}\mu}{n!\sqrt{\pi}}[\frac{2^4}{g^2}]^{n+\frac{1}{2}}e^{-\frac{4}{3g^2}} \tag{75}$$

This expression agrees, as expected,with the WKB-equivalent result[19] in the low energy limit, i.e. for $u = 0$. We also observe that under the condition $g^2(n+\frac{1}{2}) << 1, k' \to 0$ the amplitude A as well as $\triangle E_n$ grow with energy (due to the second term in W above).

These low energy results agree with those of Bachas et al.[15] who estimated the S-matrix element $S_{n\to n}$ for the vacuum instanton transition from the nth asymptotic oscillator state on one side of the barrier to the nth asymptotic oscillator state on the other side using the LSZ procedure. Thus, defining $\phi_\pm$ by

$$\phi_\pm := \pm a - \phi_c(x) \to 0, \quad x \to \pm\infty \tag{76}$$

we can construct effective boson creation and annihilation Heisenberg operators $\hat{a}^\dagger_\pm, \hat{a}_\pm$ in the wells "+" and "−" given by

$$\hat{a}_\pm = -\frac{i}{\sqrt{\mu}}e^{2i\mu t}\overleftrightarrow{\frac{\partial}{\partial t}}\phi_\pm(x=it) \stackrel{\pm\infty}{\longrightarrow} 4a\sqrt{\mu} \tag{77}$$

These operators are such that in the asymptotic limits they correspond to the harmonic oscillator operators $\hat{a}$ in

$$\phi(t) = \frac{1}{\sqrt{2\omega}}[\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}] \tag{78}$$

with $\omega = 2\mu$. The transition amplitude through the central barrier induced by a vacuum instanton is then

$$\begin{aligned} {}_+ < 1, out|1, in >_- &= < 0|\hat{a}_+\hat{a}^\dagger_-|0 > \\ &= \lim_{t\to-\infty, t'\to+\infty}(\frac{i}{\sqrt{\mu}}e^{2i\mu t'}\overleftrightarrow{\frac{\partial}{\partial t'}})(\frac{-i}{\sqrt{\mu}}e^{-2i\mu t}\overleftrightarrow{\frac{\partial}{\partial t}})G \\ &= (4a\sqrt{\mu})^2 I \\ &= S_{+,-} \end{aligned} \tag{79}$$

where $G =< 0|\phi_+(x')\phi_-(x)|0 >$ and I is the vacuum instanton tunneling propagator

High energies. High energies here means those approaching the top of the barrier,the latter generally being called the sphaleron mass. In the present context this implies $k \to 0$ and $E \to \frac{a^2\mu^2}{2}$. In this limit

$$W \approx \frac{\sqrt{2}\pi}{g^2}k^2 \to 0 \tag{80}$$

and

$$\frac{\mu\sqrt{1+u}}{K(k')} \approx \frac{\sqrt{2}\mu}{\ln(\frac{4}{k})} \to 0 \tag{81}$$

Thus at these energies the amplitude and the splitting $\triangle E$ are no longer suppressed by the typical vacuum instanton factor $\exp(-\frac{4}{3g^2})$ but by the prefactor $\frac{\mu\sqrt{1+u}}{K(k')} \rightarrow 0$.Thus in the high energy limit

$$A \sim \frac{\mu\sqrt{1+u}}{K(k')} \exp\left[-W\right] \rightarrow 0 \tag{82}$$

Similar results can be expected for various other potentials, in particular for the sine-Gordon potential which has been considered in the literature[10].

4. Nonvacuum bounces and tunneling

With methods similar to those described above one can consider an amplitude in the neighbourhood of a bounce which is the classical configuration in the case of the inverted double–well potential [16,17,7,9] and calculate the imaginary part of the energy. In this case the small fluctuation equation has three negative eigenmodes, of which two do not contribute to the amplitude in view of boundary conditions and the remaining one is responsible for the imaginary part. For further details we refer to the literature[7,9].

5. The Bogomolny-Fateyev relation and conclusions

In the case of systems with more than one classical ground state, the classical vacuum state chosen as the perturbation theory vacuum in general does not coincide with the true quantum mechanical ground state. Thus although the exact ground state is stable, the corresponding perturbation theory vacuum is only metastable due to the possibility of tunneling to the other vacuum states. For example, the double-well potential

$$V(\phi) = \frac{\lambda^2}{2}(\phi^2 - \frac{1}{\lambda^2})^2 \tag{83}$$

leads to the level splitting calculated above. But the following distorted form of the potential

$$\begin{aligned} V(\phi) &= \frac{1}{2}\lambda^2(\phi^2 - \frac{1}{\lambda^2})^2 \ \text{ for } \ \phi \leq \frac{1}{\lambda} \\ &= -\frac{1}{2}\lambda^2(\phi^2 - \frac{1}{\lambda^2})^2 \ \text{ for } \ \phi > \frac{1}{\lambda} \end{aligned} \tag{84}$$

results in an imaginary part of the energy, $\Im m E$, for a "real" metastable ground state. The shape of the potential is then similar to that of a cubic potential which is the easiest example of a potential with a bounce[18]. (The unphysical shape of $V(\phi)$ for $\phi > \frac{1}{\lambda}$ is irrelevant here;equivalently one could assume a flat behaviour or even a rising one far away so that in any case the tunneling particle would behave as free over some distance sufficiently far away from $\phi = \frac{1}{\lambda}$). Bogomolny and Fateyev[11] observed that (to leading order)

$$\triangle E = 2\pi i(\delta E)^2 \tag{85}$$

where $\triangle E$ is the discontinuity of the ground state energy at the cut $\lambda^2 \geq 0$ with $\triangle E = 2i\Im m E$ while δE is the instanton contribution to the real part of the ground state energy (i.e. for the double-well potential), namely the level shift due to quantum tunneling. The Bogomolny-Fateyev relation has been verified and extended to excited states by comparing the explicit expressions of the two quantities for both the double-well potential and the periodic potential and their appropriately distorted versions in the above sense [19]. The formula serves as a crucial test of the validity of calculating quantum tunneling effects with nonvacuum instantons and nonvacuum bounces.In the considerations above the level splitting for the excited states of the double-well potential was obtained with nonvacuum instantons . The classical solution which extremises the Euclidean action is

$$\phi_c(\tau) = \frac{kb(k)}{\lambda} sn[b(k)\tau, k] \tag{86}$$

where

$$k^2 = \frac{1-u}{1+u}, \; u = \lambda\sqrt{2E}, \; b(k) = [\frac{2}{1+k^2}]^{\frac{1}{2}} \tag{87}$$

The Jacobian elliptic function $sn(z,k)$ has period $\mathcal{T} = 4nK(k)$. In the calculation of the level splitting the solution for a half period is regarded as a nonvacuum instanton configuration. The level shift (i.e. half of the level splitting) is obtained as

$$\delta E = B\exp[-W'] \tag{88}$$

where the prefactor B is given by

$$B = \frac{[1+u]^{\frac{1}{2}}}{2K(k')} \tag{89}$$

and W' by

$$W' = \frac{4}{3\lambda^2}(1+u)^{\frac{1}{2}}[E(k) - uK(k)] \tag{90}$$

If we regard the configuration over the full period (effectively a nonvacuum instanton–anti-instanton pair) as a bounce configuration which returns to its original position (such a consideration has also been discussed by Hawking and Ross [1])we can write it

$$\tilde{\phi}_c(\tau) = \frac{kb(k)}{\lambda} sn[b(k)\tau + K(k), k] \tag{91}$$

so that this is zero at $b(k)\tau = -K(k)$, i.e. at $\tau = -T$, and at $b(k)\tau = K(k)$, i.e. at $\tau = +T$ (since snu vanishes for $u = 0, 2K(k)$). This motion is allowed for a physical system with the distorted potential above. The motion of the bounce starts at $\tau = -2T$ and ends at $\tau = +2T$. Since $sn[u + K(k), k] = cn[u,k]/dn[u.k]$ we see that this bounce is an even function of u. The zero mode ψ_0, i.e. the derivative of the bounce (which corresponds classically to its velocity), is therefore odd, i.e. a wave function with eigenvalue zero which passes through zero at $u = 0$. Thus the ground state eigenfunction of the corresponding fluctuation equation must have a negative eigenvalue. This negative eigenvalue is the one which is responsible for the instability of the configuration. The imaginary part of the energy is now obtained the way we obtained it elsewhere[7]. We then have

$$\Im m E = B\exp[-2W'] \tag{92}$$

and so

$$\Im m E = \frac{1}{B}(\delta E)^2 \tag{93}$$

In the low energy limit $\frac{1}{B} = \pi$, and the Bogomolny–Fateyev relation holds exactly. We conclude, therefore, that the consideration of classical finite energy nonvacuum configurations is applicable to numerous tunneling phenomena. Given δE one can use the Bogomolny– Fateyev relation (by inserting $\Im m E$ into the appropriate moment integral) in order to derive the behaviour of a large order term of the perturbation expansion (in Borel nonsummable cases) of the eigenvalue E in the case with splitting. The Bogomolny–Fateyev relation has also been observed in other related contexts[20] and computationally[21]. The contribution of sphalerons to the large-order behaviour of perturbation expansions in quantum mechanical models derived from nonlinear sigma models with symmetry–breaking potentials has been investigated recently by Rubakov and Shvedov[22]. We also mention that it is possible to develop a BRST-invariant approach to quantum mechanical tunneling which avoids the degeneracy problem of ill defined path integrals due to zero modes. So far we have applied this method only to the sine–Gordon potential[23]. Finally we remark that sphaleron configurations analogous to those discussed here arise also in other theories such as Skyrme-like models[24] and Yang-Mills and sigma-model theories[25].

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