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CRITICAL O(N) - VECTOR NONLINEAR SIGMA - MODELS: A RÉSUMÉ OF THEIR FIELD STRUCTURE

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ABSTRACT

The classification of quasi - primary fields is outlined. It is proved that the only conserved quasi - primary currents are the energy - momentum tensor and the O(N)-Noether currents. Derivation of all quasi - primary fields and the resolution of degeneracy is sketched. Finally the limits $d = 2$ and $d = 4$ of the space dimension are discussed. Whereas the latter is trivial the former is only almost so. (To appear in the Proceedings of the XXII Conference on Differential Geometry Methods in Theoretical Physics, Ixtapa, Mexico, September 20-24, 1993)

1 SOME GENERAL REMARKS

We have studied only a very special example of a critical field theory at dimensions $2 < d < 4$. Nevertheless we believe that the results are relevant for many critical field theories, in particular sigma models in a neighbourhood of a free theory. Our neighbourhood is defined by a $\frac{1}{N}$ expansion.

In this résumé we extract results from a series of papers on this subject [1-7] published by us in the last three years, and from earlier literature on conformal field theory in general [8, 9] or conformal sigma models in particular [10]. These results may have different status but we condense them equally into "theorems" which should not be considered as mathematical theorems but as tested conjectures. General statements of quantum field theory and group theory are thus mixed up with conclusions from low order perturbative expansions. Let us start with such a theorem which certainly disappoints many of the readers:

Theorem 0: Almost none of the structures of conformal field theory at $d = 2$ can be rediscovered at $2 < d < 4$.

2 DEFINITION OF THE MODEL

We start with the partition function

$$\mathcal{Z} \;=\; \int D[\vec{S}]D[\alpha]\exp\left\{-\int dx\left[\frac{1}{2}(\partial_\mu \vec{S})^2(x) + z^{\frac{1}{2}}\alpha(x)(\vec{S}^2(x)-1)\right]\right\} \tag{1}$$

where

$$\begin{aligned} \vec{S} &: \;\text{O}(N)\text{- vector, O}(d)\text{ - scalar;} \\ \alpha &: \;\text{O}(N)\text{- O}(d)\text{ - scalar;} \\ d = 2\mu &: \;\text{space - time dimension} \end{aligned}$$

If $\vec{S}$ and α are normalized in a standard fashion

$$\left\langle S_a(x)S_b(0)\right\rangle \;=\; \delta_{ab}\left(x^2\right)^{-\alpha} \tag{5}$$

$$\left\langle \alpha(x)\alpha(0)\right\rangle \;=\; \left(x^2\right)^{-\beta} \tag{6}$$

the critical coupling constant z becomes a computable function of N, and

$$N \to \infty : \qquad z \;=\; \mathcal{O}(\tfrac{1}{N}) \tag{7}$$

The limit $N \to \infty$ is a free field limit

$$\lim_{N\to\infty} \vec{S}(x) \;=\; \vec{s}(x) \tag{8}$$

but $\vec{s}(x)$ possesses infinitely many components which leads to problems sometimes. A saddle point expansion of (0) gives the $\frac{1}{N}$ - expansion.

A critical theory such as this is conformally covariant. Operator product expansions (OPE) generate a field algebra $\mathcal{A}(\vec{S}, \alpha)$ of the two fundamental fields $\vec{S}$ and α which is associative and possesses a commutation property connected with the crossing behaviour of n - point functions. The building blocks of $\mathcal{A}(\vec{S}, \alpha)$ are the conformal or quasiprimary fields (qp - fields).

Theorem 1: All qp - fields belong to representations of the conformal group characterized by two quantum numbers only: δ, the scaling dimension under dilatations and l, the tensor rank under space - time rotations.

These are the elementary representations. In addition the qp - fields transform irreducibly under O(N). We ascribe to them a Young frame Y.

Consider the dimension δ_ϕ of the qp - field ϕ

$$\begin{aligned} \delta_\phi &= [\delta_\phi] + \eta(\phi) && (9) \\ [\delta_\phi] &: \text{the normal dimension} \\ \eta(\phi) = \mathcal{O}(\tfrac{1}{N}) &: \text{the anomalous dimension} && (11) \end{aligned}$$

By definition

$$[\delta_\phi] = p(\mu - 1) + q, \; p, q \in \mathbb{N}_0 \tag{12}$$

$$[\delta_S] = \mu - 1 \tag{13}$$

So we expect that in the limit $N \to \infty$ ϕ tends to a normal product of p fields $\vec{S}$ with not more than q derivatives (see below).

Each elementary representation $[\delta, l]$ of a qp - field possesses a dual representation $[\delta', l']$ (" shadow representation ")

$$\delta' = d - \delta + 2l \tag{14a}$$

$$l' = l \tag{14b}$$

The two - point functions

$$\left\langle \phi_{[\delta,l]}(x)\phi_{[\delta,l]}(0) \right\rangle \qquad \text{and} \qquad \left\langle \phi_{[\delta',l']}(x)\phi_{[\delta',l']}(0) \right\rangle$$

are as kernels and up to a normalization inverse to each other. An n - point function of $\phi_{[\delta,l]}$ is transformed into an n - point function of $\phi_{[\delta',l']}$ by amputation. Therefore we have

Theorem 2: The fields $\phi_{[\delta,l]}$ and $\phi_{[\delta',l']}$ are dynamically equivalent.

So from each pair $\phi_{[\delta,l]}$, $\phi_{[\delta',l']}$ we would like to choose only one representative as basis element of $\mathcal{A}(\vec{S}, \alpha)$. We will in fact be able to do that but in an unexpected fashion.

From

$$\begin{aligned}[\delta'] &= d - \Big(p(\mu-1)+q\Big) + 2l \\ &= (2-p)(\mu-1)+2-q+2l \end{aligned} \tag{17}$$

we see that the α - field can be considered as the shadow field of

$$\left(\vec{S}^2(x)\right)_{\text{ren.}} \tag{18}$$

since

$$p=2,\ q=0,\ l=0 \text{ implies } [\delta']=2 \tag{19}$$

Inspection of the action in (0) also suggests this interpretation of α.

Next we decompose q in (11) as

$$q\ =\ l+t\ =\ l+2r \qquad (t\text{: twist}) \tag{20}$$

where r is the number of α fields bound into ϕ at $N \to \infty$ and l is the number of derivatives. p, l and r (or t) serve as quantum numbers in a neighbourhood of $N \to \infty$.

3 CLASSES OF QP - FIELDS

Construction of the qp - fields goes by OPE and harmonic analysis. This automatically orders the qp - fields according to increasing dimensions δ. From the interpretation of the quantum numbers p, l, t in (11), (3) we can naturally expect these numbers to be bounded by

$$p \geq 0,\ l \geq 0,\ t \geq 0 \tag{21}$$

Infact, this is fulfilled by our construction. Most of the shadow fields are forbidden by (3) but a few of them are still permitted. We put all qp - fields with the same Y and p into a class (Y,p). A generic class looks graphically as Fig.1. Labels may be multiply occupied by qp -fields, which are distinguished by their anomalous dimensions (" degeneracy "). Some of the simplest classes look different indeed.

(A) The class $(\square, 1)$ containing the fundamental field $\vec{S}$. At $t=0$ there is only the scalar field $\vec{S}$. At the level $t=2$, $l=0$ we would expect the shadow field $\vec{S'}$ of $\vec{S}$. But it is not found, this level is empty. The level $t=4$, $l=2$ is twofold degenerate.

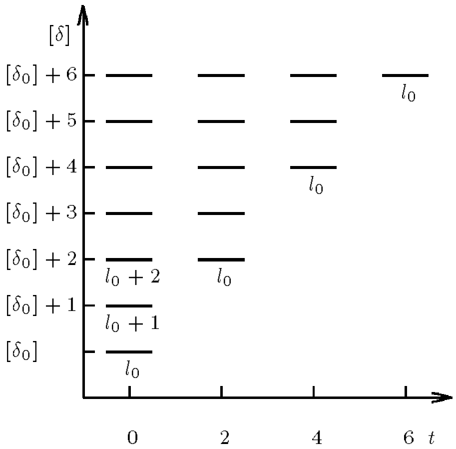


Figure 1: A generic class (Y, p)

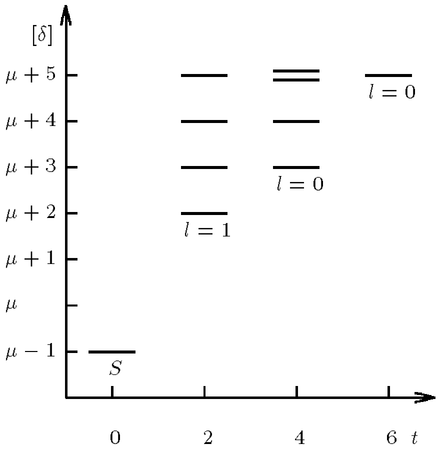


Figure 2: The class $(\Box, 1)$

(B) The class $(\emptyset, 0)$ containing the fundamental field α. At $t = 2$ we have only the α field (we start counting from $t = 2$ in this case). At $t = 4$ we have only even l and at $t \geq 6$, $l = 1$ is empty.

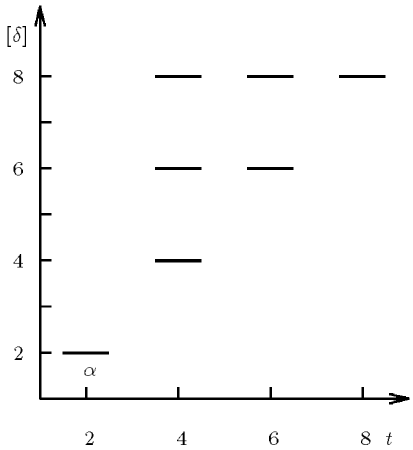


Figure 3: The class $(\emptyset, 0)$

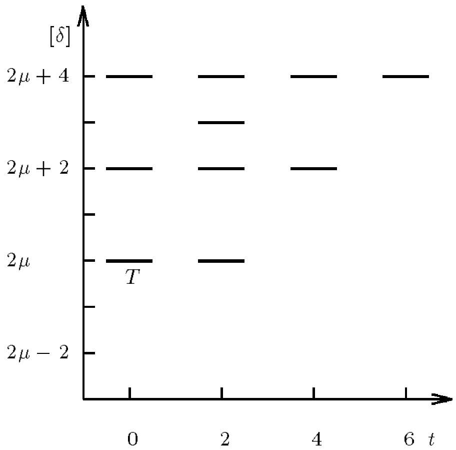


Figure 4: The class $(\emptyset, 2)$

Indeed, fusion of two qp - fields into a third one by OPE

$$A(x)B(0) \;=\; \left(x^2\right)^{\frac{1}{2}(\delta_C-\delta_A-\delta_B)} C(0) + \ldots \tag{22}$$

abbreviated as

$$A \otimes B \to C \tag{23}$$

is analogous with the formation of bound states. Two bosonic α's cannot be bound together to a state with odd l and for more than three α's $l = 1$ is also excluded by bose symmetrization.

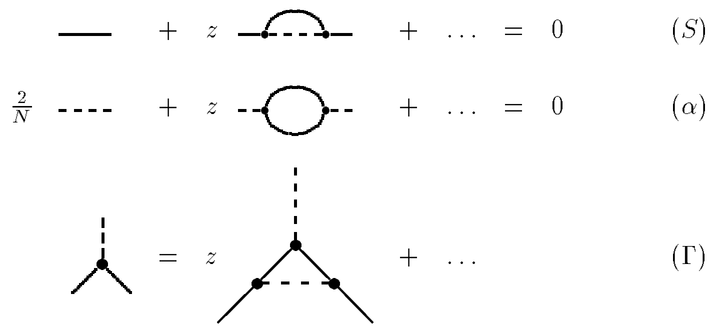
(C) The class $(\emptyset, 2)$ containing the energy - momentum tensor $T_{\mu\nu}$.

The level $t = 0$, $l = 0$ has been found unoccupied. The shadow field of α should appear on this level, or, according to our remark above, the field $\left(\vec{S}^2(x)\right)_{\text{ren.}}$. Thus the sigma - model constraint works and this field has been eliminated. The energy - momentum tensor field lies at

$$t = 0, \quad l = 2, \quad \delta = [\delta] = 2\mu = d \tag{24}$$

Looking through the classes more carefully, we recognize that the elimination of shadow fields has been completed.

In [4] we showed that elimination of the shadow field of α was directly related with a renormalization condition. Using dressed propagators and vertices (represented as Polyakov triangles $\bullet$) we have three such conditions



These three conditions suffice to determine $\eta(S)$, $\eta(\alpha)$ and z. A generalization of the argument in [4] shows validity of

Theorem 3: The requirement that one (two) shadow field(s) of the fundamental fields do(es) not show up replaces one (two) renormalization condition(s).

The status of the proof is still not satisfactory: $\mathcal{O}(\frac{1}{N^2})$ calculations at best. The theorem (" equivalence theorem ") is very powerful in practice.

The α - field produces a field algebra $\mathcal{A}(\alpha)$ which is a subalgebra of $\mathcal{A}(\vec{S},\alpha)$. It contains only O($N$)- scalars, among them the energy - momentum tensor $T_{\mu\nu}$

$$T_{\mu\nu} \in (\emptyset, 2) \tag{26}$$

Indeed

$$\alpha \otimes \alpha \to T \tag{27}$$

at $\mathcal{O}(\frac{1}{N})$, so p is not conserved at this order. Moreover

$$T \otimes T \to \alpha \tag{28}$$

so all $\mathcal{A}(\alpha)$ can be generated from T (at $d = 2$ T generates not only $Vir \times \overline{Vir}$ but W algebras as well!).

Theorem 4: The only conserved qp - currents in $\mathcal{A}(\vec{S},\alpha)$ are $T_{\mu\nu}$ and $J_{\mu,ab}$, the Noether currents of O(N)- symmetry from the class $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, 2)$.

Let us sketch the proof. Denote by $\#Y$ the number of blocks in the Young frame Y. Then

$$p - \#Y = 2n, \quad n \in \mathbb{N}_0 \tag{29}$$

This is obvious at $N = \infty$ since n is the number of contractions applied to the normal product of p vector fields $\vec{s}$. But in a neighbourhood of $N = \infty$ it remains valid due to standard arguments of harmonic analysis.

Next we use a classical lemma of conformal field theory ([5], Appendix A) for qp - fields which are symmetric tensors in spacetime. In fact for $2 < d < 4$ we have the situation of $d = 3$: symmetric tensors are sufficient. The lemma says that a qp - current is conserved if and only if

$$l \geq 1,\ \delta = [\delta] = 2\mu - 2 + l \tag{30}$$

i.e.

$$p = 2,\ l \geq 1,\ t = 0 \tag{31}$$

and

$$\eta(\phi) \;=\; 0 \tag{32}$$

This leaves as candidates the classes

$$(\square\!\square, 2), \quad (\boxminus, 2), \quad (\emptyset, 2) \tag{33}$$

In each case the $t = 0$ towers are nondegenerate with the following anomalous dimensions at leading order

$$\left.\begin{array}{lll} (\square\!\square, 2) & : & \dfrac{\eta(M_l)}{\eta(S)} \\ (\boxminus, 2) & : & \dfrac{\eta(J_l)}{\eta(S)} \end{array}\right\} \;=\; 2\frac{(l-1)(2\mu-2+l)}{(\mu-1+l)(\mu-2+l)} \qquad \begin{array}{ll} l & \text{even} \\ l & \text{odd} \end{array} \tag{34}$$

$$(\emptyset, 2) \;:\; \frac{\eta(T_l)}{\eta(S)} \;=\; \begin{cases} 0, \qquad (l = 2) \\ 2(l-1) + \\ \displaystyle\sum_{p=1}^{\frac{1}{2}l-2} \Big((p+1)!\Big)^2 \frac{(2\mu+1+p)_{l-4-2p}}{(2\mu+1)_{l-4}}, \\ \qquad\qquad (l \geq 4, \text{even}) \end{cases} \tag{35}$$

The curves for the expression (34) are presented in [6], Fig. 6. None of these functions changes sign. They vanish identically for J_1 and T_2 and are otherwise different from zero for all $2 < d < 4$. It is also important to guarantee that no empty levels are filled up at higher orders of $\frac{1}{N}$ or that degeneracy appears this way. The first is made sure by crossing symmetry, the second possibility can at present not be excluded.

4 FUSION

Each qp - field has a pedigree of fusion. The internal lines are arbitrary qp - fields which can be produced from the parents. This means that the fusion coefficients effective at a vertex must be nonzero:

$$f_{AB}^{C} \;\neq\; 0 \tag{37}$$

Theorem 5: Fusion coefficients vanish only if the corresponding Littlewood - Richardson coefficients of $\mathrm{O}(N)$ are zero or if this follows from a crossing symmmetry selection rule.

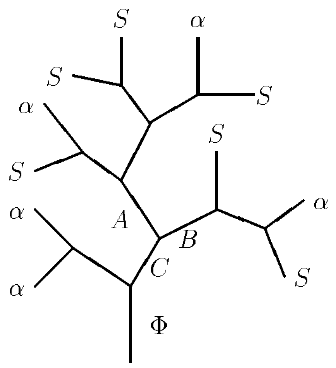


Figure 5 A pedigree of fusion

As an example let $A = B$ scalar and the Littlewood - Richardson coefficient be symmetric (antisymmetric) under exchange of A and B. Then odd (even) l are forbidden for C. Another example is the fusion

$$\alpha \otimes \vec{S} \tag{38}$$

which leads to any level of the class $(\square, 1)$ at $\mathcal{O}(\frac{1}{N})$ already, but in the class of degenerate levels only to one linear combination of qp - fields . So to resolve degeneracies we have to consider different pedigrees with the same final level.

Theorem 6: The qp - fields with $l = 0$ are never degenerate.

This corresponds to the uniqueness of a ground state in QM.

We introduce the concept of " dominant channel fusion " (DCF). This kind of fusion acts already at $\mathcal{O}(1)$ and produces scalar qp - fields of the type

$$(Y, p; [\delta], l) \;=\; (\square\!\square\!\!\!\!\boxed{}_{\,p}\,, p; p(\mu - 1) + 2r, 0) \tag{39}$$

from qp - fields of the same type. Let two such qp - fields with labels $\{p_1, r_1\}$, $\{p_2, r_2\}$ be given. The resulting field has labels $\{P, R\}$ with

$$\begin{aligned} P &= p_1 + p_2 && (40)\\ R &= r_1 + r_2 && (41) \end{aligned}$$

For DCF normal dimensions are additive and degeneracy does not occur. Only symmetric O(N) tensors are produced by definition. Pedigrees with DCF at each vertex produce a qp - field of type (7) which depends only on the numbers p of $\vec{S}$ fields and r of α fields entering and not on the form of the pedigree. In other words: DCF is abelian.

We denote the qp - fields (7) by $M_0^{\{p,r\}}$. Any qp - field on the level

$$\left(\square\!\square\!\rule{2em}{0.4pt}_{\,p}\;, p; p(\mu-1)+2r+l, l\right) \tag{42}$$

is denoted $M_{l,k}^{\{p,r\}}$ where k is introduced to take account of the degeneracy. We are interested in the fusion process

$$M_0^{\{p_1,r_1\}} \otimes M_0^{\{p_2,r_2\}} \rightarrow M_{l,k}^{\{p_1+p_2,r_1+r_2\}} \tag{43}$$

If we keep

$$P \;=\; p_1+p_2, \qquad R \;=\; r_1+r_2 \tag{44}$$

fixed but let p_1, r_1 run, we obtain different combinations of $M_{l,k}^{\{P,R\}}$ which can be resolved.

Technically one considers the four - point functions

$$\left\langle M_0^{\{p_1,r_1\}}(y_1)\, M_0^{\{p_2,r_2\}}(y_2)\, M_0^{\{p_1',r_1'\}}(y_3)\, M_0^{\{p_2',r_2'\}}(y_4)\right\rangle \tag{45}$$

with fixed

$$\begin{aligned} P &= p_1+p_2 &&= p_1'+p_2' \\ R &= r_1+r_2 &&= r_1'+r_2' \end{aligned} \tag{47}$$

On the one hand these four - point functions (7) are calculated from a $2(P+R)$ - point function involving $2P$ $\vec{S}$ fields and $2R$ α fields by OPE reduction via DCF. This is mainly a combinatorical task bringing in the " replica parameters " $p_1, r_1, p_2, r_2, p_1', r_1', p_2', r_2'$ and, at $\mathcal{O}(\frac{1}{N})$, the connected four - point functions

$$\left\langle SSSS\right\rangle_{\rm conn}, \qquad \left\langle \alpha\alpha\alpha\alpha\right\rangle_{\rm conn}, \qquad \left\langle \alpha S\alpha S\right\rangle_{\rm conn} \tag{48}$$

which are explicitly known [3,4,5]. Crossing between the unprimed factors exchanges

$$p_1 \leftrightarrow p_2, \qquad r_1 \leftrightarrow r_2 \tag{49}$$

so that we can use the crossing symmetric combinations

$$t_1 \;=\; r_1 r_2, \quad t_2 \;=\; p_1 p_2, \quad t_3 \;=\; p_1 r_2 + p_2 r_1 \tag{50}$$

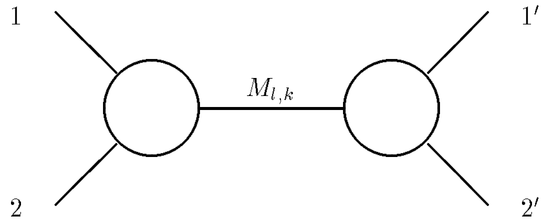


Figure 6 Conformal exchange amplitude

On the other hand we compare the four - point function (7) with conformal exchange amplitudes (this is an element of harmonic analysis). This allows us to extract expressions for

$$\sum_k f_{12}^{M_{l,k}} f_{1'2'}^{M_{l,k}} \tag{51}$$

and

$$\sum_k f_{12}^{M_{l,k}} f_{1'2'}^{M_{l,k}} \eta(M_{l,k}) \tag{52}$$

The fusion constants $f_{12}^{M_{l,k}}$ are functions of the replica parameters

$$f_{12}^{M_{l,k}} = F_{l,k}(p_1, r_1; p_2, r_2) \tag{53}$$

By a simultanous diagonalization procedure for the two expressions obtained for (6), (6) we can extract the fusion coefficients and the anomalous dimensions . The fusion coefficients are obtained in the form

$$f_{12}^{M_{l,k}} = \text{polynomial in the replica parameters giving } (-1)^l \text{ under crossing times an algebraic function depending homogenously on } t_1,\, t_2,\, t_3 \tag{54}$$

We have in fact solved the following cases [7]

$P = 0$, R arbitary > 0:	levels $0 \le l \le 6$ and $t = 2R$ in the class $(\emptyset, 0)$. Degeneracy sets in at $R \ge 4$ and $l \ge 4$,
$R = 0$, P arbitary > 0:	levels $0 \le l \le 6$ and $t = 0$ in the classes $(\square\square\cdots\square_{\,P}, P)$. Degeneracy sets in at $P \ge 4$ and $l \ge 4$.

In both cases the anomalous dimensions are

$$\frac{\eta(M_{l,k})}{\eta(S)} = \text{rational functions of } \mu \text{ at leading order.} \tag{55}$$

and the algebraic function in (53) reduces to a (nonhomogeneous) polynomial of either t_1 or t_2.

If $R\,P \neq 0$, degeneracy starts already at $R + P \geq 3$, $l \geq 2$. We resolved only the cases $0 \leq l \leq 3$. Moreover we find

$$\frac{\eta(M_{l,k})}{\eta(S)} = \text{algebraic (irrational) function of } \mu \text{ at leading order.} \tag{56}$$

Many infinite sequences of anomalous dimensions are known now and in these sequences we can study limits. Consider a tower of nondegenerate qp - fields $M_l^{\{P,R\}}$, P, R fixed, l running. Then in the DCF process (7) the pair of qp - fields on the left hand side is uniquely determined. At leading order in $\frac{1}{N}$ we find

$$\lim_{l\to\infty} \eta(M_l^{\{P,R\}}) = \eta(M_0^{\{p_1,r_1\}}) + \eta(M_0^{\{p_2,r_2\}}) \tag{57}$$

Instead in the case of degeneracy

$$\eta(M_l^{\{P,R\}}) = \mathcal{O}\Big(\frac{l^2}{N}\Big) \tag{58}$$

which makes the $\frac{1}{N}$ expansion asymptotic only if $N \gg l^2$. We could also think of keeping l fixed and letting P, R run. Then

$$\eta(M_l^{\{P,R\}}) = \mathcal{O}\Big(\frac{1}{N} \times \text{second order polynomial in } P \text{ and } R\Big) \tag{59}$$

imposing a similiar restriction on N.

We emphasize that our method of constructing the states $M_l^{\{P,R\}}$ by forcing all internal qp - fields of the pedigree to have tensor rank zero may be too restrictive for large l. In a forthcoming article we will study an alternative algorithm which remains correct at large l as well.

5 THE LIMITS $D \searrow 2$ AND $D \nearrow 4$

For any $2 < d < 4$ the limit $N \to \infty$ leads to a free field theory. In this limit each qp - field $\phi \in \mathcal{A}(\vec{S}, \alpha)$ possesses a corresponding qp - field φ in the free field algebra $\mathcal{A}_0(\vec{s})$. In Green functions involving α fields we may first amputate them and perform the limit afterwards. At the boundaries $d = 2$,

	$d=2$	$d=4$
z_1	0	2
z_2	0	2
$\eta_1(S)$	1	2
$\eta_2(S)$	1	2
$\eta_1(\phi),\ \phi \neq S$	1	1

Table 1 Table of zero orders.

$d = 4$ the behaviour of coupling constant and critical indices

$$\eta(\phi) = \sum_{k=1}^{\infty} \frac{\eta_k(\phi)}{N^k} \tag{60}$$

$$\eta_1(S) = 2\,\frac{\sin \pi\mu}{\pi}\,\frac{\Gamma(2\mu-2)}{\Gamma(\mu+1)\Gamma(\mu-2)} \tag{61}$$

$$z = \sum_{k=1}^{\infty} \frac{z_k}{N^k} \tag{62}$$

concerning their zero orders in d is listed in table 1.

All critical exponents vanish at both limits. These limits are therefore connected with free field theory.

At $d = 4$ we obtain a free field theory in the trivial sense that

$$\begin{aligned} \lim_{d \nearrow 4} \vec{S}(x) &= \vec{s}(x), \quad \triangle \vec{s}(x) \;=\; 0 \\ \vec{s}(x) &: \quad N \text{ - component O}(N) \text{ - vector field} \end{aligned} \tag{64}$$

As a test we can calculate the limit of $\left\langle \alpha \vec{S} \alpha \vec{S} \right\rangle$ after amputation. This limit $d = 4$ is assumed fieldwise and is an isomorphism of field algebras in the straightforward sense. Let $A, B, C \in \mathcal{A}(\vec{S}, \alpha)$

$$A(x)\,B(0) = \left(x^2\right)^{\frac{1}{2}(\delta_C - \delta_A - \delta_B)(\mu)} f_{AB}^{C}(\mu)\,C(0) + \ldots \tag{65}$$

Then if a, b, c are the corresponding free fields

$$a(x)\,b(0) = \left(x^2\right)^{\frac{1}{2}(\delta_C - \delta_A - \delta_B)(2)} f_{AB}^{C}(2)\,c(0) + \ldots \tag{66}$$

The limit $\mu \to 2$ is performed termwise.

This is not true at the other limit $d = 2$. First we consider the two conserved qp - currents

$$\phi \quad : \quad T_{\mu\nu} \quad \text{or} \quad J_{\mu,ab}$$

which have well defined local field limits

$$\varphi \quad : \quad t_{\mu\nu} \quad \text{or} \quad j_{\mu,ab}$$

Both T and J can be constructed from fusion of $\vec{S} \otimes \vec{S}$. We introduce the Ward identities in any ad hoc normalization and normalize the fields $\phi \ni \{T, J\}$, $\varphi \ni \{t, j\}$ relative to the same Ward identities. Conformal invariance implies the same scaling dimension and tensor structure for ϕ and φ so that

$$\Big\langle \phi(x)\, \phi(0) \Big\rangle \quad = \quad C_\phi(\mu) \Big\langle \varphi(x)\, \varphi(0) \Big\rangle \tag{69}$$

By explicit calculation we find

$$\begin{aligned} \lim_{\mu \searrow 1} C_\phi(\mu) \quad &= \quad 1 - \frac{l+1}{N} + \mathcal{O}\Big(\frac{1}{N^2}\Big) \\ l \quad &= \quad \text{tensor degree of } \phi \text{ (1 or 2)} \end{aligned} \tag{70}$$

Ward identities can be derived from the two - point functions. Instead of normalizing fields by three - point functions and comparing the two - point functions we can introduce a standard normalization of two - point functions

$$\Big\langle \phi(x)\, \phi(0) \Big\rangle \quad = \quad \Big(x^2\Big)^{-\delta_\phi} \cdot \text{tensor}(x) \tag{72}$$

with the tensor factors connected to Gegenbauer polynomials which can be submitted to an ad hoc normalization, say $C_l^{\mu-1}(1) = 1$, too. Doing that, the factors $C_\phi(\mu)$ appear in the three - point functions as fusion coefficients. It becomes clear that the appearance of such factors is quite general. Consider the fusion of n fields $\vec{S}$ by DCF into the field $M_0^{\{n,0\}}$. In the free field limit this corresponds to taking the Wick normal product

$$: \vec{s}_{\otimes}^{\,n} : \tag{73}$$

Two such fields multiply as

$$: \vec{s}_{\otimes}^{\,p_1} : (x) \;\; : \vec{s}_{\otimes}^{\,p_2} : (0) \quad = \quad : \vec{s}_{\otimes}^{\,p_1+p_2} : (0) \; + \ldots \tag{74}$$

whereas DCF yields

$$M_0^{\{p_1,0\}}(x)\, M_0^{\{p_2,0\}}(0) \quad = \quad f\left(x^2\right)^{\mathcal{O}(\mu-1)} M_0^{\{p_1+p_2,0\}}(0) + \dots \qquad (75)$$

The exponent of x^2 contains only anomalous dimensions and tends to zero at $\mu = 1$. Computation of f gives

$$f(\mu) \quad = \quad 1 + \frac{\mu}{(\mu-1)(\mu-2)} \eta(S)\, p_1 p_2 + \mathcal{O}\Big(\frac{1}{N^2}\Big) \qquad (76)$$

so that, with (60)

$$\lim_{\mu \searrow 1} f(\mu) \quad = \quad 1 - \frac{p_1 p_2}{N} + \mathcal{O}\Big(\frac{1}{N^2}\Big) \qquad (77)$$

Then we end up with a final

Theorem 7: The $d = 2$ limit is into the universality class of the polynomial algebra of free fields. Fusion coefficients are $\mathcal{O}(\frac{1}{N})$ deformed with respect to free field theory.

In particular this implies that exponential expressions of free fields (" vertex operators ") cannot arise. Moreover the $\epsilon = d - 2$ expansions (which are in the literature since about 1976) are correct only if applied to critical indices and not to amplitudes. To our knowledge this restriction has never been clearly expressed before.

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