Temperature Independent Renormalization of Finite Temperature Field Theory

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Abstract

We analyse 4-dimensional massive φ^4 theory at finite temperature T in the imaginary-time formalism. We present a rigorous proof that this quantum field theory is renormalizable, to all orders of the loop expansion. Our main point is to show that the counterterms can be chosen temperature independent, so that the temperature flow of the relevant parameters as a function of T can be followed. Our result confirms the experience from explicit calculations to the leading orders. The proof is based on flow equations, i.e. on the (perturbative) Wilson renormalization group. In fact we will show that the difference between the theories at $T > 0$ and at $T = 0$ contains no relevant terms. Contrary to BPHZ type formalisms our approach permits to lay hand on renormalization conditions and counterterms at the same time, since both appear as boundary terms of the renormalization group flow. This is crucial for the proof.

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1 Introduction

Field theories at finite temperature and density have been proposed as the fundamental underlying theory for the description of the physics of the early universe. A proposed scenario for baryogenesis is by the electroweak phase transition [1]. QCD is expected to become deconfined at high temperature. The formation of a quark gluon plasma and the phase transitions of QCD are supposed to be visible in relativistic heavy ion collision and astrophysics [2]. A modern presentation of finite temperature field theory can be found in [3].

Beyond their phenomenological implications, quantum field theories at finite temperature are very challenging also from the more theoretical point of view. There is a real-time as well as an imaginary-time formalism, the first describing dynamical and the second equilibrium properties [4]. Many fundamental issues and problems are unsolved so far or require a deeper understanding. Quantum field theories are subject to enhanced complexities compared to zero temperature and zero density. This is largely related to the presence of additional length scales, due to the interaction with a heat bath. On the various scales the properties of the theory are considerably different.

The separation of scales is widely believed to be an intrinsic property of the field theory. In QCD the scales are associated to the generation of electric and magnetic screening and plasmon masses. In the framework of perturbation theory, this manifests itself in terms of IR divergences that are "severe". They are not removable as it is the case at temperature $T = 0$ by adjusting the renormalization prescription [5]. Various elaborate resummation techniques have been proposed to (at least partially) remove the IR singularities and in addition compute screening masses in perturbation theory. In any case, all the approaches (need to) aim at a clean separation of IR and UV behaviour.

A precondition of all these considerations is renormalizability. Renormalizability is an essential requirement of any local quantum field theory, both at zero and non-zero temperature [6]. It implies that the correlation functions stay finite as the UV-cutoff Λ_0, say, is removed, $\Lambda_0 \to \infty$, and that the limit is parametrized by a set of renormalized (relevant) coupling constants. Moreover, it is crucial that renormalization can be achieved in a temperature independent way, which means that the field theory renormalized at zero temperature stays UV finite at every $T > 0$ as well. This is often taken for granted even for complicated theories, such as gauge theories. Temperature independent renormalizability is indispensable for relating bare and renormalized coupling constants in a T-independent way. It is thus required when formulating Callan-Symanzik type of equations that govern the T-dependence of observables, including correlation functions and effective masses. More generally it implies that the static and dynamic properties mediated by the interactions with a heat bath are intrinsic features of the field theory itself.

Various attempts and steps towards proving renormalizability exist [7]. In order to separate off the IR problem from the UV scale, a massive field theory is considered. Both in the real- and in the imaginary-time formalism, the investigations are commonly based on a Feynman diagrammatic approach in momentum space. In the real-time description, it is argued that the part of the propagator which depends on the temperature T or the chemical potential μ decays exponentially fast for large momenta, so it should be "innocent" of any UV problem. In the imaginary-time formalism the approach is generically more "cumbersome", but it is again argued that in the sum over the Matsubara frequencies all T- or μ-dependent UV divergences cancel out.

Experience obtained by explicit computations to leading orders of perturbation theory confirms that, once IR and UV singularities are properly disentangled, all UV divergences found are T-independent and are removed by the zero temperature counterterms. However, this is not so for non-zero chemical potential μ (associated to a finite density). A field theory that has been renormalized at $\mu = 0$ is able to generate μ-dependent UV divergences that are not removed by the $\mu = 0$ counterterms. A simple example is given by a 4-dimensional Yukawa model, with a chemical potential associated to the fermion number. In the framework of the renormalization group, the chemical potential introduces an additional relevant operator, so at least one additional renormalization condition is expected. This also indicates a possible problem for the analytic continuation from the euclidean to the real-time formulation, in agreement with a discussion [8] in the framework of axiomatic quantum field theories at finite temperature, where the problem of proving the existence of correlation functions (even at $\mu = 0$) in the real-time formalism has been stressed.

The renormalization of field theories at $T = 0$ is well understood. Strong statements and proofs on the renormalizability of various field theories relevant in physics exist, including several different regularization and renormalization schemes, see e.g. [9, 10]. Unfortunately, this sophistication does not extend to finite T so far. Rigorous proofs do not exist, to the best of our knowledge. We would like to point out, however, that recently rigorous bounds, uniform in the temperature, have been established for the perturbative correlation functions of many-fermion models. Here renormalization is necessary to obtain well-behaved bounds on the IR side, when approaching the Fermi surface, whereas the UV regularization is kept fixed. Feldman et al. [11] renormalize the many-fermion models with T-independent counterterms, as we do.

In this paper we give a mathematical proof that massive φ^4 theory at finite T, in the imaginary-time formalism, is renormalizable. More precisely, we show, to all orders of the loop expansion, that all correlation functions become UV finite at every finite T once the theory has been renormalized at $T = 0$ by (one of the) usual renormalization prescriptions.

The proof is given in the framework of Wilson's flow equation. It avoids the analysis of individual Feynman integrals (or Feynman sums), which requires the involved combinatorics encoded in the forest formula for overlapping divergences. Moreover it avoids the formulation and proof of a power counting theorem. Using flow equations, the proof of renormalizability merely amounts to establish appropriate bounds in momentum space on the correlation functions, which are viewed as coefficient functions of the associated generating functional. The proof is by induction on the number of loops.

This paper is organized as follows. In Sect. 2 we introduce our basic notations. This includes the definition of the generating functional $L^{\Lambda,\Lambda_0}(\varphi)$ of the connected, free propagator amputated Green functions on "momentum scale Λ", with $0 \le \Lambda \le \Lambda_0$, where Λ_0 denotes the UV cutoff. The dependence of L^{Λ,Λ_0} on the scale Λ is described by the so-called Wilson flow equation. We recap the basic steps of proving renormalizability of 4-dimensional φ^4 field theories at zero temperature by means of the flow equation. Renormalizability is stated in terms of uniform bounds on the (coefficient functions of the) solution $L^{\Lambda,\Lambda_0}(\varphi)$ of the flow equation and its derivative with respect to the UV-cutoff Λ_0, with boundary conditions imposed at $\Lambda = 0$ for the relevant couplings and at $\Lambda = \Lambda_0$ for the irrelevant interactions.

In Sect. 3 we show that the difference $D^{\Lambda,\Lambda_0}(\underline{\varphi};T)$ of the generating functionals at temperature $T > 0$ and $T = 0$:

$$D^{\Lambda,\Lambda_0}(\underline{\varphi};T) \equiv L^{\Lambda,\Lambda_0}(\underline{\varphi};T) - L^{\Lambda,\Lambda_0}(\underline{\varphi}) \tag{1}$$

has the properties of an irrelevant operator in the sense of the renormalization group [4]. More precisely, T-independence of the counterterms means that the boundary condition

$$D^{\Lambda_0,\Lambda_0}(\underline{\varphi};T) \equiv 0 \tag{2}$$

[4] For the definition of the momentum space field variables $\underline{\varphi}$ and their position space Fourier transform $\underline{\hat{\varphi}}$ we refer to the beginning of sect.3 : Equ. (1) should be understood in the weak sense, i.e. in a formal power series expansion w.r.t. $\hbar$ and as an identity for all coefficient functions generated by the generating functionals. For the equation to make sense as it stands the variables $\underline{\hat{\varphi}}$ have to be appropriately restricted, for instance to be smooth functions, supported in the interval $[0,\, \beta]$ in the x_0-component in position space.

holds. From this we derive strong bounds on all scales Λ for $D^{\Lambda,\Lambda_0}(\underline{\varphi};T)$. Together with the bounds on $L^{\Lambda,\Lambda_0}(\varphi)$ this proves UV finiteness of massive φ_4^4 for every finite T, that is,

$$\lim_{\Lambda_0\to\infty,\Lambda\to 0} L^{\Lambda,\Lambda_0}(\underline{\varphi};T) \tag{3}$$

exists, to all orders of the loop expansion. As an immediate consequence, the theory is also made UV finite by imposing normalization conditions on the mass, the wave function constant and on the quartic coupling constant at any fixed temperature T_0. In Sect. 4 we summarize our central statements and give a short outlook.

2 Renormalization of zero temperature φ_4^4 theory - a short reminder

Perturbative renormalizability of euclidean zero temperature φ_4^4 theory will be established by analysing the generating functional L^{Λ,Λ_0} of connected (free propagator) amputated Green functions (CAG). The upper indices Λ and Λ_0 enter through the regularized propagator

$$C^{\Lambda,\Lambda_0}(p) \;=\; \frac{1}{p^2+m^2}\{e^{-\frac{p^2+m^2}{\Lambda_0^2}} - e^{-\frac{p^2+m^2}{\Lambda^2}}\} \tag{4}$$

or its Fourier transform

$$\hat{C}^{\Lambda,\Lambda_0}(x) \;=\; \int_p C^{\Lambda,\Lambda_0}(p)\, e^{ipx}\,, \tag{5}$$

where we use the shorthand

$$\int_p \;:=\int_{\mathrm{I\!R}^4}\frac{d^4p}{(2\pi)^4}\,. \tag{6}$$

We assume

$$0\le\Lambda\le\Lambda_0\le\infty \tag{7}$$

so that the Wilson flow parameter Λ takes the role of an infrared (IR) cutoff[5], whereas Λ_0 is the ultraviolet (UV) regularization. The full propagator is recovered for $\Lambda=0$ and $\Lambda_0\to\infty$. We also introduce the convention

$$\hat{\varphi}(x) = \int_p \varphi(p)\, e^{ipx}\,,\qquad \frac{\delta}{\delta\hat{\varphi}(x)} = (2\pi)^4\int_p \frac{\delta}{\delta\varphi(p)}\, e^{-ipx}\,. \tag{8}$$

For our purposes the "fields" $\hat{\varphi}(x)$ may be assumed to live in the Schwartz space $\mathcal{S}(\mathrm{I\!R}^4)$. For finite Λ_0 and in finite volume the theory can be given rigorous meaning starting from

[5]Such a cutoff is of course not necessary in a massive theory. The IR behaviour is only modified for Λ above m.

the functional integral

$$e^{-\frac{1}{\hbar}(L^{\Lambda,\Lambda_0}(\hat\varphi)+I^{\Lambda,\Lambda_0})} = \int d\mu_{\Lambda,\Lambda_0}(\hat\phi)\, e^{-\frac{1}{\hbar}L^{\Lambda_0,\Lambda_0}(\hat\phi+\hat\varphi)}\,, \tag{9}$$

where the factors of $\hbar$ have been introduced to allow for a consistent loop expansion in the sequel. In (9) $d\mu_{\Lambda,\Lambda_0}(\hat\phi)$ denotes the (translation invariant) Gaussian measure with covariance $\hbar\hat{C}^{\Lambda,\Lambda_0}(x)$. The normalization factor $e^{-\frac{1}{\hbar}I^{\Lambda,\Lambda_0}}$ is due to vacuum contributions. It diverges in infinite volume so that we can take the infinite volume limit only when it has been eliminated [10]. We do not make the finite volume explicit here since it plays no role in the sequel.[6]

The functional $L^{\Lambda_0,\Lambda_0}(\hat\varphi)$ is the bare action including counterterms, viewed as a formal power series in $\hbar$. Its general form for symmetric[7] φ_4^4 theory is

$$L^{\Lambda_0,\Lambda_0}(\hat\varphi) = \frac{g}{4!}\int d^4x\,\hat\varphi^4(x) +$$
$$+\int d^4x\,\{\frac{1}{2}a(\Lambda_0)\hat\varphi^2(x)+\frac{1}{2}b(\Lambda_0)\sum_{\mu=0}^{3}(\partial_\mu\hat\varphi)^2(x)+\frac{1}{4!}c(\Lambda_0)\hat\varphi^4(x)\}\,, \tag{10}$$

where $g>0$ is the renormalized coupling, and the parameters $a(\Lambda_0),\ b(\Lambda_0),\ c(\Lambda_0)$ fulfill

$$a(\Lambda_0),\ b(\Lambda_0),\ c(\Lambda_0) = O(\hbar)\,. \tag{11}$$

They are directly related to the standard mass, wave function and coupling constant counterterms. Since in the flow equation framework it is not necessary to introduce bare fields in distinction to renormalized ones (our field is the renormalized one in this language), there is a slight difference, which is to be kept in mind only when comparing to other schemes. The Wilson flow equation (FE) is obtained from (9) on differentiating w.r.t. Λ. It is a differential equation for the functional L^{Λ,Λ_0} :

$$\partial_\Lambda(L^{\Lambda,\Lambda_0}+I^{\Lambda,\Lambda_0}) = \frac{\hbar}{2}\langle\frac{\delta}{\delta\hat\varphi},(\partial_\Lambda\hat{C}^{\Lambda,\Lambda_0})\frac{\delta}{\delta\hat\varphi}\rangle L^{\Lambda,\Lambda_0} - \frac{1}{2}\langle\frac{\delta}{\delta\hat\varphi}L^{\Lambda,\Lambda_0},(\partial_\Lambda\hat{C}^{\Lambda,\Lambda_0})\frac{\delta}{\delta\hat\varphi}L^{\Lambda,\Lambda_0}\rangle\ . \tag{12}$$

[6] A rigorous treatment of the thermodynamic limit requires to replace the propagator (5) by a finite volume version, e.g. $\hat{C}_V^{\Lambda,\Lambda_0}(x,y) = \chi_V(x)\,\hat{C}^{\Lambda,\Lambda_0}(x-y)\,\chi_V(y)$, where χ_V is the characteristic function of the volume V, and to regard the Gaussian measure with covariance $\hat{C}_V^{\Lambda,\Lambda_0}(x,y)$. In this case the quantity I_V^{Λ,Λ_0} is obviously well defined, at any order l in $\hbar$. Then (12) is well-defined. After decomposing L_V^{Λ,Λ_0} w.r.t. powers of $\hbar$ and of the field $\hat\varphi$, we realize that the coefficient functions $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$ are well-defined in the thermodynamic limit, since they are given as finite sums over UV-regularized connected diagrams. The existence of the thermodynamic limit is of course confirmed by the bounds on the solutions of the FE. It should also be feasible to study the thermodynamic limit itself with the aid of the FE in finite volume, by proving inductively uniform bounds on the (appropriately defined) "translational invariant part" of the finite volume Green functions and a convergence statement analogous to (18).

[7] The necessary generalizations in the nonsymmetric case will be surveyed in the end of the next section.

By $\langle\ ,\ \rangle$ we denote the standard scalar product in $L_2(\mathbb{R}^4, d^4x)$. Changing to momentum space and expanding in a formal powers series w.r.t. $\hbar$ we write with slight abuse of notation

$$L^{\Lambda,\Lambda_0}(\varphi) \;=\; \sum_{l=0}^{\infty} \hbar^l \, L_l^{\Lambda,\Lambda_0}(\varphi)\,. \tag{13}$$

From $L_l^{\Lambda,\Lambda_0}(\varphi)$ we then obtain the CAG of loop order l in momentum space as [8]

$$(2\pi)^{4(n-1)} \delta_{\varphi(p_1)} \ldots \delta_{\varphi(p_n)} L_l^{\Lambda,\Lambda_0}|_{\varphi\equiv 0} \;=\; \delta^{(4)}(p_1 + \ldots + p_n)\, \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_{n-1})\ , \tag{14}$$

where we have written $\delta_{\varphi(p)} = \delta/\delta\varphi(p)$. Note that our definition of the $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$ is such that $\mathcal{L}_{0,2}^{\Lambda,\Lambda_0}$ vanishes. The absence of 0-loop two (and one-) point functions is important for the set-up of the inductive scheme, from which we will prove renormalizability below. The FE (12) rewritten in terms of the CAG (14) takes the following form

$$\partial_\Lambda \partial^w \, \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots p_{n-1}) = \frac{1}{2} \int_k (\partial_\Lambda C^{\Lambda,\Lambda_0}(k))\, \partial^w \mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}(k, -k, p_1, \ldots p_{n-1}) \tag{15}$$

$$- \sum_{\substack{l_1+l_2=l,\; w_1+w_2+w_3=w \\ n_1+n_2=n}} \frac{1}{2} \Big[\partial^{w_1} \mathcal{L}_{l_1,n_1+1}^{\Lambda,\Lambda_0}(p_1, \ldots, p_{n_1})\, (\partial^{w_3} \partial_\Lambda C^{\Lambda,\Lambda_0}(p'))\ \partial^{w_2} \mathcal{L}_{l_2,n_2+1}^{\Lambda,\Lambda_0}(p_{n_1+1}, \ldots, p_n) \Big]_{ssym}\ ,$$

$$\text{where} \quad p' = -p_1 - \ldots - p_{n_1} \;=\; p_{n_1+1} + \ldots + p_n\ .$$

Here we have written (15) directly in a form where also momentum derivatives of the CAG (14) are performed, and we used the shorthand notation

$$\partial^w := \prod_{i=1}^{n-1} \prod_{\mu=0}^{3} \Big(\frac{\partial}{\partial p_{i,\mu}}\Big)^{w_{i,\mu}} \;\; \text{with} \;\; w = (w_{1,0}, \ldots, w_{n-1,3}),\ |w| = \sum w_{i,\mu}\ ,\ w_{i,\mu} \in \mathbb{N}_0\ . \tag{16}$$

The symbol *ssym* [9] means summation over those permutations of the momenta $p_1, \ldots, p_n$, which do not leave invariant the subsets $\{p_1, \ldots, p_{n_1}\}$ and $\{p_{n_1+1}, \ldots, p_n\}$. Note that the CAG are symmetric in their momentum arguments by definition. A simple inductive proof of the renormalizability of φ_4^4 theory has been exposed several times in the literature [10], and we will not repeat it in detail. The line of reasoning can be resumed as follows.
The induction hypotheses to be proven are :
A) Boundedness

$$|\partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\vec{p})| \leq\ (\Lambda + m)^{4-n-|w|}\, \mathcal{P}_1(log\frac{\Lambda + m}{m})\, \mathcal{P}_2(\frac{|\vec{p}|}{\Lambda + m})\ . \tag{17}$$

[8] The normalization of the $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$ is defined differently from earlier references.
[9] It is defined differently from the symbol *sym* in [10], the present conventions being slightly more elegant.

B) Convergence

$$|\partial_{\Lambda_0}\partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\vec{p})| \leq \frac{1}{\Lambda_0^2}(\Lambda+m)^{5-n-|w|}\,\mathcal{P}_3(log\frac{\Lambda_0}{m})\,\mathcal{P}_4(\frac{|\vec{p}|}{\Lambda+m})\ .^{10} \tag{18}$$

Here and in the following the $\mathcal{P}$ denote (each time they appear possibly new) polynomials with nonnegative coefficients. The coefficients depend on $l, n, |w|, m$, but not on $\vec{p}$, Λ, Λ_0. We used the shorthand $\vec{p} = (p_1, \ldots, p_{n-1})$ and $|\vec{p}| = \sup\{|p_1|, \ldots, |p_n|\}$. The statement (18) implies renormalizability, since it proves that the limits $\lim_{\Lambda_0\to\infty,\,\Lambda\to 0} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\vec{p})$ exist to all loop orders l. But the statement (17) has to be obtained first to prove (18). The inductive scheme to prove the statements proceeds upwards in l, for given l upwards in n, and for given (l, n) downwards in $|w|$, starting from some arbitrary $|w_{max}| \geq 3$. The important point to note is that the terms on the r.h.s. of the FE always are prior to the one on the l.h.s. in the inductive order. So the bound (17) may be used as an induction hypothesis on the r.h.s. Then we may integrate the FE, where terms with $n + |w| \geq 5$ are integrated down from Λ_0 to Λ, since for those terms we have the boundary conditions following from (10)

$$\partial^w \mathcal{L}_{l,n}^{\Lambda_0,\Lambda_0}(p_1, \ldots p_{n-1}) = 0\,, \quad \text{for} \quad n + |w| > 4\,, \tag{19}$$

whereas the terms with $n + |w| \leq 4$ at the renormalization point - which we choose at zero momentum for simplicity - are integrated upwards from 0 to Λ, since they are fixed by (Λ_0-independent) renormalization conditions, fixing the relevant parameters of the theory[11] :

$$\mathcal{L}_{l,2}^{0,\Lambda_0}(p) = a_l^R + b_l^R\,p^2 + O((p^2)^2)\,, \quad \mathcal{L}_{0,4}^{0,\Lambda_0}(0) = g\,,\ \mathcal{L}_{l,4}^{0,\Lambda_0}(0) = c_l^R\,,\ l \geq 1\,. \tag{20}$$

Symmetry considerations tell us that there are no other nonvanishing renormalization constants apart from a_l^R, b_l^R, c_l^R , and the Schlömilch or integrated Taylor formula permits us to move away from the renormalization point, treating first $\mathcal{L}_{l,4}^{0,\Lambda_0}$ and then the momentum derivatives of $\mathcal{L}_{l,2}^{0,\Lambda_0}$, in descending order. With these remarks on the boundary conditions, and using the bounds on the propagator and its derivatives

$$|\partial^w \partial_\Lambda C^{\Lambda,\Lambda_0}(p)| \leq \Lambda^{-3-|w|}\,\mathcal{P}(|p|/\Lambda)\,e^{-\frac{p^2+m^2}{\Lambda^2}}\ , \tag{21}$$

statement A) above is straightforwardly verified by inductive integration of the FE. Once this has been achieved statement B) follows on applying the same inductive scheme to bound the solutions of the FE, integrated over Λ and then derived w.r.t. Λ_0 .

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[10] In fact, in symmetric φ_4^4 theory $\frac{1}{\Lambda_0^2}$ can be replaced by $\frac{\Lambda}{\Lambda_0^3}$ as shown in [13].

[11] The simplest choice would be to set $a_l^R = 0$, $b_l^R = 0$, $c_l^R = 0$, in which case the renormalized coupling is identical to the connected four point function at zero momentum. A shift away from zero momentum would result in nonvanishing terms c_l^R , just to mention one example of more general choices.

3 Temperature independent renormalization of finite temperature φ_4^4 theory

We fix the notations recalling at the same time some basic facts about euclidean finite temperature field theory. The scalar field $\hat{\varphi}(x)$ becomes periodic in x_0 at finite temperature with period $\beta = 1/T$. Correspondingly position space integrals over the zero component of the coordinates are now restricted to the compact interval $[0, \beta]$. Symbols denoting finite temperature quantities will generally be underlined, thus we write

$$\underline{p} \ := (\underline{p}_0, \vec{p}) \ := (2\pi n T, \vec{p}) , \ n \in \mathbb{Z} , \quad \int_{\underline{p}} := T \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} \ . \tag{22}$$

We also introduce the convention

$$\underline{\hat{\varphi}}(x) \ := \int_{\underline{p}} \underline{\varphi}(\underline{p}) \ e^{i\underline{p}x} , \quad \underline{\varphi}(\underline{p}) \ = \int_0^\beta dx_0 \int_{\mathbb{R}^3} d^3 x \ \underline{\hat{\varphi}}(x) \ e^{-i\underline{p}x} , \tag{23}$$

$$\frac{\delta}{\delta \underline{\hat{\varphi}}(x)} = \frac{(2\pi)^3}{T} \int_{\underline{p}} \frac{\delta}{\delta \underline{\varphi}(\underline{p})} e^{-i\underline{p}x} , \quad \frac{\delta}{\delta \underline{\varphi}(\underline{p})} = \frac{T}{(2\pi)^3} \int_0^\beta dx_0 \int_{\mathbb{R}^3} d^3 x \ \frac{\delta}{\delta \underline{\hat{\varphi}}(x)} e^{i\underline{p}x} \ . \tag{24}$$

The regularized propagator now takes the form

$$C^{\Lambda,\Lambda_0}(\underline{p}) \ = \ \frac{1}{\underline{p}^2 + m^2} \{ e^{-\frac{\underline{p}^2 + m^2}{\Lambda_0^2}} - e^{-\frac{\underline{p}^2 + m^2}{\Lambda^2}} \} \ . \tag{25}$$

The generating functional of the finite temperature CAG will be called $L^{\Lambda,\Lambda_0}(\underline{\varphi}; T)$. In analogy with (14) we define the CAG through

$$\delta_{\underline{\varphi}(\underline{p}_1)} \dots \delta_{\underline{\varphi}(\underline{p}_n)} L_l^{\Lambda,\Lambda_0}(\underline{\varphi}; T)|_{\underline{\varphi} \equiv 0} \ = \tag{26}$$

$$(\frac{T}{(2\pi)^3})^{n-1} \ \delta_{0,(\underline{p}_{1,0} + \ldots + \underline{p}_{n,0})} \ \delta^{(3)}(\vec{p}_1 + \ldots + \vec{p}_n) \, \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\underline{p}_1, \ldots, \underline{p}_{n-1}; T) \ .$$

At this stage we could prove renormalizability of the finite temperature theory in the same way as for the zero temperature theory. A slight difference is that the relations (20) are to be replaced by

$$\mathcal{L}_{l,2}^{0,\Lambda_0}(\underline{p}; T) = a_l^R(T) + b_l^{R,0}(T) \, \underline{p}_0^2 + b_l^{R,1}(T) \, \vec{p}^{\,2} + \, O(\underline{p}^4) ,$$

$$\mathcal{L}_{0,4}^{0,\Lambda_0}(\underline{p} = 0; T) = g \, , \ \mathcal{L}_{l,4}^{0,\Lambda_0}(\underline{p} = 0; T) = c_l^R(T) \, , \ l \geq 1 \, , \tag{27}$$

since the space-time $O(4)$-symmetry is broken down to a $\mathbb{Z}_2 \times O(3)$-symmetry which demands a new renormalization condition. However we want to go beyond and prove temperature independent renormalizability, in the sense that the counterterms can be chosen temperature independent. To do so, it is advantageous to pass directly to the difference between the finite and zero temperature theories, which we will do now. Note in this respect that if we wanted to prove the renormalizability of the finite temperature theory, keeping the counterterms fixed at their zero temperature values, would not work within our scheme and procedure : The CAG would become arbitrarily divergent in Λ_0 with increasing loop order, since integrating relevant terms from Λ_0 to 0 (instead of integrating them from a renormalization condition fixed at $\Lambda = 0$ up to Λ_0) gives divergent integrals. Thus we rather study the difference functions

$$\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\}) := \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\};T) - \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\}) . \tag{28}$$

We only define and need the difference CAG $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ at the external momenta $(\{\underline{p}\}) := (\underline{p}_1, \ldots, \underline{p}_{n-1})$. From the FE (15) and the analogous equation for the $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\};T)$ we can derive a FE for the $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\})$ in the following form :

$$\begin{aligned}\partial_\Lambda \mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\}) =& \frac{1}{2}\int_{\underline{k}} (\partial_\Lambda C^{\Lambda,\Lambda_0}(\underline{k}))\, \mathcal{D}_{l-1,n+2}^{\Lambda,\Lambda_0}(\underline{k},-\underline{k},\{\underline{p}\}) \\ &+ \frac{1}{2}\Big\{\int_{\underline{k}} (\partial_\Lambda C^{\Lambda,\Lambda_0}(\underline{k}))\, \mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}(\underline{k},-\underline{k},\{\underline{p}\}) - \int_k (\partial_\Lambda C^{\Lambda,\Lambda_0}(k))\, \mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}(k,-k,\{\underline{p}\})\Big\} \\ &- \sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n}} \frac{1}{2}\Big\{\Big[\mathcal{L}_{l_1,n_1+1}^{\Lambda,\Lambda_0}(\underline{p}_1,\ldots,\underline{p}_{n_1};T)(\partial_\Lambda C^{\Lambda,\Lambda_0}(\underline{p}'))\; \mathcal{D}_{l_2,n_2+1}^{\Lambda,\Lambda_0}(\underline{p}_{n_1+1},\ldots,\underline{p}_n)\Big]_{ssym} \\ &+ \Big[\mathcal{D}_{l_1,n_1+1}^{\Lambda,\Lambda_0}(\underline{p}_1,\ldots,\underline{p}_{n_1})(\partial_\Lambda C^{\Lambda,\Lambda_0}(\underline{p}'))\; \mathcal{L}_{l_2,n_2+1}^{\Lambda,\Lambda_0}(\underline{p}_{n_1+1},\ldots,\underline{p}_n)\Big]_{ssym}\Big\} ,\end{aligned} \tag{29}$$

where again

$$\underline{p}' = -\underline{p}_1 - \ldots - \underline{p}_{n_1} = \underline{p}_{n_1+1} + \ldots + \underline{p}_n .$$

The boundary conditions we want to impose on the system $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ are (from the previous remarks) obviously the following ones :

$$\mathcal{D}_{l,n}^{\Lambda_0,\Lambda_0}(\underline{p}_1,\ldots,\underline{p}_{n-1}) = 0 , \quad l,n \in \mathbb{N} . \tag{30}$$

To start the induction we also note

$$\mathcal{D}_{0,n}^{\Lambda,\Lambda_0}(\underline{p}_1,\ldots,\underline{p}_{n-1}) = 0 , \quad n \in \mathbb{N} , \tag{31}$$

at the tree level all difference terms $\mathcal{D}_{0,n}^{\Lambda,\Lambda_0}$ vanish. This follows from the fact that restricted to the momenta $(\underline{p}_1, \ldots \underline{p}_{n-1})$ the tree level functions $\mathcal{L}_{0,n}^{\Lambda,\Lambda_0}(\underline{p}_1,\ldots \underline{p}_{n-1};T)$ and $\mathcal{L}_{0,n}^{\Lambda,\Lambda_0}(\underline{p}_1,\ldots \underline{p}_{n-1})$ agree. Now we would like to use the same inductive scheme proceeding upwards in l, and for given l upwards in n, to prove the finiteness of $\lim_{\Lambda_0\to\infty,\Lambda\to 0}\mathcal{D}_{0,n}^{\Lambda,\Lambda_0}$. Due to the form of (30) we *always* integrate the FE for $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ from Λ_0 down to Λ, since the boundary terms at Λ_0 always vanish. We want to prove the following

Theorem :

$$|\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(\underline{p}_1,\ldots,\underline{p}_{n-1})| \,\le\, (\Lambda+m)^{-s-n}\,\mathcal{P}_1(\log\frac{\Lambda+m}{m})\,\mathcal{P}_2(\frac{|\{\underline{p}\}|}{\Lambda+m})\,, \tag{32}$$

$$|\partial_{\Lambda_0}\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(\underline{p}_1,\ldots,\underline{p}_{n-1})| \,\le\, \frac{1}{\Lambda_0^2}(\Lambda+m)^{-s-n}\,\mathcal{P}_3(log\frac{\Lambda_0}{m})\,\mathcal{P}_4(\frac{|\{\underline{p}\}|}{\Lambda+m})\,. \tag{33}$$

The nonnegative coefficients in the polynomials $\mathcal{P}$ depend on l,n,s,m and (smoothly) on T, but not on $\{\underline{p}\}$, Λ, Λ_0. The positive integer $s\in\mathbb{N}$ may be chosen arbitrarily.
The finite temperature CAG $\mathcal{L}_{0,n}^{\Lambda,\Lambda_0}(\underline{p}_1,\ldots,\underline{p}_{n-1};T)$, when renormalized with the *same* counterterms as the zero temperature ones, satisfy the same bounds as in (17,18) restricted to $w=0$. The coefficients in the polynomials $\mathcal{P}$ may now depend on l,n,m and (smoothly) on T.

Remark : It is possible to prove the bounds (17,18) also for derivatives of the finite temperature CAG $\mathcal{L}_{0,n}^{\Lambda,\Lambda_0}(\underline{p}_1,\ldots,\underline{p}_{n-1};T)$. In the $p_{i,0}$-components differentiations then have to replaced by finite differences. However these bounds are not needed in the inductive proof, so we skip them here.

Proof : We first prove (32) and and the statement corresponding to (17) for $w=0$, using the inductive scheme indicated previously. Regarding the FE (29) we state that it is compatible with the inductive scheme and that the only term in which (32) cannot be used as an induction hypothesis is the following one :

$$\int_{\underline{k}}(\partial_\Lambda C^{\Lambda,\Lambda_0}(\underline{k}))\,\mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}(\underline{k},-\underline{k},\{\underline{p}\}) \,-\, \int_k(\partial_\Lambda C^{\Lambda,\Lambda_0}(k))\,\mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}(k,-k,\{\underline{p}\})\,. \tag{34}$$

So our sharp Λ-bound on $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ can only be verified if it holds for this difference term. Here we use (17,18) and the Euler-MacLaurin-formula, see e.g. [12]. We can rewrite (34) as

$$\frac{-2}{\Lambda^3}\int\frac{d^3\vec{k}}{(2\pi)^4}\,e^{-\frac{\vec{k}^2+m^2}{\Lambda^2}}\left[2\pi T\sum_{n\in\mathbb{Z}}g(2\pi nT)-\int_{-\infty}^{\infty}dk_0\,g(k_0)\right], \tag{35}$$

where we introduced the function

$$g(k_0) \,=\, e^{-\frac{k_0^2}{\Lambda^2}}\,\mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}(k,-k,\{\underline{p}\}) \quad\text{for}\quad \vec{k},\,\{\underline{p}\}\ \text{fixed}\,. \tag{36}$$

The Euler-MacLaurin formula for our case can be stated in the form

$$2\pi T\sum_{n\in\mathbb{Z}}g(2\pi nT)-\int_{-\infty}^{\infty}dk_0\,g(k_0) \,=\, -\pi T\,[g(\infty)-g(-\infty)] \tag{37}$$

$$+ \sum_{k=1}^{r+1} \frac{b_{2k}(2\pi T)^{2k}}{(2k)!}[g^{(2k-1)}(\infty) - g^{(2k-1)}(-\infty)] \, + \, R_{r+1} \; .$$

Here b_{2k} are the Bernoulli numbers. We observe that passing to the limit of an infinite integration interval is justified, since the function $g(k_0)$ and its derivatives vanish rapidly at infinity. The remainder R_{r+1} obeys the following bound [12]

$$|R_{r+1}| \leq \, 4 \, e^{2\pi} T^{2r+3} \int_{-\infty}^{\infty} dk_0 \, |g^{(2r+3)}(k_0)| \; , \tag{38}$$

therefore we obtain, using again (17,18)

$$|R_{r+1}| \leq \, T^{2r+3} \, \frac{(\Lambda+m)^{2-n}}{\Lambda^{2r+2}} \, \mathcal{P}_1(log\frac{\Lambda+m}{m}) \, \mathcal{P}_2(\frac{|\{k,\underline{p}\}|}{\Lambda+m}) \; . \tag{39}$$

Note that $r \in \mathbb{N}$ can be chosen arbitrarily here, and the bound for (34) is thus

$$\begin{aligned} & T^{2r+3} \, e^{-\frac{m^2}{\Lambda^2}} \, \frac{(\Lambda+m)^{2-n}}{\Lambda^{2r+2}} \, \mathcal{P}_1(log\frac{\Lambda+m}{m}) \, \mathcal{P}_2(\frac{|\{k,\underline{p}\}|}{\Lambda+m}) \\ \leq \; & T^{2r+3} \, (\Lambda+m)^{2-n-2r-2} \, \mathcal{P}_3(log\frac{\Lambda+m}{m}) \, \mathcal{P}_4(\frac{|\{k,\underline{p}\}|}{\Lambda+m}) \; . \end{aligned} \tag{40}$$

After this preparation we consider the induction process : At each loop order we first prove (32), and then (17) for finite T and corresponding momenta. This second step is trivial from (17,18) at $T=0$, from the definition (28) and from (32) [12]. We know already the theorem to be true at 0 loop order. This and the form of the FE (29) implies that we do not need a bound on any of the $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\};T)$ in the inductive bound on $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ at the given loop order l.

It is instructive to regard how the induction starts at loop order $l=1$. Treating first the case $n=2$ we find that the only nonvanishing contribution on the r.h.s. of the FE stems from (34), and it is momentum independent, so that integrating over Λ we get

$$|\mathcal{D}_{1,2}^{\Lambda,\Lambda_0}(\underline{p})| \leq \, c \, (\Lambda+m)^{-2r-1}$$

with a suitable constant c, depending on r. For $n=4$ also the last two terms on the r.h.s. of the FE contribute. Using the result for $\mathcal{D}_{1,2}^{\Lambda,\Lambda_0}(\underline{p})$, integration over Λ gives

$$|\mathcal{D}_{1,4}^{\Lambda,\Lambda_0}(\{\underline{p}\})| \leq \, (\Lambda+m)^{-2-2r-1} \, \mathcal{P}(\frac{|\{\underline{p}\}|}{\Lambda+m}) \; .$$

From this one inductively obtains the bound for $n \geq 6$

$$|\mathcal{D}_{1,n}^{\Lambda,\Lambda_0}(\{\underline{p}\})| \leq \, (\Lambda+m)^{-(n-2)-2r-1} \, \mathcal{P}(\frac{|\{\underline{p}\}|}{\Lambda+m}) \; .$$

[12] We may choose the bounds for $s=0$ from (32,33) when bounding the finite temperature CAG, so that polynomials appearing in the bounds may be chosen s-independent.

Having bounded the difference functions $\mathcal{D}_{1,n}^{\Lambda,\Lambda_0}$ we can bound the CAG $\mathcal{L}_{1,n}^{\Lambda,\Lambda_0}(T) = \mathcal{L}_{1,n}^{\Lambda,\Lambda_0}(T{=}0) + \mathcal{D}_{1,n}^{\Lambda,\Lambda_0}$, see (28). Then we may proceed inductively to higher loop orders and verify the inductive bound

$$|\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\})| \leq (\Lambda+m)^{-(n-2)-2r-1}\,\mathcal{P}_1(\log\frac{\Lambda+m}{m})\,\mathcal{P}_2(\frac{|\{\underline{p}\}|}{\Lambda+m}) \ .$$

This proves the first part of the theorem on writing $s=2r-1$ for s odd, and majorizing to obtain even s. It follows that the $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(T)$ may be bounded in agreement with (17,18).

Now we turn to the proof of the statement (33) which implies convergence of the $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ for $\Lambda_0 \to \infty$. The proof is based on the same inductive scheme and starts from the FE (29) integrated over Λ from Λ_0 to Λ, and then derived w.r.t. Λ_0. The result is of the form

$$-\,\partial_{\Lambda_0}\,\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\}) \;=\; [\text{RHS of (29)}]|_{\Lambda=\Lambda_0} \;+\; \int_{\Lambda}^{\Lambda_0} d\lambda\;\partial_{\Lambda_0}[\text{RHS of (29)}](\lambda)\,, \qquad (41)$$

and we denote the RHS of this equation shortly as

$$I_{l,n}^{\Lambda_0}(\{\underline{p}\}) \;+\; I_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\}) \ .$$

Since we have imposed $\mathcal{L}_{l,n}^{\Lambda_0,\Lambda_0}(T) \equiv \mathcal{L}_{l,n}^{\Lambda_0,\Lambda_0}$, and since moreover these terms vanish for $n \geq 6$, we find

$$I_{l,n}^{\Lambda_0}(\{\underline{p}\}) \;=\; -\delta_{n,2}\left[\int_{\underline{k}}\frac{e^{-\frac{\underline{k}^2+m^2}{\Lambda_0^2}}}{\Lambda_0^3} - \int_k \frac{e^{-\frac{k^2+m^2}{\Lambda_0^2}}}{\Lambda_0^3}\right]\mathcal{L}_{l-1,4}^{\Lambda_0,\Lambda_0} \ . \qquad (42)$$

Since $\mathcal{L}_{l-1,4}^{\Lambda_0,\Lambda_0} \equiv c_{l-1}(\Lambda_0)\,,\; l>1$ and $\mathcal{L}_{0,4}^{\Lambda_0,\Lambda_0} \equiv g$, see (10), we realize that (42) is momentum independent. The difference can be calculated explicitly or bounded again using the Euler-MacLaurin formula, and we obtain

$$|I_{l,n}^{\Lambda_0}| \leq \;\delta_{n,2}\,\Lambda_0^{-2-2r}\;\mathcal{P}(\log\frac{\Lambda_0}{m}) \qquad (43)$$

for $r \in \mathbb{N}$ and a suitable $\mathcal{P}$ depending on r. To get a bound on $I_{l,n}^{\Lambda,\Lambda_0}(\{\underline{p}\})$ we apply the derivative in (41) to all entries using the product rule (noting that when applied to $\partial_\Lambda C^{\Lambda,\Lambda_0}$ it gives zero). In any case the derivative brings down the required factor of Λ_0^{-2}, either by (18), or by (33) together with the induction hypothesis. Apart from this the bound (33) is obtained similarly as (32), using in particular the Euler-MacLaurin formula for the difference term (34) derived w.r.t. Λ_0. This proves also (33). ∎

We end this section with two remarks on possible generalizations. First the preceding analysis can be extended to nonsymmetric φ_4^4-theory. The action (10) then has to be replaced by

$$\tilde{L}^{\Lambda_0,\Lambda_0}(\hat{\varphi}) = L^{\Lambda_0,\Lambda_0}(\hat{\varphi}) + \frac{h}{3!}\int d^4x\, \hat{\varphi}^3(x) + \int d^4x\, \{\frac{1}{3!}\, d(\Lambda_0)\, \hat{\varphi}^3(x) + v(\Lambda_0)\, \hat{\varphi}(x)\} \tag{44}$$

with the tree level three-point coupling h and Λ_0-dependent parameters

$$d(\Lambda_0)\,,\ v(\Lambda_0) \,=\, O(\hbar) \tag{45}$$

implementing the counterterms necessary to renormalize the one- and three-point functions. Correspondingly we pose additional renormalization conditions

$$\mathcal{L}_{l,1}^{0,\Lambda_0} = v_l^R\,, \quad \mathcal{L}_{l,3}^{0,\Lambda_0}(0) = d_l^R \quad \text{for} \quad l \in \mathbb{N}\ , \tag{46}$$

to be joined to (20). Then the bounds (17,18) hold again, but are no more trivially fulfilled for n odd.[13] Once the theory at $T = 0$ is bounded, the differences (28) again yield the theory at $T > 0$. Bounds corresponding to (32,33) are proven proceeding as before, in the symmetric case.

As a second remark, we point out that for the existence of the large cutoff limit $\Lambda_0 \to \infty$, it is not necessary that the relevant coupling constants are subject to normalization conditions at zero temperature. Equally well we can impose normalization conditions at some temperature $T_0 > 0$. We pointed out that at finite temperature the space-time $O(4)$-symmetry is broken down to $\mathbb{Z}_2 \times O(3)$. Then the 3 independent renormalization constants a^R, b^R and c^R at $T = 0$, (20), become replaced by four parameters $a^R(T_0)$, $b^{R,0}(T_0)$, $b^{R,1}(T_0)$ and $c^R(T_0)$ at T_0, cf. (27), corresponding to four relevant couplings. However, starting from an $O(4)$-symmetric zero temperature theory we have proved that

$$L^{\Lambda,\Lambda_0}(\underline{\varphi}; T_0) - L^{\Lambda,\Lambda_0}(\underline{\varphi}) \tag{47}$$

has the properties of an irrelevant operator. This implies that for given $b^{R,0}(T_0)$ there is a unique choice for $b^{R,1}(T_0)$, or vice versa, such that the finite temperature theory stems from an $O(4)$-symmetric zero temperature theory. Any different choice would be associated to a zero temperature theory, where $O(4)$-symmetry is broken by hand through the renormalization conditions. Note that the $O(4)$-symmetric choice is generally not the one where $b^{R,0}(T_0) = b^{R,1}(T_0)$: Integration over Λ, starting from the same counterterms (the $O(4)$-symmetric ones) will lead to a finite difference at $\Lambda = 0$, since $O(4)$-invariance

[13] These bounds can be improved by replacing n by $\hat{n}$, defined to be the smallest even integer greater or equal to n.

is broken in the propagator. Otherwise stated, the fact that the finite temperature theory stems from an $O(4)$-symmetric zero temperature theory, can be simply recognized on inspection of the counterterms, but not on the renormalization conditions.

4 Summary

We have presented a proof for the perturbative renormalizability of massive finite temperature φ_4^4-theory. The starting point are the bounds (17,18) which prove the renormalizability of the zero temperature theory. In the flow equation framework they serve at the same time as induction hypotheses for the inductive proof. Bounds of this type have by now been rigorously established for nearly all theories of physical interest, including gauge theories, where the restoration of the Ward identities in the final theory pose an additional problem, to be solved by a suitable restriction on the renormalization conditions. Taking due care of the exceptional momentum problem, corresponding bounds can also be established for theories with massless particles.

To extend the bounds to the corresponding finite temperature theories presents no really new problems for the practitioner. The main problem to be solved rather is that the existence of the correlation functions in the large cutoff limit should be proven without changing the counterterms. In our setup this corresponds to posing the boundary conditions (30) for the difference Green functions $\mathcal{D}$ between the $T > 0$ and the $T = 0$ theories. The anounced result is contained in the bounds (32,33). The main new technical tool used to get there is the Euler-MacLaurin formula, generalized to an infinite integration interval for a rapidly decaying integrand. It is applied to the difference terms appearing in the flow equations for the functions $\mathcal{D}$ that are not bounded by the induction hypothesis alone, (see (34)- (40)). Here it comes to our help that the bounds (17,18) are sufficiently powerful so as to transform momentum derivatives into negative powers of Λ. Via the Euler-MacLaurin formula it is then possible to gain an arbitrary power in Λ paying the corresponding power in T (see 39). This achieves (far more than) showing that all difference functions $\mathcal{D}$ are irrelevant. For the latter a gain of a power of $\Lambda^{2+\varepsilon}$ would have sufficed. We emphasize again that our result agrees with the experience and intuition gained from explicit perturbative calculations.

Renormalization is a central issue that is strongly related to the fundamental principles of local quantum field theory. Renormalizability of a field theory gives it a meaning beyond some low energy effective model. The techniques we have presented here for proving renormalizability of a field theory at finite temperature mainly rely on two properties. The first property is renormalizability at zero temperature. The second one is that the difference between the theory at finite and zero temperature act like an irrelevant operator that does not spoil renormalizability. Renormalization group flow equations provide an appropriate tool to put this statement on a strong basis and prove renormalizability for finite temperature. We expect that these methods generalize appropriately to apply to more realistic and complex field theories such as QCD, where both the UV and the IR scale problem are to be attacked.

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