

# Matrices with the consecutive ones property, interval graphs and their applications

## Master Thesis

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# Introduction

Matrices with the consecutive ones property and interval graphs are important notations in the field of applied mathematics.

We give a theoretical picture of them in first part. We present the earliest work in interval graphs and matrices with the consecutive ones property pointing out the close relation between them. We pay attention to Tucker's structure theorem on matrices with the consecutive ones property as an essential step that requires a deep considerations. Later on we concentrate on some recent work characterizing the matrices with the consecutive ones property and matrices related to them in the terms of interval digraphs as the latest and most interesting outlook on our topic. Within this framework we introduce a classification of matrices with consecutive ones property and matrices related to them.

We describe the applications of matrices with the consecutive ones property and interval graphs in different fields. We make sure to give a general view of application and their close relation to our studying phenomena. Sometimes we mention algorithms that work in certain fields.

In the third part we give a polyhedral approach to matrices with the consecutive ones property. We present the weighted consecutive ones problem and its relation to Tucker's matrices. The constraints of the weighted consecutive ones problem are improved by introducing stronger inequalities, based on the latest theorems on polyhedral aspect of consecutive ones property. Finally we implement a separation algorithm of Oswald and Reinhelt on matrices with the consecutive ones property.

We would like to mention that we give a complete proof to the theorems when we consider important within our framework. We prove theorems partially when it is worthwhile to have a closer look, and we omit the proof when there are is only an intersection with our studying phenomena.

# Chapter 1

## Basic theoretical results

The aim of this chapter is to give an important part of basic theoretical work on interval graphs, interval digraphs and their connection to consecutive ones matrices(C1P).

### 1.1 Interval graphs, their characterization and connection with consecutive ones property

We take the reference of this section from [R1] and [S1] (see the bibliography). A part of figures are taken from [S1]. We begin by giving the definition of **intersection graph**. Throughout we denote in the following graphs by  $G = (V(G), E(G))$ , where  $V(G)$  is the set of vertices and  $E(G)$  is the set of its edges. Let  $M$  be the family of sets.

**Definition 1.1** *Then the **intersection graph**  $G$  of  $M$  satisfies:  $V(G) = M$  and if  $S, T \in M$  with  $S \neq T$ , then.*

$$S, T \in E(G) \iff S \cap T \in \emptyset$$

Example. See Fig 1.1

$M = S1, S2, S3, S4$  with

$S1 = (a, b, c, d), S2 = (e, f, g, h), S3 = (a, b, g, h), S4 = (a, d, e, f)$

A graph  $G$  is an **interval graph** if it is the intersection graph of some family of intervals on the real line.

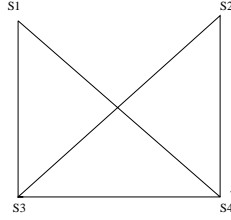


Figure 1.1: An intersection graph

**Definition 1.2** That is,  $G = (V, E)$  is an **interval graph** if and only if there is an assignment  $J$  of a real line  $J(u)$  to each  $u \in V$  such that  $\forall (u \neq v) \in V$ .

$$(u, v) \in E(G) \iff J(u) \cap J(v) \neq \emptyset$$

**Definition 1.3** If the intervals of a real line have unit lengths, then we call the graph a **unit interval graph**

**Definition 1.4** A **proper interval graph** is constructed from a family of intervals on a line such that no interval properly contains another.

In 1969 Roberts showed that that a graph is unit interval graph iff it is a proper interval graph.

Example. See Fig.1.2

Note that Fig.1.3 is an important example, *an interval assignment for a cycle graph with number of arcs  $k \geq 4$  does not exist.*

Consider the following generalizations of the notion of intervals in line, **circular-arc graphs**.

**Definition 1.5** We call a graph a **circular-arc graph** if it is an intersection graph of a family of arcs in a circle.



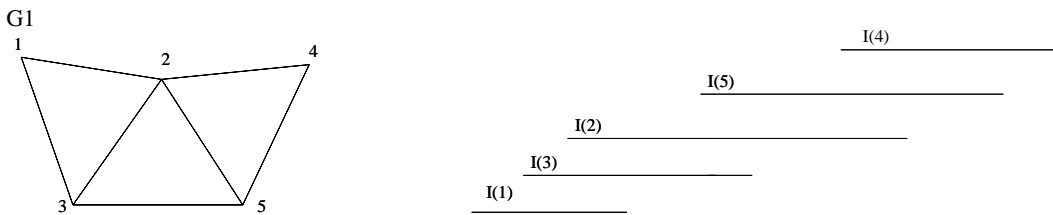


Figure 1.2: An interval graph and interval assignment for it

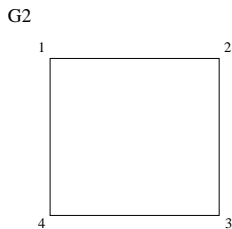


Figure 1.3: An interval assignment for graph G2 does not exist

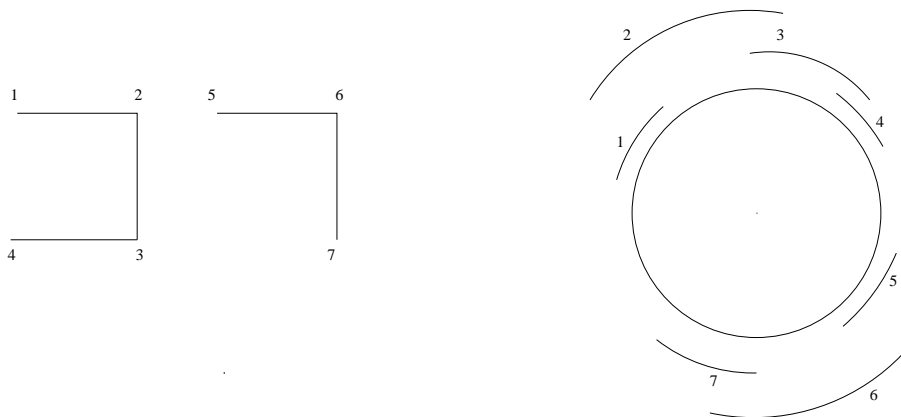


Figure 1.4: A circular-arc graph and interval assignment of it

We give an illustration in Fig.1.4.

Interval graphs possess a lot of properties. Here we mention some of them. *The hereditary property:* An induced graph of an interval graph is an interval graph.

**Definition 1.6** *The triangulated graph property:* Every simple cycle of length strictly greater than three possesses a chord. Graphs which satisfy this property are **triangulated graphs**.

**Theorem 1.1** *Hajos 1958.* An interval graph satisfies the triangulated graph property

**Definition 1.7** *The transitive orientation property:* Graph  $G=(V,E)$  satisfies the following condition:

$$(ab \in E \text{ and } bc \in E) \text{ imply } ac \in E \quad (\forall a, b, c \in V)$$

If the graph satisfies the transitive orientation property it is called **comparability digraph**.

**Theorem 1.2** *Ghouila, Hourii 1968.* The complement of an interval graph satisfies the transitive orientation property

Now we will meet the first important theorem on interval graphs.

**Theorem 1.3** *Gilmore, Hoffman 1964.* An undirected graph  $G$  is an interval graph iff  $G$  is a triangulated graph and its complement  $\overline{G}$  is a comparability graph.

We illustrate the theorem in the Fig.1.5.

Let explain the notations that we will use in the next theorem.

**Definition 1.8** A graph is **complete** if every two distinct nodes are adjacent.

**Definition 1.9** A complete subgraph or graph is usually denoted by  $K_n$ , and is called **a clique** of size  $n$ .

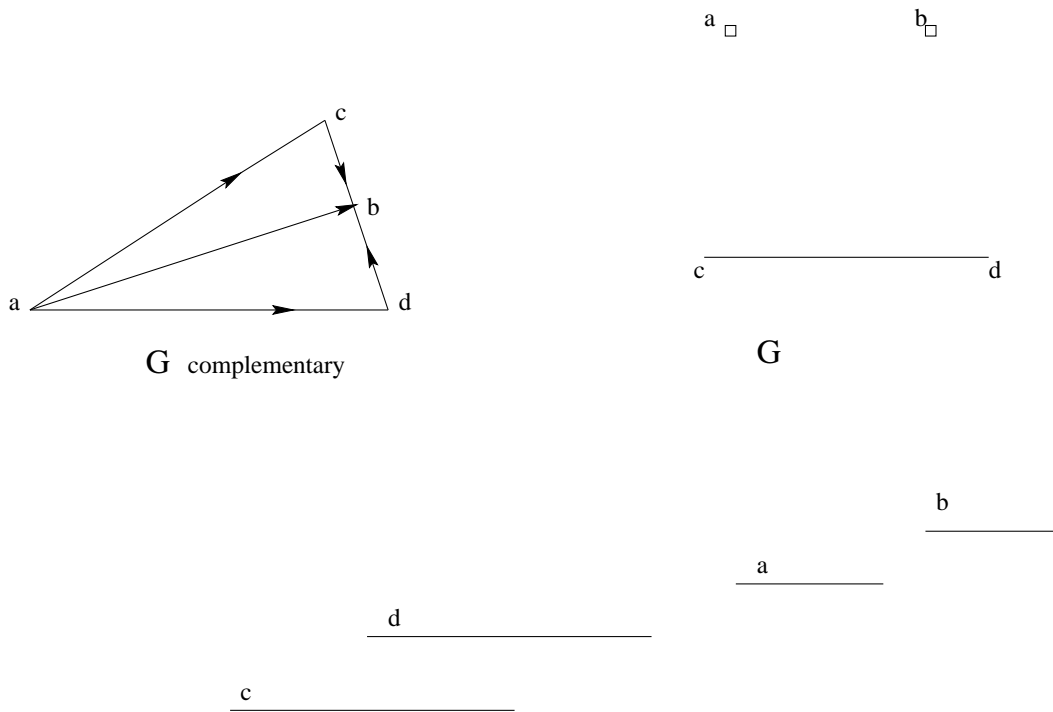


Figure 1.5: Graph  $G$ , transitive orientation of the complementary of  $G$ , and interval assignment of  $G$ .

**Definition 1.10** *The clique matrix of graph  $G=(V,E)$  is a  $(0,1)$  matrix that satisfies the following; if we put the vertices  $V=1,2,\dots,n$  in the columns and  $C_1, C_2, \dots, C_n$  that are all the maximal cliques numbered arbitrarily in the rows then  $a_{ij} = 1$  iff  $j \in C_i$ .*

See the Fig 1.8 for example of *clique matrix*.

**Definition 1.11** *A  $(0,1)$  matrix  $A$  has the **consecutive ones property** (for rows) if ones in each row appear consecutively or the columns of  $A$  can be permuted so that ones in each row appear consecutively.*

For convenience we say that  $A$  is **C1P**. We give one example of the matrices with consecutive ones property in Fig 1.6.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1.6: An example of matrix with consecutive ones property for rows

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Figure 1.7: An example of matrix with circular ones property

**Theorem 1.4** *Fulkerson, Gross 1965. An undirected graph is interval iff its clique matrix has the consecutive ones property.*

We give a simple illustration of the theorem in Fig 1.8.

**Definition 1.12** *A  $(0,1)$  matrix  $A$  has the **circular ones property** (for rows) if ones in each row appear in a circular consecutive order or the columns of  $A$  can be permuted so that ones in each row appear circular in a consecutive order.*

In order to understand clearly the definition above, *imagine a matrix wrapped around a cylinder.* We give one example of matrix with circular ones property in Fig 1.7. We give only one theorem on circular-arc interval graphs as we will mention them very rarely in the upcoming study.

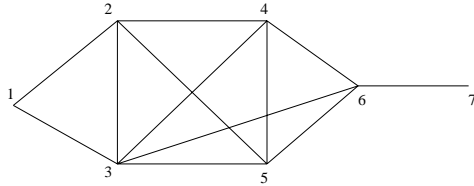
**Definition 1.13** *The adjacency matrix  $A(G)$  of graph  $G = (V, E)$  and a vertex ordering of  $V$  is the  $(0,1)$  matrix where each entry  $a_{ij}$  satisfies:*

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 0 & \text{otherwise} \end{cases}$$

**Definition 1.14** If  $M$  is an  $n$  by  $n$  matrix and the entries of main diagonal are all 0, then **the augmented matrix  $M^*$  to  $M$**  is obtained by  $M$  by adding 1's along the main diagonal.

**Theorem 1.5** Tucker 1970. If the augmented adjacency matrix  $A^*(G)$  of  $G$  has the circular ones property for columns, then  $G$  is a circular-arc graph.

No proof, but note that the converse is not true.



vertices	1	2	3	4	5	6	7
{1,2,3}	1	1	1	0	0	0	0
{2,3,4,5}	0	1	1	1	1	0	0
{3,4,5,6}	0	0	1	1	1	1	0
{6,7}	0	0	0	0	0	1	1

Figure 1.8: A graph and its clique matrix that has the consecutive ones property

**Definition 1.15** In undirected graph  $G$  is, a triplet of vertices  $(x,y,z)$  is called **asteroidal triplet** if  $G_{(x,y,z)}$  is **stable** (= vertices  $(x,y,z)$  are not adjacent with each other) and for each pair from  $(x,y,z)$  it exists a path between these two vertices which does not pass through neighbors (=set of adjacent vertices) of the third.

We have defined *triangulated graphs*. Triangulated graphs can be also called *chordal graphs*.

**Theorem 1.6** *Lekkerkerker, Boland 1962.  $G$  is interval graph iff it is chordal and asteroidal triplet free.*

*Proof: Only  $\Rightarrow$*

Suppose we have an interval graph  $G$  and  $(x,y,z)$  is an asteroidal triplet. As  $G$  interval there exists a sequence of the intervals  $z = v_1, \dots, v_k = x$  such that  $\forall 1 \leq i \leq k-1$  we have  $I_{v_i} \cap I_{v_{i+1}} \neq \emptyset$ . On the other hand if  $\exists 2 \leq j \leq k-1$  such that  $I_y \cap I_{v_j} \neq \emptyset$ . So there is no path between  $x$  and  $z$  that does not cross the neighbor of  $y$ . This is equivalent to the fact that  $(x,y,z)$  is not a asteroidal triplet, so we have a contradiction. We illustrate the proof in the Fig.1.7



Figure 1.9: Example of Lekkerkerker-Boland theorem

## 1.2 Tucker structure theorem on matrices with consecutive ones property and interval graphs

The reference and figures of this section are taken from [T2]. We have already defined what is the matrices with C1P. Let give another formulation by simply considering bipartite graphs and asteroidal triplets. Given an  $m \times n$  (0,1) matrix  $M$ , we can associate with it an un-oriented bipartite graph  $G = (V_1, V_2, A)$  where  $V_1$  and  $V_2$  are sets of vertices and  $A$  is the symmetric adjacency matrices. relation defined in  $V_1 V_2$ , such that,

**Definition 1.16**  *$A$  is the symmetric adjacency matrix. The relation defined in  $V_1 V_2$  is such that,*

$$x_i \in V_1, y_j \in V_2 \iff x_i A y_j \iff a_{i,j} = 1$$

**Definition 1.17** *The consecutive ones property for columns has an equivalent formulation in  $G$ , namely, the vertices of  $V_1$  can be ordered so that, for each  $x$  in  $V_2$ ,  $N(x) = y : x A y$  is a consecutive set of vertices in  $V_1$  or  $N(x) = \emptyset$ . We call such an ordering  $V_1$  **consecutive arrangement of  $G$***

We will give some simple notations and go on with some important theorems. Given the connected bipartite graph  $G = (V_1, V_2, A)$ , we define  $d(x,y)$  to be the length of the shortest path from  $x$  to  $y$ . We define  $\delta(G)$  the  $V_1$ -diameter of  $G$ , to be.

$$\delta(G) = \sup_{x, y \in V_1} d(x, y)$$

When  $x, y \in V_1$  and  $d(x, y) = \delta(G)$ , then  $x$  and  $y$  are called  $V_1$  diameter points of  $G$ . Note that the  $V_1$  diameter is always even as we are moving in the bipartite graph among vertices of  $V_1$ . We write  $G-x$  to denote the subgraph of  $G$  obtained by deleting vertex  $x$  and edges adjacent to  $x$ .

**Lemma 1.1** *Tucker 1972. If  $G = (V_1, V_2, A)$  is a connected bipartite graph and  $x$  is a  $V_1$ diameter point of  $G$ , then  $G-x$  is connected.*

**Lemma 1.2** *Tucker 1972. Let  $G = (V_1, V_2, A)$  be a connected bipartite graph where  $v_1$  contains no asteroidal triple of  $G$ , and let  $p$  be a  $V_1$ -diameter point of  $G$ . Suppose  $G-p$  has  $V_1$ -consecutive arrangement  $R$  in which  $x$  and  $y$  are the left and right end vertices respectively. Then,  $d(x, p) = \delta(G)$  or  $d(y, p) = \delta(G)$ .*

*We omit the proof*

**Theorem 1.7** *Tucker 1972. A bipartite graph  $G = (V_1, V_2, A)$  has a  $V_1$  consecutive arrangement iff  $V_1$  contains no asteroidal triplet of  $G$ .*

*We omit the proof but see the similarity with Lekkerkerker-Boland Theorem 1.6*

**Theorem 1.8** *Tucker 1972. A bipartite graph  $G = (V_1, V_2, A)$ ,  $V_1$  contains no asteroidal triplet of  $G$  iff  $V_1$  contains none of the bipartite subgraphs  $A_n, B_n, C_n, D, E$  in Fig 1.10.*

*Proof:*

First we need some definitions:

**Definition 1.18** A graph  $G(V, A)$  is said to be **strongly connected** if  $\forall x, y \in V$  there exists a path  $\mu_1(x, y)$  and a path  $\mu_2(y, x)$ .

**Definition 1.19** A graph  $G$  is called **minimal** if it is strongly connected and the removal of any arcs destroy the strongly connected property.

We will give the proof partially and try to clarify *the general mechanism of it*.

The necessity is obvious. For sufficiency, it is clearly enough to prove the theorem for a minimal  $G$ . That is, we assume that in every proper bipartite subgraph  $G' = (V'_1, V'_2, A)$  of  $G$ ,  $V'_1$  contains no asteroidal triple.

Let  $P_{xy}$  be a primitive (chordless) path from  $x$  to  $y$  not adjacent to  $z$ , so an asteroidal triple. Define  $P_{xz}$  and  $P_{yz}$  in the same way. By minimality of  $G$ , these three paths contain all vertices in  $G$ . The main idea of the proof is: **this asteroidal triple has to be one of the form of Fig 1.10.**

Let  $(r, P_1, s_1, s_2, P_2, t)$  denote the path (or circuit if rAt) obtained by following the path  $P_1$  (perhaps in reverse direction) from  $r$  to  $s_1$ , then passing to  $s_2$ , and then along  $P_2$  to  $t$ .

Observe that the minimality of  $G$  excludes the possibility of the sort: for some  $r$  in  $P_{yz}$ ,  $x, y, z$  are asteroidal triple of  $G - r$  because there is a path  $P$  in  $G - r$  from  $y$  to  $z$  not adjacent to  $x$ , for short we say “ $P$  replaces  $P_{yz}$  in  $G - r$ ”.

Let  $x, y, z$  be named so that  $|P_{yz}| \geq \max(|P_{xy}|, |P_{xz}|)$ .

1) It is obvious then if  $|P_{yz}| = 2$  there results an  $I_1$  graph of Fig 1.10.

2) So we assume that  $|P_{yz}| > 2$  and let  $(q_0, r_1, q_1, r_2, \dots, r_n, q_n)$  where  $q_0 = y$  and  $q_n = z$ . Suppose a vertex  $p_1$  on  $P_{xy} - P_{xz}$  is adjacent to some  $p_2$  on  $P_{xz} - P_{xy}$ . Let such  $p_1$  be as close as possible to  $x$ . Then the circuit  $(p_1, P_{xy}, x, P_{yz}, p_2)$  is an  $I_n$  graph of Fig 1.10 if it does not happen that  $p_1$  (or  $p_2$ ) is adjacent to  $x$  and the other is distance two from  $x$ .

3) Suppose  $p_2 A x$  and  $P_{yz} = (x, t, p_1, \dots, p_1, y)$ . See Fig 1.11. Now  $p_2 A z$  (i.e.,  $P_{xz} = (x, p_2, z)$ ) or else  $(x, p_2, p_1, P_{xy}, y)$  replaces  $P_{xy}$  in  $G - t$ . Next note that  $p_1$  is adjacent to  $P_{yz}$  or else  $p_1, y, z$  are an asteroidal triple in  $G - x$ . But if  $p_1 A r_i$  for  $i > 1$ , we then get a path which replaces  $P_{xz}$  in  $G - p_2$ . Thus  $p_1 A r_1$  and we have an  $I_n$  (see Fig 1.10) subgraph of the form  $(p_1, r_1, P_{yz}, q_k, p_2)$  or else  $p_2 A q_i$ ,  $0 < i \leq n$  and then  $(x, t, p_1, p_2, y, r_1, q_1, r_2, q_2, y)$  form a V graph, (see Fig 1.10) with  $y, x, q_2$  being the asteroidal triple.

The rest of the proof is going to deal with *all possible positions in graph  $G$  of  $p_1$  and  $p_2$* , proving **that any way comes out to a graph of Fig 1.10** so we *omit the rest of the proof*.



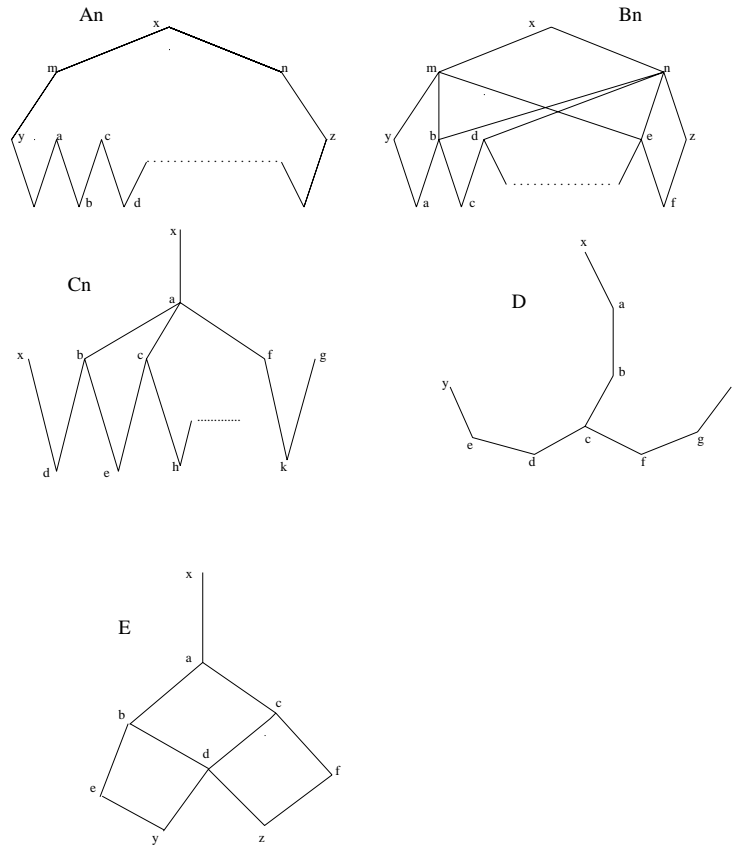


Figure 1.10:  $x, y, z$  are asteroidal triples in each graph

**Theorem 1.9** *Tucker 1972. A bipartite graph  $G = (V_1, V_2, A)$  has  $V_1$  consecutive arrangement iff  $G$  contains none of the bipartite subgraphs  $An, Bn, Cn, D, E$  in Fig 1.10.*

*No proof*

**Theorem 1.10** *Tucker 1972. The  $(0,1)$  matrix  $M$  has the consecutive ones property for columns iff no sub-matrix of  $M$  is a member of configurations  $M(An), M(Bn), M(Cn), M(D), M(E)$  in Fig 1.12. .*

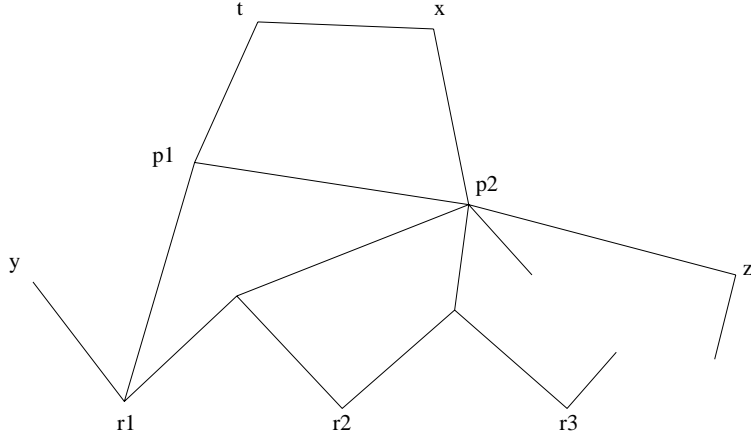


Figure 1.11: Bipartite graph containing a V bipartite subgraph

The main idea of the theorem is to use Theorem 8 and the Fulkerson Gross theorem on interval graphs while keeping in mind that  $M_1, M_2, M_3, M_4, M_5$  are the clique-vertex matrix of graphs that contain asteroidal triple of Fig. 1.10. According to the Lekkerker Boland theorem this is a condition for not being interval graph.

**Theorem 1.11** *Tucker 1972. A bipartite graph  $G = (V_1, V_2, A)$  has  $V_1$  and  $V_2$  consecutive arrangement iff  $G$  contains none of the bipartite subgraphs  $A_n, B_1, B_2, C_1, C_2, C_3$  or their transposes in Fig 1.10.*

Observe that the  $A_n$  subgraph for  $n \geq 2$ , the  $B_n$  graphs for  $n \geq 3$  and the  $C_4$  subgraph all contain the transpose of the  $A_1$  subgraph. Hence this theorem follows from Theorem 1.8.

**Theorem 1.12** *Tucker 1972. The  $(0,1)$  matrix  $M$  has the consecutive ones property for rows and columns iff no submatrix of  $M$ , or transpose of  $M$  is a member of configuration  $M(A_n), M(B_1), M(B_2), M(C_1), M(C_2), M(C_3)$  in Fig 1.12*

It comes out immediately from from Theorem 1.10 and Theorem 1.11.

$$\mathbf{M}(\mathbf{A}_n) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & \dots & \dots & 1 & 1 \end{pmatrix} \quad \mathbf{M}(\mathbf{B}_n) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & \dots & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}$$

$$\mathbf{M}(\mathbf{C}_n) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & 1 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$\mathbf{M}(\mathbf{D}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathbf{M}(\mathbf{E}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 1.12: Forbidden Tucker's matrices

### 1.3 Characterization of unit interval graphs and proper interval graphs

The reference and figures of this section are from [S1]. We have defined before what *unit and proper interval graphs* are. Now we will mentioned some essential results on *unit and proper interval graphs* and their connection to matrix with consecutive ones property.

**Theorem 1.13** *Roberts 1969.* Let  $G = (V, E)$  be an undirected graph. The following are equivalent:

- (1) There exists a real valued function  $u : V \rightarrow \mathbb{R}$  such that  $\forall xy \in E, x \neq y, xy \in E \iff |u(x) - u(y)| < 1$
- (2)  $G$  is a proper interval graph
- (3)  $G$  is a unit interval graph
- (4)  $G$  is an interval graph that doesn't contain an induced  $K_1^3$ .

We will partially give the proof.

1  $\Rightarrow$  6

Let  $u$  be a function that satisfies (1). We will match to each vertex  $x \in V$  an open interval  $I_x = [u(x) - 1/2, u(x) + 1/2]$ . Then:

$$\forall x, y | x \neq y, I_x \cap I_y \neq \emptyset \iff |u(x) - u(y)| < 1 \iff xy \in E$$

Therefore the  $[I_x]_{x \in V}$  is a unit interval realization of  $G$ .

2  $\Rightarrow$  3

In a unit interval realization of  $G$ , no interval can properly contain another, because they all have lengths 1. Therefore this realization will be proper.

2  $\Rightarrow$  4

Let  $[I_x]_{x \in V}$  be a proper interval realization of  $G$ . Suppose  $G$  contains an induced subgraph  $G_{(y, z_1, z_2, z_3)}$  isomorphic to  $K_1^3$  where  $(z_1, z_2, z_3)$  is a stable set and  $y$  is adjacent to each  $z_i$ . Without loss of generality, suppose that  $I_{z_1} < I_{z_2} < I_{z_3}$ . Then  $I_y$  must properly contain  $I_{z_2}$ , a contradiction. Thus  $G$  can not contain an induced copy of  $K_1^3$ .

*We omit the rest of the proof.*

Let  $G(V, E)$  be a unit interval graph. Then all the following are equivalent:

- The clique matrix of  $G$  satisfies the consecutive ones property for rows and columns, Roberts 1968
- Let  $A^*(G)$  be the adjacency matrix of  $G$  when 1's are added on the diagonal, then  $A^*(G)$  satisfies then consecutive 1's property for columns, Roberts 1968.
- $G$  is triangulated and it doesn't contain as an induced subgraph any of the graphs of Fig. 1.10, Wegner 1967

No proof.

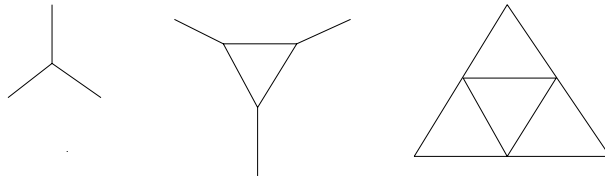


Figure 1.13: Forbidden graphs of Wegner,1967

## 1.4 Interval digraphs matrices with the consecutive ones property and matrices related to them

The reference of this section and figures belong to [SSRW],[SDRW] and [LSW]. In the following part we will introduce **interval digraphs** or **interval directed graphs** and their connection to matrices with the consecutive property or matrices related to them.

**Definition 1.20** A digraph is **Ferrers digraphs** or **Ferrers relations** iff when  $a,b$  and  $c,d$  edge of Ferrers digraph implies  $a,d$  or  $c,b$  is edge of digraph too.

Note that being a Ferrers digraphs is equivalent with the fact that, *the rows and columns of the adjacency matrix of Ferrers digraph can be independently permuted so that the positions of 1's forms a Ferrers diagram.* We explain that the ones in Ferrers diagram can be placed in a low left corner in such a way that every position below or left to each 1's is again 1.

**Theorem 1.14** *Sen, Das, Row, West 1989. For a digraph D the following are equivalent:*

- (a) *D is a interval digraph*
- (b) *The rows of  $A(D)$  can be (independently) permuted so that each 0 can be replaced by one of  $(R, C)$  in such a way that every R has only R's to the right and every C has only C's below it.*
- (c) *D is the intersection of two Ferrers digraphs whose union is complete.*

The matrix that satisfies the (b) has **partitionable zeros property**.  $A(D)$  is just *the adjacency matrix* mentioned before.

We give a complete proof of the theorem above as it is especially important for *connection of the digraphs with matrices with consecutive property and matrices related to them.* Throughout the proof we will use different symbols so we will again give the definition of *intersection graph and interval graph.* We define the *intersection digraph* as the family of ordered pairs of sets in the digraph such that:

$$u, v \in E \iff S_u \cap T_v \neq \emptyset \text{ when}$$

$$S_u \text{ is source set}$$

$$T_v \text{ is the terminal set or sink set}$$

In the case of *an interval digraph*  $S_u$  and  $T_v$  are sets of intervals on real line.

*Proof:*

$$1 \ A \Rightarrow B$$

Consider an interval representation of D, with vertex v assigned to source set  $S_v [a(v), b(v)]$  and sink set  $T_u [c(v), d(v)]$ . Order the rows and columns of  $\mathbf{A}(D)$  as  $u_1, \dots, u_n$  and  $w_1, \dots, w_n$ . If the entry  $i, j$  of the resulting matrix is 0, then  $a(u_i) > d(w_j)$  or  $c(w_j) < b(u_i)$ , but  $a(v) \leq b(v)$  and  $c(v) \leq d(v)$  implies that the two first inequalities can not both hold. Relabel the position by C

if  $a(u_i) > d(w_j)$  and R if  $c(w_j) < b(u_i)$ . Since a's and c's increase with index the first inequality will continue to hold below this entry and the second to the right of this entry.

2  $B \Rightarrow C$

Let  $Q_1, Q_2$  denote the sets of edges of  $\bar{D}$  corresponding to positions of M labelled R,C, respectively. Let  $F_1, F_2$  be digraphs defined by  $E(F_i) = D \cup Q_i$ . Then  $F_1, F_2$  are Ferrer's digraphs whose intersection is D and union is complete.

3.  $C \Rightarrow A$

We will work with the complements of  $F_1, F_2$ , which we shall call  $H_1, H_2$ , whose union is  $\bar{D}$  and the intersection is empty. Note from the

adjacency matrix that the complement of Ferrers digraph is again a Ferrers

digraph.

Let  $(A_0, \dots, A_{p-1})$  be the source partition for  $H_1$  and let  $(D_1, \dots, D_p)$  be its terminal partition. Note that  $A_0, D_p$  can be empty. Let  $(C_0, \dots, C_{q-1})$  be the source partition for  $H_2$  let  $(B_1, \dots, B_p)$  be its terminal partition. Note that  $C_0, B_p$  can be empty. It is important to notice that:

$$\begin{aligned} u, v \in H_1 &\iff i \geq l \text{ where } u \in A_i \text{ and } v \in B_l \\ u, v \in H_2 &\iff j \leq k \text{ where } u \in B_j \text{ and } v \in C_k \end{aligned}$$

We will construct an interval representation of  $D$  by assigning to the vertex  $v$  the intervals  $S_v = [a(v), b(v)], T_v = [c(v), d(v)]$  where  $[a(v), b(v), c(v), d(v)] = (a_i, b_j, c_k, d_l)$  if  $v \in A_i, B_j, C_k, D_l$ . The  $S_v, T_v$  are intervals that have to satisfy:

- $a_i < b_j$  and  $c_k < d_l$  if  $v \in A_i \cup B_j$  and  $v \in C_k \cup D_l$
- $u, v \in D \iff a(u) \leq d(v)$  and  $b(u) \geq c(v)$
- $u, v \notin H_1 \cup H_2 \iff u, v \in \bar{H}_1 \cap \bar{H}_2 \iff u, v \in D \iff$   
 $\iff (a, d \text{ satisfy } a_i \leq d_l \text{ for } i < l \text{ and } b, c \text{ satisfy } b_j \geq c_k \text{ for } j > l)$

To achieve the third request it suffices that a,b,c,d are strictly increasing sequences with  $a_i = d_i + 1$  and  $c_i = b_i + 1 \forall i \geq 1$ . To construct the sequences of  $S_v, T_v$  we create a *directed graph*  $M$  corresponding to set of partitions. Begin with directed paths  $(A_0, \dots, A_{p-1}), (B_1, \dots, B_p), (C_0, \dots, C_{q-1})$  and  $(D_1, \dots, D_p)$ .

Add edges  $A_i B_j$  when  $A_i \cap B_j \neq \emptyset, C_k D_l$  when  $C_k \cap D_l \neq \emptyset, B_i C_i$  for  $1 \leq i \leq q-1, D-iA_i$  for  $1 \leq i \leq p-1$ . It is essential to bear in mind that  $M$  is acyclic.

*Note:* Try to prove that for having a cycle we need  $H_1$  and  $H_2$  to be connected but this is a contradiction. Since  $M$  is acyclic there exists an integer numbering:  $f : V(M) \rightarrow N$  called a “topological ordering” ( $N$  is natural number set) such that  $xy \in E(M) \Rightarrow f(y) > f(x)$ .

It remains only to show that  $f(A_i) = f(D_i)+1$  and  $f(C_i) = f(B_i)+1$ ; the desired sequences a,b,c,d then appear in value of  $f$ . The natural algorithm is to assign a vertex  $X$  the number  $t$  if the longest path ending at  $X$  has  $t$  vertices. We prove that  $f(C_i) = f(B_i) + 1$ ; the same argument works to show  $f(A_i) = f(D_i) + 1$  also. For any  $C^i$  with  $i \geq 1$  the predecessors are  $C_{i-1}$  and  $B_i$ . For any path ending at  $B^i$  we can extend it to  $C_i$ , and we have  $f(C_i) = f(B_i) + 1$ . For the opposite inequality, we consider a longest path ending at  $C_i$ . We can go back to  $B_i$  as  $B_i C_i$  is an edge so  $f(B_i) = f(C_i) - 1$ . The proof is completed.

We give some definitions of new notations and later introduce some important theorems concerning the consecutive ones property.

**Definition 1.21** *A generalized complete bipartite subdigraph (abbreviated GBS) is a subgraph generated by vertex sets  $X, Y$  whose edges are all  $xy$  such that  $x \in X, y \in Y$ .*

We say **generalized** because sets  $X, Y$  need not to be disjoint which means loop may arise.

Let  $B = (X_k, Y_k)$  be a collection of GBS’s whose union is  $D$ . We define the *vertex-source incidence matrix for  $B$*  (abbreviated  $V, X$ -matrix) to be the incidence matrix between the vertices and source sets  $X_k$ . We define the *vertex-terminus incidence matrix for  $B$*  (abbreviated  $V, Y$ -matrix) to be the incidence matrix between the vertices and termin sets  $Y_k$ . Now we give an important characterization of interval digraphs.

**Theorem 1.15** *Sen, Das, Row, West 1989.  $D$  is an interval iff there is numbering of GBS’s in some covering  $B$  of  $D$  such that ones in rows appears consecutively for both  $V, X$ -matrix and  $V, Y$ -matrix of  $D$ .*

The main idea of sufficiency (proof) comes out in this way. Look closely the  $V, X$ -matrix and  $V, Y$ -matrix of  $D$ . If they possess the consecutive ones



$$\mathbf{L} = \begin{pmatrix} 1 & R & R & R & R & R \\ 1 & 1 & R & R & R & R \\ C & 1 & 1 & R & R & R \\ C & C & 1 & 1 & R & R \\ C & C & C & 1 & 1 & R \\ C & C & C & C & 1 & 1 \end{pmatrix}$$

Figure 1.14: Example of the matrix that has the monotone consecutive arrangement

property then the digraph that come out is the interval digraph For necessity try to see that every GBS can be seen as an interval digraph that brings the consecutive ones property for  $V, X$ -matrix and  $V, Y$ -matrix of  $D$ .

To obtain a proper set of intervals we need the resulting matrices to have the *proper consecutive ones property*(for rows).

**Definition 1.22** *The matrices have the proper consecutive ones property for rows if there exists a column ordering so that the 1's in each row appear consecutively and do not properly contain 1's in another row.*

**Theorem 1.16** *Sen, Sanyal 1994. A digraph is a proper interval digraph iff its edges can be covered by collection  $B$  of GBS's that can be indexed so that ones in the rows of the  $V, X$ -matrix and in the rows  $V, Y$ -matrix exhibits the proper consecutive ones property.*

**Definition 1.23** *Matrices with  $(0,1)$  has **the monotone consecutive arrangement** if the rows of it can be (independently) permuted so that each 0 can be replaced by one of  $(R, C)$  in such way that every  $R$  has only  $R$ 's to the right and above it and every  $C$  has only  $C$ 's to the left and below it.*

We give a example of a matrix that has *the monotone consecutive arrangement* in Fig 1.14:

**Lemma 1.3** *Sen,Sanyal 1994. A 0,1-matrix with nonzero rows has a monotone consecutive arrangement iff it has independent row permutations such that 1's appear consecutively in each row and the values  $(a_i)$  and  $(b_i)$  denoting the initial column and final column of interval of 1's in row  $i$  satisfy  $a_1 \leq \dots \leq a_n$  and  $b_1 \leq \dots \leq b_n$*

Let describe the main idea of the lemma. Let  $a_{ij} = 0$  and let label it C if  $j < a_i$  and R if  $j > b_i$ . By construction, the labelling is satisfied in rows, and monotonicity of sequences guarantees that it is satisfied in columns.

**Theorem 1.17** *Sen,Sanyal 1994.If  $D(V,E)$  is a digraph, the following conditions are equivalent:*

- (a) *D is a unit interval digraph.*
- (b) *D is a proper interval digraph.*
- (c) *The adjacency matrix of D has the monotone consecutive arrangement.*

See the similarity with the Theorem 1.13.

*Proof:*

We prove only  $a \Rightarrow b$  and  $b \Rightarrow c$  as the rest of proof is lengthy so we will omit it.

$a \Rightarrow b$

In unit interval realization of D, no interval can properly contain another, because they all have lengths 1. Therefore this realization will be proper.

$b \Rightarrow c$

By previous theorem we have a collection  $B = (X_k, Y_k)$  of GBS's that cover D such that, for V,X and V,Y-matrix the 1's in each row appear consecutively and do not properly contain 1's in another row. We may assume that each row has 1's otherwise we put in the end rows only with 0 so that they do not affect the monotone consecutive arrangement.

Let  $a(v), b(v)$  denote the first and last column containing 1 in the row of the V,X-matrix corresponding to  $v$ ; similarly define  $c(v), d(v)$  from the V,Y-matrix.

Index the vertices  $u_1, \dots, u_n$  so that  $a(u_1) \leq \dots \leq a(u_n)$ ; the proper consecutive ones property also implies that  $b(u_1) \leq \dots \leq b(u_n)$ . Similarly, index them as  $v_1, \dots, v_n$  so that  $c(u_1) \leq \dots \leq c(u_n)$  and  $d(u_1) \leq \dots \leq d(u_n)$ . Let  $a_i = a(u_i), b_i = b(u_i), c_i = c(u_i), d_i = d(u_i)$ .

$$\mathbf{F1} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \mathbf{F2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \mathbf{F3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Figure 1.15: Three forbidden matrices of theorem Lin,West 1995

We may assume that all sink sets and all source sets are not empty. We claim that the row ordering  $u_1, \dots, u_n$  and column ordering  $v_1, \dots, v_n$  of the adjacency matrix exhibits the monotone consecutive arrangement. The edges with tail  $u_i$  are covered by the GBS's with source sets  $X_{a_i} \cup \dots \cup X_{b_i}$ ; hence the successors of  $u_i$  are  $Y_{a_i} \cup \dots \cup Y_{b_i}$ ; Let  $\alpha_i = \min[j : a_i \in (c_j, d_j)]$  and  $\beta_i = \max[j : b_i \in (c_j, d_j)]$ . The proper consecutive ones property of the V,Y-matrix (lack of zeros except possibly only at the end) implies that the ones in the row  $i$  of the adjacency matrix are  $[j : \alpha_i \leq j \leq \beta_i]$ . It also implies that  $\alpha_i \leq \alpha_{i+1}$  because  $a_i \leq a_{i+1}$  and  $\beta_i \leq \beta_{i+1}$  because  $b_i \leq b_{i+1}$ . That completes the part of the proof that we intended to do.

**Theorem 1.18** *Lin, West 1995. A zero partitionable matrix has an monotone consecutive arrangement(MCA) iff does not contain one of the three forbidden submatrices or their transposes in Fig 1.15*

No proof, but we would like only to stress that matrix F3 is the incidence matrix for the graph  $K_1^3$  mention before in theorem of Roberts, 1968 that is *interval graph* so has the *consecutive ones property for columns but not for rows* as  $K_1^3$  is not **unit interval graph**.

We have already mentioned the matrices that are *zero partitionable (or weak monotone consecutive arrangement(MCA))* and we have given definition for matrices *with the monotone consecutive arrangement*. We will repeat the definition for matrices that are *zero partitionable* as we want to introduce a classification of Lin, Sen, West (1997) of above matrices and connection with matrices with *consecutive ones property for rows, columns or for both rows and columns*. It is small difference of classification of Lin, Sen, West (1997) and our classification, we include in our classification a new type of matrices the so-called **zero partitionable transpose**, see Def 1.25.

$$\mathbf{Q} = \begin{pmatrix} 1 & R & R & R & R & R \\ 1 & 1 & R & R & R & R \\ C & 1 & 1 & 1 & 1 & R \\ C & 1 & 1 & 1 & R & R \\ C & 1 & C & 1 & 1 & R \\ C & C & C & C & 1 & 1 \end{pmatrix}$$

Figure 1.16: Example of matrix that is zero partitionable

**Definition 1.24** *Matrices with  $(0,1)$  is **zero partitionable** if the rows of it can be (independently) permuted so that each 0 can be replaced by one of  $(R,C)$  in such way that every  $R$  has only  $R$ 's to the right and every  $C$  has only  $C$ 's below it.*

We give a example of a matrix that is *zero partitionable* in Fig 1.16:

Note that a *matrix with consecutive ones property for rows is zero partitionable*, for this we need only to list the rows in nondecreasing order of the position of their left most ones. It is simple to note that the inverse is not true, the example in Fig 1.16 makes it clear.

We will introduce another type of matrices, the *transpose matrices of zero partitionable*. We call this matrix **the zero partitionable transpose**.

**Definition 1.25** *Matrices with  $(0,1)$  is **zero partitionable transpose** if the rows of it can be (independently) permuted so that each 0 can be replaced by one of  $(R,C)$  in such way that every  $R$  has only  $R$ 's above it and every  $C$  has only  $C$ 's to left of it.*

We are not going to give an example on matrices that are zero partitionable transpose, but transpose of matrix in Fig 1.16 is *zero partitionable transpose* Note that a *matrix with consecutive ones property for columns is zero partitionable transpose*, for this we need only to list the rows in nondecreasing order of the position of their left most ones. It is simple to note that the inverse is not true, check the transpose of matrix in Fig 1.16.

See Def 1.23 for matrices with  $(0,1)$  that have **the monotone consecutive arrangement**.

$$\mathbf{N} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Figure 1.17: Matrix N

*It is worthwhile to note that monotone consecutive arrangement implies the consecutive ones property for rows and columns but the inverse does not hold. We illustrate it with one example.*

The matrix N in Fig 1.17 has consecutive ones property for rows and columns but it does not have the monotone consecutive arrangement property. Let give a direct and simple proof of it. By symmetry of  $v_1$  and  $v_3$  in the digraph represented by matrix N, we may assume that 0 position (1,3) is labelled R and 0 in position (3,1) is labelled C. *Note that row 2 has all 1's so must be under row 1 and over row 3 in any monotone consecutive arrangement.* It is clear that position (3,4) can not be neither C nor R that complete the proof.

Let give a symbol to every class of matrices:

**I** zero partitionable

**T** zero partitionable transpose

**R** having the consecutive ones property for rows

**C** having the consecutive ones property for columns

**B** having the consecutive ones property for rows and columns

**U** having the monotone consecutive arrangement

Based in what we have said up to know we can say:

$$\mathbf{I, T} \supset \mathbf{R, C} \supset \mathbf{B} \supset \mathbf{U}$$

We introduce the classification in diagram of Fig 1.18:

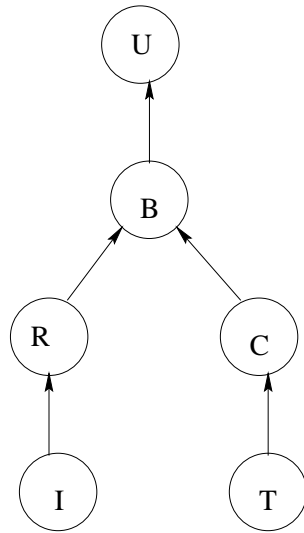


Figure 1.18: Classification diagram for matrices with the consecutive ones property and matrices related to them

# Chapter 2

## Application of Interval Graphs

The aim of this chapter is to give a general view of the application of interval graphs and the consecutive ones property on different fields

### 2.1 Application to traffic light control

We refer in this section to [R1],[R3] and [St]. Let us consider the problem of phasing traffic signals at simple intersection. We can assign to each *traffic stream* a given interval time during which it has green light. This can be done by *circular-arc interval graphs*, recall Def. 1.5. We shall suppose only the simple situation where the light is either green or red. Now certain streams are compatible: those which are not in collision course or there are not too much of hazard or causes for delay. For example in Fig 2.1 traffic stream 1 is compatible with traffic stream 2 but not with traffic stream 5 that would collide with it.

It is up to traffic engineer to make decisions about compatibility before traffic lights are phased. Made up this decision the main point is that if two streams are compatible, the corresponding green lights may overlap. An assignment to each traffic stream of an arc to clock circle is called a *feasible green light assignment* if only compatible traffic streams get overlapping arcs.

*How does one find a feasible assignment?*

Before describing the proceeding to find a feasible assignment we define what is *spanning subgraph*.

**Definition 2.1** *The graph  $H$  is called spanning subgraph of  $G$  if they have the same set of vertices.*

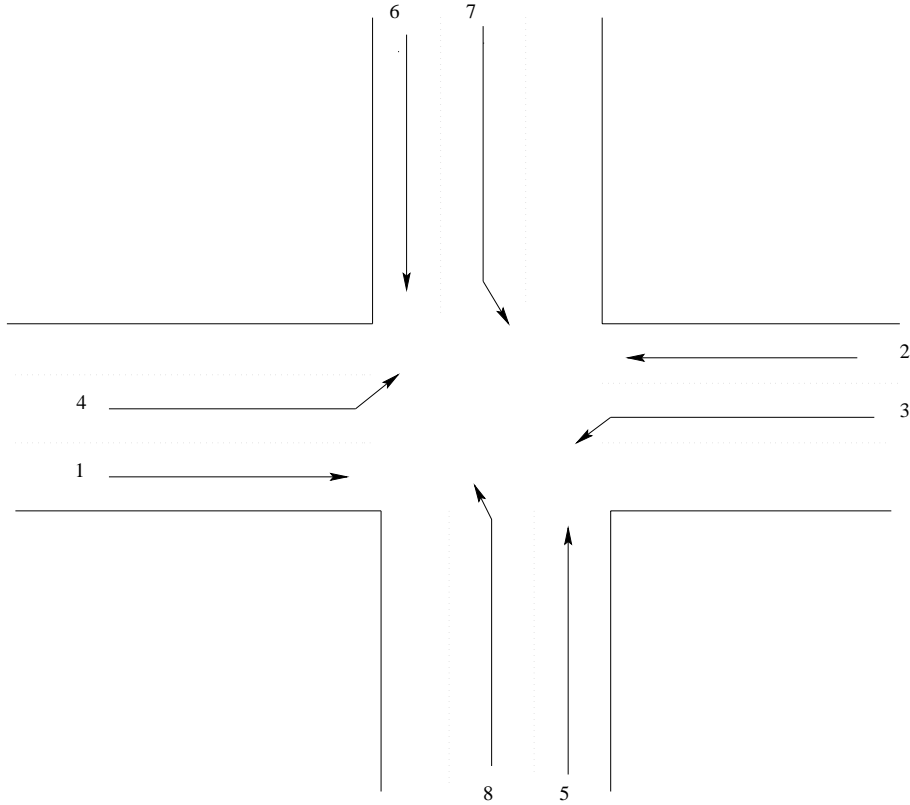


Figure 2.1: Sketch of a traffic intersection

We can find a feasible assignment in three following steps:

1) Let us draw a graph *compatible graph*  $G$  such that, the vertices of it represent the traffic streams, and two vertices are joined if and only if they represent the compatible traffic stream.

2) We get the spanning subgraph  $H$  of  $G$  in such way that we can have a *circular-arc interval graph representation of  $H$* . Note the  $G$  can not happen to be interval graph as in Fig 2.2.

3) Draw the circular-arc representation of  $G$  that is a *feasible assignment too*.

We illustrate these three steps in Fig. 2.2.

But not every feasible assignment is *efficient*. First we have to define what does mean *efficiency* for us. In the case of isolated traffic intersection efficiency can be the minimization of total waiting times that equals the



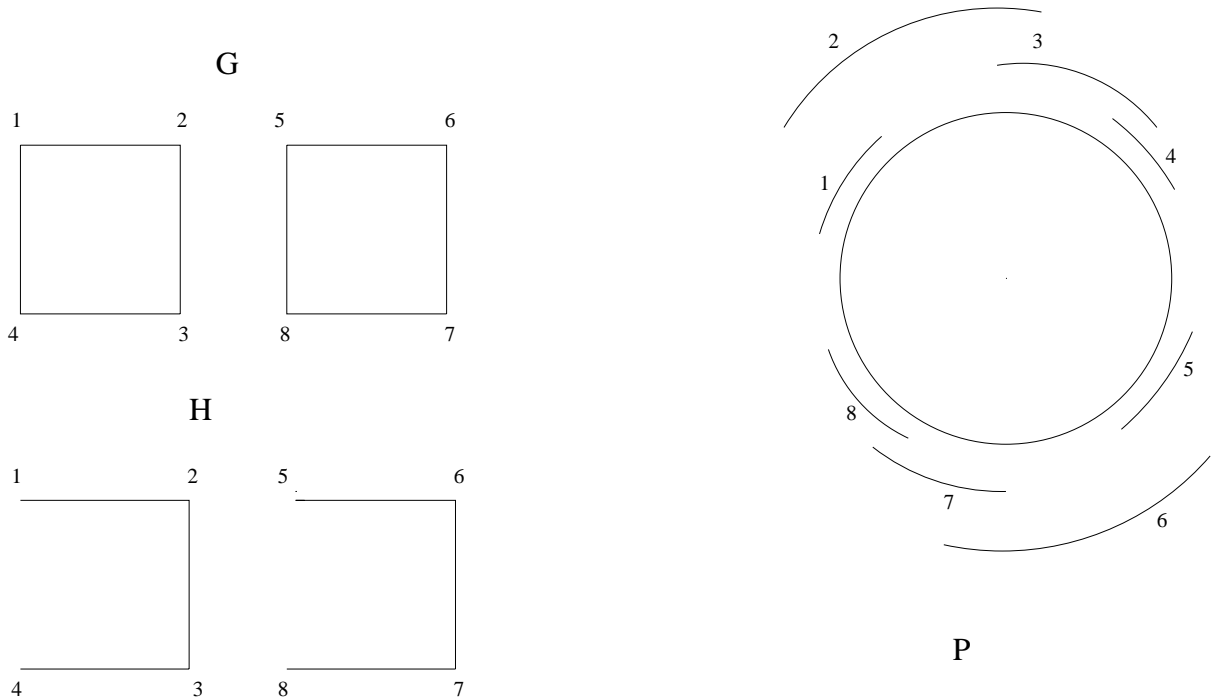


Figure 2.2: Compatible graph  $G$  of traffic intersection of Fig. 2.1, graph  $H$  that is spanning graph of  $G$  and graph  $P$  that is circular-arc interval graph representation of  $H$

minimization of the sum of the total red light in given cycle.

In any case, to find an efficient green light assignment, an effective procedure is the following:

- 1) Given the compatibility graph  $G$ , find all the circular-arc interval spanning subgraphs  $H$  of graph  $G$ .
- 2) For each such subgraph  $H$ , find a green light assignment which is the most efficient in the sense described above (minimization of total waiting times)
- 3) Find the most efficient of these assignment.

Step (1) and (3) are trivial, so will go into details for step (2). In order to find the to find the most efficient green light assignment, first we find all maximal cliques of  $H$ ,  $K_1, K_2, \dots, K_j$ . We put the maximal cliques in *consecutive ranking* such that:

u is an edge and  $u \in K_i, u \in K_j$  and  $i < j$  then  $\forall(i < r < j) \Rightarrow u \in K_r$

Note that each maximal cliques  $K_i$  correspond to *phase during which all traffic streams making up it receive green light*. Phase  $K_i$  has a certain duration  $d_i$ , which has to be determined. If we know that traffic stream u appears in all consecutive maximal cliques  $K_i, K_{i+1} \dots K_j$ . Thus u receives the green light just as a phase  $K_i$  begins and remains green all the phases through  $K_j$  equals to  $d_i + d_{i+1} + \dots d_j$ . If  $K_1$  starts at time 0, then  $K_i$  starts at time  $d_1 + d_2 + \dots d_{i-1}$ , and so traffic stream u receives a green light interval:

$$(d_1 + d_2 + \dots d_{i-1}, d_1 + d_2 + \dots d_j)$$

The most efficient assignment for  $d_i$  for a given ranking of maximal cliques is discovered by linear programming procedure. It is easy to see that  $d_i$ 's has to satisfy certain inequalities (constrains) and we wish to minimize the quantities expressible in terms of  $d_i$ 's that is total waiting times. (objective function). That makes clear that we come out with linear programming problem.

We illustrate the procedure with one example.

Consider the graph H of Fig. 2.2. The maximal cliques for this graph are  $K=(1,2), L=(2,3), M=(3,4), N=(5,6), O=(6,7), P=(7,8)$ . It is easy to see that the maximal cliques are in the consecutive ranking. We wish to assign duration  $d_1$  to K,  $d_2$  to L,  $d_3$  to M,  $d_4$  to N,  $d_5$  to O,  $d_6$  to P. So the total green time for all phases is equal to  $(d_1 + d_2 + d_3 + d_4 + d_5 + d_6)$ . Traffic stream 1 is only in maximal cliques K, so it is green during interval  $(0, d_1)$ . The traffic stream 2 is in L and M, so 2 is green during interval  $(d_1, d_1 + d_2)$ . Other green lights are shown in the Table 2.1. In the Table 2.1 we use the following symbols: Ts.=Traffic stream, Maximal cliques=Max.Cl, t=time.

Note in the table that Total red time in all streams =  $6d_1 + 6d_2 + 6d_3 + 6d_4 + 6d_5 + 6d_6$  If we find efficient that each traffic stream is at least 8 seconds and total red time for all traffic streams is 64 seconds we come out with linear programming:

$$\begin{aligned} d_1 &\geq 8 && \text{(total green time for traffic stream 1)} \\ d_1 + d_2 &\geq 8 && \text{(total green time for traffic stream 2)} \\ d_2 + d_3 &\geq 8 && \text{(total green time for traffic stream 3)} \\ d_3 &\geq 8 && \text{(total green time for traffic stream 4)} \\ d_4 &\geq 8 && \text{(total green time for traffic stream 5)} \end{aligned}$$

$$\begin{aligned}
d_4 + d_5 &\geq 8 && \text{(total green time for traffic stream 6)} \\
d_5 + d_6 &\geq 8 && \text{(total green time for traffic stream 7)} \\
d_6 &\geq 8 && \text{(total green time for traffic stream 8)} \\
d_1 + d_2 + d_3 + d_4 + d_5 + d_6 &= 64 && \text{(total green time for all phases )} \\
\min[6 * (d_1 + d_2 + d_3 + d_4 + d_5 + d_6)] &&& \text{(total red time for all traffic streams)}
\end{aligned}$$

Table 2.1. Green Light Interval for Subgraph H of Fig. 2.2.

Ts.u	Max.Cl having u	Green interval t for u	Total green t	Total red t
1	K	$(0, d_1)$	$d_1$	$d_2 + ..d_6$
2	K,L	$(0, d_1 + d_2)$	$d_1 + d_2$	$d_3 + ..d_6$
3	L,M	$(d_1, d_1 + d_2 + d_3)$	$d_2 + d_3$	$d_1 + d_4.. + d_6$
4	M	$(d_1 + d_2, d_1 + ..d_3)$	$d_3$	$d_1 + d_2 + d_4 + .. + d_6$
5	N	$(d_1 + .. + d_3, d_1 + .. + d_4)$	$d_4$	$d_1 + .. + d_3 + d_5 + d_6$
6	N,P	$(d_1 + .. + d_3, d_1 + .. + d_5)$	$d_4 + d_5$	$d_1 + .. + d_3 + d_6$
7	P,O	$(d_1 + .. + d_4, d_1 + .. + d_6)$	$d_5 + d_6$	$d_1 + .. + d_4$
8	O	$(d_1 + .. + d_5, d_1 + .. + d_6)$	$d_6$	$d_1 + .. + d_5$

## 2.2 Application to action and events to interval temporal logic

The reference for this section (and figures) is taken from [AF] and [A]. First we would like to say that the topic of *action and events to interval temporal logic is deep and wide* but we are going to give only a *general idea of it and connection to interval graphs*.

The first issue is what an event is. The events are the way by which we classify certain useful and relevant patterns of change. A word "action" is used many different senses. For us, an action refers to something that person or robot might do.

Representing and reasoning about dynamic aspects of the world essentially about actions and events is a problem of interests to many different disciplines. A good understanding of action and events is important for following tasks:

- Prediction: Given a general order of action and events try to for-cast what is most likely to happen.

- Planning: Given a explanation of the world and the desired goal, find a course of action what is most likely to achieve the goal.
- Explanation: Given a set of observation about the world, find the best explanation of the data.

In the Fig 2.3 we give some axioms concerning *structure of time (time periods)* and later on introduce one simple examples.

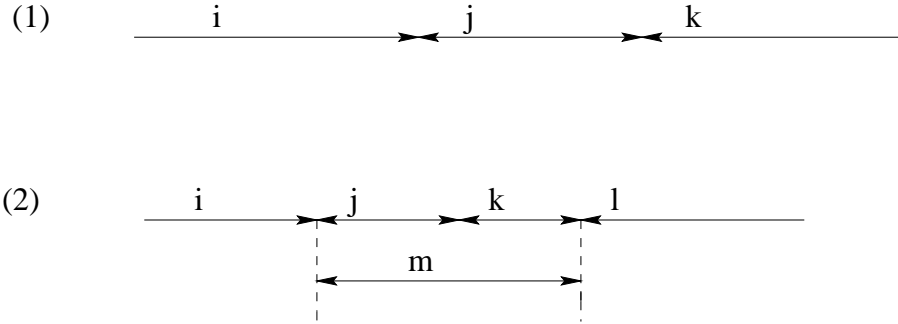


Figure 2.3: The axiomatizations of time periods

First there are no beginning and ending of time and there are no semi infinite or infinite time of periods. The axioms (1) just says that every period has a period that meets it and another that it meets. It can be expressed in following way:

$$\forall i \quad \exists j, k \text{ Meets}(j, i) \wedge \text{Meets}(j, k)$$

The axioms (2), periods can compose to produce a larger period. In particular , for any two periods that meet, there is another period that is “concatenation” of them. This can be axiomatized as follows:

$$\forall i, j, k, l. \text{Meets}(i, j) \wedge \text{Meets}(j, k) \wedge \text{Meets}(k, l) \quad \exists m \text{ Meets}(i, m) \wedge \text{Meets}(m, l)$$

It is clear in Fig 2.3 the similarity between the representations of the time periods and interval graphs. There are more axioms on structure of time but we are not interested on going further as it is *obvious the connections between structure of time (time periods axioms) and the interval graphs.*

We would like to introduce events as objects in interval time logic. Suppose we want *to stack block x on to block y.* We will give some new notation that we need in our example.

*pre* - "preconditions", conditions that must hold prior to event occurrence.  
*eff* - conditions that must hold following the event.  
*con* - conditions that must hold during the event.  
*t* - time, *On* - the block *x* is already on block *y* *e* - event  
*STACK*, *Clear* and *Holding* - have the same meaning as in usual language.

In temporal structure, the axiom defining the necessary conditions for event's occurrence can be expressed as follows:

$$\begin{aligned}
 & \forall x, y, t, e. \text{STACK}(x,y,t,e) \supset \\
 & \text{Clear}(x, \text{pre1}(e)) \wedge \text{Holding}(x, \text{con1}(e)) \wedge \text{Clear}(x, \text{eff1}(e)) \wedge \\
 & \text{Clear}(y, \text{pre2}(e)) \wedge \text{On}(x, y, \text{eff2}(e)).
 \end{aligned}$$

This axiom is shown graphically in Fig 2.4.

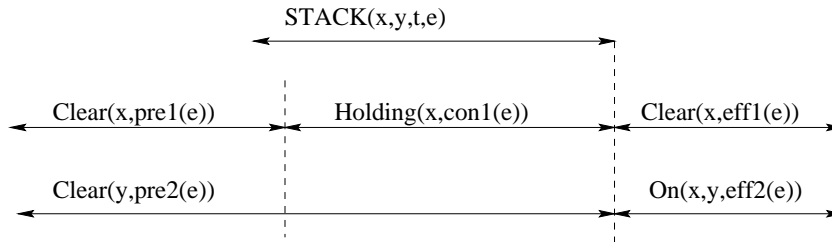


Figure 2.4: The necessary conditions (axioms) for stacking *x* on *y*

The graphics resembles the interval graphs and here can be applied *all theoretical principles on interval graphs* that we already have introduced.

## 2.3 Application of interval graphs on DNA physical mapping

We take reference for this section (and figures) from [S2] and [W]. *Physical mapping* is the process of determining the relative position of landmarks along the chromosome segments. The resulting maps are used as basis for the DNA sequencing, and for the isolation and characterization of individual genes or other DNA regions of interest.

Given a long DNA segment it is relatively easy to produce large group of DNA fragments known as clones. The process of creating the clones consists of breaking several copies of DNA sequences in many locations, and then cloning each of the fragments. One of the problem with the cloning process is that the resulting fragments are obtained "out of order". This means that it is difficult to reassemble the fragments in order to get a map of original sequence. Moreover the cloning process does not ensure that a *continuous sequence of DNA can be reconstructed* from the fragments.

An *STS (Sequenced Tagged Site) probe or STS discriminator* is a filter that can uniquely determines whether or not a specific short sequence of DNA appears along any given (longer) sequence. The filter can identify the existence of the short sequence but provides no information to its location. Running a number of STS probes against numerous clones results in a matrix cell  $M_{i,j}$  with the entry 1 representing a positive result of probe j against clone i, and with the entry 0 representing a negative result of probe j against clone i. In Fig 2.5 we have a example of ordered clones and the corresponding STS probes. In Table 2.2 we represent the STS matrix.

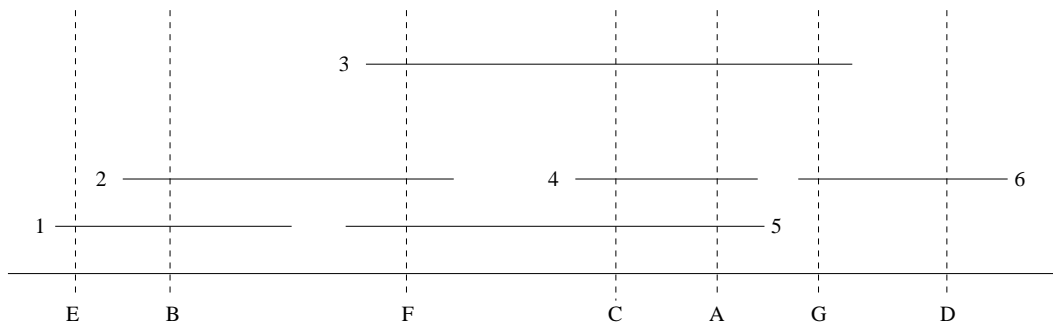


Figure 2.5: Ordered clones and several STS probes

Clones/Probes	A	B	C	D	E	F	G
1	0	1	0	0	1	0	0
2	0	1	0	0	0	1	0
3	1	0	1	0	0	1	1
4	1	0	1	0	0	0	0
5	1	0	1	0	0	1	0
6	0	0	0	1	0	0	1

Table 2.2 Resulting STS matrix

The problem can be formulated in this way:

We have a set of elements  $U(\text{probes})$  and a collection of subsets  $\varphi = (S_1, S_2, \dots, S_n), \forall : S_i \subset U$ . Find **the set  $\Pi(\varphi)$  of all permutations over  $U$  along which every  $S_i$ , is continuous.** The formulation above is called **unique probes mapping problem.**

The *unique probes mapping problem* is equivalent to the rearranging the columns of the **STS result matrix** so that 1's in the rows are consecutive. So we meet again the **consecutive ones property**. It is worthwhile to notice *unique probes mapping problem* is quite the same with *the Tucker's formulation in Section 1.2*. The problem of finding the set of permutations of  $\Pi(\varphi)$  is well known problem in the computer science. A linear time algorithm was presented in 1976 by Both and Lueker, the so called *PQ-Tree algorithm*. The algorithm works good if there is no "noise" (errors). Unfortunately due to the "real life" measurement errors the input matrix usually has either extra or missing of 1's entries. In such cases the PQ-Tree algorithm does not produce the best (minimum error) solution available. Another algorithm represented by Wen, (1997) can work in presence of certain errors too.

Intuitively, it is clear that the problem of checking if graph is **interval graph** is closely related of finding **the set  $\Pi(\varphi)$  of all permutations over  $U$** . The algorithm is based in Theorem 1.4 of Fulkerson, Gross, 1965. Recall the the *clique matrix in illustration of Theorem 1.4*. By using one part of PQ-Tree algorithm we can find a permutation of clique order satisfying the requirement of Theorem 1.4 equivalently the clique matrix satisfying the consecutive ones property. Note that construction of the clique matrix M use the following property: *An interval graph has  $O(n)$  maximal cliques and these cliques can be found in  $O(n)$  time.*

As mentioned above, solving the unique mapping problem can be done quite easily in absence of noise. In the case of either missing edges (probe not

identified) or extra edges (probe identified where it should not have been) the resulting graph might not be an interval graph. The problem of creating an interval graph from existing graph is known as the **interval graph editing problem**.

## 2.4 Application of interval graph to ecology, food webs and competition graphs

### 2.4.1 Ecological Phase Space

The reference of this section (and figures) is from [R2], [Ra,Ro],[R1],[C] and [DG]. The normal, healthy environment of a species of a species or animals or plant can be characterized by considering upper and lower bounds on various dimensions such as temperature,moisture,pH, etc. If  $k$  different dimensions are used, and upper and lower bounds are set in each dimension, the region in  $k$ -dimensional Euclidean space of all points lying within the bounds on every dimension is a  $k$ -dimensional rectangle with sides parallel to coordinate axes.

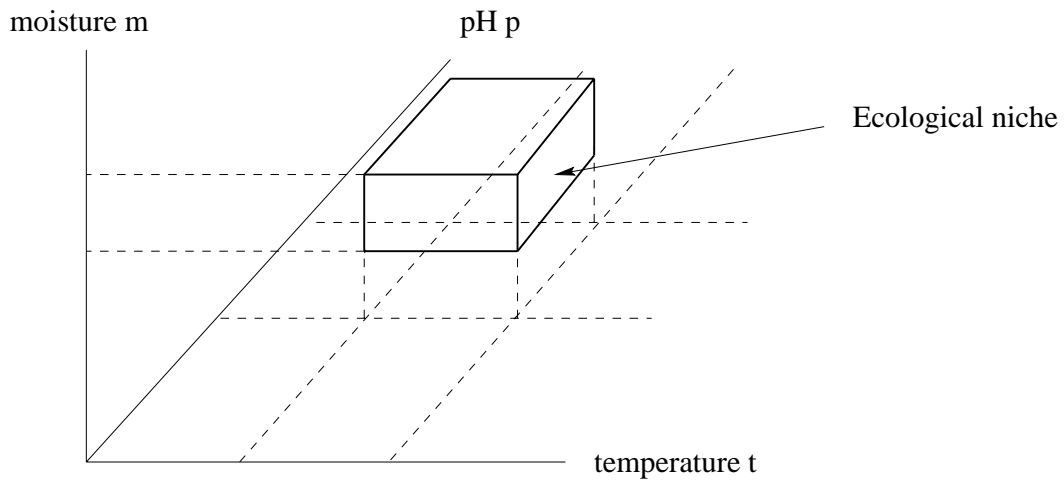


Figure 2.6: An example of ecological niche in three dimensions

Following Danzer and Grunbaun, we call such a region a **box**. This region corresponds to what is called the species **ecological niche**. The  $k$ -dimensional ecological space is called **ecological phase space**. A basic



ecological principle is that two species *compete* if and only if their *ecological niches overlap*. Joel Cohen has suggested the following problem: if we start with an independent notion of competition, what is the *minimum number of dimensions* which must be used to describe the ecological phase space in which at least the *notion of competition is captured*, i.e., in which exactly the competing species have overlapping niches ?

## 2.4.2 Food webs, competition graph and boxity of it

We begin with one definition:

**Definition 2.2** *A competition graph is undirected graph  $G = (V, E)$  that satisfies:*

- 1) *have as vertices the species in an ecosystem*
- 2) *have an edge between two vertices only if the species represented by those vertices compete*

Joel Cohen suggested defining competition in an ecosystem as follows. Start with **food web** for the ecosystem.

**Definition 2.3** *A food web is a directed graph whose vertices are the species of the system and has an arc from vertex  $u$  to vertex  $v$  if and only if  $u$  preys on  $v$ .*

Food webs will be assumed to be acyclic digraphs. Given a food web  $F$ , we say that species  $u$  and  $v$  *compete in food web* if and only if they have the same common prey. We illustrate with an example in Fig 2.7.

We assigned to every species want to find a number a *box (ecological phase space)*  $B(u)$  in Euclidean  $k$ -space.

**Definition 2.4** *The boxity of competition graph  $G$  is called the smallest natural number  $k$  (the dimensions of Euclidean space) that satisfies :*

$$\forall u, v \in V | u \neq v \Rightarrow (u, v) \in G \iff B(u) \cap B(v) \neq \emptyset$$

We can generalize the notation of boxity  $k$  for general graph, then it is essential to note that **interval graphs are just a special case of graph with boxity  $k = 1$** . Let remember the graph in Fig 1.3 that is just a circle with length 4. Due to Theorem of Fulkerson and Gross is not an interval

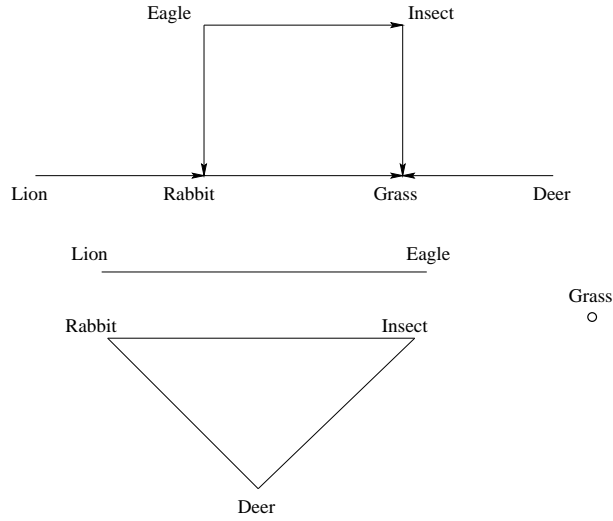


Figure 2.7: A food web and competition graph of it

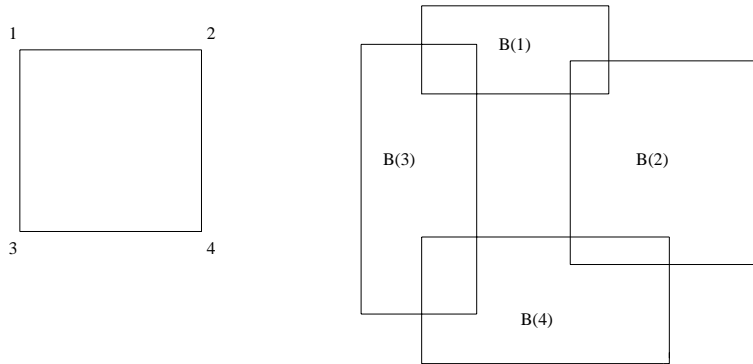


Figure 2.8: A graph circle 4 that is not interval graph but can be represented as graph with boxicity  $k = 2$

graph, but it can be represented as graph with boxicity  $k = 2$ . We illustrate with Fig 2.8.

Cohen and others have analyzed a number of competition graphs from food webs. Remarkably each of them has led to a competition graph with boxicity at most 1 (interval graph). But there are competition graphs with boxicity greater than 1 even competition graphs *can have arbitrary large boxicity* but it is not our goal to go further at this point.

## 2.5 Application of interval graphs to scheduling problems

The reference of this section (and figures) belongs to [PY] and [S1] but we want to note that the sketch of the proof of Lemma 2.1 is done in our own way. We begin by giving a general view of notations needed.

**Definition 2.5** When  $E$  is set of edges of graph  $G$ ,  $E^{-1}$  is set of edges **re-*versal*** to set of edges  $E$ .

We construct two graphs, an oriented graph  $H = (V, P)$  and an unoriented graph  $G = (V, E)$  as follows:

$$\begin{aligned} \forall x, y | x \neq y \text{ then we have } (x, y) \in P &\iff x \text{ is preferred over } y. \\ (x, y) \in E &\iff \text{indifference is felt between } x \text{ and } y. \end{aligned}$$

By definition graph  $(V, P + P^{-1} + E)$  is complete. We define  $H$  as having no *cycles* and having the *transition property* (see Def. 1.7). Note that having cycles brings confusion for decision makers who must go around in circle to find the best decision. On the other hand if  $x$  is preferred over  $y$  and  $y$  is preferred over  $z$  it is very likely that  $x$  is preferred over  $z$  (=transition property). Thus graph  $H$  is **partial order**.

**Definition 2.6** The *incomparability graph* of partial order  $H = (V, P)$  is a graph  $G = (V, E)$  where,  $(u, v) \in E \iff (u, v), (v, u) \notin P$ .

The complement of *incomparability graph* obviously has the *transition orientation property*.

The problem we are going to face is as follows; given a set of  $n$ -unit time tasks, each with release time and deadline, find a feasible schedule-in which a task never starts before its release time or finishes after its deadline-minimizing the total number of steps during which at least one processor is operating.

**Lemma 2.1** A partial order  $G(V_1, E_1)$  is interval order iff its incomparability graph  $C(V_2, E_2)$  is chordal.

For partial order we refer to what we mentioned at the very beginning of the section and in Def 2.6.

*Sketch of the proof:*

We know that all *complements of incomparability graph* have the transitive orientation property. On the other hand our comparability graph  $C$  is *chordal*. We are in condition of Gilmore-Hoffman Theorem 1.3 that brings that our incomparability graph  $C$  is **an interval graph**.

We will point out that ( $C$  is interval graph  $\Rightarrow G$  is partial order of interval order). This come out as two conditions below are equivalent:

1.  $[(x, y) \in E_2 \Rightarrow \text{we can assign intervals } x, y \text{ that intersects with each other}]$  that means  $C$  is an interval graph.
2.  $[(x, y) \notin E_2 \Rightarrow (x, y) \in E_1] \Rightarrow$  (in interval representation of  $C$ , intervals  $x$  and  $y$  are not intersected), that means  $G$  is a partial order, recall Def 2.6.

Note that important conclusion is that *the interval representation of  $C$  is partial order representation for  $G$ , ( in sense of Def 2.6) then **we can schedule our tasks***

We will introduce an example with graph  $G$  and graph  $C$  that is an incomparability graph of graph  $G$  see Fig 2.9. It is worthwhile to see that :

1. each vertex represents a unit task.
2. in graph  $G$  if the vertex 2 is the endpoint of edges  $(1,2)$  and  $(3,2)$ , this means that task 2 has to be scheduled at least with one of tasks 3 or 1. That means it exists one partial order that tasks 2 can be scheduled. From now and on in stead of vertex sometime we use *vertex-task*.
3. in graph  $G$  each groups of the vertex-tasks that has to be scheduled in queue can be represented by intervals in real line that do not intersect (that means the time of those duties that have to be execute do not overlap) with each other, that means an interval order among vertex-tasks exists. Lemma 2.1 make sure us for existence of an interval order between vertex-tasks.
4. interval representation of incomparability graph  $C$  is partial order for our task schedule.

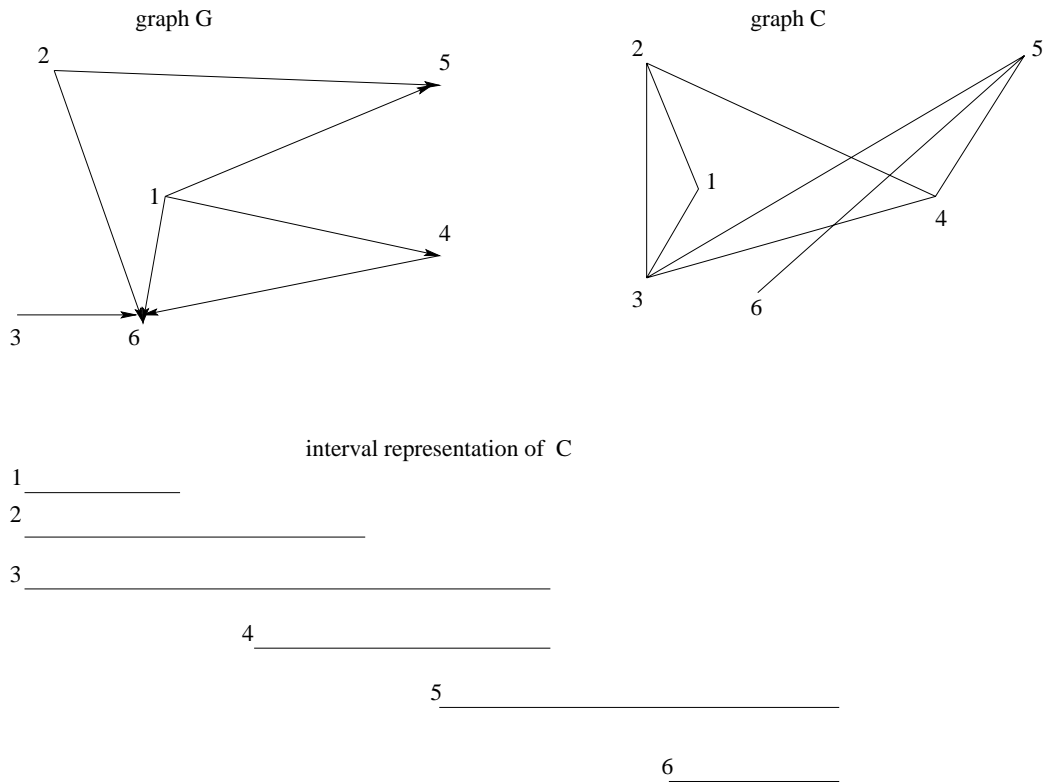


Figure 2.9: Graph G, graph C that is the incomparability graph of G and interval representation of C

Note that we introduce the graph C (incomparability of G) in order to make clear that we are in conditions of Lemma 2.1, that means we are sure that *the partial order G has an interval order*. Our example is a simple application of Lemma 2.1 that *the interval representation of C is a partial order representation for vertex-task G, ( in sense of Def 2.6) then **we can schedule our vertex-tasks***.

## 2.6 Application of interval graphs to radiation therapy

The reference for this section and part of figures are taken from [L]. We want to mention that in the end of the section we present the similarity between the *interleaf motion constrains, consecutive ones matrices and interval graphs* by ourselves. Radiation therapy is commonly used means to fight cancer when the tumor can be localized and metastases have not been yet started to form. Its purpose is to supply enough energy to tumor such that either clonogenic cells are destroyed or tumor can at least be controlled in its growth. The fact that the tumor surrounding organs (organs *at risk*) are in general very sensitive to radiation therapy the treatment must be carefully handle.

In order to treat the patient with radiation therapy, the equipment called multileaf collimator (MLC) are used. In order to control the irradiating area piece of metal with low thickness (around 5-7 mm) are used. Those piece of metal are called *leaves*. Two of those leaves are placed opposite to each other. Each of the leaves is connected with linear motor by a metal band and can move in the direction towards the other leaf or away of it. Such two leaves are called a *channel or a row*. Placing several leaf pairs as described above adjacent to each other we can shape a two dimensional area. The blocked area does, theoretically, does not receive nay radiation at all. We believe that Fig 2.6 and Fig 2.12 make clear what we said in this paragraph.

There are two different methods of the MLC's usage which now will be explained, dynamic and static.

### 2.6.1 Dynamic mode

The dynamic mode can be described as follows: The leaf pairs are positioned at initial state and then the beam is turned on. Then the leaves move with calculated, not necessarily constant, speed while the beams remains switch on. Arranging the speed and the time that leaves move as well as the doserate of radiation in time we can get the desired intensity profile, for example we can get the following intensity profile  $(0, 1, 2, 1, 0, 0)$ , first by putting initially the left leaf to the left of columns 2 and right leaf to the right of column 4. Move the leaves in the same time towards each other, one for 1 sec the other for 2 sec,  $Speed = 0.5columns/sec, doseerate = 1Gy/sec$ .

The dynamic mode is very hard problem and beyond our subject so we

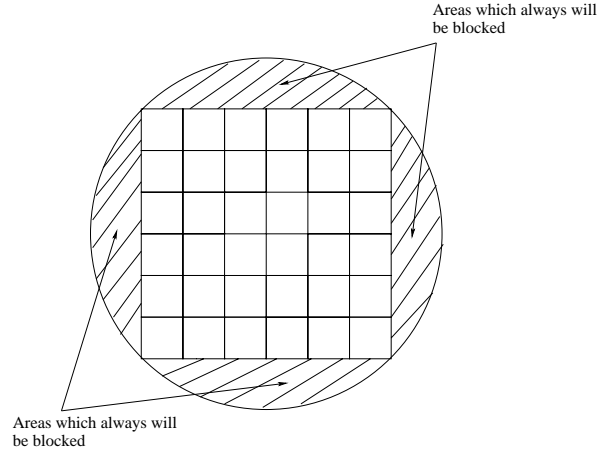


Figure 2.10: The discretized area seen from the beam's eye through the MLC when all leaves are totally retracted. In this figure the leaves are retracted to the left and right side respectively.

are concerned with static mode.

## 2.6.2 Static mode

The mode is different from the dynamic mode and exploit the fact that doses delivered at different times to the destination some up linearly. The idea is to switch off the beam while the leafs are moved to desired position. Then, keeping the leaf pairs at this position the beam is switched on for a certain time in order to irradiate area, which is not blocked by the MLC leaf pairs. The procedure is repeated until the required intensity profile has been delivered.

Thus we are looking for a feasible decomposition of  $I$  such that:

$$I = \sum_k M^k, \quad M^k \in (0, 1)^{m \times n}$$

entry  $m_{ij}^k = 0$  means that either the right or the left leaf of channel  $i$  covers the column  $j$ . **The underlying constrains to create the matrix  $M^k$**  will be explained later. A first example shall help in understanding the relation between matrices and MLC leaf sequencing.

Let take matrix  $I$  in Fig 2.11. This can be written as sum of  $M^1 + M^2 + M^3$ , as follows in Fig 2.11. Then we get the the following MLC leaf sequence in

$$\mathbf{I} = \begin{pmatrix} 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$I = M^1 + M^2 + M^3$$

$$\mathbf{I} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2.11: Matrix I and its decomposition

order to shape this intensity profile. The areas with L signs are covered by left leaf, the last L gives the terminate position of left leaf. The areas with R signs are covered by right leaf, the last R gives the terminate position of right leaf. See Fig 2.12.

It is clear that matrices  $M^1, M^2, M^3$  have the consecutive ones property. It is essential to note that in *previous application like DNA, scheduling problem etc we try to organize the data according to the interval graphs or the consecutive ones property, in radio therapy we have matrices of decomposition that often have the the consecutive ones property, the data are organized themselves according to consecutive ones property and we want to find the best decomposition matrices.* What means "the best" has still to be defined.

The decomposition of matrix I into shape matrices  $M^k$  is a combinatorial optimization problem, with certain restrictions. One of the restriction that we will is the **interleaf motion constrains**. Interleaf motion constrains prohibit the movement of a left leaf further to the right than any of its opposing right leaves in the adjacent channels.



L	L			R	R
L	L	L			R
L	L	L		R	R
L	L				R
L	L	L			
L	L	L			

L	L				R
L				R	R
L	L				R
				R	R
L	L				R
L					

L	L	L	L		R
L	L	L		R	R
L	L			R	R
L		R	R	R	R
L				R	R
L	L			R	R

Figure 2.12: MLC sequencing for decomposition of matrix I

$$\mathbf{J} = \begin{pmatrix} L & L & L & L & 1 & R \\ L & 1 & 1 & R & R & R \end{pmatrix}$$

Figure 2.13: Matrix J that is not allowed to happen as submatrix in decomposition matrices, the L's are used for left leaves and R's are used for right leaves

See matrix J in Fig 2.13 to understand clearly what shape of matrices is not allowed as submatrices to happen thanks to *interleaf motion constrains*.

Observe that if in matrix J we switch L's and R's to 0's, we get matrix  $J_1$  given in Fig 2.14.

We want to say that forbidden matrix,  $J_1$  (because of interleaf motion constrain) can be described in the terms of the bipartite graphs as we characterize the consecutive ones property in Def 1.17. This is equivalent with the fact that, given a bipartite graph  $G = (V_1, V_2, A)$  and two consecutive vertices  $y_j, y_{j-1} \in V_2$  then; if the  $x_b \in V_1$  is the biggest index vertices connect with  $y_j$  and  $x_s \in V_1$  is the smallest index vertices connect with  $y_{j-1}$  then we *must request*  $b - s \leq 1$ .

$$\mathbf{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 2.14: Matrix  $J_1$  taken by switching L's and R's to 0's

# Chapter 3

## Polyhedral approach to consecutive ones property

We take the reference and figures from [OR]. We would like to mention that we did by ourselves the implementation of the algorithm of Oswald and Reinhelt. The discussion will be presented in three parts.

- On Integer Programming formulation of the given problem: given matrix  $B$  of  $(0,1)$  and matrix  $A$  that have the consecutive ones property (C1P), minimize the switches from 0 to (or from 1 to 0) in order to transform matrix  $B$  to a matrix  $A$ .
- On presenting another formulation of the consecutive ones property by introducing the so called consecutive ones polytopes.
- On separation problem solved in a branch and cut approach based on the second formulation.

### 3.1 An Integer Programming formulation of the weighted consecutive ones problem

We begin with some definitions and notations and go on with our main problem.

**Definition 3.1** *Given matrix  $B$  of  $(0,1)$  and matrix  $A$  that have consecutive ones property (abbreviated **C1P**), the task to minimize the switches from 0*

to (or from 1 to 0) in order to transform matrix  $B$  to a matrix  $A$  is called **weighted consecutive one problem**(abbreviated **WC1P**).

We introduce one new notation:

$$A \circ x_{IJ} = \sum_{i=1}^n \sum_{j=1}^m a_{ij}x_{ij} \quad (3.1)$$

where  $A$  and  $x$  are  $m \cdot n$  matrices. Note that notation above just means the multiplication between entries with the same index of  $A$  and  $x$ .

We are going to formulate the WC1P as Integer Programming. Let 0/1-matrix  $B$  be given as input. We are looking for matrix  $x$  that resembles  $B$  as closely as possible. If  $x$  contains a 1 where  $B$  contains a 0, we add a penalty  $c_0$ , if  $x$  contains 0, where 1 would be preferred, we add a penalty  $c_z$ . Therefore the following objective function value (penalty) is associated with  $x$ :

$$\begin{aligned} f &= c_0 \sum_{b_{ij}=0} x_{ij} + c_z \sum_{b_{ij}=1} (1 - x_{ij}) \\ f &= c_0 \sum_{b_{ij}=0} x_{ij} + c_z \sum_{b_{ij}=1} x_{ij} - \sum_{b_{ij}=1} c_z \end{aligned}$$

then if:

$$c_{ij} = \begin{cases} -c_z & \text{if } b_{ij} = 1 \\ c_0 & \text{if } b_{ij} = 0 \end{cases}$$

it comes out our objective function to be:

$$f = \min \sum_{ij} c_{ij}x_{ij}$$

In order to get the constrains we use the Theorem 1.10 of Tucker and recall Fig 1.12. Seeing closely the structure of Tucker matrices, and using the notation in equation 3.1 that is just the multiplication between entries of the same index of two matrices. Now we can write our constrains:

$$M(An) \circ x_{IJ} \leq 2k + 3 \text{ for matrices } M(An),x \text{ with } (k+2,k+2) \text{ dimensions}$$

$$M(Bn) \circ x_{IJ} \leq 4k + 5 \text{ for matrices } M(Bn),x \text{ with } (k+3,k+3) \text{ dimensions}$$

$$M(Cn) \circ x_{IJ} \leq 3k + 2 \text{ for matrices } M(Cn),x \text{ with } (k+2,k+3) \text{ dimensions}$$

$$M(D) \circ x_{IJ} \leq 8 \text{ for matrices } M(D),x \text{ with } (4,6) \text{ dimensions}$$

$$M(E) \circ x_{IJ} \leq 10 \text{ for matrices } M(E),x \text{ with } (4,5) \text{ dimensions}$$

As we have the *the objective function and constrain* so we have an Integer programming formulation of WC1P.

## 3.2 The consecutive ones property and the consecutive ones polytopes

During this section we are to introduce *stronger constrains for the WC1P* based on characterization of the consecutive ones polytopes. We give some useful definitions and later go on with some important theorems.

**Definition 3.2** *The consecutive ones polytope  $P_{C1}^{m,n}$  satisfies:*

$$P_{C1}^{m,n} = \text{conv}(M \mid M \text{ is } (m,n) \text{ matrix with C1P})$$

The conv is abbreviation of convex. Note that  $P_{C1}^{m,n}$  has full dimension  $m \cdot n$ . We consider that zero matrix has the C1P. From the other side all the matrix with 1 in position  $ij$  for every  $1 \leq i \leq m$  and  $1 \leq j \leq n$  are C1P. So will have in total  $m \cdot n + 1$  *independent affinely matrices* that means that consecutive ones polytope  $P_{C1}^{m,n}$  has full dimension.

**Definition 3.3** *If  $a^T x \leq a_0$  be valid inequality for  $P_{C1}^{m,n}$  and inequality  $\bar{a}^T x \leq a_0$  valid inequality for  $P_{C1}^{m',n'}$  (with  $m' \geq m$  and  $n' \geq n$ ) is obtained by first one by **trivial lifting** if:*

$$\begin{aligned} a_{i,j}^t &= a_{i,j} \text{ if } i \leq m \text{ and } j \leq n. \\ a_{i,j}^t &= 0 \text{ otherwise} \end{aligned}$$

Now we will present two theorems on the consecutive ones polytopes close with the notion of the trivial lifted inequality.

**Theorem 3.1** *Let  $a^T x \leq a_0$  be facet defining inequality for  $P_{C1}^{m,n}$  and let  $m' \geq m$  and  $n' \geq n$ . If  $a^T x \leq a_0$  is trivially lifted then the resulting inequality defines a facet of  $P_{C1}^{m',n'}$ .*

*Proof*

Let matrix  $\bar{x}$  satisfies the equality  $a^T \bar{x} = a_0$ . Without loss of a generality we take  $m' = m + 1$  and  $n = n'$ . We add one row to  $\bar{x}$  of zeros except one 1 in position  $j$  for  $1 \leq j \leq n$  and form  $\bar{x}_j$ . All generated matrices are C1P and we have  $\bar{a}^T \bar{x}_j = a_0$ .

By this construction we can form  $n$  more affinely independent, totally  $m(n+1)$  affinely independent matrices satisfying  $\bar{a}^T \bar{x} = a_0$  proving that the trivial lifted inequality is also facet defining.

If trivial lifting is possible, this means, *that larger polytopes inherit the faces of smaller polytopes.*

$$\mathbf{F}_{1k} = \begin{pmatrix} + & + & - & - & - & - & - \\ - & + & + & - & - & - & - \\ - & 0 & + & + & - & - & - \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ - & 0 & 0 & 0 & + & + & - \\ - & 0 & 0 & 0 & 0 & + & + \\ + & 0 & 0 & 0 & 0 & - & + \end{pmatrix} \quad \mathbf{F}_{2k} = \begin{pmatrix} + & + & - & - & - & - & - \\ - & + & + & - & - & - & - \\ - & 0 & + & + & - & - & - \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ - & 0 & 0 & 0 & + & + & - \\ - & 0 & 0 & 0 & + & + & - \\ - & 0 & 0 & 0 & + & - & + \end{pmatrix}$$

Figure 3.1: Matrices  $F_{1k}$  and  $F_{2k}$

**Theorem 3.2** For all  $m \geq 1, n \geq 1, 1 \leq j \leq n, 1 \leq j \leq n$ , the inequalities  $x_{ij} \geq 0$  and  $x_{ij} \leq 1$  defines facets of  $P_{C1}^{m,n}$ .

*Proof*

$$P_{C1}^{1,1} = \text{conv}(0,1)$$

So  $x_{ij} \geq 0$  and  $x_{ij} \leq 1$  are facet defining for  $P_{C1}^{1,1}$ . The general result holds due to trivial lifting property.

We are going to introduce two important theorems on facet of the consecutive ones polytopes that involve a certain class of matrices  $F_{1k}$  and  $F_{2k}$ , see Fig 3.1. Note that *matrix*  $F_{1k}$  has  $(k+2) \cdot (k+2)$  dimensions  $F_{2k}$  has  $(k+2) \cdot (k+3)$  dimensions.

**Theorem 3.3** The inequalities  $F_{1k} \circ x_{IJ} \leq 2k+3, k \geq 1$ , for all  $(k+2, k+2)$  index sets, are facet defining for  $P_{C1}^{m,n}$  for all  $m \geq k+2$  and  $n \geq k+2$ .

*Proof*

We need to show it only for  $m = k+2$  and  $n = k+2$  and due to Theorem 1 we can extend for  $m \geq k+2$  and  $n \geq k+2$ .

First we show that inequality  $F_{1k} \circ x_{IJ} \leq 2k+3$  is valid. Let  $x$  be  $(0,1)$  matrix. Let prove that inequality  $F_{1k} \circ x_{IJ} \geq 2k+3$  is impossible for  $x$  being C1P. In order to satisfy the second inequality we have only one possibility entries of  $x$  has to be:

- 1 where entries of  $F$  has to be 1
- 0 where entries of  $F$  are -1

We see that  $x$  has the same form as the transpose of first Tucker matrix of Fig 1.12 that is not C1P. We can argue in different way too. The status of matrix from "No C1P" to "C1P" can change iff:

a) in one we have bad row that has two bad entries one 0 between ones one 1 between 0.

b) for each pair of columns of x there is even number of rows (0 or 2) with the property a)

Since  $P_{C_1}^{mn}$  is a full dimensional, the facet defining inequalities of the same facet only differ by multiplication with positive scalar. Let  $a^T x = a_0$  and  $b^T x = a_0$  be two different equalities that defines the same facet. We have to show that  $b = \lambda a$  for  $\lambda \geq 0$ . We call the C1P matrices x solutions if they satisfy the facet defining equality  $b^T x = a_0$ . Take  $b_{11} = \beta$ .

We will see three cases:

1) Take two solutions  $x^1$  and  $x^2$  matrices with  $2k+3$  ones 1 and 0 otherwise such that:  $x_{11}^1 = 0$  and  $x_{12}^2 = 0$  then it comes out

$$0 = b^T x_1 - b^T x_2 = b_{12} - b_{11} \text{ that brings } b_{12} = b_{11} = \beta$$

Using appropriate x matrices we can show that  $b_{ij} = \beta$  for every "+" positions.

2) Take two solutions  $x^1$  and  $x^2$  matrices such that:

$x_{k+2,k+2}^1 = 0$  take  $x^2$  the same as  $x^1$  except for 1 in 0 position closed to another 1 in any row i. Let this position be 'il'. So it comes out

$$0 = b^T x_1 - b^T x_2 = -b_{il}$$

It means that extending the chain of 1's in matrix x to 0's positions shows that  $b_{is} = 0$  for all zero positions of row i.

3) Take two solutions  $x^1$  and  $x^2$  matrices such that:

$x_{11}^1 = 0$ . Construct  $x^2$  from  $x^1$  in this way Let  $x_{11}^2 = 0$  and  $x_{1k+2}^2 = 0$ . Note that matrix  $x_2$  is not a C1P matrices but it can be transformed to C1P by some permutations. Now:

$$0 = b^T x_1 - b^T x_2 = -b_{11} - b_{1,k+2} \text{ so it results that}$$

$$b_{1,k+2} = -b_{11} = -\beta$$

Using appropriate x matrices we can show that  $b_{ij} = -\beta$  for every "-1" positions. Thus we have shown that  $b = \beta a$  for  $\beta \geq 0$ .

**Theorem 3.4** *The inequalities  $F_{2k} \circ x_{IJ} \leq 2k+3$ ,  $k \geq 1$ , for all  $(k+2, k+3)$  index sets, are facet defining for  $P_{C_1}^{m'n'}$  for all  $m \geq k+2$  and  $n \geq k+3$ .*

The proof is very similar with the previous theorem so we omit it.

$$\left( \begin{array}{cccccccc} i & \dots & \dots & j & c_2 & \dots & c_d & h \\ \dots & + & + & - & - & - & - & - \\ \dots & 0 & + & + & - & - & - & - \\ \dots & 0 & 0 & + & + & - & - & - \\ r_1 & - & 0 & 0 & + & + & - & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ r_{d-1} & - & 0 & - & 0 & + & + & 0 \\ r_d & - & 0 & - & 0 & 0 & + & + \\ l & + & 0 & - & 0 & 0 & 0 & + \end{array} \right) \circ x \leq 2k + 3$$

Figure 3.2:  $\bar{F}_{1k}$  inequality

### 3.3 Separation problem

We will discuss the separation of  $\bar{F}_{1k}$  but for a more general class of inequalities. These inequalities can be obtained by observing that *the "-1" entry in the last row can be moved to any position* changing the first and last column in appropriate way. The more general class of inequalities can be visualized in the Fig 3.2. The main task of algorithm is to *identify the row  $l$  and the columns  $i, j$  and  $h$  and to sum appropriate coefficient for row and columns between.*

We construct a undirected bipartite graph  $G^i$  for matrix  $x$  with  $n$  columns and  $n$  rows as two node sets. With every edge 'cr' we associate the weight:

$$w_{cr}^i = 1 - x_{rc} + \frac{1}{2}x_{ri} \quad (3.2)$$

In every weighted graph  $G_i$  we now compute for every pair  $j, j \neq i$  and  $h, h \neq i, h \neq j$  of columns a shortest path between  $j$  and  $h$  with respect to assigned edge weights. This way obtained shortest lengths  $p_{jh}^i$ , see Fig 3.3.

For every shortest path  $p_{jh}^i$  and  $p_{ji}^h$  we can write. It is essential to note that the path  $p_{ji}^h$  can be built in the same way as the path  $p_{jh}^i$  in Fig 3.3 .

$$p_{jh}^i + p_{ji}^h - x_{li} + x_{lj} - x_{lh} = 2k + 2 - \bar{F}_{1k} \circ x \geq -1 \quad (3.3)$$

For every expression that has values less than -1 we have the violated  $\bar{F}_{1k}$  inequality that can be computed by using the shortest path calculated above.

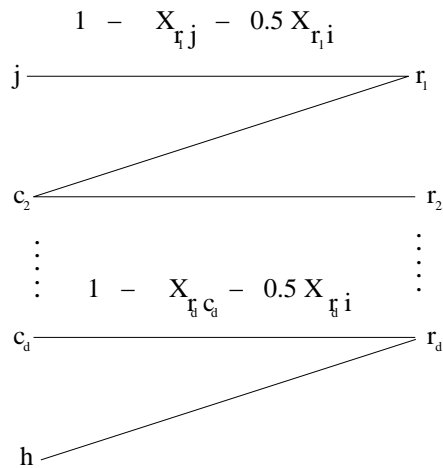


Figure 3.3: Path between columns  $j$  and  $h$ .

So finally we reach at the polynomial algorithm of complexity  $O(n^3(n+m))$ .

We have prepared a short program in C++ in order to implement the separation algorithm, the first part of the program is prepared in Matlab and Pascal too. We are going to introduce the code in the Appendix.



# Chapter 4

## Conclusion and further suggestion

First we would like to say that it can be said more on *matrices with the consecutive ones property and interval graphs* as the subject is too wide. But staying in our framework we find reasonable to stop here.

1. It is interesting and useful from theoretical point of view to go further in structure theorems of Lin 1995, and Lin, Sen, West 1997. Together with the structure theorem of Tucker 1972, they can give a beautiful and complete picture on *structure theorems of matrices with the consecutive ones property*. We stress that Muller 1997 invented a interesting algorithm *for recognizing interval digraphs and unit interval digraphs* that carry further work of Lin, Sen, West and Sanyal.
2. We just touch the notion of *circular-arc interval graphs and matrices with the circular ones property*. We like to mention only that *there are important theorems on circular-arc interval graphs and matrices the with circular ones property too*, (similar to structure theorems of Tucker on interval graphs and matrices with the consecutive ones property) presented by Tucker 1971, that helps getting better idea on them.
3. It is explained in our study the relation between the scheduling problem and interval graphs. We emphasize that there are powerful algorithms on treating scheduling problems presented by Radermacher and Mohring 1984, as well as further work and algorithms of University of California, Irvine by Irani, Leung 1997 on *scheduling with conflicts*.

The word conflicts is used to say that studying phenomena can not happen together.

4. Finally we note that the importance of algorithm for the DNA problem that works in presence of errors (the advantage towards previous algorithms) presented by Wen 1997. It is essential in biomathematics a fully understanding as well as implementation of it.

# Appendix

## Algorithm:

that check if matrix  $X$  of  $(-1,1,0)$  can satisfy the  $\bar{F}_{1k}$  inequality in Fig 3.2, that is equivalent with fact that matrix can have the consecutive ones property.

*Input:*  $(n \times n)$  matrix  $X$  of  $(-1,1,0)$  note as values of matrix  $X$  are changing in every part of algorithm one has to give it four times as input matrix  $X$  with four different matrix notations. We use  $X, B, C, D$ .

*Output:* checking if matrix  $X$  is satisfying  $\bar{F}_{1k}$  inequality.

*Part 1:* applies the algorithm to main diagonal and lower triangle part of matrix, by calculating  $p_{jh}^i$  for all positions of  $i$ .

*Step 1:* Create the vector  $m_1$  with  $n_1$  elements equal to 0, that will store the calculated value of  $x_{ij}$ . The  $n_1$  equal to number of elements in lower triangular part of matrix  $X$ ,  $n_1 = n(n - 1)$ .

$$t_1 = 1$$

$$q_1 = 0$$

*Step 2:*

for  $i = 0$  to  $n$   $\leftarrow$  the loop working with matrix  $X$

for  $j = 0$  to  $(i - 1)$   $\leftarrow$  the loop working with matrix  $X$

$x_{ii} = 1 - x_{ii} + 0.5 * x_{ij}$   $\leftarrow$  main step of algorithm see equation 3.2

$m_1[t_1] = x_{ii}$   $\leftarrow$  store the value  $x_{ii}$  in vector  $m_1$

$x_{ii} = \text{previousvalue}$   $\leftarrow$  note that after each loop value of entry  $x_{ii}$  is changing, so we make it equal to the previous value (input value) again as algorithm works with input value of  $x_{ii}$

$$t_1 = t_1 + 1$$

endfor

endfor  
for  $l_1 = 1$  to  $n_1$   
 $q_1 = q_1 + m_1[l_1]$   $\leftarrow$  the sum of equation 3.3 that check if matrix X can satisfy the consecutive ones property as well  $\bar{F}_{1k}$  inequality. If  $q_1 < -1 \implies$  that matrix X can violate the consecutive ones property.

endfor

*Part 2:* applies the algorithm to upper diagonal and lower triangle part of matrix, by calculating  $p_{jh}^i$  for all positions of  $i$ , note that here we have to introduce matrix B that is the same with matrix X, as the algorithm has already change the value of X.

**As part 2 is quiet the same with part 1 we omit it.** *Note that in the end of Part 2, we have value of  $q_2$  (similar with  $q_1$ ) that tells us if matrix X can satisfy the consecutive ones property*

*Part 3:* applies the algorithm to main diagonal and upper triangle part of matrix, by calculating  $p_{ji}^h$  for all positions of  $h$ . Note that we have to introduce matrix C the same as matrix X.

*Step 1:* Create the vector  $m_3$  with  $n_1$  elements equal to 0, that will store the calculated value of  $c_{ij}$ . The  $n_1$  equal to number of elements in lower triangular part of matrix C,  $n_1 = n(n - 1)$ .

$t_3 = 1$

$q_3 = 0$

*Step 2:*

for  $i = (n - 2)$  to 0  $\leftarrow$  the loop working with matrix C

for  $j = n - j = (i + 2)$   $\leftarrow$  the loop working with matrix C

$c_{ii} = 1 - c_{ii} + 0.5 * c_{ij}$   $\leftarrow$  main step of algorithm see equation 3.2

$m_3[t_3] = c_{ii}$   $\leftarrow$  store the value  $c_{ii}$  in vector  $m_1$

$c_{ii} = previousvalue$   $\leftarrow$  note that after each loop value of entry  $c_{ii}$  is changing, so we make it equal to the previous value (input value) again as algorithm works with input value of  $c_{ii}$

$t_3 = t_3 + 1$

endfor

endfor

for  $l_3 = 1$  to  $n_1$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 0 & -1 & 1 \end{pmatrix}$$

Figure 4.1: The matrix A, as example where we apply the separation algorithm

$q_3 = q_3 + m_3[l_3]$  ← the sum of equation 3.3 that check if matrix C a can satisfy the consecutive ones property as well  $\bar{F}_{1k}$  inequality. If  $q_3 < -1 \implies$  that matrix C can violate the consecutive ones property.

endfor

*Part 4:* applies the algorithm to upper diagonal and upper triangle part of matrix, by calculating  $p_{ji}^h$  for all positions of  $h$ . Note that we have to introduce matrix D the same as matrix X.

**As part 4 is quiet the same with part 3 we omit it.** Note that in the end of Part 4, we have value of  $q_4$  (similar with  $q_3$ ) that tells us if matrix D can satisfy the consecutive ones property

$q = q_1 + q_2 + q_3 + q_4$  ← the total sum of equation 3.3 that finally check if matrix X a can satisfy the consecutive ones property as well  $\bar{F}_{1k}$  inequality. If  $q < -1 \implies$  that matrix X can violate the consecutive ones property.

We present below one simple example. We apply our algorithm to matrix A in Fig 4.1.

When we apply the separation algorithm we get  $q_1 = -4.5 < -1 \implies$  matrix A does not satisfy the  $\bar{F}_{1k}$  inequality in Fig 3.2, that is equivalent with fact that matrix can not have the consecutive ones property. We got an conclusion but for illustration we give the value of  $m_1, m_3$  and  $q_2, q_3, q_4$ .

$$m_1 = [-0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5, 0, -0.5]$$

$$m_3 = [-0.5, 0, -0.5, -0.5, 0, -0.5]$$

$$q_2 = -3, q_3 = -2, q_4 = -2.$$

It is clear that the example is so simple that values can be checked only by looking in matrix A in Fig 4.1.

# Bibliography

- [A] Allen, F.J (1991) Planning as temporal reasoning *Proc. of the Second International Conference on Principles of Knowledge Representation and Reasoning, Cambridge.*
- [AF] Allen, F.J, Ferguson, G. (1994) Actions and Events in Interval Temporal Logic *The University of Rochester, Computer Science Department, Rochester, New York 14627, Technical Report 521.*
- [C] Cohen, E.J. (1978) Food Webs and Niche Space. *Princeton University Press, Princeton, N.J.*
- [DG] Danzer, L, Grunbaun, B. (1967) Intersection properties of boxes in  $R^d$ . *Mimeographed, Department of Mathematics, Univ. of Washington, Seattle.*
- [FG] Fulkerson, D.R, Gross, A.O. (1965) Incidence matrices and interval graphs *Pacif.J of Math.* 15 835-855.
- [IL] Irany, S, Leung, V. (1997) Scheduling with conflicts. *Technical Report UCI-ICS-97-21, Department of Information and Computer Science, University of California, Irvine, California 92697-3425.*
- [L] Lenzen.F. (2000) An Integer Programming Approach to the Multi-leaf Collimator Problem *Diploma Thesis, Supervisor Hamacher. W.H, University of Kaiserslautern, Dept. of Mathematics.*
- [LSW] Lin.J.I, Sen.K.M, West.B.D. (1997) Classes of interval digraphs and 0,1-matrices *28th SE Conf.Comb.Graph.Th. and Comp.Congr.Numer.* 201-209.
- [M] Muller.H, (1997) Recognizing interval digraphs and interval bigraphs in polynomial time. *Discrete Applied Mathematics, 78* 189-205.

- [MR] Mohring.H.R, Radermacher.J.F. (1984) Scheduling problem with resource-duration interaction *Methods of Oper.Res.*48 423-452.
- [OR] Oswald.M, Reinhelt.G, (2000) Polyhedral Aspects of the Consecutive Ones Problem.*D.-Z.Du et al. Eds: COCOON 2000, LNCS 1858, Springer-Verlag Berlin Heidelberg* 373-382.
- [PY] Papadimitriou.H.C, Yannakakis.M. (1979) Scheduling Interval Ordered Tasks. *SIAM J. Computing, Vol. 8(3)*. 405-409.
- [R1] Roberts.F.S. (1976) Discrete Mathematical Models, with Applications to Social, Biological and Environmental Problems *Prentice-Hall, Englewood Cliffs, NJ*.
- [R2] Roberts.F.S. (1978a) Food webs competition graph and the boxity of ecological phase space. *In Alavi.Y,Lick.D editions,Theory and Application of Graph, Springer-Verlag, New York* 477-490.
- [R3] Roberts.F.S. (2000) Some application of graph theory. *In editions of Department of Mathematics and DIMACS, Rutgers University, USA (draft)*.
- [RaRo] Raychaudhuri.A, Roberts.F.S. (1985) Generalized competition graph and their application.*Methods of Operations Research* 49, *Konigstein, Germany* 477-490.
- [SSRW] Sanyal.K.B, Sen.M, Roy.B.A, West.B.D. (1994) Indifference digraphs: Generalization of indifference graphs and semiorders *SIAM J.Discrete Mathematics, Vol.7,No.2*. 157-162.
- [SDRW] Sen.M, Das.S. (1989) Interval digraphs: An analogue of interval graphs, *Journal of graph theory,Vol.13,No.2*. 189-202.
- [S1] Shamir.R. (1994) Advanced topics in graph algorithms (Lecture notes) *Tel Aviv University*.
- [S2] Shamir.R. (1999) Algorithms for molecular biology (Lecture notes) *Tel Aviv University*.
- [St] Stoffers.K.E (1968) Scheduling of traffic lights - a new approach *Transportation Research* 2. 199-234.

- [T1] Tucker.A, (1971) Matrix characterization of circular-arc graphs *Pacific J. Math*, 39 535-545.
- [T2] Tucker.A, (1972) A structure theorem for the consecutive ones property *Journal of Combinatorial Theory*,12(B). 153-162.
- [W] Wen.L.H, (1997) On physical mapping algorithms: An error tolerant test for the consecutive ones property *COCOON 1997* 242-250.