Homological Mirror Symmetry in Dimension One

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Abstract. In this paper we complete the proof began by A. Polishchuk and E. Zaslow [**PZ**] of a weak version of Kontsevich's homological mirror symmetry conjecture for elliptic curves.

1. Introduction

This note grew out of an attempt to understand the details in the paper of A. Polishchuk and E. Zaslow [**PZ**] on the homological mirror symmetry conjecture of M. Kontsevich. This conjecture was formulated in his ICM talk [**Ko**] in Zürich in 1994. Roughly speaking, this conjecture describes a close relation between two seemingly totally unrelated mathematical structures defined in dependence of two Calabi-Yau manifolds, a so-called mirror pair. Although the conjecture is mostly interesting in dimension three, it is mathematically interesting also in other dimensions.

Of course, one encounters the easiest situation in dimension one. A one dimensional Calabi-Yau manifold is just a two dimensional torus (elliptic curve). In [**PZ**] precisely this case is investigated. There exist attempts to generalize their results to higher dimensional tori, undertaken by C. van Enckevort [**En**] and K. Fukaya [**F2**]. These papers show that the conjecture is very hard, even in the presumably simple case of a torus. To date there is no complete proof of the conjecture, even in dimension one. In [**PZ**] a weaker modification of Kontsevich's conjecture is studied, but it is not completely clear how to obtain a similar result in the higher dimensional case without restricting additionally the objects in the categories compared.

The contribution of this paper to the conjecture is a complete proof and a correct formulation of the weakened version of Kontsevich's conjecture described in [**PZ**]. We will not repeat all steps of the proof. Here we restrict to those details whose proofs seem to require more explanation or are lacking. The author hopes to contribute in this way to a better understanding of the beautiful results of [**PZ**]. The availability of [**En**] and [**F2**] was particularly helpful in preparing this paper.

The homological mirror symmetry conjecture, as stated by M. Kontsevich in his ICM talk in Zürich in 1994 [**Ko**] can be formulated as follows:

For every Calabi-Yau (3-)manifold X there exists a mirror partner X° with a symplectic form ω and an equivalence of (A_∞)-categories:

$$\mathcal{D}^b(X) \cong \mathcal{F}(X^\circ).$$

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We denote by $\mathcal{D}^b(X)$ the derived category of the abelian category of coherent sheaves on X ([**GM**], [**H1**]). By $\mathcal{F}(X^\circ)$ we denote Fukaya's category, which actually is not a category in the usual sense. It is a so-called A_∞-category. To give the conjecture a precise meaning, one has to either equip $\mathcal{D}^b(X)$ with an A_∞-structure or to modify Fukaya's category in such a way that it becomes a triangulated (ordinary) category. In this paper we follow [**PZ**] and replace $\mathcal{F}(X^\circ)$ by its cohomology and compare it with $\mathcal{D}^b(X)$ as an additive category (forgetting the triangulated structure). To prove deeper versions of mirror symmetry in the one dimensional case involving the A_∞-structure, it will probably be helpful to have a detailed proof in this simple case. A proof of the equivalence of two A_∞-structures on the category of vector bundles on an elliptic curve was given in [**P2**]. One of these structures is induced via the equivalence with the cohomology of Fukaya's category and the second A_∞-structure was introduced in [**P1**], inspired by [**Me**]. But in [**P2**] only so-called transversal structures are considered, non-transversal Lagrangians are not taken into account.

In Section 2 we collect basic facts about the structure of the bounded derived category of coherent sheaves on a smooth projective curve necessary for the proof of the main theorem. Because the construction of our equivalence of categories relies heavily on the use of a version of Serre duality for coherent sheaves on an elliptic curve, a proof of it is presented here.

In Section 3 we recall the definition of Fukaya's category as given in [**PZ**]. To overcome the fact that [**PZ**] do not define spaces of morphisms between arbitrary pairs of objects, we define the cohomology $\mathcal{F}^0(X)$ of Fukaya's category directly, i.e. without giving a definition of Fukaya's A_∞-category $\mathcal{F}(X)$. To be able to prove the main theorem we need to enlarge this category to get an additive category $\mathcal{FK}^0(X)$. We study some functors between such categories and collect important properties of them in analogy to the statements in Section 2.

Section 4 is devoted to the proof of the main theorem stating an equivalence of categories

$$\mathcal{D}^b(X) \cong \mathcal{FK}^0(X^\circ).$$

We focus on those parts of the proof which are not contained in the paper of A. Polishchuk and E. Zaslow to keep the paper short.

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2. The derived category

One reason, that makes possible the proof of a weak version of the homological mirror symmetry conjecture in dimension one, is the fact that the bounded derived category $\mathcal{D}^b(X) := \mathcal{D}^b(\mathrm{Coh}_X)$ of a smooth projective curve X has a well understood structure. To become more specific let us recall [**KS**] that the homological dimension $\operatorname{hd}\mathcal{A}$ of an abelian category $\mathcal{A}$ is by definition the smallest non-negative integer k such that for all $j > k$ and all objects A, B in $\mathcal{A}$: $\quad \mathrm{Ext}^j(A, B) = 0$.

We know, for example, that the category $\mathcal{A} = \mathrm{Coh}_X$ of coherent sheaves on a smooth projective variety X has homological dimension $\operatorname{hd}(\mathrm{Coh}_X) = \dim X$, because any coherent sheaf on X has a resolution of length at most equal to $\dim X$ whose members are vector bundles of finite rank.

On the other hand, the following lemma, which is an easy exercise (see [**KS**]), provides the information we are interested in. Here we denote as usual by $A[-k]$ the object in $\mathcal{D}^b(\mathcal{A})$ given by the complex having the object A of the abelian category $\mathcal{A}$ at position k and the zero object elsewhere.

LEMMA 2.1. *If $\mathcal{A}$ is an abelian category with $\mathrm{hd}(\mathcal{A}) \leq 1$ and $A^\bullet$ an object of $\mathcal{D}^b(\mathcal{A})$, then $A \cong \bigoplus_k H^k(A^\bullet)[-k]$ in $\mathcal{D}^b(\mathcal{A})$.*

This gives the following structure result for the bounded derived category of a curve:

COROLLARY 2.2. *If X is a smooth projective curve and $A^\bullet$ an object of $\mathcal{D}^b(X)$, then $A \cong \bigoplus_k H^k(A^\bullet)[-k]$ in $\mathcal{D}^b(X)$.*

Here we denote by $H^k(A^\bullet)$ the k-th cohomology sheaf of the complex $A^\bullet$ (considered as an object in the derived category). It is not the sheaf cohomology of the members of this complex.

Call a coherent sheaf indecomposable if it is not the direct sum of two non-trivial coherent sheaves. Now we can formulate the following result.

PROPOSITION 2.3. *Let X be a smooth projective curve. By taking finite direct sums of objects of the form $A[n]$, where A is an indecomposable coherent sheaf on X, we obtain all objects of $\mathcal{D}^b(X)$ up to isomorphism.*

By using standard results from category theory [**ML**] we can reformulate this in the following way:

PROPOSITION 2.4. *Let X be a smooth projective curve. The full subcategory of $\mathcal{D}^b(X)$ formed by all objects which are finite direct sums of objects of the form $A[n]$, where A is an indecomposable coherent sheaf on X, is equivalent to $\mathcal{D}^b(X)$.*

We will work with this subcategory instead of the bounded derived category itself, but for simplicity we will not always mention this explicitly. Note this subcategory is closed under shifts and finite direct sums.

On a smooth projective curve X for any coherent sheaf A the torsion free sheaf A/A_{tor} is locally free and because a coherent sheaf with zero dimensional support has vanishing first cohomology, any coherent sheaf is the direct sum of a locally free sheaf and a torsion sheaf: $A \cong A/A_{\mathrm{tor}} \oplus A_{\mathrm{tor}}$. In particular, any indecomposable coherent sheaf is either an (indecomposable) torsion sheaf supported at one point or an (indecomposable) vector bundle. This observation will be used frequently.

We need a more explicit description of the indecomposable coherent sheaves on elliptic curves. We denote by E_τ the elliptic curve defined by the lattice $\Gamma_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ where τ is a complex number lying in the upper half plane. By abuse of language we will speak about "the point $a\tau + b$" on E_τ if we mean the point $a\tau + b \mod \Gamma_\tau$.

All line bundles on E_τ are obtained in the following way: Let $\varphi : \mathbb{C} \to \mathbb{C}^*$ be a holomorphic function satisfying $\varphi(z+1) = \varphi(z)$ for all $z \in \mathbb{C}$. The line bundle $\mathcal{L}(\varphi)$ is by definition $(\mathbb{C} \oplus \mathbb{C})/\Gamma_\tau$ where the action of the lattice is defined by $\tau \cdot (z, v) = (z + \tau, \varphi(z)v)$ (and $1 \in \Gamma_\tau$ acts trivially). If it is important to stress the dependence on τ we write $\mathcal{L}_\tau(\varphi)$ instead of $\mathcal{L}(\varphi)$. A special role is played by the function $\varphi_0(z) = \exp(-\pi i\tau - 2\pi i z)$. The line bundle $\mathcal{L}_\tau(\varphi_0)$ corresponds to the divisor of degree one supported at the point $\frac{1+\tau}{2}$. We obtain all line bundles on E_τ by considering functions $\varphi(z) = t_x^*\varphi_0 \cdot \varphi_0^{n-1}$, where $x = a\tau + b$ is a point on E_τ, t_x is the translation by x (i.e. $t_x^*\varphi(z) = \varphi(z+x)$) and n is an integer. The global sections of $\mathcal{L}(\varphi)$ are given by holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ fulfilling the identities $f(z+\tau) = \varphi(z)f(z)$ and $f(z+1) = f(z)$ for all $z \in \mathbb{C}$. A basis of the space of global sections $H^0(\mathcal{L}(t_x^*\varphi_0^n))$ is formed by the theta functions with characteristics $t_x^*\theta\big[\frac{k}{n}, 0\big](n\tau, nz)$ where $0 \le k < n$. In particular, the dimension of this space of sections is n if $n > 0$. According to Mumford [**Mu**] we use here

$$\theta\big[a,b\big](\tau,z) := \sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2 + 2\pi i(n+a)(z+b)).$$

To describe vector bundles of higher rank we use the notion of a unipotent vector bundle. A vector bundle F is called *unipotent* if it has a filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_r = F$ with $F_{i+1}/F_i \cong \mathcal{O}_{E_\tau}$. We will consider the following unipotent bundles: Let V be a (finite dimensional) complex vector space and $A \in \mathrm{GL}(V)$, then the bundle $F(V,A)$ is by definition $(\mathbb{C} \times V)/\Gamma_\tau$ where the action of Γ is given by $\tau \cdot (z,v) = (z+\tau, Av)$ (and $1 \in \Gamma$ acting trivially). In the particular case where $A = \exp(N)$ with a nilpotent endomorphism $N \in \mathrm{End}(V)$ we obtain a unipotent bundle $F(V,\exp(N)) = F_\tau(V,\exp(N))$. If the kernel of N is of dimension one, then $F(V,\exp(N))$ is indecomposable. A nilpotent endomorphism with one dimensional kernel will be called *cyclic*, because it defines the structure of a cyclic $\mathbb{C}[T]$-module on V.

Later we shall need the following lemma, which follows easily from the fact that holomorphic sections of a flat vector bundle are covariantly constant.

LEMMA 2.5. *If $A \in \mathrm{GL}(V)$ then*

$$H^0(E_\tau, F_\tau(V,A)) = \ker(\mathbf{1}_V - A).$$

In particular,

$$\mathrm{Hom}(F(V_1,A_1), F(V_2,A_2)) = \{f \in \mathrm{Hom}(V_1,V_2) \mid f \circ A_1 = A_2 \circ f\}.$$

Thanks to the following result, which follows from work of M.F. Atiyah [**At**], this is enough to have an explicit description of all indecomposable vector bundles on an elliptic curve. To formulate the result we need the notion of an isogeny. This is a surjective homomorphism of algebraic groups with finite kernel. We need only the isogenies: $\pi_r : E_{r\tau} \to E_\tau$ coming from the inclusion $\Gamma_{r\tau} \subset \Gamma_\tau$ and the identity map on $\mathbb{C}$.

THEOREM 2.6. *Any indecomposable vector bundle on an elliptic curve E_τ is of the form $\pi_{r*}(\mathcal{L}_{r\tau}(\varphi) \otimes F_{r\tau}(V,\exp(N)))$, with N a cyclic nilpotent endomorphism.*

An indecomposable coherent sheaf which is not a vector bundle is a torsion sheaf supported at one point $x = a\tau + b$. The space V of global sections of such a sheaf is a finite dimensional complex vector space with the structure of an $\mathcal{O}_{E_\tau,x}$-module. Since we are on a smooth curve, we can choose a generator of the maximal ideal of $\mathcal{O}_{E_\tau,x}$ (a uniformizing parameter). The module structure is then determined by the action of this generator on V. Because we constructed E_τ as a quotient of $\mathbb{C}$ we can use the function $z - x$ as a generator, where z denotes the usual holomorphic coordinate on $\mathbb{C}$. Thus, a coherent sheaf supported at one point is determined by the point x supporting it, a vector space V and a nilpotent endomorphism $N \in \mathrm{End}(V)$. We denote it by $S(x,V,N)$ or sometimes also by $S_\tau(x,V,N)$. Again, the sheaf is indecomposable if and only if N has one dimensional kernel.

Because the author was not able to find the following version of Serre duality in the literature, we give a proof here.

LEMMA 2.7. *Let A_1, A_2 be two coherent sheaves on the elliptic curve E_τ, then we have a functorial isomorphism*

$$\mathrm{Ext}^1(A_2, A_1) \cong \mathrm{Hom}(A_1, A_2)^*.$$

PROOF. Recall that the dualizing sheaf of a smooth elliptic curve is the structure sheaf. Using Serre duality as presented in [**H2**, III.7.6] and using [**H2**, III.6.3, III.6.7] we obtain the statement of the Lemma with the additional assumption that A_2 is locally free.

Let now A_2 be an arbitrary coherent sheaf. Because E_τ is a smooth projective curve, there exists an exact sequence $0 \to F' \to F \to A_2 \to 0$ with locally free sheaves F and F'. By applying the covariant functors $\mathrm{Ext}^1(A_1, \cdot)$ and $\mathrm{Hom}(\cdot, A_1)^*$ we obtain the following diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Ext}^1(A_1, F') & \longrightarrow & \mathrm{Ext}^1(A_1, F) & \longrightarrow & \mathrm{Ext}^1(A_1, A_2) & \longrightarrow & 0 \\
\cong\big\downarrow & & \cong\big\downarrow & & \exists\big\downarrow & & \\
\mathrm{Hom}(F', A_1)^* & \longrightarrow & \mathrm{Hom}(F, A_1)^* & \longrightarrow & \mathrm{Hom}(A_2, A_1)^* & \longrightarrow & 0
\end{array}$$

The commutativity of the left square ensures the existence of the right vertical arrow, which is an isomorphism by the five lemma. By construction, this isomorphism is functorial with respect to A_1. To get functoriality with respect to A_2 we have to show commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Ext}^1(A_1, A_2) & \longrightarrow & \mathrm{Ext}^1(A_1, \tilde{A}_2) \\
\big\downarrow\cong & & \big\downarrow\cong \\
\mathrm{Hom}(A_2, A_1)^* & \longrightarrow & \mathrm{Hom}(\tilde{A}_2, A_1)^*
\end{array} \tag{2.1}$$

arising from a morphism of coherent shaves $A_2 \to \tilde{A}_2$ and resolutions

$$0 \to F' \to F \to A_2 \to 0$$

and

$$0 \to \tilde{F}' \to \tilde{F} \to \tilde{A}_2 \to 0$$

as above. A priori the vertical isomorphism in the diagram (2.1) might depend on the resolutions chosen. But once we have shown commutativity for the identity $A_2 \to A_2$ (but arbitrary resolutions) we know that the isomorphism $\mathrm{Ext}^1(A_1, A_2) \cong \mathrm{Hom}(A_2, A_1)^*$ does not depend on the resolution of A_2. In particular, if A_2 is locally free, this is already known.

To show commutativity of (2.1) one usually lifts $A_2 \to \tilde{A}_2$ to a morphism of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & A_2 & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \tilde{F}' & \longrightarrow & \tilde{F} & \longrightarrow & \tilde{A}_2 & \longrightarrow & 0
\end{array}$$

but this will not be possible in general.

We consider first the situation where $A_2 = \tilde{A}_2$ is a torsion sheaf, $\tilde{F} = F \otimes L$ and $\tilde{F}' = F' \otimes L$ for a line bundle L on E_τ. Observe that $A_2 \otimes L \cong A_2$ since A_2 is a torsion sheaf. If the degree of L is large enough we have $\mathrm{Ext}^1(F, F' \otimes L) = H^1(F^\vee \otimes F' \otimes L) = 0$ and obtain a lift for any morphism $A_2 \to A_2$. This shows that we obtain the same Serre duality isomorphism, if we twist the resolution of a torsion sheaf by a line bundle of sufficiently high degree.

If now $\tilde{A}_2$ is an arbitrary torsion sheaf we can apply the same argument to show that we can lift any morphism $A_2 \to \tilde{A}_2$ to a morphism from

$$0 \to F' \to F \to A_2 \to 0$$

to a twisted resolution

$$0 \to \tilde{F}' \otimes L \to \tilde{F} \otimes L \to \tilde{A}_2 \to 0$$

where L is of sufficiently high degree. This shows commutativity of diagram (2.1) in case $\tilde{A}_2$ is torsion and so also the independence on the resolution of the Serre duality isomorphism if A_2 is a torsion sheaf.

If $\tilde{A}_2$ is locally free, any torsion direct summand of A_2 will be mapped to zero in $\tilde{A}_2$. So we can assume A_2 to be locally free in this case. Then the known functoriality of the Serre duality isomorphism implies the commutativity of (2.1). Using the additivity of the functors involved, we obtain the required commutativity in general. □

To understand morphisms and their composition in $\mathcal{D}^b(E_\tau)$ we recall the functorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_1[m], A_2[n]) = \mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_1, A_2[n-m]) = \mathrm{Ext}^{n-m}(A_1, A_2)$$

for objects A_1, A_2 in Coh_{E_τ}. Since E_τ is a curve this vanishes if $n - m \notin \{0, 1\}$.

So we have $\mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_1, A_2) = \mathrm{Hom}(A_1, A_2)$, which means that Coh_{E_τ} is a full subcategory of $\mathcal{D}^b(E_\tau)$ if we send A to $A[0]$. On the other hand, if we combine the above isomorphism with Serre duality (Lemma 2.7) we obtain a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_1, A_2[1]) = \mathrm{Ext}^1(A_1, A_2) = \mathrm{Hom}(A_2, A_1)^*.$$

using functoriality in the first argument we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_1, A_2) \times \mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_2, A_3[1]) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_1, A_3[1]) \\
\big\downarrow{=} & & \big\downarrow{=} \\
\mathrm{Hom}(A_1, A_2) \times \mathrm{Ext}^1(A_2, A_3) & \longrightarrow & \mathrm{Ext}^1(A_1, A_3) \\
\big\downarrow{\cong} & & \big\downarrow{\cong} \\
\mathrm{Hom}(A_1, A_2) \times \mathrm{Hom}(A_3, A_2)^* & \longrightarrow & \mathrm{Hom}(A_3, A_1)^*
\end{array}$$

where the arrow in the bottom row sends (f, ψ) to $\psi(f \circ \cdot) \in \mathrm{Hom}(A_3, A_1)^*$. The map in the second row is usually called Yoneda pairing. Similarly, using functoriality in the second argument, we translate the composition

$$\mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_1, A_2[1]) \times \mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_2[1], A_3[1]) \to \mathrm{Hom}_{\mathcal{D}^b(E_\tau)}(A_1, A_3[1])$$

to the map

$$\mathrm{Hom}(A_2, A_1)^* \times \mathrm{Hom}(A_2, A_3) \to \mathrm{Hom}(A_3, A_1)^*$$

sending (φ, g) to $\varphi(\cdot \circ g) \in \mathrm{Hom}(A_3, A_1)^*$.

3. A version of Fukaya's category

In his formulation of homological mirror symmetry Kontsevich used a modified version of Fukaya's A_∞-category. In Fukaya's original construction [**F1**] the objects were certain Lagrangian submanifolds of a fixed symplectic manifold. The morphisms in this A_∞-category are defined with the aid of Floer complexes for Lagrangian intersections. Inspired by some developments in physics (D-branes), Kontsevich proposed to take as objects special Lagrangian submanifolds equipped with unitary local systems. In the paper [**F2**], Fukaya considers Hamiltonian isotopy classes of Lagrangian submanifolds equipped with flat line bundles. At the moment it doesn't seem to be clear which one is the final definition. We will follow here the paper [**PZ**] and use as objects pairs consisting of a special Lagrangian submanifold and a local system on it whose monodromy has eigenvalues of modulus one. It is an interesting question to study the relations between all these different versions of Fukaya's category, even in the case of a torus.

There are some attempts to define an A_∞-category for any symplectic manifold (equipped with a B-field) [**F1**], [**F2**]. But there are some unclear technical details in the general case. In the case of a torus of dimension one the situation is simple enough to be able to study the cohomology of this A_∞-category. In particular, the graded spaces of morphisms between transversal Lagrangians consist of just one graded piece and so the differential on this complex is trivial. Taking cohomology is then the same as taking the zeroth graded piece. The appearance of this simple structure seems to be a second reason for the accessibility of the proof.

We are not treating the A_∞-structure here, because the space of homomorphisms between objects with the same underlying Lagrangian submanifold has more that one graded piece and is determined up to homotopy only. Results about Floer homology suggest (see [**F2**]) what the correct cohomology space of such a complex should be and so we are able to define $\mathcal{F}^0(E^\tau)$ without having $\mathcal{F}(E^\tau)$. See also [**En**].

We consider the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and equip it with a symplectic structure by using the volume form $A(dx^2 + dy^2)$, where x and y are real coordinates and $A > 0$ is a positive real number. The Kähler form of this metric is $Adx \wedge dy$. We choose, in addition, a so-called B-field which is an element of $H^2(\mathbb{T}^2, \mathbb{R})/H^2(\mathbb{T}^2, \mathbb{Z})$. It can be represented by a two form $Bdx \wedge dy$ with $B \in \mathbb{R}$. Choosing such a representative we obtain a complex number $\tau = B + iA$ lying in the upper half plane. The B-field determines τ modulo $\mathbb{Z}$. The torus $\mathbb{T}^2$ equipped with τ will be denoted by E^τ. The two form $\tau(dx \wedge dy)$ is also called the *complexified Kähler form.*

Let us recall the definition of $\mathcal{F}^0(E^\tau)$, the cohomology of Fukaya's A_∞-category $\mathcal{F}(E^\tau)$ for the symplectic torus. We will not consider A_∞-structures here. We follow [**PZ**] to define the objects and morphisms in the transversal case. In contrast with the definition of Polishchuk and Zaslow we define morphisms between any pair of objects, not only in the transversal case.

The objects are triples (Λ, α, M) where $\Lambda \subset E^\tau$ is a closed submanifold given by an affine line in $\mathbb{R}^2$. We describe such a submanifold usually by a parametrization $(x(t), y(t))$, where the two components depend linearly on the real parameter t. To get a closed submanifold, this line must have rational slope.

Such submanifolds equipped with an orientation are precisely the special Lagrangian submanifolds of E^τ. For a discussion of the concept of special Lagrangian submanifolds see e.g. [**HL**].

To get a shift functor in Fukaya's category we have to equip the Lagrangian submanifolds with an additional structure called a grading. A detailed study of this concept can be found in [**Se**]. In general, this additional structure is necessary to define the Maslov index, which is used to introduce a grading on the spaces of morphisms in Fukaya's A_∞-category.

Because we consider special Lagrangian submanifolds of a two torus, things simplify a lot. The additional structure is given by the choice of a logarithm α of the slope of Λ. More precisely, α is a real number with the property that there exists $t \in \mathbb{R}$ with $x(0) + iy(0) + \exp(\pi i \alpha) = x(t) + iy(t)$.

Finally, M stands for a local system on Λ whose monodromy has eigenvalues with unit modulus. This is slightly more general than considering only unitary local systems.

By a local system we mean here a locally constant sheaf of complex vector spaces. A local system can be described equivalently by a representation of the fundamental group of the underlying manifold or by a complex vector bundle equipped with a flat connection. Because $\Lambda = S^1$ is a circle, its fundamental group is a free abelian group with one generator. If we fix an orientation on Λ, a generator of the fundamental group is determined and a local system on Λ is given by a vector space V and an automorphism $M \in \mathrm{GL}(V)$. If we change the orientation and consider M^{-1}, we obtain an isomorphic local system on Λ. Two conjugate automorphisms define isomorphic local systems. The operator M is called the monodromy of the local system and we are considering here only monodromy automorphisms whose eigenvalues have modulus one.

Considering a triple (Λ, α, M) and speaking about the local system M we mean the local system on Λ determined by $M \in \mathrm{GL}(V)$ and the orientation on Λ given by the "direction" $\exp(\pi i \beta)$, where β is the unique real number with $-\frac{1}{2} < \beta \le \frac{1}{2}$ and $\beta - \alpha \in \mathbb{Z}$.

Sometimes it is convenient to replace the stalk M_x of a local system M by the vector space V. To do this in a coherent way we introduce the following convention: If $\Lambda = S^1$ is equipped with an orientation, the local system given by $M \in \mathrm{GL}(V)$ can be described as the sheaf of (local) solutions s of the differential equation $\nabla s = 0$ on the vector bundle $\tilde{\Lambda} \times V/\sim$. Here we let $\tilde{\Lambda} \to \Lambda$ be the universal cover, which is the affine line in $\mathbb{R}^2$ defining $\Lambda \subset E^\tau$ in our special situation. By $\lambda \in \mathbb{R}^2$ we denote the unique vector parallel to $\tilde{\Lambda}$ representing a lift of the positive generator of $\pi_1(\Lambda)$. The equivalence relation $\sim$ is generated by $(l, v) \sim (l + \lambda, Mv)$. If $U \subset \Lambda$ is open and $s : U \to V$ defines a local section of $\tilde{\Lambda} \times V/\sim$, we define $\nabla s = ds$. This means, the local system is formed by locally constant maps $U \to V$. If $x \in \Lambda$ and we choose $\tilde{x} \in \tilde{\Lambda}$ over x we get from this construction an isomorphism $V \xrightarrow{\sim} M_x$ between V and the stalk of the local system M at x. Our convention will be to choose once an for all an interval of the form $\{\tilde{x}_0 + t\lambda \mid 0 \le t \le 1\} \subset \tilde{\Lambda}$ for any Λ and use the isomorphism $V \xrightarrow{\sim} M_x$ defined by the point in this particular interval which lies over x. The advantage of this choice is that in case $M = \exp(M')$ the monodromy of the local system M along a path $\gamma : [0,1] \to \Lambda$ is given by $\exp(sM') \in \mathrm{GL}(V)$ via the identifications $V \xrightarrow{\sim} M_{\gamma(0)}$ and $V \xrightarrow{\sim} M_{\gamma(1)}$ chosen above. By s we denote here the unique real number such that $\tilde{\gamma}(t) := \tilde{x}_0 + ts\lambda \in \tilde{\Lambda}$ defines a lift of γ.

The shift functor on objects of $\mathcal{F}^0(E^\tau)$ is defined by

$$(\Lambda, \alpha, M)[1] := (\Lambda, \alpha + 1, M).$$

Let $(\Lambda_\nu, \alpha_\nu, M_\nu)$ with $\nu = 1, 2$ be two objects of $\mathcal{F}^0(E^\tau)$. If $\Lambda_1 \neq \Lambda_2$ we define

$$\operatorname{Hom}_{\mathcal{F}^0(E^\tau)}((\Lambda_1, \alpha_1, M_1), (\Lambda_2, \alpha_2, M_2)) = \\ = \begin{cases} 0 & \text{if } \alpha_2 - \alpha_1 \notin [0, 1), \\ \bigoplus_{x \in \Lambda_1 \cap \Lambda_2} \operatorname{Hom}((M_1)_x, (M_2)_x) & \text{if } \alpha_2 - \alpha_1 \in [0, 1). \end{cases}$$

If, however, $\Lambda_1 = \Lambda_2$, we automatically have $\alpha_1 - \alpha_2 \in \mathbb{Z}$ and we define

$$\operatorname{Hom}_{\mathcal{F}^0(E^\tau)}((\Lambda_1, \alpha_1, M_1), (\Lambda_2, \alpha_2, M_2)) = \\ = \begin{cases} 0 & \text{if } \alpha_2 - \alpha_1 \notin \{0, 1\}, \\ H^0(\Lambda_1, \mathcal{H}om(M_1, M_2)) & \text{if } \alpha_2 = \alpha_1, \\ H^1(\Lambda_1, \mathcal{H}om(M_1, M_2)) & \text{if } \alpha_2 = \alpha_1 + 1. \end{cases}$$

Here we denote by $\mathcal{H}om(M_1, M_2)$ the sheaf of homomorphisms of the two local systems defined by the vector spaces V_ν and the automorphisms $M_\nu \in \mathrm{GL}(V_\nu)$. The vector space for this local system is $V = \operatorname{Hom}(V_1, V_2)$ and the automorphism $M \in \mathrm{GL}(V)$ is given by $M(f) = M_2 \circ f \circ M_1^{-1}$ for $f \in V$, because we choose always the same orientation on Λ_1.

Since Λ_1 is a circle, it is easy to compute the sheaf cohomology appearing in the definition. The result is the following: $H^0(\Lambda_1, M) \cong \ker(M - \mathbf{1}_V)$ and $H^1(\Lambda_1, M) \cong \operatorname{coker}(M - \mathbf{1}_V)$. This gives, in particular,

$$H^0(\Lambda_1, \mathcal{H}om(M_1, M_2)) \cong \{f \in \operatorname{Hom}(V_1, V_2) \mid M_2 \circ f = f \circ M_1\}$$

and

$$H^1(\Lambda_1, \mathcal{H}om(M_1, M_2)) \cong \operatorname{Hom}(V_1, V_2)/M_2 \circ \operatorname{Hom}(V_1, V_2) \circ M_1^{-1}.$$

As an application we obtain for $k = 0, 1$

$$H^k(\Lambda, \mathcal{H}om(M_1, M_2)) \neq 0 \iff M_1 \text{ and } M_2 \text{ have a common eigenvalue.}$$

Since $\ker(M - \mathbf{1}_V)^* \cong \operatorname{coker}({}^tM - \mathbf{1}_V) = \operatorname{coker}({}^tM^{-1} - \mathbf{1}_V)$ we obtain a canonical isomorphism $H^0(\Lambda_1, M)^* \cong H^1(\Lambda_1, M^\vee)$ (where * denotes the dual of a vector space, ${}^tM$ the transposed homomorphism and $M^\vee$ the dual local system, which is given by the automorphism ${}^tM^{-1}$.)

Combining this with our definitions we obtain the following "Symplectic Serre Duality":

LEMMA 3.1. *If $(\Lambda_i, \alpha_i, M_i)$ are objects in $\mathcal{F}^0(E^\tau)$ then there exists a canonical isomorphism*

$$\operatorname{Hom}((\Lambda_1, \alpha_1, M_1), (\Lambda_2, \alpha_2, M_2)[1]) \cong \operatorname{Hom}((\Lambda_2, \alpha_2, M_2), (\Lambda_1, \alpha_1, M_1))^*.$$

To complete the definition of the category $\mathcal{F}^0(E^\tau)$ we have to define the composition of morphisms. This will be the first place where dependence on τ occurs. Because this is not yet done, we cannot speak about functoriality in Lemma 3.1.

Let $(\Lambda_1, \alpha_1, M_1) \xrightarrow{u} (\Lambda_2, \alpha_2, M_2) \xrightarrow{v} (\Lambda_3, \alpha_3, M_3)$ be two non-zero morphisms in $\mathcal{F}^0(E^\tau)$. The definitions imply $\alpha_1 \leq \alpha_2 \leq \alpha_3, \alpha_2 \leq \alpha_1 + 1$ and $\alpha_3 \leq \alpha_2 + 1$. To define $v \circ u$ we have to distinguish several cases.

Case (i): $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_1 + 1$.

This is the case with three different underlying Lagrangians. Only this case is considered in [**PZ**] and their definition is the following:

Assume $u \in \mathrm{Hom}((M_1)_{x_1}, (M_2)_{x_1}) \subset \mathrm{Hom}((\Lambda_1, \alpha_1, M_1), (\Lambda_2, \alpha_2, M_2))$ and $v \in \mathrm{Hom}((M_2)_{x_2}, (M_3)_{x_2})$ with $x_1 \in \Lambda_1 \cap \Lambda_2, x_2 \in \Lambda_2 \cap \Lambda_3$. For any $x_3 \in \Lambda_1 \cap \Lambda_3$, the component of $v \circ u$ in $\mathrm{Hom}((M_1)_{x_3}, (M_3)_{x_3})$ will be given by the following expression:

$$\sum_{\phi} \exp(2\pi i \int \phi^*(\tau dx \wedge dy)) P(M_3) \circ v \circ P(M_2) \circ u \circ P(M_1) \tag{3.1}$$

where the sum is performed over equivalence classes of quadruples $(\phi; z_1, z_2, z_3)$ where $\phi : D \to E^\tau$ is a holomorphic map, $D \subset \mathbb{C}$ denotes the unit disc and $\{z_1, z_2, z_3\}$ are distinct points on the boundary of D such that the order (z_3, z_2, z_1) coincides with the usual orientation of $\mathbb{C}$. We require $\phi(z_\nu) = x_\nu$ and that ϕ maps the arc connecting z_ν with $z_{\nu-1}$ to Λ_ν. Equivalent quadruples are obtained by applying automorphisms of D to ϕ and the z_ν. By $P(M_\nu) : (M_\nu)_{x_{\nu-1}} \to (M_\nu)_{x_\nu}$ we denote the parallel transport on the local system M_ν. Using the identification $(M_\nu)_{x_\nu} \cong V_\nu$ chosen above and assuming $M_\nu = \exp(M'_\nu) \in \mathrm{GL}(V_\nu)$, the map $P(M_\nu)$ is identified with $\exp(sM'_\nu) \in \mathrm{GL}(V_\nu)$ where s is the unique real number such that $[0,1] \ni t \mapsto \tilde{x}_{\nu-1} + ts\lambda_\nu \in \mathbb{R}^2$ is a lift to $\mathbb{R}^2$ of the path connecting $x_{\nu-1}$ with x_ν which is the image under ϕ of the corresponding arc on the boundary of D. The weights in the sum can be computed as $\exp(2\pi i \tau A_\phi)$ where A_ϕ denotes the Euclidean area of a triangle in $\mathbb{R}^2$ whose image in E^τ is $\phi(D)$. (The edges of this triangle can be used as the lifts of the paths connecting the x_ν as above.)

Case (ii): $\alpha_3 > \alpha_1 + 1$.

We define $v \circ u = 0$.

Case (iii): Precisely two of the α_ν coincide and $\alpha_1 + 1 < \alpha_3$.

If $\alpha_1 = \alpha_2 < \alpha_3$ we have $\Lambda_1 = \Lambda_2 \neq \Lambda_3$. Let us abbreviate $\mathbf{\Lambda}_\nu = (\Lambda_\nu, \alpha_\nu, M_\nu)$ and define the composition via the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2) \otimes \mathrm{Hom}(\mathbf{\Lambda}_2, \mathbf{\Lambda}_3) & \longrightarrow & \mathrm{Hom}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_3) \\
\big\downarrow = & & \big\downarrow = \\
H^0(\Lambda_1, \mathcal{H}om(M_1, M_2)) \otimes \bigoplus_x \mathrm{Hom}((M_2)_x, (M_3)_x) & \longrightarrow & \bigoplus_x \mathrm{Hom}((M_1)_x, (M_3)_x)
\end{array}$$

The sums in the bottom row are performed over all $x \in \Lambda_1 \cap \Lambda_3$ and the morphism sends $\varphi \otimes (f_x)_x$ to $(f_x \circ \varphi_x)_x$ where $\varphi_x : (M_1)_x \to (M_2)_x$ denotes the map on stalks induced by φ. The definition in the case $\alpha_1 < \alpha_2 = \alpha_3$ is similar.

Case (iv): $\alpha_1 = \alpha_2 = \alpha_3$.

Here we have $\Lambda_1 = \Lambda_2 = \Lambda_3$ and we can use the composition of homomorphisms of local systems to define the composition in $\mathcal{F}^0(E^\tau)$.

Case (v): We are not in case (ii) and two of the α_ν differ by 1 or equivalently $\alpha_3 = \alpha_1 + 1$.

By using the canonical isomorphism $\mathrm{Hom}(U \otimes V^*, W^*) = \mathrm{Hom}(W \otimes U, V)$ (where U, V, W are vector spaces) and symplectic Serre duality we reduce this situation to other cases as follows: If $\alpha_1 < \alpha_2 < \alpha_3 = \alpha_1 + 1$ we are led by the diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\mathbf{\Lambda}_1,\mathbf{\Lambda}_2)\otimes\mathrm{Hom}(\mathbf{\Lambda}_2,\mathbf{\Lambda}_3) & \longrightarrow & \mathrm{Hom}(\mathbf{\Lambda}_1,\mathbf{\Lambda}_3) \\
\downarrow = & & \downarrow = \\
\mathrm{Hom}(\mathbf{\Lambda}_1,\mathbf{\Lambda}_2)\otimes\mathrm{Hom}(\mathbf{\Lambda}_3[-1],\mathbf{\Lambda}_2)^* & \longrightarrow & \mathrm{Hom}(\mathbf{\Lambda}_3[-1],\mathbf{\Lambda}_1)^*
\end{array}$$

to a composition of the following kind

$$\mathrm{Hom}(\mathbf{\Lambda}_3[-1],\mathbf{\Lambda}_1)\otimes\mathrm{Hom}(\mathbf{\Lambda}_1,\mathbf{\Lambda}_2) \longrightarrow \mathrm{Hom}(\mathbf{\Lambda}_3[-1],\mathbf{\Lambda}_2)$$

which was studied in case (iii). If $\alpha_2=\alpha_1+1$ we have $\alpha_3=\alpha_2$ (otherwise we would be in case (ii)) and thus $\Lambda_1=\Lambda_2=\Lambda_3$. Similarly, if $\alpha_3=\alpha_2+1$ we have $\alpha_1=\alpha_2$ and $\Lambda_1=\Lambda_2=\Lambda_3$. With the aid of Lemma 3.1 both cases are reduced to case (iv).

Remark 3.2. Since symplectic Serre duality does not involve τ, the definition of the composition of morphisms depends on τ in case (i) only.

Remark 3.3. The definition, especially case (v), is made precisely to get functoriality in Lemma 3.1.

Because the derived category is additive and contains in particular finite direct sums and a zero object, we would have no chance to find an equivalence of categories if we don't have direct sums in Fukaya's category. By construction, the category $\mathcal{F}^0(E^\tau)$ is an **Ab**-category (also called a preadditive category), but it does not contain all direct sums (biproducts). To see this let $(\Lambda_\nu,\alpha_\nu,M_\nu)$ be two objects of $\mathcal{F}^0(E^\tau)$ and assume their direct sum $(\Lambda,\alpha,M)=(\Lambda_1,\alpha_1,M_1)\oplus(\Lambda_2,\alpha_2,M_2)$ exists in $\mathcal{F}^0(E^\tau)$. Then we have projections $p_\nu:(\Lambda,\alpha,M)\to(\Lambda_\nu,\alpha_\nu,M_\nu)$ and embeddings $i_\nu:(\Lambda_\nu,\alpha_\nu,M_\nu)\to(\Lambda,\alpha,M)$ fulfilling $p_\nu i_\nu=\mathbf{1}_{(\Lambda_\nu,\alpha_\nu,M_\nu)}$ and $i_1p_1+i_2p_2=\mathbf{1}_{(\Lambda,\alpha,M)}$. Assuming the $(\Lambda_\nu,\alpha_\nu,M_\nu)$ are not zero objects, we obtain that neither p_ν nor i_ν is the zero homomorphism.

If we would have $\Lambda_1\neq\Lambda_2$, using $\mathrm{Hom}((\Lambda_1,\alpha_1,M_1),(\Lambda_2,\alpha_2,M_2))\neq 0$ we would obtain directly from the definitions $\mathrm{Hom}((\Lambda_2,\alpha_2,M_2),(\Lambda_1,\alpha_1,M_1))=0$. Hence $\Lambda_1=\Lambda_2$ and we conclude that $\mathcal{F}^0(E^\tau)$ contains the direct sum of the objects as above only if $\Lambda_1=\Lambda_2$.

We use the following general construction to enlarge our category to one with direct sums.

Let $\mathcal{A}$ be an **Ab**-category. We define the category $\underline{\mathcal{A}}$ to have as objects the ordered k-tuples of objects of $\mathcal{A}$ where $k\geq 0$ runs through the integers. The morphisms are formed by matrices of morphisms of $\mathcal{A}$ and their composition is given by usual matrix multiplication. More formally $\mathrm{ob}(\underline{\mathcal{A}}):=\coprod_{k\geq 0}\prod_{\nu=1}^{k}\mathrm{ob}(\mathcal{A})$. A 0-tuple is here considered to be a unique object of $\underline{\mathcal{A}}$ serving as a zero object. We denote it by $\underline{0}$. If $\underline{A}=(A_1,\ldots,A_k)$ and $\underline{B}=(B_1,\ldots,B_l)$ are objects of $\underline{\mathcal{A}}$ with $k,l>0$, we define

$$\mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{A},\underline{B}):=\prod_{(i,j)}\mathrm{Hom}_{\mathcal{A}}(A_i,B_j).$$

If $\underline{A}=\underline{0}$ or $\underline{B}=\underline{0}$, we define $\mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{A},\underline{B}):=0$. This makes $\underline{0}$ a zero object (this means it is initial and terminal). If the category $\mathcal{A}$ contains already a zero object 0, then all k-tuples ($k\geq 0$) of the form $(0,\ldots,0)$ are isomorphic to $\underline{0}$. The composition of morphisms in $\underline{\mathcal{A}}$ is defined by the usual formula for matrix multiplication. The composition $\underline{A} \to \underline{0} \to \underline{B}$ is by definition the morphism whose components are zero morphisms. It is easy to see that $\underline{\mathcal{A}}$ is an additive category and that the additive functor $\mathcal{A} \to \underline{\mathcal{A}}$ sending an object to the 1-tuple consisting of the same object, is a universal functor from $\mathcal{A}$ to an additive category.

We apply this construction to $\mathcal{A} = \mathcal{F}^0$ and define the (cohomology of the) *Fukaya–Kontsevich* category $\mathcal{FK}^0 := \underline{\mathcal{F}^0}$. This is an additive category and the correctly formulated result from A. Polishchuk and E. Zaslow [**PZ**] is Theorem 4.1.

Now, having the correct category, we are going to define some functors on it. The study of $\mathcal{FK}^0(E^\tau)$ requires to consider the complex tori $E^{r\tau}$ for all positive integers r. We denote by $p_r : E^{r\tau} \to E^\tau$ the map defined by $p_r(x,y) = (rx,y)$. It corresponds somehow to the isogeny $\pi_r : E_{r\tau} \to E_\tau$ on the holomorphic side of the mirror story. Next we define additive functors $p_{r*} : \mathcal{FK}^0(E^{r\tau}) \to \mathcal{FK}^0(E^\tau)$ and $p_r^* : \mathcal{FK}^0(E^\tau) \to \mathcal{FK}^0(E^{r\tau})$ which are compatible with shifts. They are analogous to the functors π_{r*} and π_r^*.

Let (Λ, α, M) be an object in $\mathcal{FK}^0(E^{r\tau})$. We define

$$p_{r*}(\Lambda, \alpha, M) := (p_r(\Lambda), \alpha', p_{r*}M)$$

where α' is the unique possible value lying in the same interval $(k - \frac{1}{2}, k + \frac{1}{2}]$ with $k \in \mathbb{Z}$ as α lies and $p_{r*}M$ is the direct image of the local system M in the sense of sheaves of complex vector spaces. This means, if $p_r : \Lambda \to p_r(\Lambda)$ is of degree d and $M \in \mathrm{GL}(V)$ then $p_{r*}M \in \mathrm{GL}(V^{\oplus d})$ is given by $(p_{r*}M)(v_1, \dots, v_d) = (v_2, \dots, v_d, Mv_1)$.

If $p_r(\Lambda_1) \neq p_r(\Lambda_2)$ we obtain

$$p_{r*} : \mathrm{Hom}((\Lambda_1, \alpha_1, M_1), (\Lambda_2, \alpha_2, M_2)) \to \mathrm{Hom}(p_{r*}(\Lambda_1, \alpha_1, M_1), p_{r*}(\Lambda_2, \alpha_2, M_2))$$

in an obvious way from a bijection between the direct summands of the stalk $(p_{r*}M_\nu)_x = \bigoplus_{y \in \Lambda_\nu, p_r(y) = x} (M_\nu)_y$ with the stalks of M_ν at the preimages of x on Λ_ν.

If $\Lambda_1 = \Lambda_2$, we use the canonical homomorphism of sheaves

$$p_{r*}\,\mathcal{H}\mathrm{om}(M_1, M_2) \to \mathcal{H}\mathrm{om}(p_{r*}M_1, p_{r*}M_2)$$

and the fact that p_r is a local homeomorphism to get the required map.

If $\Lambda_1 \neq \Lambda_2$ but $p_r(\Lambda_1) = p_r(\Lambda_2)$ we necessarily have $\Lambda_1 \cap \Lambda_2 = \emptyset$, hence p_{r*} is zero in this case.

To verify compatibility of the functor p_{r*} with compositions, we consider three objects $(\Lambda_\nu, \alpha_\nu, M_\nu)$ in $\mathcal{FK}^0(E^{r\tau})$. If at least two of the Λ_ν coincide, compatibility is easily obtained from the definitions. If $\Lambda_1 \neq \Lambda_2 \neq \Lambda_3 \neq \Lambda_1$ we have to compare two sums over certain holomorphic maps $\phi^\tau : D \to E^\tau$ and $\phi^{r\tau} : D \to E^{r\tau}$ respectively (see (3.1)). The map p_r defines a bijection between the images of the lifts of these maps to $\mathbb{R}^2$. These images are triangles and their Euclidean areas A_{ϕ^τ} and $A_{\phi^{r\tau}}$ fulfill $A_{\phi^\tau} = rA_{\phi^{r\tau}}$. This implies the equality we want.

Let now (Λ, α, M) be an object in $\mathcal{FK}^0(E^\tau)$. Assume the preimage $p_r^{-1}(\Lambda)$ consists of n connected components $\Lambda^{(1)}, \dots, \Lambda^{(n)}$. Then the restrictions $p_r^{(k)} : \Lambda^{(k)} \to \Lambda$ are of degree $d := r/n$ for all k. We define

$$p_r^*(\Lambda, \alpha, M) := \bigoplus_{k=1}^{n} (\Lambda^{(k)}, \alpha', (p_r^{(k)})^* M),$$

where α' is in the same interval $(k - \frac{1}{2}, k + \frac{1}{2}]$ with $k \in \mathbb{Z}$ as α is, and $(p_r^{(k)})^*M$ denotes the sheaf on $\Lambda^{(k)}$ obtained by pulling back the local system M considered as a locally constant sheaf of complex vector spaces. This means more explicitly that this local system is defined by the vector space V and the automorphism $M^d \in \mathrm{GL}(V)$. Observe that p_r^* does not have values in $\mathcal{F}^0(E^\tau)$, because we need direct sums if $n > 1$.

To define p_r^* on morphisms we have to distinguish two cases. In the first case we assume $\Lambda := \Lambda_1 = \Lambda_2$. From sheaf theory we know that there is a canonical homomorphism $\mathcal{H}\mathrm{om}((p_r^{(k)})^* M_1, (p_r^{(k)})^* M_2) \to (p_r^{(k)})^* \mathcal{H}\mathrm{om}(M_1, M_2)$. By taking cohomology, we obtain p_r^*.

In the second case we assume $\Lambda_1 \neq \Lambda_2$. The definition is now straightforward: If $f \in \mathrm{Hom}((\Lambda_1, \alpha_1, M_1), (\Lambda_2, \alpha_2, M_2)) = \bigoplus_{x \in \Lambda_1 \cap \Lambda_2} \mathrm{Hom}((M_1)_x, (M_2)_x)$ has components $f_x \in \mathrm{Hom}((M_1)_x, (M_2)_x)$ with $x \in \Lambda_1 \cap \Lambda_2$, we define the component $(p_r^* f)_y$ of $p_r^* f$ corresponding to $y \in p_r^{-1}\Lambda_1 \cap p_r^{-1}\Lambda_2$ by

$$(p_r^* f)_y = f_{p_r(y)} \in \mathrm{Hom}((M_1)_{p_r(y)}, (M_2)_{p_r(y)}).$$

We used here the canonical isomorphism $((p_r^{(k)})^* M)_y = M_{p_r(y)}$.

A similar reasoning as for the functor p_{r*} leads to compatibility of p_r^* with compositions. This establishes that we really defined two functors p_{r*} and p_r^*.

We shall need also the pull-back under translations. If $(x_0, y_0) \in \mathbb{R}^2$ we let $t(x,y) = (x - x_0, y - y_0)$ be a translation on E^τ. We define $t^*(\Lambda, \alpha, M) := (t^{-1}(\Lambda), \alpha, t^* M)$. Since t is an isomorphism the corresponding map t^* on morphisms and compatibility with composition is obvious. Similarly, we define t_*. Directly from the definitions we obtain $(t \circ p_r)_* = t_* \circ p_{r*}$ and $(t \circ p_r)^* = p_r^* \circ t^*$.

LEMMA 3.4. *Let $p_r : E^{r\tau} \to E^\tau$ be as above and $t : E^\tau \to E^\tau$ a translation of the form $t(x,y) = (x + \frac{m}{n}, y)$, with m, n integers. Define $p = t \circ p_r : E^{r\tau} \to E^\tau$. Let $(\Lambda_1, \alpha_1, M_1), (\Lambda_2, \alpha_2, M_2)$ be objects in $\mathcal{FK}^0(E^\tau)$ and $\mathcal{FK}^0(E^{r\tau})$ respectively. Then there are functorial isomorphisms:*

$$\mathrm{Hom}(p^*(\Lambda_1, \alpha_1, M_1), (\Lambda_2, \alpha_2, M_2)) \cong \mathrm{Hom}((\Lambda_1, \alpha_1, M_1), p_*(\Lambda_2, \alpha_2, M_2))$$

and

$$\mathrm{Hom}(p_*(\Lambda_2, \alpha_2, M_2), (\Lambda_1, \alpha_1, M_1)) \cong \mathrm{Hom}((\Lambda_2, \alpha_2, M_2), p^*(\Lambda_1, \alpha_1, M_1)).$$

PROOF. Since $(t \circ p_r)^* = p_r^* \circ t^*$ and $(t \circ p_r)_* = t_* \circ p_{r*}$ it is enough to prove the claim in case $p = p_r$ and $p = t$ separately. But in the case $p = t$ the statements are obvious. For the rest of the proof we assume $p = p_r$.

If $\Lambda_1 = p_r(\Lambda_2)$ the first statement is the usual adjointness of p^* and p_* for sheaves of vector spaces. To obtain the second statement in this case, we have to use that p_r is a local homeomorphism.

If $\Lambda_1 \neq p_r(\Lambda_2)$ we have

$$\begin{aligned}
\mathrm{Hom}((\Lambda_1, \alpha_1, M_1), p_{r*}(\Lambda_2, \alpha_2, M_2)) &= \bigoplus_{x \in \Lambda_1 \cap p_r(\Lambda_2)} \mathrm{Hom}((M_1)_x, p_{r*}(M_2)_x) = \\
\bigoplus_{\substack{x \in \Lambda_1 \cap p_r(\Lambda_2) \\ y \in \Lambda_2, p_r(y) = x}} \mathrm{Hom}((M_1)_x, (M_2)_y) &= \bigoplus_{y \in p_r^{-1}(\Lambda_1) \cap \Lambda_1} \mathrm{Hom}((M_1)_{p_r(y)}, (M_2)_y) = \\
\bigoplus_{y \in p_r^{-1}(\Lambda_1) \cap \Lambda_1} \mathrm{Hom}((p_r^* M_1)_y, (M_2)_y) &= \mathrm{Hom}(p_r^*(\Lambda_1, \alpha_1, M_1), (\Lambda_2, \alpha_2, M_2))
\end{aligned}$$

where we used the canonical isomorphisms $(p_{r*}M_2)_x = \bigoplus_{y\in\Lambda_2, p_r(y)=x}(M_2)_y$ and $(p_r^*M_1)_y = (M_1)_{p_r(y)}$.

Suppose $(\Lambda_1', \alpha_1', M_1')$ is another object in $\mathcal{FK}^0(E^\tau)$ with $\Lambda_1' \neq \Lambda_1$ and $\Lambda_1' \neq p_r(\Lambda_2)$. Let a morphism $(\Lambda_1', \alpha_1', M_1') \to (\Lambda_1, \alpha_1, M_1)$ be given and consider the induced diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(p_r^*(\Lambda_1,\alpha_1,M_1),(\Lambda_2,\alpha_2,M_2)) & \xrightarrow[\cong]{} & \mathrm{Hom}((\Lambda_1,\alpha_1,M_1),p_{r*}(\Lambda_2,\alpha_2,M_2)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(p_r^*(\Lambda_1',\alpha_1',M_1'),(\Lambda_2,\alpha_2,M_2)) & \xrightarrow[\cong]{} & \mathrm{Hom}((\Lambda_1',\alpha_1',M_1'),p_{r*}(\Lambda_2,\alpha_2,M_2)).
\end{array}$$

To get functoriality we have to show commutativity of this diagram. This involves the definition of composition in $\mathcal{FK}^0$. Assume the given morphism is represented simply by $f : (M_1')_x \to (M_1)_x$ where $x \in \Lambda_1 \cap \Lambda_1'$. The left vertical arrow sends the morphism $g : (p_r^*M_1)_y = (M_1)_{p_r(y)} \to (M_2)_y$ with $y \in p_r^{-1}(\Lambda_1)\cap\Lambda_2$ to $g\circ p_r^*(f) \in \bigoplus_{z\in\Lambda_2, p_r(z)=x} \mathrm{Hom}((p_r^*M_1')_z, (M_2)_z)$ whose component corresponding to z is $\sum_\phi \exp(2\pi i r\tau A_\phi)P(M_2)\circ g\circ P(p_r^*M_1)\circ(p_r^*f)_z\circ P(p_r^*M_1')$. The sum is performed over equivalence classes of certain maps $\phi : D \to E^{r\tau}$ sending a boundary point to z and $(p_r^*f)_z$ is obtained from f by composition with the identification $(p_r^*M_1')_z = (M_1')_x$.

Our isomorphism sends this to the morphism on the right hand side whose component in

$$\begin{aligned}
\mathrm{Hom}((\Lambda_1',\alpha_1',M_1'),p_r^*(\Lambda_2,\alpha_2,M_2)) &= \\
&= \bigoplus_{x\in p_r(\Lambda_2)\cap\Lambda_1'} \mathrm{Hom}((M_1')_x,(p_{r*}M_2)_x) = \\
&= \bigoplus_{y\in\Lambda_2\cap p_r^{-1}(\Lambda_1')} \mathrm{Hom}((M_1')_{p_r(y)},(M_2)_y)
\end{aligned}$$

corresponding to $z \in p_r^{-1}\Lambda_1'\cap\Lambda_2$ is given by the above sum. This coincides with the sum we would write down for the composition on the right hand side, because p_r defines a bijection between the images of the lifts of the holomorphic maps ϕ to the universal covers $\mathbb{R}^2$ of E^τ and $E^{r\tau}$ respectively. These images are triangles and if we denote their Euclidean area by A_ϕ^τ and $A_\phi^{r\tau}$ we obtain from the definition of p_r the equation $A_\phi^\tau = rA_\phi^{r\tau}$ ant this shows that both sums have the same weights.

To get functoriality in general we have to consider situations involving both cases considered above. In all such situations the commutativity of the corresponding diagram follows immediately from the definition of composition in $\mathcal{FK}^0$. A similar consideration gives the second statement. □

REMARK 3.5. The same statement is true for π_{r*} and π_r^*.

To prepare the proof of Theorem 4.1 let us consider the following Cartesian diagram

$$\begin{array}{ccc}
\tilde{E} & \xrightarrow[\tilde{p}_{r_1}]{} & E^{r_2\tau} \\
{\scriptstyle\tilde{p}_{r_2}}\downarrow & & \downarrow{\scriptstyle p_{r_2}} \\
E^{r_1\tau} & \xrightarrow[p_{r_1}]{} & E^\tau
\end{array}$$

where $\tilde{E} = E^{r\tau} \times \mathbb{Z}/d\mathbb{Z}$ with $d = \gcd(r_1, r_2)$ and $r = \frac{r_1 r_2}{d}$. We denote by $\tilde{p}_{r_i,\nu}$ the restriction of $\tilde{p}_{r_i}$ to $E^{r\tau} \times \{\nu\}$, which is the composition of $p_{\frac{r_{3-i}}{d}} : E^{r\tau} \to E^{r_i\tau}$ with a translation of the form $(x, y) \mapsto (x - n, y)$ on $E^{r_i\tau}$. In our application these translations are determined by corresponding translations on the elliptic curves $E_{r_i\tau}$. Under these assumptions the following lemma holds:

LEMMA 3.6. *There exists a functorial isomorphism*

$$\begin{aligned}\operatorname{Hom}(p_{r_1*}(\Lambda_1, \alpha_1, M_1), p_{r_2*}(\Lambda_2, \alpha_2, M_2)) &\cong \\ &\cong \bigoplus \operatorname{Hom}(\tilde{p}^*_{r_2,\nu}(\Lambda_1, \alpha_1, M_1), \tilde{p}^*_{r_1,\nu}(\Lambda_2, \alpha_2, M_2))\end{aligned}$$

PROOF. If we apply Lemma 3.4 to p_{r_1} we obtain a functorial isomorphism:

$$\begin{aligned}\operatorname{Hom}(p_{r_1*}(\Lambda_1, \alpha_1, M_1), p_{r_2*}(\Lambda_2, \alpha_2, M_2)) &\cong \\ &\cong \operatorname{Hom}((\Lambda_1, \alpha_1, M_1), p^*_{r_1} p_{r_2*}(\Lambda_2, \alpha_2, M_2)).\end{aligned}$$

On the other hand, by applying Lemma 3.4 to the maps $\tilde{p}_{r_{1,\nu}}$ we obtain:

$$\begin{aligned}\operatorname{Hom}(\tilde{p}^*_{r_2,\nu}(\Lambda_1, \alpha_1, M_1), \tilde{p}^*_{r_1,\nu}(\Lambda_2, \alpha_2, M_2)) &\cong \\ \cong \bigoplus_{\nu} \operatorname{Hom}((\Lambda_1, \alpha_1, M_1), \tilde{p}_{r_2,\nu*}\tilde{p}^*_{r_1,\nu}(\Lambda_2, \alpha_2, M_2)) &\cong \\ &\cong \operatorname{Hom}((\Lambda_1, \alpha_1, M_1), \tilde{p}_{r_2*}\tilde{p}^*_{r_1}(\Lambda_2, \alpha_2, M_2)).\end{aligned}$$

To get the claim we need to see that the functors $\tilde{p}_{r_2*} \circ \tilde{p}^*_{r_1}$ and $p^*_{r_1} \circ p_{r_2*}$ are isomorphic, but this follows easily from the definitions because p_{r_1} and $\tilde{p}_{r_1}$ are local homeomorphisms. □

4. The equivalence

THEOREM 4.1. $\Phi_\tau : \mathcal{D}^b(E_\tau) \to \mathcal{FK}^0(E^\tau)$ *is an equivalence of additive categories compatible with the shift functors.*

The rest of this section consists of the proof of this theorem.

Recall (Proposition 2.4) that we replace $\mathcal{D}^b(E_\tau)$ by the equivalent subcategory whose objects are finite direct sums of shifted indecomposable coherent sheaves on E_τ.

To define $\Phi = \Phi_\tau$ on objects it suffices to define the objects $\Phi(A)$ where $A = A[0]$ is an indecomposable coherent sheaf on E_τ.

We repeat here the definition given in [**PZ**]. The strategy is the following. First they define $\Phi_{r\tau}(A)$ for all positive integers r and all vector bundles A being the tensor product of a line bundle with a unipotent bundle on E_τ. In the second step they extend this to all indecomposable vector bundles by using Theorem 2.6. Finally, they write down the definition of $\Phi_\tau(A)$ for torsion sheaves A. These definitions are more or less dictated by the description of mirror symmetry as T-duality [**SYZ**], [**F2**]. The definitions are the following:

If $A = \mathcal{L}(\varphi) \otimes F(V, \exp(N))$ with $\varphi = t^*_{a\tau+b}\varphi_0 \cdot \varphi_0^{n-1}$ and V a finite dimensional complex vector space and $N \in \operatorname{End}(V)$ a cyclic nilpotent endomorphism (we call a nilpotent endomorphism cyclic, if the corresponding $\mathbb{C}[T]$-module structure on V is cyclic, which is equivalent to $\dim\ker N = 1$), we define

$$\Phi(A) = (\Lambda, \alpha, M)$$

where Λ is given by $(a+t,(n-1)a+nt)$, α is the unique real number with $-\frac{1}{2}<\alpha\leq\frac{1}{2}$ and $\exp(i\pi\alpha)=\frac{1+in}{\sqrt{1+n^2}}$ (i.e. $i\pi\alpha$ is a logarithm of the slope of Λ) and $M=\exp(-2\pi ib\mathbf{1}_V+N)$.

As a first and useful observation we note here that

$$t'^*\Phi(A)=\Phi(t^*A)$$

if A is a vector bundle on E_τ as before, $t: E_\tau\to E_\tau$ denotes the translation by $\frac{m}{n}\tau$ (i.e. $t(z)=z+\frac{m}{n}\tau$) with integers m,n and $t': E^\tau\to E^\tau$ is the corresponding translation on the symplectic side, given by $t'(x,y)=(x-\frac{m}{n},y)$.

If $\pi_r: E_{r\tau}\to E_\tau$ is an isogeny and A is a vector bundle on $E_{r\tau}$ of the form considered above and $p_r: E^{r\tau}\to E^\tau$ is the corresponding morphism on the symplectic side, we define

$$\Phi_\tau(\pi_{r*}A):=p_{r*}\Phi_{r\tau}(A).$$

More explicitly, this means for $A=\mathcal{L}(\varphi)\otimes F(V,\exp(N))$ as above $\Phi_\tau(\pi_{r*}A)=(\Lambda,\alpha,M)$ where Λ is given by the line with x-intercept $\frac{ra}{n}$ and slope $\frac{n}{r}$, $-\frac{1}{2}<\alpha\leq\frac{1}{2}$ is the corresponding logarithm and $M=\exp(-2\pi ib\mathbf{1}_V+N)$ because p_r defines an isomorphism between the underlying Lagrangian submanifolds.

Finally, let $A=S(a\tau+b,V,N)$ be a torsion sheaf on E_τ supported at $x=a\tau+b$. Then $\Phi_\tau(A):=(\Lambda,\frac{1}{2},\exp(2\pi ib\mathbf{1}_V+N))$ where Λ is defined by $(-a,t)$.

To get a functor Φ we have to define its effect on morphisms. Using compatibility with finite direct sums and with shifts, this amounts to define

$$\Phi_\tau:\operatorname{Hom}_{\mathcal{D}^b(E_\tau)}(A_1,A_2[n])\to\operatorname{Hom}_{\mathcal{FK}^0(E^\tau)}(\Phi_\tau(A_1),\Phi_\tau(A_2)[n]).$$

Since both sides vanish if $n\notin\{0,1\}$, we are concerned with $n=0$ and $n=1$ only. Using Serre duality (Lemmas 2.7 and 3.1) and $\operatorname{Hom}_{\mathcal{D}^b(E_\tau)}(A_1,A_2[1])=\operatorname{Ext}^1(A_1,A_2)$ we obtain the isomorphism Φ_τ for all n from its definition in case $n=0$.

In the paper [**PZ**] the considerations are restricted to those cases where $\Phi(A_1)$ and $\Phi(A_2)$ have distinct underlying Lagrangian submanifolds. We shall extend the definition to all cases.

Let us repeat the definition of [**PZ**] in the situation $A_i=\mathcal{L}(\varphi_i)\otimes F(V_i,N_i)$ and $\Phi_\tau(A_i)=(\Lambda_i,\alpha_i,M_i)$ with $\Lambda_1\neq\Lambda_2$. As usual, we write $\varphi_i=t^*_{x_i}\varphi_0\cdot\varphi_0^{n_i-1}$ with $x_i=a_i\tau+b_i$. Using the isomorphism $H^0(\mathcal{L}(\varphi))\otimes V\to H^0(\mathcal{L}(\varphi)\otimes F(V,\exp(N)))$ explicitly given in [**PZ**, Proposition 2] and assuming $n_1<n_2$, we obtain an isomorphism $\operatorname{Hom}(A_1,A_2)\cong H^0(\mathcal{L}(\varphi_2\varphi_1^{-1}))\otimes\operatorname{Hom}(V_1,V_2)$. Because Φ becomes complex linear it suffices to define the images of elements of the form $\theta\otimes f$ with $f\in\operatorname{Hom}(V_1,V_2)$ and θ a basis element of $H^0(\mathcal{L}(\varphi_2\varphi_1^{-1}))$. The basis of this vector space we are using is formed by the theta functions $t^*_{x_{12}}\theta\big[\frac{k}{n_2-n_1},0\big]((n_2-n_1)\tau,(n_2-n_1)z)$. Here we use the abbreviation $x_{12}=\frac{x_2-x_1}{n_2-n_1}$ and $0\leq k<n_2-n_1$. Their number (n_2-n_1) is equal to the number of intersection points of Λ_1 and Λ_2. The definition of the isomorphism

$$\Phi_\tau:\operatorname{Hom}_{\mathcal{D}^b(E_\tau)}(A_1,A_2)\to\operatorname{Hom}_{\mathcal{FK}^0(E^\tau)}(\Phi_\tau(A_1),\Phi_\tau(A_2))$$

is now obtained by specifying a bijection between these theta functions and the intersection points. The choice of [**PZ**] is to send $t^*_{x_{12}}\theta\big[\frac{k}{n_2-n_1},0\big]((n_2-n_1)\tau,(n_2-n_1)z)$ to the point $e_k=(\frac{a_2-a_1+k}{n_2-n_1},\frac{n_1a_2-n_2a_1+n_1k}{n_2-n_1})$.

If $n_1\geq n_2$ but $\mathcal{L}(\varphi_1)\not\cong\mathcal{L}(\varphi_2)$ both spaces of homomorphisms are zero.

Let us extend the definition to the case $\Lambda_1 = \Lambda_2$. Under this assumption we have $a_1 = a_2$ and $\deg \mathcal{L}(\varphi_1) = \deg \mathcal{L}(\varphi_2)$. In particular, $\mathcal{L}(\varphi_2\varphi_1^{-1})$ is of degree zero. Because the structure sheaf is the only line bundle of degree zero with nontrivial sections, we have $H^0(\mathcal{L}(\varphi_2\varphi_1^{-1})) = 0$ if $\varphi_1 \neq \varphi_2$. But $\varphi_1 = \varphi_2$ is equivalent to $b_1 = b_2$ which in turn is equivalent to the existence of a common eigenvalue of M_1 and M_2. This is equivalent to the non-vanishing of $\mathrm{Hom}_{\mathcal{F}^0}((\Lambda_1,\alpha_1,M_1),(\Lambda_1,\alpha_1,M_2))$. So the spaces of morphisms are zero on both sides, if $\varphi_1 \neq \varphi_2$.

If $\varphi_1 = \varphi_2$ the operators M_i have the same eigenvalue $\exp(-2\pi i b_1)$. This implies $H^0(\Lambda_1, \mathcal{H}om(M_1, M_2)) = \{f : V_1 \to V_2 \mid M_1 \circ f = f \circ M_2\} = \{f : V_1 \to V_2 \mid \exp(N_1) \circ f = f \circ \exp(N_2)\}$. The composition of all these identifications with Lemma 2.5 gives the definition of Φ.

The next step is the extension of the definition of Φ to homomorphisms between arbitrary vector bundles on E_τ. Let r_1, r_2 be two positive integers (possibly equal to one) and $\mathcal{E}_i = \mathcal{L}(\varphi_i) \otimes F(V_i, \exp(N_i))$ vector bundles on $E_{r_i\tau}$. Define $E := E_{r_1\tau} \times_{E_\tau} E_{r_2\tau}$ and denote the projections by $\tilde{\pi}_{r_i} : E \to E_{r_i\tau}$. The curve E is a disjoint union of $d = \gcd(r_1, r_2)$ elliptic curves and the restrictions $\tilde{\pi}_{r_i,\nu}$ of the projections $\tilde{\pi}_{r_i}$ onto the components are isogenies composed with translations. More precisely, $E = E_{r\tau} \times \mathbb{Z}/d\mathbb{Z}$ with $r := \frac{r_1 r_2}{d}$ and $\tilde{\pi}_{r_i,\nu} : E_{r\tau} \times \{\nu\} \to E_{r_i\nu}$ is the composition of the isogeny $E_{r\tau} \to E_{r_i\tau}$ defined by $\Gamma_{r_i\tau} \subset \Gamma_{r\tau}$ with the translation by $\nu\tau$ on $E_{r_1\tau}$ and with the identity on $E_{r_2\tau}$. We could use the translation by $s_i\tau$ on $E_{r_i\tau}$ for any pair of integers (s_1, s_2) with $s_1 - s_2 \equiv \nu \mod d$. Two such choices differ by a translation on $E_{r\tau}$ by an integer multiple of τ.

Using the flat base change theorem and the adjointness properties of π_* and π^*, we obtain a canonical isomorphism

$$\mathrm{Hom}(\pi_{r_1*}\mathcal{E}_1, \pi_{r_2*}\mathcal{E}_2) \cong \mathrm{Hom}(\tilde{\pi}_{r_1}^*\mathcal{E}_1, \tilde{\pi}_{r_2}^*\mathcal{E}_2) \cong \bigoplus_{\nu=1}^{d} \mathrm{Hom}(\tilde{\pi}_{r_1,\nu}^*\mathcal{E}_1, \tilde{\pi}_{r_2,\nu}^*\mathcal{E}_2).$$

Because the situation on the symplectic side is similar (Lemma 3.6), we can define Φ_τ by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\pi_{r_1*}\mathcal{E}_1, \pi_{r_2*}\mathcal{E}_2) \arrow[r, "\sim"] \arrow[d, "\Phi_\tau"] & \bigoplus \mathrm{Hom}(\tilde{\pi}_{r_1,\nu}^*\mathcal{E}_1, \tilde{\pi}_{r_2,\nu}^*\mathcal{E}_2) \arrow[d, "\oplus\Phi"] \\
\mathrm{Hom}(\Phi(\pi_{r_1*}\mathcal{E}_1), \Phi(\pi_{r_2*}\mathcal{E}_2)) \arrow[d, "\cong"] & \bigoplus \mathrm{Hom}(\Phi(\tilde{\pi}_{r_1,\nu}^*\mathcal{E}_1), \Phi(\tilde{\pi}_{r_2,\nu}^*\mathcal{E}_2)) \arrow[d, "\cong"] \\
\mathrm{Hom}(p_{r_1*}\Phi(\mathcal{E}_1), p_{r_2*}\Phi(\mathcal{E}_2)) \arrow[r, "\sim"] & \bigoplus \mathrm{Hom}(\tilde{p}_{r_1,\nu}^*\Phi(\mathcal{E}_1), \tilde{p}_{r_2,\nu}^*\Phi(\mathcal{E}_2))
\end{tikzcd}$$

In addition to the definition of $\Phi(\mathcal{E}_i)$ we use $\Phi_{r\tau}(\pi^*(\mathcal{E})) \cong p^*(\Phi_\tau(\mathcal{E}))$ if $\mathcal{E} \cong \mathcal{L}(\varphi) \otimes F(V, \exp(N))$ (see [**PZ**, Prop. 4]) for isogenies π and the compatibility of translations with Φ observed earlier.

We still have to deal with the cases where A_1 or A_2 is a torsion sheaf. The easy observation is that $\mathrm{Hom}(A_1, A_2) = \mathrm{Ext}^1(A_2, A_1) = 0$ if A_1 is a torsion sheaf and A_2 is locally free. This corresponds nicely to our definition on the symplectic side where $\mathrm{Hom}((\Lambda_1,\alpha_1,M_1),(\Lambda_2,\alpha_2,M_2)) = 0$ if $\alpha_1 > \alpha_2$. So we have to investigate just the cases where $A_2 = S(a_2\tau + b_2, V_2, N_2)$ is a torsion sheaf. If $A_1 = \mathcal{L}(\varphi_1) \otimes F(V_1, N_1)$ the definition of [**PZ**] is the following:

There are canonical isomorphisms $\mathrm{Hom}(A_1, A_2) = \mathrm{Hom}(V_1, V_2) = V_1^* \otimes V_2$ and $\mathrm{Hom}(\Phi(A_1), \Phi(A_2)) = \mathrm{Hom}(V_1, V_2) = V_1^* \otimes V_2$ because $\Lambda_1 \cap \Lambda_2$ is one point. The isomorphism $\Phi_\tau : V_1^* \otimes V_2 \to V_1^* \otimes V_2$ is by definition

$$\exp(-\pi i\tau(na_2^2 + 2a_1a_2) - 2\pi i(a_2b_1 + a_1b_2 + na_2b_2)) \cdot \\ \cdot \exp(-(a_1 + na_2)\mathbf{1}_{V_1^*} \otimes N_2 + a_2{}^tN_1 \otimes \mathbf{1}_{V_2}).$$

To extend the definition to arbitrary indecomposable locally free sheaves A_1, assume $A_1 = \pi_{r*}\mathcal{E}_1$ with $\mathcal{E}_1 = \mathcal{L}(\varphi_1) \otimes F(V_1, \exp(N_1))$ and A_2 a torsion sheaf as above. We define Φ_τ by the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{E}_1, \pi_r^* A_2) \arrow[r, "\sim"] \arrow[d, "\cong"', "\Phi_{r\tau}"] & \mathrm{Hom}(\pi_{r*}\mathcal{E}_1, A_2) \arrow[d, "\Phi_\tau"] \\
\mathrm{Hom}(\Phi(\mathcal{E}_1), \Phi(\pi_r^* A_2)) \arrow[r, "\sim"] \arrow[d, "\cong"'] & \mathrm{Hom}(\Phi(\pi_{r*}\mathcal{E}_1), \Phi(A_2)) \arrow[d, "\cong"'] \\
\mathrm{Hom}(\Phi(\mathcal{E}_1), p_r^*\Phi(A_2)) \arrow[r, "\sim"] & \mathrm{Hom}(p_{r*}\Phi(\mathcal{E}_1), \Phi(A_2)).
\end{tikzcd}$$

Here we used the definition of Φ on objects, Lemma 3.4 and $\Phi(\pi_r^* A_2) = p_r^*\Phi(A_2)$ which follows easily from the definitions.

Note that we need here the existence of direct sums in the category $\mathcal{FK}^0(E^{r\tau})$ because $\pi_r^* A_2$ is not indecomposable, it is the direct sum of skyscraper sheaves supported at all preimages of the support on A_2.

Finally, we have to deal with two indecomposable torsion sheaves A_ν. If they have distinct support, then $\mathrm{Hom}(A_1, A_2) = \mathrm{Ext}^1(A_1, A_2) = 0$. With $A_\nu = S(a_\nu\tau + b_\nu, V_\nu, N_\nu)$ we have $\Phi(A_\nu) = (\Lambda_\nu, \frac{1}{2}, M_\nu)$ where Λ_ν is given by $(-a_\nu, t)$ and $M_\nu = \exp(2\pi i b_\nu \mathbf{1}_{V_\nu} + N_\nu)$. If $a_1 \neq a_2$ we get $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\mathrm{Hom}(\Phi(A_1), \Phi(A_2)[n]) = 0$. If $a_1 = a_2$ but $b_1 \neq b_2$ then $\Lambda_1 = \Lambda_2$ but M_1 and M_2 have no common eigenvalues, hence $H^n(\Lambda_1, \mathcal{H}\mathrm{om}(M_1, M_2)) = 0$ for $n = 0, 1$. If finally A_1 and A_2 have the same support, we obtain $\mathrm{Hom}(A_1, A_2) = \mathrm{Hom}_{\mathcal{O}_{E_\tau, x_1}}((V_1, N_1), (V_2, N_2)) = \{f \in \mathrm{Hom}(V_1, V_2) \mid f \circ N_1 = N_2 \circ F\} = \{f \in \mathrm{Hom}(V_1, V_2) \mid f \circ M_1 = M_2 \circ f\} = H^0(\Lambda_1, \mathcal{H}\mathrm{om}(M_1, M_2))$.

This completes the definition of the functor Φ. The most important part of the proof is to show that Φ is indeed a functor. This amounts to showing compatibility of Φ with compositions.

We have to show the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}(A_1, A_2[k]) \otimes \mathrm{Hom}(A_2[k], A_3[l]) \arrow[r] \arrow[d, "\Phi_\tau \otimes \Phi_\tau"] & \mathrm{Hom}(A_1, A_3[l]) \arrow[d, "\Phi_\tau"] \\
\mathrm{Hom}(\Phi(A_1), \Phi(A_2)[k]) \otimes \mathrm{Hom}(\Phi(A_2)[k], \Phi(A_3)[l]) \arrow[r] & \mathrm{Hom}(\Phi(A_1), \Phi(A_3)[l])
\end{tikzcd}$$

where the horizontal arrows are the compositions in the respective categories and $0 \leq k \leq l \leq 1$. We write $\Phi(A_\nu) = (\Lambda_\nu, \alpha_\nu, M_\nu)$. If $l = 1$ we use functoriality of Serre duality in both categories and the definition of Φ_τ to reduce it to the case $k = l = 0$.

Assume $k = l = 0$. We first deal with the case where we have $A_\nu = \mathcal{L}(\varphi_\nu) \otimes F(V_\nu, \exp(N_\nu))$ for all ν.

If $\Lambda_1 \neq \Lambda_2 \neq \Lambda_3 \neq \Lambda_1$, the commutativity of the diagram was derived in [**PZ**] from an addition formula for theta functions.

If $\Lambda_1 = \Lambda_2 \neq \Lambda_3$ we can assume $\varphi_1 = \varphi_2$ since otherwise $\mathrm{Hom}(A_1, A_2) = 0$. This implies, using Lemma 2.5:

$$\begin{aligned}\mathrm{Hom}(A_1, A_2) &= \mathrm{Hom}(F(V_1, \exp(N_1)), F(V_2, \exp(N_2))) = \\ &= \{f \in \mathrm{Hom}(V_1, V_2) \mid f \circ N_1 = N_2 \circ f\},\end{aligned}$$

which is identified by Φ_τ with

$$\begin{aligned}\mathrm{Hom}(\Phi(A_1), \Phi(A_2)) &= H^0(\Lambda_1, \mathcal{H}\mathrm{om}(M_1, M_2)) = \\ &= \{f \in \mathrm{Hom}(V_1, V_2) \mid f \circ M_1 = M_2 \circ f\}.\end{aligned}$$

On the other hand, if we use the explicit description of the isomorphism

$$\mathrm{Hom}(A_2, A_3) \cong H^0(\mathcal{L}(\varphi_2^{-1}\varphi_3)) \otimes \mathrm{Hom}(V_2, V_3)$$

given in [**PZ**, Prop. 2], we see immediately that the composition translates to the map

$$\begin{gathered}\{f \in \mathrm{Hom}(V_1, V_2) \mid f \circ N_1 = N_2 \circ f\} \otimes H^0(\mathcal{L}(\varphi_2^{-1}\varphi_3)) \otimes \mathrm{Hom}(V_2, V_3) \\ \downarrow \\ H^0(\mathcal{AL}(\varphi_1^{-1}\varphi_3)) \otimes \mathrm{Hom}(V_1, V_3)\end{gathered}$$

sending $f \otimes \theta \otimes g$ to $\theta \otimes (g \circ f)$. Now, the commutativity of the above diagram is clear from the definition of Φ_τ. If $\Lambda_1 \neq \Lambda_2 = \Lambda_3$ the proof is the same, so we skip it.

If $\Lambda_1 = \Lambda_3$ we have seen above that we can have a nonzero composition only if $\Lambda_1 = \Lambda_2 = \Lambda_3$ or we can reduce the claim to the previously discussed situation, using Serre duality. If $\Lambda_1 = \Lambda_2 = \Lambda_3$ we can assume $\varphi_1 = \varphi_2\varphi_3$ and again from Lemma 2.5 and the definition of Φ_τ the commutativity is obtained.

The case where $A_\nu = \pi_{r_\nu *}(\mathcal{L}_{r_\nu \tau}(\varphi_\nu) \otimes F_{r_\nu \tau}(V_\nu, \exp(N_\nu)))$ are arbitrary indecomposable vector bundles on E_τ was studied in [**PZ**]. Finally, we have to show commutativity of the diagram above in case at least one of the sheaves A_ν is an irreducible torsion sheaf. But again by putting together the descriptions and definitions of the spaces involved the result follows immediately.

We have established that

$$\Phi_\tau : \mathcal{D}^b(E_\tau) \to \mathcal{FK}^0(E^\tau)$$

is a functor. Just by definition this functor is additive, fully faithful and compatible with shifts.

To finish the proof of Theorem 4.1 it is sufficient to show that any indecomposable object in $\mathcal{FK}^0(E^\tau)$ is isomorphic to an object of the form $\Phi_\tau(A)$. Let (Λ, α, M) be an indecomposable object in $\mathcal{FK}^0(E^\tau)$. Indecomposability means that the local system M is indecomposable. Since we assume that the eigenvalues of the monodromy are of modulus one, we can describe the local system up to isomorphism by

$$M = \exp(-2\pi i b + N) \in \mathrm{GL}(V)$$

with a (cyclic) nilpotent endomorphism N and a real number b. Because Φ_τ is compatible with shifts, we may assume $-\frac{1}{2} < \alpha \leq \frac{1}{2}$. If $\alpha = \frac{1}{2}$ we obtain $(\Lambda, \alpha, M) = \Phi_\tau(S(-a\tau - b, V, N))$, where $-1 < a \leq 0$ is the x-intercept of a line in $\mathbb{R}^2$ representing Λ. If $\alpha < \frac{1}{2}$ we let (r, n) be the pair of relatively prime nonnegative integers such that $r + in$ is a real multiple of $\exp(i\pi\alpha)$. If we denote by $0 \leq \frac{ra}{n} < \frac{1}{n}$

the smallest possible nonnegative x-intercept of a line representing Λ, we see easily $(\Lambda, \alpha, M) = \Phi_\tau(\pi_{r*}(\mathcal{L}_{r\tau}(\varphi) \otimes F_{r\tau}(V, \exp(N))))$ with $\varphi = t^*_{ar\tau+b}\varphi_0 \cdot \varphi_0^{n-1}$. Observe that replacing a by $a + \frac{u}{r}$ yields the same direct image vector bundle on E_τ, if $0 \leq u < r$ is an integer. This completes the proof of Theorem 4.1.

References

[At] M.F. Atiyah: *Vector bundles over an elliptic curve*, Proc. London Math. Soc. **7** (1957) 414–452

[En] C. van Enckevort: *Mirror Symmetry and T-duality*, Thesis, Utrecht, January 2000

[F1] K. Fukaya: *Morse Homotopy, A^∞-category and Floer homologies*, Proc. of the 1993 GARC Workshop on Geometry and Topology (H.J. Kim ed.), Lecture Notes series, vol. 18, Seoul Nat. Univ., Seoul 1993, 1-102

[F2] K. Fukaya: *Mirror Symmetry of Abelian Varieties and Multi-theta Functions*, Preprint 2000

[GM] S.I. Gelfand, Yu.I. Manin: *Methods of Homological Algebra*, Springer 1996

[H1] R. Hartshorne: *Residues and Duality*, LNM 20, Springer 1966

[H2] R. Hartshorne: *Algebraic Geometry*, GTM 52, Springer 1977

[HL] R. Harvey, H.B. Lawson: *Calibrated geometries*, Acta Math. **148**, 47–157 (1981)

[KS] M. Kashiwara, P. Shapira: *Sheaves on Manifolds*, Grundlehren der mathematischen Wissenschaften 292, Springer 1990

[Ko] M. Kontsevich: *Homological Algebra of Mirror Symmetry*, Proceedings of ICM 1994, Zürich, 120–139; alg-geom/9411018

[ML] S. Mac Lane: *Categories for the Working Mathematician*, GTM 5, Second edition, Springer 1998

[Me] S. Merkulov: *Strong homotopy algebras of a Kähler manifold*, Internat. Math. Res. Notices **3** (1999) 153–164; math.AG/9809172

[Mu] D. Mumford: *Tata Lectures on Theta I*, Progress in Mathematics 28, Birkhäuser 1983

[P1] A. Polishchuk: *Homological mirror symmetry with higher products*, math.AG/9901025

[P2] A. Polishchuk: *A_∞-structures on an elliptic curve*, math.AG/0001048

[PZ] A. Polishchuk, E. Zaslow: *Categorical Mirror Symmetry: The Elliptic Curve*, Adv. Theor. Math. Phys. **2** (1998) 443-470; math.AG/9801119

[Se] P. Seidel: *Graded Lagrangian submanifolds*, math.SG/9903049

[SYZ] A. Strominger, S.-T. Yau, E. Zaslow: *Mirror Symmetry is T-duality*, Nucl. Phys. B**479** (1996) 243–259; hep-th/9606040

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