Wavelets Generated by Layer Potentials

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Abstract

By means of the limit and jump relations of classical potential theory the framework of a wavelet approach on a regular surface is established. The properties of a multiresolution analysis are verified, and a tree algorithm for fast computation is developed based on numerical integration. As applications of the wavelet approach some numerical examples are presented, including the zoom–in property as well as the detection of high frequency perturbations. At the end we discuss a fast multiscale representation of the solution of (exterior) Dirichlet's or Neumann's boundary–value problem corresponding to regular surfaces.

Key words. regular surface, potential operators, limit and jump relations, wavelets, multiscale analysis, pyramid scheme, (exterior) boundary–value problems of potential theory

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1 Introduction

Wavelets are known as mathematical means for breaking up a complicated function (signal) into many simple pieces at different scales and positions. Thus wavelets have become a powerful and flexible tool for scientific computation and data handling. Basically, wavelet analysis is done by convolving the function under consideration against 'dilated' and 'shifted' versions of one fixed function, viz. the 'mother wavelet'. Traditionally, applications of wavelets have been signal analysis, image processing, noise cancellation, etc, but there is also a growing interest in the numerical treatment of partial differential equations. However, wavelet methods are known for unfolding their computational economy and efficiency when applied to problems on Euclidian space, the sphere or the torus. Strategies of extending the applicability of wavelet techniques to boundary–value problems corresponding to (general) regular surfaces have only rarely been attempted. The usual procedure is to transform the partial differential equation problem into an integral equation over a parametrizable boundary surface such that standard techniques of the (iterated one–dimensional) wavelet concept for solving this integral equation become applicable (see e.g. [1]). Another approach is motivated by the Runge–Walsh theorem of constructive approximation. The wavelet concept (see [8]) established in this way naturally arises as a result of scale discretization of wavelets on e.g. a sphere in connection with approximate integration techniques relating an integrand over the sphere to the boundary values on the regular surface.

In this paper we follow the standard procedure in potential theory by transforming a boundary–value problem corresponding to a general (regular) surface (like sphere, ellipsoid, spheroid, Earth's surface, etc) into a Fredholm integral equation of the second kind. More explicitly, we choose the double layer potential for the Dirichlet problem and the single layer potential for the Neumann problem. However, instead of applying conventional wavelet constructions oriented on Euclidian theory for discretizing the integral equations in accordance with a collocational, Galerkin or least squares procedure we use the kernels of the layer potentials themselves to establish a new class of wavelets on (general) regular surfaces. In other words, a new wavelet theory will be developed on (general) regular surfaces that arises naturally as a result of scale discretization of the limit and jump relations of potential theory.

The outline of this paper is as follows: First we introduce the notations and preliminaries that are needed for our wavelet approach. We define regular surfaces on which our whole theory is established. Moreover, we recapitulate the Fourier theory of square–integrable scalar fields in terms of spherical (outer) harmonics on the sphere as well as on regular surfaces. Then we introduce potential operators which are the main ingredients of this work. We develop the limit and jump relations of the potential operators formulated in the framework of the Hilbert space of square–integrable functions. The setup of a multiresolution analysis (i.e. scaling functions, scale spaces, wavelets, detail spaces) is defined by interpreting the kernel functions of the limit and jump integral operators as scaling functions on regular surfaces. The distance to the parallel surfaces of the regular surface under consideration thereby represents the scale level in the scaling function. After scale discretizing the continuous theory we show that our wavelet setup fulfills the properties of a multiresolution analysis. As a significant aspect of scientific computing we present a pyramid scheme (tree algorithm) providing fast wavelet transform (FWT). Furthermore, we dicuss some numerical examples. In particular we are concerned with the zoom–in property, the detection of a high frequency perturbation and the technique of data compression which are typical applications within a wavelet framework. At the end we deal with the already mentioned discretization of Fredholm integral equations in order to give a multiscale representation of the solution of boundary–value problems in three dimensions corresponding to regular surfaces.

2 Preliminaries

At first we want to introduce some basic notations which we need for our wavelet approach. As usual, $\mathbb{R}^3$ denotes three–dimensional Euclidian space. For $x, y \in \mathbb{R}^3, x = (x_1, x_2, x_3)^T, y = (y_1, y_2, y_3)^T$ the inner product is defined by

$$x \cdot y = x^T y = \sum_{i=1}^{3} x_i y_i. \tag{1}$$

For elements $x \in \mathbb{R}^3, x = (x_1, x_2, x_3)^T$, different from the origin, we have

$$x = r\xi, \quad r = |x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + x_3^2}, \tag{2}$$

where $\xi = (\xi_1, \xi_2, \xi_3)^T$ is the uniquely determined directional unit vector of x. The unit sphere in $\mathbb{R}^3$ is denoted by Ω. If the vectors $\epsilon^1, \epsilon^2, \epsilon^3$ form the canonical orthonormal basis in $\mathbb{R}^3$, the points $\xi \in \Omega$ may be represented in polar coordinates by

$$\begin{array}{l} \xi = t\epsilon^3 + \sqrt{1-t^2}\left(\cos\varphi\epsilon^1 + \sin\varphi\epsilon^2\right) \\ t = \cos\vartheta, \vartheta \in [0,\pi], \varphi \in [0, 2\pi) \quad . \end{array} \tag{3}$$

The *spherical harmonics* Y_n of degree n are defined as the everywhere on Ω infinitely differentiable eigenfunctions of the Beltrami operator Δ^* corresponding to the eigenvalues $(\Delta^*)^\wedge(n) = -n(n+1), n = 0, 1, \ldots$, where the Beltrami operator is the angular part of the Laplace operator Δ in $\mathbb{R}^3$. As it is well–known, the functions $H_n : \mathbb{R}^3 \to \mathbb{R}$ defined by $H_n(x) = r^n Y_n(\xi)$, $x = r\xi$, $\xi \in \Omega$, are homogeneous polynomials in rectangular coordinates which satisfy the Laplace equation $\Delta_x H_n(x) = 0$, $x \in \mathbb{R}^3$. Conversely, every homogeneous harmonic polynomial of degree n when restricted to Ω is a spherical harmonic of degree n. The *Legendre polynomials* $P_n : [-1, +1] \to [-1, +1]$ are the only everywhere in $[-1, +1]$ infinitely differentiable eigenfunctions of the Legendre operator $L_t =$

$(d/dt)(1-t^2)(d/dt)$, which satisfy $P_n(1)=1$. Apart from a multiplicative constant, the 'ϵ^3-Legendre function' $P_n(\epsilon^3 \cdot) : \Omega \to [-1,+1]$, $\xi \mapsto P_n(\epsilon^3 \cdot \xi)$, $\xi \in \Omega$, is the only spherical harmonic of degree n which is invariant under orthogonal transformations leaving ϵ^3 fixed. The linear space $\mathrm{Harm}_n(\Omega)$ of all spherical harmonics of degree n is of dimension $\dim(\mathrm{Harm}_n(\Omega)) = 2n+1$. Thus, there exist $2n+1$ linearly independent spherical harmonics $Y_{n,1}, \dots, Y_{n,2n+1}$ in $\mathrm{Harm}_n(\Omega)$. Throughout the remainder of this paper we assume this system to be orthonormal in the sense of the $\mathcal{L}^2(\Omega)$–inner product

$$(Y_{n,j}, Y_{n,k})_{\mathcal{L}^2(\Omega)} = \int_\Omega Y_{n,j}(\eta) Y_{n,k}(\eta) d\omega(\eta) = \delta_{n,m}\delta_{j,k} \tag{4}$$

($d\omega$ denotes the surface element). An outstanding result of the theory of spherical harmonics is the *addition theorem*

$$\sum_{k=1}^{2n+1} Y_{n,k}(\xi) Y_{n,k}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta), \quad (\xi, \eta) \in \Omega \times \Omega\,. \tag{5}$$

The connection between the orthogonal invariance and the addition theorem is established by the *Funk–Hecke formula*

$$\int_\Omega H(\xi \cdot \eta) P_n(\zeta \cdot \eta)\, d\omega(\eta) = (H(\xi\cdot), P_n(\zeta\cdot)_{\mathcal{L}^2(\Omega)} = H^\wedge(n) P_n(\xi \cdot \zeta), \tag{6}$$

$H \in \mathrm{L}^1[-1,+1]$, $\xi, \zeta \in \Omega$, where the *Legendre transform* $LT : H \to (LT)(H)$, $H \in \mathrm{L}^1[-1,1]$, is given by

$$(LT)(H)(n) = H^\wedge(n) = 2\pi \int_{-1}^{+1} H(t) P_n(t)\; dt, \quad n = 0, 1, \dots\ .$$

The sequence $\{H^\wedge(n)\}_{n \in \mathbb{N}_0}$ is called the *symbol* of H. For more details about the theory of spherical harmonics the reader is referred, for example, to [7], [20]. In accordance with the notation used in [7] we let

$$Harm_{0,\dots,m} = \underset{\substack{n=0,\dots,m \\ k=1,\dots,2n+1}}{span} (Y_{n,k})\,.$$

Moreover,

$$Harm_{0,\dots,m} = \bigoplus_{n=0}^{m} Harm_n$$

so that

$$dim(Harm_{0,\dots,m}) = \sum_{n=0}^{m} (2n+1) = (m+1)^2.$$

As it is well known, $K_{Harm_{0,\dots,m}} : \Omega \times \Omega \to \mathbb{R}$ defined by

$$K_{Harm_{0,\dots,m}(\Omega)}(\xi, \eta) = \sum_{n=0}^{m} \sum_{k=1}^{2n+1} Y_{n,k}(\xi) Y_{n,k}(\eta) = \sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta) \tag{7}$$

is the only reproducing kernel of the space $Harm_{0,\ldots,m}$ with respect to $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$. For later use it is worth mentioning that

$$\begin{aligned} Y(\xi) &= \int_\Omega K_{Harm_{0,\ldots,m+1}(\Omega)}(\xi,\eta)Y(\eta)\ d\omega(\eta) \\ &= \int_\Omega K_{Harm_{0,\ldots,m}(\Omega)}(\xi,\eta)Y(\eta)\ d\omega(\eta) \end{aligned}$$

for all $\xi \in \Omega$ and all $Y \in Harm_{0,\ldots,m}$.
By $\mathrm{L}^2(\Omega)$ we denote the Hilbert space of all square–integrable functions on the unit sphere Ω equipped with the inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$. $\mathrm{L}^2(\Omega)$ is the completion of $\mathrm{C}^{(0)}(\Omega)$ with respect to the norm $\|\cdot\|_{\mathrm{L}^2(\Omega)}$. Any function of class $\mathrm{L}^2(\Omega)$ of the form $H_\xi : \Omega \to \mathbb{R}$, $\eta \mapsto H_\xi(\eta) = H(\xi\cdot\eta)$, $\eta \in \Omega$, is called a *ξ–zonal function* on Ω. Zonal functions are constant on the sets of all $\eta \in \Omega$, with $\xi\cdot\eta = h$, $h \in [-1,+1]$. The set of all ξ–zonal functions is isomorphic to the set of functions $H : [-1,+1] \to \mathbb{R}$. This gives rise to interpret the space $\mathrm{L}^2[-1,+1]$ with norm defined correspondingly by

$$\|H\|_{\mathrm{L}^2[-1,+1]} = \left(2\pi \int_{-1}^{+1} |H(t)|^2\ dt\right)^{1/2} = \|H(\varepsilon^3\cdot)\|_{\mathrm{L}^2(\Omega)}, \quad H \in \mathrm{L}^2[-1,+1]\ .$$

as subspace of $\mathrm{L}^2(\Omega)$.
The *spherical Fourier transform* $H \mapsto (FT)(H)$, $H \in \mathrm{L}^2(\Omega)$, is given by

$$\left((FT)(H)\right)(n,k) = H^\wedge(n,k) = (H, Y_{n,k})_{\mathrm{L}^2(\Omega)},\ \ n = 0,1,\ldots; k = 1,\ldots,2n+1.$$

FT forms a mapping from $\mathrm{L}^2(\Omega)$ onto the space $l^2(\mathcal{N})$ of all sequences $\{W_{n,k}\}_{(n,k)\in\mathcal{N}}$ satisfying

$$\sum_{(n,k)\in\mathcal{N}} W_{n,k}^2 = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} W_{n,k}^2 < \infty,$$

where we have used the abbreviation

$$\mathcal{N} = \{(n,k) | n = 0,1,\ldots; k = 1,\ldots,2n+1\}\ .$$

The series

$$\sum_{(n,k)\in\mathcal{N}} F^\wedge(n,k)Y_{n,k} = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} F^\wedge(n,k)Y_{n,k}$$

is called the *spherical Fourier expansion* of F (with Fourier coefficients $F^\wedge(n,k)$, $(n,k) \in \mathcal{N}$). For all $F \in \mathrm{L}^2(\Omega)$ we have

$$\lim_{N\to\infty} \left\| F - \sum_{n=0}^{N}\sum_{k=1}^{2n+1} F^\wedge(n,k)Y_{n,k} \right\|_{\mathrm{L}^2(\Omega)} = 0.$$

The system $\left\{ H^{\alpha}_{-n-1,k}(\cdot) \right\}$, $(n,k) \in \mathcal{N}$, of *outer harmonics* is defined by

$$H^{\alpha}_{-n-1,k}(x) = \frac{1}{\alpha} \left(\frac{\alpha}{|x|} \right)^{n+1} Y_{n,k} \left(\frac{x}{|x|} \right), \; x \in \mathbb{R}^3 \backslash \{0\}, \;\; \alpha > 0 \tag{8}$$

The system defined above satisfies the following properties:

- $H^{\alpha}_{-n-1,k}$ is of class $\mathcal{C}^{(\infty)}(\mathbb{R}^3 \backslash \{0\})$,
- $H^{\alpha}_{-n-1,k}$ satisfies the Laplace equation $\Delta_x H^{\alpha}_{-n-1,k}(x) = 0$ for $x \in \mathbb{R}^3 \backslash \{0\}$,
- $H^{\alpha}_{-n-1,k}|_{\Omega_\alpha} = \frac{1}{\alpha} Y_{n,k}$,
- $\int_{|x|=\alpha} H^{\alpha}_{-n-1,k}(x) H^{\alpha}_{-m-1,l}(x) \; d\omega(x) = \delta_{n,m} \delta_{k,l}$.

Clearly the *addition theorem for outer harmonics* reads as follows:

$$\sum_{k=1}^{2n+1} H^{\alpha}_{-n-1,k}(x) H^{\alpha}_{-n-1,k}(y) = \frac{2n+1}{4\pi\alpha^2} \left(\frac{\alpha^2}{|x|\,|y|} \right)^{n+1} P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \tag{9}$$

$(x,y) \in \left(\mathbb{R}^3 \backslash \{0\} \right) \times \left(\mathbb{R}^3 \backslash \{0\} \right)$.

3 Basic Concepts

Next we introduce some settings which are standard in potential theory (see, for example, [13], [17], [22], [23]).

3.1 Regular Surfaces

We begin our considerations by introducing the notation of a regular surface:

Definition 3.1. A surface $\Sigma \subset \mathbb{R}^3$ is called *regular*, if it satisfies the following properties:

(i) Σ divides the three-dimensional Euclidean space $\mathbb{R}^3$ into the bounded region Σ_{int} (*inner space*) and the unbounded region Σ_{ext} (*outer space*) defined by $\Sigma_{\text{ext}} = \mathbb{R}^3 \backslash \overline{\Sigma_{\text{int}}}$, $\overline{\Sigma_{\text{int}}} = \Sigma_{\text{int}} \cup \Sigma$,

(ii) Σ_{int} contains the origin,

(iii) Σ is a closed and compact surface free of double points,

(iv) Σ has a continuously differentiable unit normal field ν pointing into the outer space Σ_{ext}.

Geoscientifically regular surfaces Σ are, for example, sphere, ellipsoid, spheroid, geoid, (regular) Earth's surface.
Given a regular surface, then there exist positive constants α, β such that

$$\alpha < \sigma^{\inf} = \inf_{x \in \Sigma} |x| \leq \sup_{x \in \Sigma} |x| = \sigma^{\sup} < \beta. \tag{10}$$

As usual, $A_{\rm int}$, $B_{\rm int}$ (resp. $A_{\rm ext}$, $B_{\rm ext}$) denote the inner (resp. outer) space of the sphere A (resp. B) around the origin with radius α (resp. β). $\Sigma^{\inf}_{\rm int}$, $\Sigma^{\sup}_{\rm int}$ (resp. $\Sigma^{\inf}_{\rm ext}$, $\Sigma^{\sup}_{\rm ext}$) denote the inner (resp. outer) space of the sphere $\Sigma^{\inf}$ (resp. $\Sigma^{\sup}$) around the origin with radius $\sigma^{\inf}$ (resp. $\sigma^{\sup}$).
The set

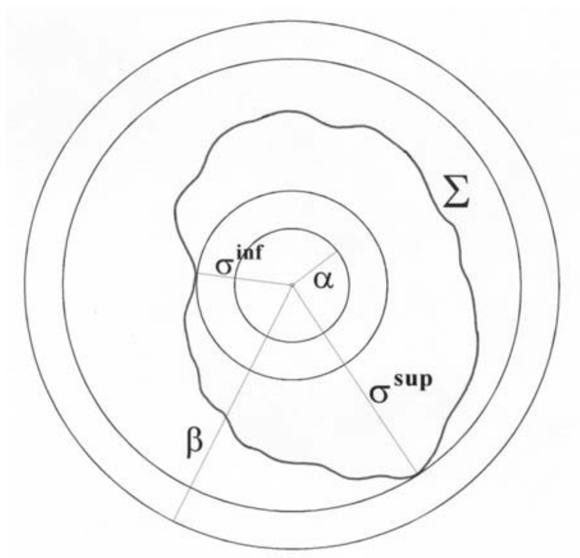


Figure 1: Regular surface (geometrical illustration)

$$\Sigma(\tau) = \{x \in \mathbb{R}^3 | x = y + \tau\nu(y), y \in \Sigma\} \tag{11}$$

generates a *parallel surface* which is exterior to Σ for $\tau > 0$ and interior for $\tau < 0$. It is well known from differential geometry (see e.g. [21]) that if $|\tau|$ is sufficiently small, then the surface $\Sigma(\tau)$ is regular, and the normal to one parallel surface is a normal to the other. According to our regularity assumptions imposed on Σ the functions

$$\begin{array}{rcl} (x,y) & \mapsto & \frac{|\nu(x)-\nu(y)|}{|x-y|}, \ (x,y) \in \Sigma \times \Sigma, \ x \neq y \\ (x,y) & \mapsto & \frac{|\nu(x)\cdot(x-y)|}{|x-y|^2}, \ (x,y) \in \Sigma \times \Sigma, \ x \neq y \end{array} \tag{12}$$

are bounded. Hence, there exists a constant $M > 0$ such that

$$\begin{array}{rcl} |\nu(x) - \nu(y)| & \leq & M|x-y|, \\ |\nu(x)\cdot(x-y)| & \leq & M|x-y|^2, \end{array} \tag{13}$$

for all $(x,y) \in \Sigma \times \Sigma$. Moreover, it is easy to see that

$$\inf_{x,y\in\Sigma} |x + \tau\nu(x) - (y + \sigma\nu(y))| = |\tau - \sigma|$$

provided that $|\tau|$, $|\sigma|$ are sufficiently small.

3.2 Function Spaces

In what follows we discuss function spaces that are of particular significance in our approach.
Let Σ be a regular surface. $\mathrm{Pot}(\Sigma_{\mathrm{int}})$ denotes the space of all functions $U \in \mathrm{C}^{(2)}(\Sigma_{\mathrm{int}})$ satisfying Laplace's equation in Σ_{int}, while $\mathrm{Pot}(\Sigma_{\mathrm{ext}})$ denotes the space of all functions $U \in \mathrm{C}^{(2)}(\Sigma_{\mathrm{ext}})$ satisfying Laplace's equation in Σ_{ext} and being regular at infinity (that is, $|U(x)| = O(|x|^{-1})$, $|(\nabla U)(x)| = O(|x|^{-2})$ for $|x| \to \infty$ uniformly with respect to all directions).

For $k = 0, 1, \ldots$ we denote by $\mathrm{Pot}^{(k)}(\overline{\Sigma_{\mathrm{int}}})$ the space of all $U \in \mathrm{C}^{(k)}(\overline{\Sigma_{\mathrm{int}}})$ such that $U|\Sigma_{\mathrm{int}}$ is of class $\mathrm{Pot}(\Sigma_{\mathrm{int}})$. Analogously, $\mathrm{Pot}^{(k)}(\overline{\Sigma_{\mathrm{ext}}})$ is the space of all $U \in \mathrm{C}^{(k)}(\overline{\Sigma_{\mathrm{ext}}})$ such that $U|\Sigma_{\mathrm{ext}}$ is of class $\mathrm{Pot}(\Sigma_{\mathrm{ext}})$.
In shorthand notation,

$$\mathrm{Pot}^{(k)}(\overline{\Sigma_{\mathrm{int}}}) = \mathrm{Pot}(\Sigma_{\mathrm{int}}) \cap \mathrm{C}^{(k)}(\overline{\Sigma_{\mathrm{int}}}), \tag{14}$$

$$\mathrm{Pot}^{(k)}(\overline{\Sigma_{\mathrm{ext}}}) = \mathrm{Pot}(\Sigma_{\mathrm{ext}}) \cap \mathrm{C}^{(k)}(\overline{\Sigma_{\mathrm{ext}}}). \tag{15}$$

Let U be of class $\mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{int}}})$. Then the maximum/minimum principle of potential theory states

$$\sup_{x\in\overline{\Sigma_{\mathrm{int}}}} |U(x)| \le \sup_{x\in\Sigma} |U(x)| \ . \tag{16}$$

Let U be of class $\mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$. Then the maximum/minimum principle gives

$$\sup_{x\in\overline{\Sigma_{\mathrm{ext}}}} |U(x)| \le \sup_{x\in\Sigma} |U(x)|. \tag{17}$$

In $\mathrm{C}^{(0)}(\Sigma)$ we have the inner product

$$(F,H)_{L^2(\Sigma)} = \int_\Sigma F(x)H(x)\ d\omega(x), \tag{18}$$

where $d\omega$ denotes the surface element. The inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Sigma)}$ implies the norm

$$\|F\|_{\mathrm{L}^2(\Sigma)} = \left((F,F)_{\mathrm{L}^2(\Sigma)}\right)^{1/2}. \tag{19}$$

The space $(\mathrm{C}^{(0)}(\Sigma), (\cdot,\cdot)_{\mathrm{L}^2(\Sigma)})$ is a pre-Hilbert space. For every $F \in \mathrm{C}^{(0)}(\Sigma)$ we have the norm-estimate

$$\|F\|_{\mathrm{L}^2(\Sigma)} \le \sqrt{\|\Sigma\|}\ \|F\|_{\mathrm{C}^{(0)}(\Sigma)}, \tag{20}$$

where

$$\|\Sigma\| = \int_{\Sigma} d\omega(x) \ . \tag{21}$$

By $L^2(\Sigma)$ we denote the *space of (Lebesgue) square-integrable functions* on the regular surface Σ. $L^2(\Sigma)$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{L^2(\Sigma)}$ and a Banach space with respect to the norm $\|\cdot\|_{L^2(\Sigma)}$. $L^2(\Sigma)$ is the completion of $C^{(0)}(\Sigma)$ with respect to the norm $\|\cdot\|_{L^2(\Sigma)}$:

$$\overline{C^{(0)}(\Sigma)}^{\|\cdot\|_{L^2(\Sigma)}} = L^2(\Sigma) \ . \tag{22}$$

4 Limit Formulae and Jump Relations

Let F be a continuous function on a regular surface Σ. Then the functions $U_n : \mathbb{R}^3 \backslash \Sigma \to \mathbb{R}$, $n = 1, 2, \ldots$, defined by

$$U_n(x) = \int_{\Sigma} F(y) (\frac{\partial}{\partial \nu(y)})^{n-1} \frac{1}{\mid x - y \mid} \ d\omega(y) \tag{23}$$

are infinitely often differentiable and satisfy the Laplace equation in Σ_{int} and Σ_{ext}. In addition, the functions U_n are regular at infinity.
The function U_1 given by

$$U_1(x) = \int_{\Sigma} F(y) \frac{1}{|x-y|} \ d\omega(y) \tag{24}$$

is called the *potential of the single layer* on Σ, while U_2 given by

$$U_2(x) = \int_{\Sigma} F(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \ d\omega(y) \tag{25}$$

is called the *potential of the double layer* on Σ.

4.1 Formulation in $(C(\Sigma), \|\cdot\|_{C(\Sigma)})$

For $F \in C^{(0)}(\Sigma)$, the functions U_n can be continued continuously to the surface Σ, but the limits depend from which parallel surface (inner or outer) the points x tend to Σ. On the other hand, the functions $U_n, n = 1, 2$, also are defined on the surface Σ, i.e., the integrals (24), (25) exist and are continuous for $x \in \Sigma$. Furthermore, the integral

$$U_1'(x) = \int_{\Sigma} F(y) \frac{\partial}{\partial \nu(x)} \frac{1}{|x-y|} \ d\omega(y) \tag{26}$$

exists for all $x \in \Sigma$ and can be continued continuously to Σ. However, the integrals do not coincide, in general, with the inner or outer limits of the potentials (cf. [19]).

From classical potential theory (see, for example, [13]) it is known that for all $x \in \Sigma$ and $F \in \mathrm{C}^{(0)}(\Sigma)$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} U_1(x \pm \tau\nu(x)) = U_1(x), \tag{27}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \frac{\partial U_1}{\partial \nu}(x \pm \tau\nu(x)) = \mp 2\pi F(x) + U_1'(x), \tag{28}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} U_2(x \pm \tau\nu(x)) = \pm 2\pi F(x) + U_2(x), \tag{29}$$

('limit relations')

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(U_1(x + \tau\nu(x)) - U_1(x - \tau\nu(x))\right) = 0, \tag{30}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\frac{\partial U_1}{\partial \nu}(x + \tau\nu(x)) - \frac{\partial U_1}{\partial \nu}(x - \tau\nu(x))\right) = -4\pi F(x), \tag{31}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(U_2(x + \tau\nu(x)) - U_2(x - \tau\nu(x))\right) = 4\pi F(x), \tag{32}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\frac{\partial U_2}{\partial \nu}(x + \tau\nu(x)) - \frac{\partial U_2}{\partial \nu}(x - \tau\nu(x))\right) = 0 \tag{33}$$

('jump relations').

In addition, O.D. Kellogg (1929), J. Schauder (1931) proved that the above relations hold uniformly with respect to all $x \in \Sigma$. This means that

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Sigma} \left|U_1(x \pm \tau\nu(x)) - U_1(x)\right| = 0, \tag{34}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Sigma} \left|\frac{\partial U_1}{\partial \nu}(x \pm \tau\nu(x)) \pm 2\pi F(x) - U_1'(x)\right| = 0, \tag{35}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Sigma} \left|U_2(x \pm \tau\nu(x)) \mp 2\pi F(x) - U_2(x)\right| = 0 \tag{36}$$

and

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Sigma} \left|U_1(x + \tau\nu(x) - U_1(x - \tau\nu(x))\right| = 0, \tag{37}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Sigma} \left|\frac{\partial U_1}{\partial \nu}(x + \tau\nu(x)) - \frac{\partial U_1}{\partial \nu}(x - \tau\nu(x)) + 4\pi F(x)\right| = 0, \tag{38}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Sigma} \left|U_2(x + \tau\nu(x)) - U_2(x - \tau\nu(x)) - 4\pi F(x)\right| = 0, \tag{39}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Sigma} \left|\frac{\partial U_2}{\partial \nu}(x + \tau\nu(x)) - \frac{\partial U_2}{\partial \nu}(x - \tau\nu(x))\right| = 0\ . \tag{40}$$

Here we have written, as usual,

$$\frac{\partial U}{\partial \nu}(x \pm \tau\nu(x)) = \nu(x) \cdot (\nabla U)(x \pm \tau\nu(x)) \ . \tag{41}$$

For $\tau \neq \sigma$ with $|\tau|, |\sigma|$ sufficiently small, the functions

$$(x, y) \mapsto \frac{1}{|x + \tau\nu(x) - (y + \sigma\nu(y))|}, \ (x, y) \in \Sigma \times \Sigma, \tag{42}$$

are continuous. Thus the *potential operators* $P(\tau, \sigma)$ defined by

$$P(\tau, \sigma)F(x) = \int_\Sigma F(y) \frac{1}{|x + \tau\nu(x) - (y + \sigma\nu(y))|} \ d\omega(y) \tag{43}$$

form mappings from $\mathrm{L}^2(\Sigma)$ into $\mathrm{C}^{(0)}(\Sigma)$ and are continuous with respect to $\| \cdot \|_{\mathrm{C}^{(0)}(\Sigma)}$. For all $\tau \neq \sigma$ the restrictions of $P(\tau, \sigma)$ on $\mathrm{C}^{(0)}(\Sigma)$ are bounded with respect to $\| \cdot \|_{\mathrm{L}^2(\Sigma)}$.
By formal operations we obtain for $F \in \mathrm{C}^{(0)}(\Sigma)$

$$P(\tau, 0)F(x) = \int_\Sigma F(y) \frac{1}{|x + \tau\nu(x) - y|} \ d\omega(y) \tag{44}$$

($P(\tau, 0)$*: operator of the single–layer potential on* Σ *for values on* $\Sigma(\tau)$),

$$\begin{aligned} P_{|2}(\tau, 0)F(x) &= \frac{\partial}{\partial \sigma} P(\tau, \sigma)F(x)|_{\sigma=0} & (45) \\ &= \int_\Sigma F(y) \left(\frac{\partial}{\partial \nu(y)} \frac{1}{|x + \tau\nu(x) - (y + \sigma\nu(y))|} \right)_{\sigma=0} d\omega(y) \\ &= \int_\Sigma F(y) \frac{\nu(y) \cdot (x + \tau\nu(x) - y)}{|x + \tau\nu(x) - y|^3} \ d\omega(y) & (46) \end{aligned}$$

($P_{|2}(\tau, 0)$*: operator of the double–layer potential on* Σ *for values on* $\Sigma(\tau)$).
The notation $P_{|i}$ indicates differentiation with respect to the i-th variable. Analogously we get

$$\begin{aligned} P_{|1}(\tau, 0)F(x) &= \frac{\partial}{\partial \tau} P(\tau, \sigma)F(x)\big|_{\sigma=0}, & (47) \\ &= -\int_\Sigma F(y) \frac{\nu(x) \cdot (x + \tau\nu(x) - y)}{|x + \tau\nu(x) - y|^3} \ d\omega(y) & (48) \end{aligned}$$

and

$$P_{|2|1}(\tau, 0)F(x) = \frac{\partial^2}{\partial \tau \partial \sigma} P(\tau, \sigma)F(x)\big|_{\sigma=0} \tag{49}$$

for the *operators of the normal derivatives.*
If $\tau = \sigma = 0$, the kernels of the potentials have weak singularities. The integrals *formally* defined by

$$P(0, 0)F(x) = \int_\Sigma F(y) \frac{1}{|x - y|} \ d\omega(y), \tag{50}$$

$$P_{|2}(0,0)F(x) = \int_\Sigma F(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \; d\omega(y), \tag{51}$$

$$P_{|1}(0,0)F(x) = \frac{\partial}{\partial \nu(x)} \int_\Sigma F(y) \frac{1}{|x-y|} \; d\omega(y), \tag{52}$$

however, exist and define linear bounded operators in $\mathrm{L}^2(\Sigma)$. $P(0,0)$, $P_{|1}(0,0)$ and $P_{|2}(0,0)$ map $\mathrm{C}^{(0)}(\Sigma)$ into itself (see [19]). Furthermore, the operators are continuous (even compact) with respect to $\| \cdot \|_{\mathrm{C}^{(0)}(\Sigma)}$.
The operator $P(\tau,\sigma)^*$ satisfying

$$(F, P(\tau,\sigma)G)_{\mathrm{L}^2(\Sigma)} = (P(\tau,\sigma)^* F, G)_{\mathrm{L}^2(\Sigma)} \tag{53}$$

for all $F, G \in \mathrm{L}^2(\Sigma)$ is called the *adjoint operator* of $P(\tau,\sigma)$ with respect to $(\cdot,\cdot)_{\mathrm{L}^2(\Sigma)}$. According to Fubini's theorem it follows that

$$\begin{aligned} &(F, P(\tau,\sigma)G)_{\mathrm{L}^2(\Sigma)} \\ &= \int_\Sigma F(x) \Big(\int_\Sigma \frac{G(y)}{|x+\tau\nu(x)-(y+\sigma\nu(y))|} \; d\omega(y) \Big) \; d\omega(x) \\ &= \int_\Sigma G(y) \Big(\int_\Sigma \frac{F(x)}{|x+\tau\nu(x)-(y+\sigma\nu(y))|} \; d\omega(x) \Big) \; d\omega(y) \\ &= (P(\sigma,\tau)^* F, G)_{\mathrm{L}^2(\Sigma)} \; . \end{aligned} \tag{54}$$

By comparison we thus have

$$\begin{aligned} P(\tau,0)^* F(x) &= P(\tau,\sigma)^* F(x)|_{\sigma=0} \\ &= \int_\Sigma F(y) \frac{1}{|y+\tau\nu(y)-x|} \; d\omega(y). \end{aligned} \tag{55}$$

Analogously we obtain expressions of $P_{|1}(\tau,0)^*$ and $P_{|2}(\tau,0)^*$:

$$P_{|1}(\tau,0)^* F(x) = -\int_\Sigma F(y) \frac{\nu(y)\cdot(y+\tau\nu(y)-x)}{|y+\tau\nu(y)-x|^3} \; d\omega(y), \tag{56}$$

$$P_{|2}(\tau,0)^* F(x) = \int_\Sigma F(y) \frac{\nu(x)\cdot(y+\tau\nu(y)-x)}{|y+\tau\nu(y)-x|^3} \; d\omega(y) \; . \tag{57}$$

Elementary calculations show that

$$\begin{aligned} &P_{|1}(0,0)^* F(x) \\ &= -\int_\Sigma F(y) \frac{\nu(y)\cdot(y-x)}{|y-x|^3} \; d\omega(y) = P_{|2}(0,0)F(x) \end{aligned} \tag{58}$$

and

$$\begin{aligned} &P_{|2}(0,0)^* F(x) \\ &= \int_\Sigma F(y) \frac{\nu(x)\cdot(y-x)}{|y-x|^3} \; d\omega(y) = P_{|1}(0,0)F(x) \; . \end{aligned} \tag{59}$$

The potential operators now enable us to give concise formulations of the classical *limit formulae* and *jump relations* in potential theory. Let I be the identity operator in $\mathrm{L}^2(\Sigma)$. Suppose that, for all sufficiently small values $\tau > 0, L_i^{\pm}(\tau), i = 1,2,3$, and $J_i(\tau), i = 1,...,6$, respectively, define the following operators:

$$\begin{aligned} \mathrm{L}_1^{\pm}(\tau) &= P(\pm\tau,0) - P(0,0), && (60)\\ \mathrm{L}_2^{\pm}(\tau) &= P_{|1}(\pm\tau,0) - P_{|1}(0,0) \pm 2\pi I, && (61)\\ \mathrm{L}_3^{\pm}(\tau) &= P_{|2}(\pm\tau,0) - P_{|2}(0,0) \mp 2\pi I, && (62)\end{aligned}$$

$$\begin{aligned} J_1(\tau) &= P(\tau,0) - P(-\tau,0), && (63)\\ J_2(\tau) &= P_{|1}(\tau,0) - P_{|1}(-\tau,0) + 4\pi I, && (64)\\ J_3(\tau) &= P_{|2}(\tau,0) - P_{|2}(-\tau,0) - 4\pi I, && (65)\\ J_4(\tau) &= P_{|2|1}(\tau,0) - P_{|2|1}(-\tau,0), && (66)\\ J_5(\tau) &= P_{|1}(\tau,0) + P_{|1}(-\tau,0) - 2P_{|1}(0,0), && (67)\\ J_6(\tau) &= P_{|2}(\tau,0) + P_{|2}(-\tau,0) - 2P_{|2}(0,0). && (68)\end{aligned}$$

Then, for $F \in \mathrm{C}^{(0)}(\Sigma)$, the main results of classical potential theory may be formulated by

$$\begin{aligned} &\lim_{\substack{\tau\to 0\\ \tau>0}} \| L_i^{\pm}(\tau)F \|_{\mathrm{C}^{(0)}(\Sigma)} = 0, \qquad \lim_{\substack{\tau\to 0\\ \tau>0}} \| J_i(\tau)F \|_{\mathrm{C}^{(0)}(\Sigma)} = 0,\\ &\lim_{\substack{\tau\to 0\\ \tau>0}} \| L_i^{\pm}(\tau)^*F \|_{\mathrm{C}^{(0)}(\Sigma)} = 0, \qquad \lim_{\substack{\tau\to 0\\ \tau>0}} \| J_i(\tau)^*F \|_{\mathrm{C}^{(0)}(\Sigma)} = 0. \end{aligned} \tag{69}$$

4.2 Formulation in $(\mathrm{L}^2(\Sigma), \| \cdot \|_{\mathrm{L}^2(\Sigma)})$

The relations (69) can be generalized to the Hilbert space $\mathrm{L}^2(\Sigma)$ (see [3], [14]):

THEOREM 4.1. *For all* $F \in \mathrm{L}^2(\Sigma)$

$$\begin{aligned} &\lim_{\substack{\tau\to 0\\ \tau>0}} \| \mathrm{L}_i^{\pm}(\tau)F \|_{\mathrm{L}^2(\Sigma)} = 0, \qquad \lim_{\substack{\tau\to 0\\ \tau>0}} \| J_i(\tau)F \|_{\mathrm{L}^2(\Sigma)} = 0,\\ &\lim_{\substack{\tau\to 0\\ \tau>0}} \| \mathrm{L}_i^{\pm}(\tau)^*F \|_{\mathrm{L}^2(\Sigma)} = 0, \qquad \lim_{\substack{\tau\to 0\\ \tau>0}} \| J_i(\tau)^*F \|_{\mathrm{L}^2(\Sigma)} = 0. \end{aligned} \tag{70}$$

Proof. Denote by $T(\tau)$ one of the operators $\mathrm{L}_i^{\pm}(\tau), i = 1,2,3,\ J_i(\tau), i = 1,\ldots,6$. Then, by virtue of the norm estimate,

$$\| F \|_{\mathrm{L}^2(\Sigma)} \le \sqrt{\| \Sigma \|}\, \| F \|_{\mathrm{C}^{(0)}(\Sigma)},\ \| \Sigma \| = \int_\Sigma d\omega,\ F \in \mathrm{C}^{(0)}(\Sigma), \tag{71}$$

we obtain

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \| T(\tau)F \|_{\mathrm{L}^2(\Sigma)} = 0,\ \lim_{\substack{\tau\to 0\\ \tau>0}} \| T(\tau)^*F \|_{\mathrm{L}^2(\Sigma)} = 0 \tag{72}$$

for all $F \in \mathrm{C}(\Sigma)$. Therefore there exists a constant $\mathrm{C}(F) > 0$ such that

$$\| T(\tau)F \|_{\mathrm{L}^2(\Sigma)} \le \mathrm{C}(F), \quad \| T(\tau)^*F \|_{\mathrm{L}^2(\Sigma)} \le \mathrm{C}(F) \tag{73}$$

for all $\tau \le \tau_0$ (τ_0 sufficiently small). The uniform boundedness principle of functional analysis (see e.g. [12]) then shows us that there exists a constant $M > 0$ such that

$$\| T(\tau)|\mathrm{C}^{(0)}(\Sigma) \|_{\mathrm{L}^2(\Sigma)} \le M, \quad \| T(\tau)^*|\mathrm{C}^{(0)}(\Sigma) \|_{\mathrm{L}^2(\Sigma)} \le M \tag{74}$$

for all $\tau \le \tau_0$. The operators $(T(\tau)^* T(\tau))$ are self-adjoint, and their restrictions to the Banach space $\mathrm{C}^{(0)}(\Sigma)$ are continuous. We now modify a technique due to [15]. According to the Cauchy-Schwarz inequality we get for $F \in \mathrm{C}^{(0)}(\Sigma)$

$$\begin{aligned}(\| T(\tau)F \|_{\mathrm{L}^2(\Sigma)})^2 &= (T(\tau)F, T(\tau)F)_{\mathrm{L}^2(\Sigma)} \\ &= (F, (T(\tau)^* T(\tau))F)_{\mathrm{L}^2(\Sigma)} \\ &\le \| F \|_{\mathrm{L}^2(\Sigma)} \| (T(\tau)^* T(\tau))F \|_{\mathrm{L}^2(\Sigma)} .\end{aligned} \tag{75}$$

Consequently it follows that

$$\begin{aligned}(\| T(\tau)F \|_{\mathrm{L}^2(\Sigma)})^{2^2} &\le (\| F \|_{\mathrm{L}^2(\Sigma)})^2 (\| T(\tau)^* T(\tau)F \|_{\mathrm{L}^2(\Sigma)})^2 \\ &\le (\| F \|_{\mathrm{L}^2(\Sigma)})^2 \| F \|_{\mathrm{L}^2(\Sigma)} \| (T(\tau)^* T(\tau))^2 F \|_{\mathrm{L}^2(\Sigma)} .\end{aligned} \tag{76}$$

Induction yields

$$(\| T(\tau)F \|_{\mathrm{L}^2(\Sigma)})^{2^n} \le (\| F \|_{\mathrm{L}^2(\Sigma)})^{2^n - 1} \| (T(\tau)^* T(\tau))^{2^{n-1}} F \|_{\mathrm{L}^2(\Sigma)} \tag{77}$$

for all positive integers n. According to the norm estimate (71) and the boundedness of the operators $T(\tau), T(\tau)^*$ for all $\tau \le \tau_0$ there exists a positive constant K such that

$$(\| T(\tau)F \|_{\mathrm{L}^2(\Sigma)})^{2^n} \le \sqrt{\| \Sigma \|}\, K^{2^n} (\| F \|_{\mathrm{L}^2(\Sigma)})^{2^n - 1} \| F \|_{\mathrm{C}^{(0)}(\Sigma)} \quad . \tag{78}$$

Therefore, for positive integers n and all $F \in \mathrm{C}^{(0)}(\Sigma)$ with $F \neq 0$, we find

$$\frac{\| T(\tau)F \|_{\mathrm{L}^2(\Sigma)}}{\| F \|_{\mathrm{L}^2(\Sigma)}} \le K \left(\frac{\sqrt{\| \Sigma \|}\ \| F \|_{\mathrm{C}^{(0)}(\Sigma)}}{\| F \|_{\mathrm{L}^2(\Sigma)}} \right)^{2^{-n}} . \tag{79}$$

Letting n tend to infinity we obtain for all $F \neq 0$

$$\lim_{n\to\infty} \Big(\frac{\sqrt{\| \Sigma \|}\ \| F \|_{\mathrm{C}^{(0)}(\Sigma)}}{\| F \|_{\mathrm{L}^2(\Sigma)}} \Big)^{2^{-n}} = 1. \tag{80}$$

This shows us that the norm $\| T(\tau) \|_{\mathrm{L}^2(\Sigma)}$ of the operator $T(\tau), \tau \le \tau_0$, can be estimated by K, i.e.

$$\| T(\tau)F \|_{\mathrm{L}^2(\Sigma)} \le K \| F \|_{\mathrm{L}^2(\Sigma)} \tag{81}$$

for all $F \in \mathrm{C}^{(0)}(\Sigma)$ and all $\tau \le \tau_0$. The same argument holds true for the adjoint operators, i.e.

$$\| T(\tau)^* F \|_{\mathrm{L}^2(\Sigma)} \le K \| F \|_{\mathrm{L}^2(\Sigma)} \tag{82}$$

for all $F \in \mathrm{C}^{(0)}(\Sigma)$ and all $\tau \leq \tau_0$. The space $\mathrm{C}^{(0)}(\Sigma)$ is as a linear subspace dense in $\mathrm{L}^2(\Sigma)$. Thus, by the Hahn-Banach theorem (cf. [12]), we can extend the operators $T(\tau)$ and $T(\tau)^*$ from $\mathrm{C}^{(0)}(\Sigma)$ to $\mathrm{L}^2(\Sigma)$ without enlarging their norms. Therefore, $T(\tau)$ and $T(\tau)^*, \tau \leq \tau_0$, are bounded with respect to $\mathrm{L}^2(\Sigma)$. To be more specific,

$$\| T(\tau) \|_{\mathrm{L}^2(\Sigma)} \leq (\| T(\tau) \|_{\mathrm{C}^{(0)}(\Sigma)} \| T(\tau)^* \|_{\mathrm{C}^{(0)}(\Sigma)})^{\frac{1}{2}}, \tag{83}$$

$$\| T(\tau)^* \|_{\mathrm{L}^2(\Sigma)} \leq (\| T(\tau) \|_{\mathrm{C}^{(0)}(\Sigma)} \| T(\tau)^* \|_{\mathrm{C}^{(0)}(\Sigma)})^{\frac{1}{2}}. \tag{84}$$

But this immediately leads to Theorem 4.1 . □

5 $L^2(\Sigma)$–Closure of Outer Harmonics

We begin our consideration with the following lemma concerning outer harmonics.

LEMMA 5.1. *Let Σ be a regular surface such that (10) holds true. Then the following statements are valid:*

(i) $\left(H^\alpha_{-n-1,k}|\Sigma\right)_{\substack{n=0,1,\ldots\\ k=1,\ldots,2n+1}}$ *is linearly independent,*

(ii) $\left(\frac{\partial H^\alpha_{-n-1,k}}{\partial \nu_\Sigma}\right)_{\substack{n=0,1,\ldots\\ k=1,\ldots,2n+1}}$ *is linearly independent.*

Proof. In order to verify the statement (i) we have to derive that for any linear combination H of the form

$$H = \sum_{n=0}^{m} \sum_{k=1}^{2n+1} a_{n,k} H^\alpha_{-n-1,k}, \tag{85}$$

the condition $H|\Sigma = 0$ implies $a_{0,1} = ... = a_{m,1} = ... = a_{m,2m+1} = 0$. From the uniqueness theorem of the exterior Dirichlet problem we know that $H|\Sigma = 0$ yields $H|\overline{\Sigma_{\mathrm{ext}}} = 0$. Therefore, for every sphere Γ around the origin with radius $\gamma > \sigma^{\mathrm{sup}} = \sup_{x\in\Sigma} |x|$, it follows that

$$\int_\Gamma H^\alpha_{-n-1,k}(x) H(x) \; d\omega(x) = 0 \tag{86}$$

for $n = 0, \ldots, m; j = 1, \ldots, 2n+1$. Inserting (85) into (86) gives us in connection with the completeness property of the spherical harmonics $a_{n,k} = 0$ for $n = 0, ..., m; j = 1, ..., 2n + 1$, as required for statement (i).

For the proof of statement (ii) we start from the homogeneous boundary condition

$$\frac{\partial H}{\partial \nu_\Sigma} = \sum_{n=0}^{m} \sum_{k=1}^{2n+1} a_{n,k} \frac{\partial H^\alpha_{-n-1,k}}{\partial \nu_\Sigma} = 0 \tag{87}$$

on Σ. The uniqueness theorem of the exterior Neumann problem then yields $H|\overline{\Sigma_{\text{ext}}} = 0$. This gives us $a_{n,k} = 0$ for $n = 0, \ldots, m$; $j = 1, \ldots, 2n+1$, as required for statement (ii). $\square$

Next our purpose is to prove completeness and closure theorems in $\mathrm{L}^2(\Sigma)$.

THEOREM 5.2. *Let Σ be a regular surface such that (10) is satisfied. Then the following statements are valid:*

(i) $\left(H^{\alpha}_{-n-1,k}|\Sigma\right)_{\substack{n=0,1,\ldots\\ k=1,\ldots,2n+1}}$ *is complete in* $L^2(\Sigma)$,

(ii) $\left(\frac{\partial H^{\alpha}_{-n-1,k}}{\partial \nu_\Sigma}\right)_{\substack{n=0,1,\ldots\\ k=1,\ldots,2n+1}}$ *is complete in* $L^2(\Sigma)$.

Proof. We restrict our attention to statement (i). Suppose that $F \in \mathrm{L}^2(\Sigma)$ satisfies

$$(F, H^{\alpha}_{-n-1,k}|\Sigma)_{\mathrm{L}^2(\Sigma)} = \int_\Sigma F(y) H^{\alpha}_{-n-1,k}(y)\, d\omega(y) = 0 \tag{88}$$

for all $n = 0, 1, \ldots; j = 1, \ldots, 2n+1$. We have to show that $F = 0$ on $L^2(\Sigma)$.

We remember that the series expansion

$$\frac{1}{|x-y|} = 4\pi \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{|x|^n}{|y|^{n+1}} \sum_{k=1}^{2n+1} Y_{n,k}(\xi) Y_{n,k}(\eta), \tag{89}$$

$x = |x|\xi$, $y = |y|\eta$, is analytic in the domain A_{int} with $\alpha < \sigma^{\text{inf}}$ (cf. 10). For all $x \in A_{\text{int}}$ we thus find by virtue of (88)

$$\begin{aligned} U_1(x) &= \int_\Sigma F(y) \frac{1}{|x-y|}\, d\omega(y) \\ &= \sum_{n=0}^{\infty} \frac{4\pi\alpha}{2n+1} \sum_{k=1}^{2n+1} H^{\alpha}_{n,k}(x) \int_\Sigma F(y) H^{\alpha}_{-n-1,k}(y)\, d\omega(y) \\ &= 0. \end{aligned} \tag{90}$$

Analytic continuation shows that the single-layer potential U_1 vanishes at each point $x \in \Sigma_{\text{int}}$. In other words, the equations

$$U_1(x - \tau\nu(x)) = 0, \tag{91}$$

$$\frac{\partial U_1}{\partial \nu}(x - \tau\nu(x)) = 0 \tag{92}$$

hold for all $x \in \Sigma$ and all sufficiently small $\tau > 0$. This yields using the relations of Theorem 4.1

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_\Sigma \left| U_1\left(x + \tau\nu(x)\right)\right|^2 d\omega(x) = 0, \tag{93}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_\Sigma \left| \frac{\partial U_1}{\partial \nu(x)}(x + \tau\nu(x)) + 4\pi F(x)\right|^2 d\omega(x) = 0, \tag{94}$$

and

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_\Sigma \left| U_1'(x) + 2\pi F(x) \right|^2 \, d\omega(x) = 0. \tag{95}$$

The last equation can be rewritten in the explicit form

$$-\frac{1}{2\pi} \int_\Sigma F(y) \frac{\partial}{\partial\nu(x)} \frac{1}{|x-y|} \, d\omega(y) = F(x) \tag{96}$$

in the sense of $\mathrm{L}^2(\Sigma)$. However, the left hand side of (96) is a continuous function of the variable x (see e.g. [13], [19]). Thus, the function F can be replaced by a function $\tilde{F} \in \mathrm{C}^{(0)}(\Sigma)$ satisfying $F = \tilde{F}$ in the sense of $\mathrm{L}^2(\Sigma)$. For the continuous function $\tilde{F}$, however, the classical limit relations and jump formulae are valid:

$$\lim_{\substack{\tau\to 0\\ \tau>0}} U_1(x+\tau\nu(x)) = 0, \quad x \in \Sigma, \tag{97}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \frac{\partial U_1}{\partial\nu}(x+\tau\nu(x)) = -4\pi\tilde{F}(x), \quad x \in \Sigma. \tag{98}$$

The uniqueness theorem of the exterior Dirichlet problem then shows us that $U_1(x) = 0$ for all $x \in \Sigma_{\mathrm{ext}}$. But this means that $\tilde{F} = 0$ on the surface Σ, as required.
The remaining statement (ii) follows by analogous arguments. □

From functional analysis (see e.g. [2]) we know that the properties of completeness and closure are equivalent in a Hilbert space such as $\mathrm{L}^2(\Sigma)$. This leads us to the following corollary.

COROLLARY 5.3. *Under the assumptions of Theorem 5.2 the following statements are valid:*

(i) $\left(H^\alpha_{-n-1,k}|\Sigma\right)_{\substack{n=0,1,\ldots,\\ k=1,\ldots,2n+1}}$ *is closed in* $\mathrm{L}^2(\Sigma)$, *i.e.: for given* $F \in \mathrm{L}^2(\Sigma)$ *and arbitrary* $\varepsilon > 0$ *there exist a linear combination*

$$H_m = \sum_{n=0}^{m} \sum_{k=1}^{2n+1} a_{n,k} H^\alpha_{-n-1,k}|\Sigma \tag{99}$$

such that

$$\|F - H_m\|_{\mathrm{L}^2(\Sigma)} \le \varepsilon \ . \tag{100}$$

(ii) $\left(\frac{\partial H^\alpha_{-n-1,k}}{\partial\nu_\Sigma}\right)_{\substack{n=0,1,\ldots,\\ k=1,\ldots,2n+1}}$ *is closed in* $\mathrm{L}^2(\Sigma)$, *i.e.: for given* $F \in \mathrm{L}^2(\Sigma)$ *and arbitrary* $\varepsilon > 0$ *there exist a linear combination*

$$S_m = \sum_{n=0}^{m} \sum_{k=1}^{2n+1} a_{n,k} \frac{\partial H^\alpha_{-n-1,k}}{\partial\nu_\Sigma} \tag{101}$$

such that

$$\|F - S_m\|_{\mathrm{L}^2(\Sigma)} \le \varepsilon \ .$$

6 Multiscale Modelling in $(\mathrm{L}^2(\Sigma), \|\cdot\|_{\mathrm{L}^2(\Sigma)})$

Writing out the limit and jump relations (Theorem 4.1) we obtain the following corollary.

COROLLARY 6.1. *For* $F \in \mathrm{L}^2(\Sigma)$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_\Sigma \Phi^i_\tau(\cdot,y)\, F(y)\; d\omega(y)$$
$$= \begin{cases} F & , \quad i = 2,3,5,6 \\ 0 & , \quad i = 4,7 \\ \int_\Sigma \frac{1}{|\cdot - y|} F(y)\; d\omega(y) & , \quad i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(\cdot)} \frac{1}{|\cdot - y|} F(y)\; d\omega(y) & , \quad i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)} \frac{1}{|\cdot - y|} F(y)\; d\omega(y) & , \quad i = 9, \end{cases}$$

where

$$\Phi^1_{\pm\tau}(x,y) = \frac{1}{|x \pm \tau\nu(x) - y|},$$

$$\Phi^2_{\pm\tau}(x,y) = \frac{1}{2\pi}\left(\frac{(x \pm \tau\nu(x) - y)\cdot\nu(x)}{|x \pm \tau\nu(x) - y|^3} - \frac{(x-y)\cdot\nu(x)}{|x-y|^3}\right),$$

$$\Phi^3_{\pm\tau}(x,y) = \frac{1}{2\pi}\left(\frac{(x \pm \tau\nu(x) - y)\cdot\nu(y)}{|x \pm \tau\nu(x) - y|^3} - \frac{(x-y)\cdot\nu(y)}{|x-y|^3}\right),$$

$$\Phi^4_{\tau}(x,y) = \frac{1}{|x + \tau\nu(x) - y|} - \frac{1}{|x - \tau\nu(x) - y|}$$

$$\Phi^5_{\tau}(x,y) = \frac{1}{4\pi}\left(\frac{(x + \tau\nu(x) - y)\cdot\nu(x)}{|x + \tau\nu(x) - y|^3} - \frac{(x - \tau\nu(x) - y)\cdot\nu(x)}{|x - \tau\nu(x) - y|^3}\right),$$

$$\Phi^6_{\tau}(x,y) = \frac{1}{4\pi}\left(\frac{(x + \tau\nu(x) - y)\cdot\nu(y)}{|x + \tau\nu(x) - y|^3} - \frac{(x - \tau\nu(x) - y)\cdot\nu(y)}{|x - \tau\nu(x) - y|^3}\right),$$

$$\begin{aligned}\Phi^7_{\tau}(x,y) = {} & \frac{\nu(x)\cdot\nu(y)}{|x + \tau\nu(x) - y|} - \frac{\nu(x)\cdot\nu(y)}{|x - \tau\nu(x) - y|^3} \\ & -3\frac{((x + \tau\nu(x) - y)\cdot\nu(y))((x + \tau\nu(x) - y)\cdot\nu(y))}{|x + \tau\nu(x) - y|^5} \\ & +3\frac{((x - \tau\nu(x) - y)\cdot\nu(y))((x + \tau\nu(x) - y)\cdot\nu(y))}{|x - \tau\nu(x) - y|^5},\end{aligned}$$

$$\Phi^8_{\tau}(x,y) = \frac{1}{2}\left(\frac{(x + \tau\nu(x) - y)\cdot\nu(x)}{|x + \tau\nu(x) - y|^3} + \frac{(x - \tau\nu(x) - y)\cdot\nu(x)}{|x - \tau\nu(x) - y|^3}\right),$$

$$\Phi^9_{\tau}(x,y) = \frac{1}{2}\left(\frac{(x + \tau\nu(x) - y)\cdot\nu(y)}{|x + \tau\nu(x) - y|^3} + \frac{(x - \tau\nu(x) - y)\cdot\nu(y)}{|x - \tau\nu(x) - y|^3}\right),$$

$\tau > 0, (x,y) \in \Sigma \times \Sigma$.

From the limit and jump relations for the dual operators we obtain the following result.

Corollary 6.2. *For* $F \in \mathrm{L}^2(\Sigma)$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_\Sigma \Phi_\tau^i\,(y,\cdot)\,F(y)\;d\omega(y)$$

$$= \begin{cases} F & , \quad i = 2,3,5,6 \\ 0 & , \quad i = 4,7 \\ \int_\Sigma \frac{1}{|\cdot - y|} F(y)\;d\omega(y) & , \quad i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(\cdot)} \frac{1}{|\cdot - y|} F(y)\;d\omega(y) & , \quad i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)} \frac{1}{|\cdot - y|} F(y)\;d\omega(y) & , \quad i = 9. \end{cases}$$

6.1 Scaling and Wavelet Functions

For $\tau > 0$ and $i \in \{1,\ldots,9\}$, the family $\{\Phi_\tau^i\}_{\tau>0}$ of kernels $\Phi_\tau^i : \Sigma\times\Sigma \to \mathbb{R}$ is called a Σ*–scaling function of type* i. Moreover, $\Phi_1^i : \Sigma\times\Sigma\to\mathbb{R}$ (i.e.: $\tau = 1$) is called the *mother kernel of the* Σ*–scaling function of type* i.
Correspondingly, for $\tau > 0$ and $i \in \{1,\ldots,9\}$, the family $\{\Psi_\tau^i\}_{\tau>0}$ of kernels $\Psi_\tau^i : \Sigma\times\Sigma\to\mathbb{R}$ given by

$$\Psi_\tau^i(x,y) = -\alpha(\tau)^{-1}\frac{d}{d\tau}\Phi_\tau^i(x,y), \qquad x,y\in\Sigma, \tag{102}$$

is called a Σ*–wavelet function of type* i.
In the remainder of this paper we particularly choose $\alpha(\tau) = \tau^{-1}$ (of course, other weight functions than $\alpha(\tau) = \tau^{-1}$ can be chosen in (102)). Moreover, $\Psi_1^i : \Sigma\times\Sigma\to\mathbb{R}$ (i.e.: $\tau = 1$) is called the *mother kernel of the* Σ*–wavelet function of type* i.

The differential equation (102) is called the (scale continuous) Σ*–scaling equation of type* i.

Definition 6.3. Let $\{\Phi_\tau^i\}_{\tau>0}$ be a Σ–scaling function of type i. Then the associated Σ*–wavelet transform of type* i is defined by

$$(WT)^{(i)} : \mathrm{L}^2(\Sigma) \to \mathrm{L}^2((0,\infty)\times\Sigma)$$

with

$$(WT)^{(i)}(F)(\tau,x) = \int_\Sigma \Psi_\tau^i(x,y)F(y)\;d\omega(y)\ .$$

In accordance with our construction we have

$$\begin{aligned}
\Psi_\tau^1(x,y) &= \frac{\tau(x+\tau\nu(x)-y)\cdot\nu(x)}{|x+\tau\nu(x)-y|^3} \\
\Psi_\tau^2(x,y) &= \frac{-\tau}{2\pi}\left(\frac{1}{|x+\tau\nu(x)-y|^3} - 3\frac{((x+\tau\nu(x)-y)\cdot\nu(x))^2}{|x+\tau\nu(x)-y|^5}\right), \\
\Psi_\tau^3(x,y) &= \frac{\tau}{2\pi}\left(\frac{\nu(y)\cdot\nu(x)}{|x+\tau\nu(x)-y|^3}\right) \\
&\quad - \frac{3\tau}{2\pi}\left(\frac{((x+\tau\nu(x)-y)\cdot\nu(x))((x+\tau\nu(x)-y)\cdot\nu(y))}{|x+\tau\nu(x)-y|^5}\right), \\
\Psi_\tau^4(x,y) &= -\tau\left(\frac{(x+\tau\nu(x)-y)\cdot\nu(x)}{|x+\tau\nu(x)-y|^3} + \frac{(x-\tau\nu(x)-y)\cdot\nu(x)}{|x-\tau\nu(x)-y|^3}\right), \\
\Psi_\tau^5(x,y) &= \frac{-\tau}{4\pi}\left(\frac{1}{|x-\tau\nu(x)-y|^3} + \frac{1}{|x-\tau\nu(x)-y|^3}\right) \\
&\quad + \frac{3\tau}{4\pi}\left(\frac{((x+\tau\nu(x)-y)\cdot\nu(x))^2}{|x+\tau\nu(x)-y|^5} + \frac{((x-\tau\nu(x)-y)\cdot\nu(x))^2}{|x-\tau\nu(x)-y|^5}\right), \\
\Psi_\tau^6(x,y) &= \frac{-\tau}{4\pi}\left(\frac{\nu(x)\cdot\nu(y)}{|x+\tau\nu(x)-y|^3} + \frac{\nu(x)\cdot\nu(y)}{|x-\tau\nu(x)-y|^3}\right) \\
&\quad + \frac{3\tau}{4\pi}\left(\frac{((x+\tau\nu(x)-y)\cdot\nu(x))((x+\tau\nu(x)-y)\cdot\nu(y))}{|s+\tau\nu(x)-y|^5}\right. \\
&\quad + \left.\frac{((x-\tau\nu(x)-y)\cdot\nu(x))((x-\tau\nu(x)-y)\cdot\nu(y))}{|x-\tau\nu(x)-y|^5}\right),
\end{aligned}$$

for $x, y \in \Sigma$. For simplicity, we omit the representations of $\Psi_\tau^7(x,y), \Psi_\tau^8(x,y)$ and $\Psi_\tau^9(x,y)$, $x, y \in \Sigma$.

6.2 Scale Continuous Reconstruction Formula

It is not difficult to see that the wavelets Ψ_τ^i, $i \in \{1,\ldots,9\}$, behave like $O(\tau^{-1})$, hence, the convergence of the following integrals in the *reconstruction theorem* is guaranteed.

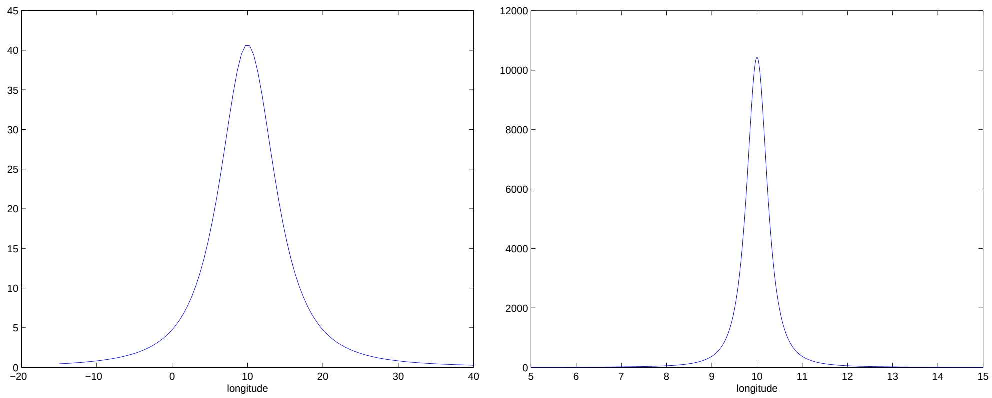
THEOREM 6.4. *Let* $\{\Phi_\tau^i\}_{\tau>0}$ *be a* Σ*–scaling function of type* i*. Suppose that* F *is of class* $\mathrm{L}^2(\Sigma)$*. Then the reconstruction formula*

$$\int_0^\infty (WT)^i(F)(\tau,\cdot)\frac{d\tau}{\tau} = \begin{cases} F & , \quad i = 2,3,5,6 \\ 0 & , \quad i = 4,7 \\ \int_\Sigma \frac{1}{|\cdot-y|}F(y)\, d\omega(y) & , \quad i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(x)}\frac{1}{|\cdot-y|}F(y)\, d\omega(y) & , \quad i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)}\frac{1}{|\cdot-y|}F(y)\, d\omega(y) & , \quad i = 9 \end{cases}$$

holds in the sense of $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$.

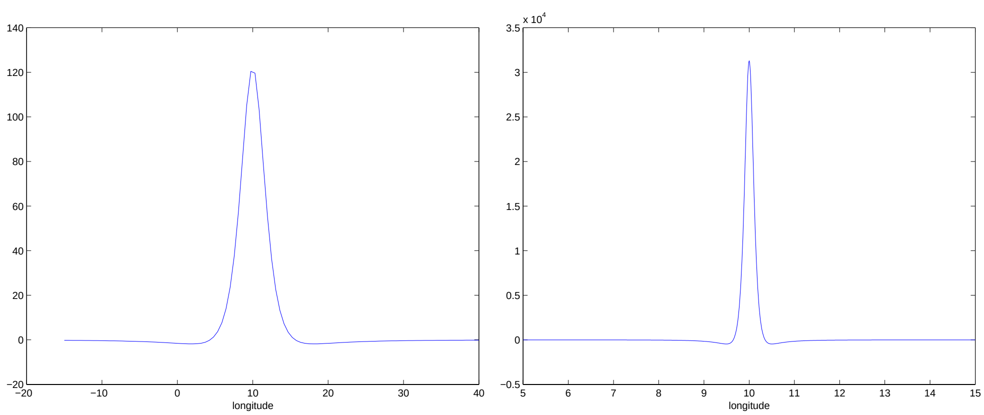
Proof. Let $R > 0$ be arbitrary. By observing Fubini's theorem and the identity

$$\Phi_R^i(x,y) = \int_R^\infty \Psi_\tau^i(x,y)\frac{d\tau}{\tau}, \qquad (x,y) \in \Sigma\times\Sigma,$$



(a) $\tau = 2^{-4}$ (b) $\tau = 2^{-8}$

Figure 2: Scaling–function Φ_τ^6 (sectional illustration) for two values of τ



(a) $\tau = 2^{-4}$ (b) $\tau = 2^{-8}$

Figure 3: Wavelet–function Ψ_τ^6 (sectional illustration) for two values of τ

we obtain

$$\begin{aligned}\int_R^\infty (WT)^i(F)(\tau,\cdot)\frac{d\tau}{\tau} &= \int_R^\infty \int_\Sigma \Psi^i_\tau(\cdot,y)F(y)\ d\omega(y)\frac{d\tau}{\tau} \\ &= \int_\Sigma \int_R^\infty \Psi^i_\tau(\cdot,y)F(y)\ d\omega(y)\frac{d\tau}{\tau} \\ &= \int_\Sigma \Phi^i_R(\cdot,y)F(y)\ d\omega(y)\ .\end{aligned} \tag{103}$$

The limit $R \to 0$ in connection with Theorem 6.1 yields the desired result. $\square$

Next our interest is to reformulate the wavelet transform and the reconstruction theorem by use of dilated and shifted versions of the mother kernel. For that purpose we introduce the x–translation and the τ–dilation operator of a mother kernel as follows:

$$T_x : \Psi^i_1 \mapsto T_x\Psi^i_1 = \Psi^i_{1;x} = \Psi^i_1(x,\cdot), \qquad x \in \Sigma, \tag{104}$$

$$D_\tau : \Psi^i_1 \to D_\tau\Psi^i_1 = \Psi^i_\tau, \quad \tau > 0\ . \tag{105}$$

Consequently it follows that

$$T_x D_\tau \Psi^i_1 = T_x\Psi^i_\tau = \Psi^i_{\tau;x} = \Psi^i_\tau(x,\cdot), \tag{106}$$

$i = 1,\ldots,9$. In other words,

$$(WT)^i(F)(\tau;x) = \int_\Sigma \Psi_{\tau;x}(y)F(y)\ d\omega(y), \quad x \in \Sigma, \tau > 0\ . \tag{107}$$

Moreover, we have the following limit results.

THEOREM 6.5. *For $x \in \Sigma$ and $F \in \mathrm{L}^2(\Sigma)$*

$$\lim_{\substack{R\to 0\\ R>0}} \int_\Sigma \Phi^i_{R;x}(y)F(y)\ d\omega(y) = \begin{cases} F(x) & ,\ i = 2,3,5,6 \\ 0 & ,\ i = 4,7 \\ \int_\Sigma \frac{1}{|x-y|}F(y)\ d\omega(y) & ,\ i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(x)}\frac{1}{|x-y|}F(y)\ d\omega(y) & ,\ i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}F(y)\ d\omega(y) & ,\ i = 9 \end{cases}$$

and

$$\int_0^\infty \int_\Sigma \Psi_{\tau;x}(y)F(y)\ d\omega(y)\frac{d\tau}{\tau} = \begin{cases} F(x) & ,\ i = 2,3,5,6 \\ 0 & ,\ i = 4,7 \\ \int_\Sigma \frac{1}{|x-y|}F(y)\ d\omega(y) & ,\ i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(x)}\frac{1}{|x-y|}F(y)\ d\omega(y) & ,\ i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}F(y)\ d\omega(y) & ,\ i = 9\ . \end{cases}$$

Note that the properties of the Σ–wavelets of type i (analogously to variants of spherical wavelets developed in [7], [8]) do not presume the zero–mean property of Ψ^i_τ. The wavelets constructed in this way, therefore, do not satisfy a substantial condition of the Euclidean concept. However, it should be pointed out that a construction of wavelets possessing the zero-mean property (see [7]), is obvious and will not be discussed here.

6.3 Scale Discretized Reconstruction Formula

Until now we were concerned with a scale continuous approach to wavelets. In what follows, scale discrete Σ–scaling functions and wavelets of type i will be introduced. We start with the choice of a sequence which divides the continuous scale interval $(0, \infty)$ into discrete pieces. More explicitly, $(\tau_j)_{j\in\mathbb{Z}}$ denotes a sequence of real numbers satisfying

$$\lim_{j\to\infty} \tau_j = 0 \tag{108}$$

and

$$\lim_{j\to-\infty} \tau_j = \infty \ . \tag{109}$$

Remark. For example, one may choose $\tau_j = 2^{-j}$, $j \in \mathbb{Z}$ (note that in this case, $2\tau_{j+1} = \tau_j$, $j \in \mathbb{Z}$).

Given a Σ–scaling function $\{\Phi^i_\tau\}_{\tau>0}$ of type i, then we clearly define the (scale) *discretized* Σ*–scaling function* of type i by $\{\Phi^i_{\tau_j}\}_{j\in\mathbb{Z}}$.
In doing so, by Theorem 6.5, we immediately get the following result.

THEOREM 6.6. *For* $F \in \mathrm{L}^2(\Sigma)$

$$\lim_{j\to\infty} \int_\Sigma \Phi^i_{\tau_j}(\cdot, y) F(y) \ d\omega(y) \tag{110}$$

$$= \begin{cases} F & , \quad i = 2,3,5,6 \\ 0 & , \quad i = 4,7 \\ \int_\Sigma \frac{1}{|\cdot - y|} F(y) d\omega(y) & , \quad i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(\cdot)} \frac{1}{|\cdot - y|} F(y) \ d\omega(y) & , \quad i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)} \frac{1}{|\cdot - y|} F(y) \ d\omega(y) & , \quad i = 9 \end{cases} \tag{111}$$

holds in the $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$*–sense.*

Our procedure canonically leads us to the following type of scale discretized wavelets.

DEFINITION 6.7. Let $\{\Phi_{\tau_j}\}_{j\in\mathbb{Z}}$ be a discretized Σ–scaling function of type i. Then the (scale) discretized Σ–*wavelet function of type* i is defined by

$$\Psi^i_{\tau_j}(\cdot, \cdot) = \int_{\tau_{j+1}}^{\tau_j} \Psi^i_\tau(\cdot, \cdot) \frac{d\tau}{\tau}, \quad j \in \mathbb{Z} \tag{112}$$

In connection with (102) it follows that

$$\begin{aligned} \Psi^i_{\tau_j}(\cdot,\cdot) &= -\int_{\tau_{j+1}}^{\tau_j} \tau \frac{d}{d\tau} \Phi^i_\tau(\cdot,\cdot) \frac{d\tau}{\tau} \\ &= \Phi^i_{\tau_{j+1}}(\cdot,\cdot) - \Phi^i_{\tau_j}(\cdot,\cdot). \end{aligned} \tag{113}$$

Formula (113) is called *(scale) discretized* Σ*–scaling equation of type* i.
Assume now that F is a function of class $\mathrm{L}^2(\Sigma)$. Observing the discretized Σ–scaling equation of type i we get for $J \in \mathbb{Z}$ and $N \in \mathbb{N}$

$$\begin{aligned} \int_\Sigma \Phi^i_{\tau_{J+N}}(\cdot,y)F(y)\ d\omega(y) &= \int_\Sigma \Phi^i_{\tau_J}(\cdot,y)F(y)\ d\omega(y) \\ &+ \sum_{j=J}^{J+N-1} \int_\Sigma \Psi^i_{\tau_j}(\cdot,y)F(y)\ d\omega(y) \ . \end{aligned}$$

Therefore we are able to formulate the following corollary.

Corollary 6.8. *Let* $\{\Phi^i_{\tau_j}\}_{j\in\mathbb{Z}}$ *be a (scale) discretized* Σ*–scaling function of type* i. *Then the multiscale representation of a function* $F \in \mathrm{L}^2(\Sigma)$

$$\begin{aligned} &\sum_{j=-\infty}^{+\infty} \int_\Sigma \Psi^i_{\tau_j}(\cdot,y)F(y)\ d\omega(y) \\ &= \begin{cases} F & ,\quad i = 2,3,5,6 \\ 0 & ,\quad i = 4,5 \\ \int_\Sigma \frac{1}{|\cdot - y|} F(y)\ d\omega(y) & ,\quad i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(\cdot)} \frac{1}{|\cdot - y|} F(y)\ d\omega(y) & ,\quad i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)} \frac{1}{|\cdot - y|} F(y)\ d\omega(y) & ,\quad i = 9 \end{cases} \end{aligned}$$

holds in the $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$*–sense.*

Corollary 6.8 admits the following reformulation.

Corollary 6.9. *Under the assumption of Corollary 6.8*

$$\begin{aligned} &P^i_{\tau_J}(F) + \sum_{j=J}^{+\infty} \int_\Sigma \Psi^i_{\tau_j}(\cdot,y)F(y)\ d\omega(y) \\ &= \begin{cases} F & ,\quad i = 2,3,5,6 \\ 0 & ,\quad i = 4,5 \\ \int_\Sigma \frac{1}{|\cdot - y|} F(y)\ d\omega(y) & ,\quad i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(\cdot)} \frac{1}{|\cdot - y|} F(y)\ d\omega(y) & ,\quad i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)} \frac{1}{|\cdot - y|} F(y)\ d\omega(y) & ,\quad i = 9 \end{cases} \end{aligned}$$

for every $J \in \mathbb{Z}$ *(in the sense of the* $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$*–norm), where* $P^i_{\tau_J}(F)$ *is given by*

$$P^i_{\tau_J}(F) = \int_\Sigma \Phi^i_{\tau_J}(\cdot,y)F(y)\ d\omega(y) \ .$$

The scale discretized Σ–wavelets allow the following formulation

$$T_x D_{\tau_j} \Psi_1^i = T_x \Psi_{\tau_j}^i = \Psi_{\tau_j;x}^i = \Psi_{\tau_j}^i(x,\cdot) \tag{114}$$

for $i = 1,\ldots,9$ and $x \in \Sigma$.

The *(scale) discretized Σ–wavelet transform of type i* is defined by

$$(WT)^i : \mathrm{L}^2(\Sigma) \mapsto \left\{ H : \mathbb{Z}\times\Sigma \to \mathbb{R} \middle| \sum_{j=-\infty}^{\infty} \int_\Sigma (H(j;y))^2 \; d\omega(y) < \infty \right\}$$

with

$$(WT)^i(F)(\tau_j;x) = \int_\Sigma \Psi_{\tau_j;x}^i(y) F(y) \; d\omega(y) \quad .$$

THEOREM 6.10. *Let $\{\Phi_{\tau_j}^i\}_{j\in\mathbb{Z}}$ be a (scale) discretized Σ–scaling function of type i. Then, for all $F \in \mathrm{L}^2(\Sigma)$, the reconstruction formula*

$$\sum_{j=-\infty}^{+\infty} (WT)^i(F)(\tau_j;\cdot) = \begin{cases} F & , \quad i = 2,3,5,6 \\ 0 & , \quad i = 4,7 \\ \int_\Sigma \frac{1}{|\cdot - y|} F(y)\; d\omega(y) & , \quad i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(\cdot)} \frac{1}{|\cdot - y|} F(y)\; d\omega(y) & , \quad i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)} \frac{1}{|\cdot - y|} F(y)\; d\omega(y) & , \quad i = 9 \end{cases}$$

holds in $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$–sense.

6.4 Scale and Detail Spaces

Comparing this result with the continuous analogue (6.4) we notice that the subdivision of the continuous scale interval $(0,\infty)$ into discrete pieces means substitution of the integral over τ by an associated discrete sum.

As in the spherical theory of wavelets (see [5], [6]), the operators $R_{\tau_j}^i$, $P_{\tau_j}^i$ defined by

$$R_{\tau_j}^i(F) = \int_\Sigma \Psi_{\tau_j}^i(\cdot,y) F(y) \; d\omega(y), \quad F \in \mathrm{L}^2(\Sigma), \tag{115}$$

$$P_{\tau_j}^i(F) = \int_\Sigma \Phi_{\tau_j}^i(\cdot,y) F(y) \; d\omega(y), \quad F \in \mathrm{L}^2(\Sigma) \tag{116}$$

may be understood as band pass and low pass filter, respectively. The *scale spaces* $\mathcal{V}_{\tau_j}^i$ and the *detail spaces* $W_{\tau_j}^i$ *of type* i are defined by

$$\mathcal{V}_{\tau_j}^i = P_{\tau_j}^i\left(\mathrm{L}^2(\Sigma)\right) = \left\{ P_{\tau_j}^i(F) \middle| F \in \mathrm{L}^2(\Sigma) \right\}, \tag{117}$$

$$\mathcal{W}_{\tau_j}^i = R_{\tau_j}^i\left(\mathrm{L}^2(\Sigma)\right) = \left\{ R_{\tau_j}^i(F) \middle| F \in \mathrm{L}^2(\Sigma) \right\}, \tag{118}$$

respectively. From the identity

$$\begin{aligned} \int_\Sigma \Phi^i_{\tau_{J+1}}(\cdot,y)F(y)\ d\omega(y) &= \int_\Sigma \Phi^i_{\tau_J}(\cdot,y)F(y)\ d\omega(y) \\ &+ \int_\Sigma \Psi^i_{\tau_J}(\cdot,y)F(y)\ d\omega(y) \end{aligned} \tag{119}$$

i.e.

$$P^i_{\tau_{J+1}}(F) = P^i_{\tau_J}(F) + R^i_{\tau_J}(F) \tag{120}$$

for all $J \in \mathbb{Z}$ it easily follows that

$$\mathcal{V}^i_{\tau_{j+1}} = \mathcal{V}^i_{\tau_j} + \mathcal{W}^i_{\tau_j}\ . \tag{121}$$

However, it should be remarked that the sum (121) generally is neither direct nor orthogonal.
The equation (121) may be interpreted in the following way: The set $\mathcal{V}^i_{\tau_j}$ contains a $P^i_{\tau_j}$−filtered version of a function belonging to the class $\mathrm{L}^2(\Sigma)$. The lower the scale, the stronger the intensity of filtering. By adding '$R^i_{\tau_j}$−details' contained in the space $\mathcal{W}^i_{\tau_j}$ the space $\mathcal{V}^i_{\tau_{j+1}}$ is created, which consists of a filtered versions at resolution $j+1$. Obviously, for $i = 2,3,5,6$,

$$\overline{\bigcup_{j=-\infty}^{\infty} \mathcal{V}^i_{\tau_j}}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = \mathrm{L}^2(\Sigma)\ .$$

Moreover,

$$\overline{\bigcup_{j=-\infty}^{\infty} \mathcal{V}^1_{\tau_j}}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = P(0,0)(\mathrm{L}^2(\Sigma)), \tag{122}$$

$$\overline{\bigcup_{j=-\infty}^{\infty} \mathcal{V}^8_{\tau_j}}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = P_{|1}(0,0)(\mathrm{L}^2(\Sigma)), \tag{123}$$

$$\overline{\bigcup_{j=-\infty}^{\infty} \mathcal{V}^9_{\tau_j}}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = P_{|2}(0,0)(\mathrm{L}^2(\Sigma))\ . \tag{124}$$

6.5 Multiresolution Analysis

Our purpose is to establish a multiresolution analysis for the Σ−wavelet function.

DEFINITION 6.11. A family of subspaces $\{V^i_\tau(\Sigma)\}_{\tau\in(0,\infty)} \subset \mathrm{L}^2(\Sigma)$, $i \in \{1,\ldots,9\}$, is called a *multiresolution* analysis if it satisfies the following properties:

(i) $\{0\} \subset V^i_\tau(\Sigma) \subset V^i_{\tau'}(\Sigma) \subset \mathrm{L}^2(\Sigma)$ for $0 \le \tau' \le \tau \le \infty$,

(ii) $\{\lim_{\tau\to\infty}\left(\int_\Sigma \Phi^i_\tau(.,y)F(y)\ \ d\omega(y)\right)|F \in \mathrm{L}^2(\Sigma)\} = \{0\}$,

(iii) $\overline{\{F \in \mathrm{L}^2(\Sigma) | F \in V_\tau^i(\Sigma) \text{ for some } \tau \in (0,\infty)\}}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = \mathrm{L}^2(\Sigma)$.

The following lemma summarizes results which were listed in the previous section.

LEMMA 6.12. *For the scale spaces* V_τ^i, $i = 5, 6$, *of the* $\Sigma-$*scaling function of type* 5 *and* 6 *defined in* (117)*, respectively, the following statements are true:*

(i) $V_\tau^i \subset \mathrm{L}^2(\Sigma)$ *for all* $\tau \in (,\infty)$,

(ii) $\{\lim_{\tau\to\infty} \left(\int_\Sigma \Phi_\tau^i(.,y)F(y)\ d\omega(y)\right) | F \in \mathrm{L}^2(\Sigma)\} = \{0\}$,

(iii) V_τ^i *is a linear subspace of* $\mathrm{L}^2(\Sigma)$,

(iv) $\overline{\{F \in \mathrm{L}^2(\Sigma) | F \in V_\tau^i(\Sigma) \text{ for some } \tau \in (0,\infty)\}}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = \mathrm{L}^2(\Sigma)$.

Proof. Statement (i) is clear by the definition of the scale spaces. Moreover, statement (ii) follows from the fact that the $\Sigma-$scaling functions of type 5 and 6 tend to 0 for $\tau \to \infty$. Finally, property (iii) is a result of the linearity of the integral, while (iv) has been shown in the last section. □

We are interested in discussing the multiresolution analysis for the spherical case and the case of a regular surface separately. From now on, we restrict ourselves to the types $i = 5, 6$.

6.5.1 Spherical Case

First we prove the following lemma which forms the bridge to the spherical wavelet theory of [6], [7].

LEMMA 6.13. *The family of scale spaces* $\{V_\tau^i(\Omega_\alpha)\}_{\tau\in(0,\infty)}$, $i \in \{5,6\}$, *fulfills the properties of a multiresolution analysis in* $\mathrm{L}^2(\Omega_\alpha)$ *for arbitrary* $\alpha > 0$.

Proof. We embed our considerations in the theory of product kernels and spherical multiscale theory as presented in [7]. By using the identity

$$\frac{1}{|x-y|} = \frac{1}{|y|}\sum_{n=0}^{\infty}\left(\frac{|x|}{|y|}\right)^n P_n(\xi\cdot\eta) \tag{125}$$

with $x = |x|\,\xi, y = |y|\,\eta,\ \ \xi,\eta \in \Omega, |x| \le |y|$ and

$$|x + \tau\nu(x)| = \alpha + \tau, \quad x \in \Omega_\alpha,\ \ \tau \in (-\alpha, \infty) \tag{126}$$

we first get by some easy calculations

$$\Phi_\tau^5(x,y) \quad = \quad \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} G_{\tau,5}^\wedge(\alpha,n) P_n(t), \tag{127}$$

$$\Phi_\tau^6(x,y) \quad = \quad \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} G_{\tau,6}^\wedge(\alpha,n) P_n(t), \tag{128}$$

with $x, y \in \Omega_\alpha$, $t = \frac{x \cdot y}{\alpha^2} \in [-1, 1]$ and

$$G^\wedge_{\tau,5}(\alpha, n) = \frac{1}{2n+1}\left(\frac{n(\alpha-\tau)^{n-1}}{\alpha^{n-1}} + \frac{(n+1)\alpha^{n+2}}{(\alpha+\tau)^{n+2}}\right), \quad (129)$$

$$G^\wedge_{\tau,6}(\alpha, n) = \frac{1}{2n+1}\left(\frac{n\alpha^{n-1}}{(\alpha+\tau)^{n+1}} + \frac{(n+1)(\alpha-\tau)^{n}}{\alpha^{n+2}}\right), \quad (130)$$

$n \in \mathbb{N}_0$, $\tau \in (0, \alpha)$ and $\alpha > 0$.
This is the standard spectral representation for product kernels as used in [7]. However, in the approach presented here, elementary representations in explicit form are additionally available for the scaling functions and the wavelets.
It should be noted that the restriction $\tau < \alpha$ does not really matter, since we are interested in the limit $\tau \to 0$.
For the symbol $G^\wedge_{\tau,i}(\alpha, n)$ of the spherical kernels Φ^5_τ and Φ^6_τ, respectively, we easily obtain the following properties:

(i) $G^\wedge_{\tau,5}(\alpha, 0) = \frac{\alpha^2}{(\alpha+\tau)^2}$ for all $\tau \in (0, \alpha)$,

(ii) $G^\wedge_{\tau,6}(\alpha, 0) = 1$ for all $\tau \in (0, \alpha)$,

(iii) $\lim_{\substack{\tau \to 0 \\ \tau > 0}} G^\wedge_{\tau,i}(\alpha, n) = 1$ for all $n \in \mathbb{N}_0$ and $i \in \{5, 6\}$,

(iv) $G^\wedge_{\tau,i}(\alpha, n)$ is monotonically decreasing in τ for all $n \in \mathbb{N}_0$, $\tau \in (0, \alpha)$ and $i \in \{5, 6\}$.

The first three points are clear from our construction, the fourth point can easily be verified by the facts that

$$\begin{aligned}
\frac{\partial}{\partial \tau} G^\wedge_{\tau,5}(\alpha, n) &= \frac{1}{2n+1}\left(-n(n+1)\frac{(\alpha-\tau)^{n-2}}{\alpha^{n-1}} - (n+1)(n+2)\frac{\alpha^{n+2}}{(\alpha+\tau)^{n+3}}\right) \\
&= -\frac{(n+1)}{2n+1}\left(\frac{n(\alpha-\tau)^{n-2}}{\alpha^{n-1}} + \frac{(n+2)\alpha^{n+2}}{(\alpha+\tau)^{n+3}}\right) < 0, \\
\frac{\partial}{\partial \tau} G^\wedge_{\tau,6}(\alpha, n) &= \frac{1}{2n+1}\left(-n(n+1)\frac{\alpha^{n+1}}{(\alpha+\tau)^{n+2}} - n(n+1)\frac{(\alpha-\tau)^{n-1}}{\alpha^{n}}\right) \\
&= -\frac{n(n+1)}{2n+1}\left(\frac{\alpha^{n+1}}{(\alpha+\tau)^{n+2}} + \frac{(\alpha-s)^{n-1}}{\alpha^{n}}\right) < 0,
\end{aligned}$$

for $n \in \mathbb{N}_0$ and $\tau \in (0, \alpha)$. By these properties we are immediately able to deduce that the scale spaces $V^5_\tau(\Omega_\alpha)$ and $V^6_\tau(\Omega_\alpha)$ form a multiresolution analysis in $\mathrm{L}^2(\Omega_\alpha)$. (For a more detailed discussion concerning the idea of continuous multiresolution analysis the reader is referred to [6], [7].) □

For the sake of completeness we next present the symbols of the $\Sigma-$wavelet function of type 5 and 6, respectively

$$\Psi^5_\tau(x,y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \psi^\wedge_{\tau,5}(\alpha,n) P_n(t), \tag{131}$$

$$\Psi^5_\tau(x,y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \psi^\wedge_{\tau,6}(\alpha,n) P_n(t), \tag{132}$$

with $x,y \in \Omega_\alpha$, $t = \frac{x\cdot y}{\alpha^2}$ and

$$\psi^\wedge_{\tau,5}(\alpha,n) = \tau \frac{(n+1)}{2n+1} \left(\frac{n(\alpha-\tau)^{n-2}}{\alpha^{n-1}} + \frac{(n+2)\alpha^{n+2}}{(\alpha+\tau)^{n+3}} \right), \tag{133}$$

$$\psi^\wedge_{\tau,6}(\alpha,n) = \tau \frac{n(n+1)}{2n+1} \left(\frac{\alpha^{n+1}}{(\alpha+\tau)^{n+2}} + \frac{(\alpha-s)^{n-1}}{\alpha^{n}} \right), \tag{134}$$

for $n \in \mathbb{N}_0$.

6.5.2 Non-spherical Case

Let Σ be a regular surface. Then, as shown in Lemma 6.12, the scale spaces V^6_τ fulfill the properties of a multiresolution analysis up to one condition, viz. the inclusion property

$$V^6_\tau(\Sigma) \subset V^6_{\tau'}(\Sigma) \quad \text{for} \quad \tau' \leq \tau. \tag{135}$$

In what follows we discuss the inclusion property in more detail. The point of departure is the observation that the system $\left(H^\alpha_{-n-1,k}|\Sigma\right)_{\substack{n=0,1,\ldots,\\ k=1,\ldots,2n+1}}$ is closed and complete in $\mathrm{L}^2(\Sigma)$, i.e. a countable Hilbert basis in $\mathrm{L}^2(\Sigma)$. Clearly, by means of the well-known Gram-Schmidt orthonormalization process, this system can be orthonormalized resulting in the countable system $\left(H^\alpha_{-n-1,k}(\Sigma,.)\right)_{\substack{n=0,1,\ldots,\\ k=1,\ldots,2n+1}}$ which is complete and closed in $\mathrm{L}^2(\Sigma)$ and fulfills the orthonormality property

$$\int_\Sigma H^\alpha_{-n-1,k}(\Sigma,x)\, H^\alpha_{-m-1,j}(\Sigma,x) \quad d\omega(x) = \delta_{n,m}\delta_{k,j}. \tag{136}$$

The $\Sigma-$scaling function of type 6 can now be expanded in terms of this $\mathrm{L}^2(\Sigma)-$orthonormal system. In other words, there exist functions $c^\tau_{n,k} \in \mathrm{L}^2(\Sigma)$ such that

$$\Phi^6_\tau(x,\,.\,) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} c^\tau_{n,k}(x) \;\; H^\alpha_{-n-1,k}(\Sigma,\,.\,), \quad x \in \Sigma. \tag{137}$$

The function $c^\tau_{n,k} \in \mathrm{L}^2(\Sigma)$ itself can then be expanded into a Fourier series and we get

$$\Phi^6_\tau(\,.\,,\,.\,) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \sum_{m=0}^{\infty} \sum_{j=1}^{2m+1} \lambda^\tau_{n,k,m,j} \;\; H^\alpha_{-n-1,k}(\Sigma,\,.\,) \;\; H^\alpha_{-m-1,j}(\Sigma,\,.\,). \tag{138}$$

This leads us to the following result.

LEMMA 6.14. *Let* $\left\{H^{\alpha}_{-n-1,k}(\Sigma,.)\right\}_{\substack{n=0,1,\ldots,\\ k=1,\ldots,2n+1}}$ *be an* $\mathrm{L}^2(\Sigma)-$*orthonormal system as constructed above. Then the tensorial product system*

$$\{H^{\alpha}_{-n-1,k}(\Sigma,\,.\,)H^{\alpha}_{-m-1,j}(\Sigma,\,.\,)\}_{\substack{n=0,1,\ldots;k=1,\ldots,2n+1\\ m=0,1,\ldots;j=1,\ldots 2m+1}} \tag{139}$$

is an $\mathrm{L}^2(\Sigma\times\Sigma)-$*orthonormal system.*

Proof. By using Fubini's theorem and the $\mathrm{L}^2(\Sigma)-$orthonormality of $H^{\alpha}_{-n-1,k}(\Sigma,\,.\,)$ we find

$$\begin{aligned} & \left(H^{\alpha}_{-n-1,k}(\Sigma,\,.\,)H^{\alpha}_{-m-1,j}(\Sigma,\,.\,), H^{\alpha}_{-n'-1,k'}(\Sigma,\,.\,)H^{\alpha}_{-m'-1,j'}(\Sigma,\,.\,)\right)_{\mathrm{L}^2(\Sigma\times\Sigma)} \\ = & \left(\int_{\Sigma} H^{\alpha}_{-n-1,k}(\Sigma,x)H^{\alpha}_{-n'-1,k'}(\Sigma,x)\ d\omega(x)\right) \\ & \qquad \left(\int_{\Sigma} H^{\alpha}_{-m-1,j}(\Sigma,y)H^{\alpha}_{-m'-1,j'}(\Sigma,y)\ d\omega(y)\right) \\ = & \ \delta_{n,n'}\delta_{k,k'}\delta_{m,m'}\delta_{j,j'}\,. \end{aligned}$$

□

In other words, observing the identity (138) and Parseval's theorem we are led to the following representation of the $\mathrm{L}^2(\Sigma\times\Sigma)-$norm of the $\Sigma-$scaling function of type 6

$$\left\|\Phi^6_{\tau}\right\|_{\mathrm{L}^2(\Sigma\times\Sigma)} = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}\sum_{m=0}^{\infty}\sum_{j=1}^{2m+1}\left(\lambda^{\tau}_{n,k,m,j}\right)^2 . \tag{140}$$

Based on these auxiliary tools we are therefore able to formulate the following result.

THEOREM 6.15. *If the* $\Sigma-$*scaling function of type* 6, Φ^6_{τ}, *satisfies*

$$\left\|\Phi^6_{\tau}\right\|_{\mathrm{L}^2(\Sigma\times\Sigma)} \leq \left\|\Phi^6_{\tau'}\right\|_{\mathrm{L}^2(\Sigma\times\Sigma)}, \tag{141}$$

then the scale spaces $V^6_{\tau}(\Sigma)$ *of the* $\Sigma-$*scaling function of type* 6 *satisfy the inclusion*

$$V^6_{\tau}(\Sigma) \subset V^6_{\tau'}(\Sigma)\ . \tag{142}$$

Proof. Let a function G in $V^6_{\tau}(\Sigma)$ be given. Then there exists a function $F \in \mathrm{L}^2(\Sigma)$ such that $G = \int_{\Sigma}\Phi^6_{\tau}(\,.\,,y)F(y)\ d\omega(y)$. Translated in the spectral language in terms of the system $\left\{H^{\alpha}_{-n-1,k}(\Sigma,.)\right\}_{\substack{n=0,1,\ldots,\\ k=1,\ldots,2n+1}}$ this equivalently means that

$$G^{\wedge}(m,j) = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}F^{\wedge}(n,k)\lambda^{\tau}_{n,k,m,j} \tag{143}$$

for all $m = 0, 1, \ldots; j = 1, \ldots, 2m+1$. Thus we obtain

$$\begin{aligned}(G^\wedge(m,j))^2 &= \left(\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} F^\wedge(n,k)\lambda^\tau_{n,k,m,j}\right)^2 \\ &\leq \|F\|^2_{\mathrm{L}^2(\Sigma)} \sum_{n=1}^{\infty} \left(\lambda^\tau_{n,k,m,j}\right)^2 \end{aligned} \tag{144}$$

Keeping this result in mind we find the following relation:

$$G \in V^6_\tau(\Sigma) \Leftrightarrow \frac{\|G\|_{\mathrm{L}^2(\Sigma)}}{\|\Phi^6_\tau\|_{\mathrm{L}^2(\Sigma\times\Sigma)}} < \infty \ . \tag{145}$$

Observing the condition (141) it follows that

$$\frac{\|G\|_{\mathrm{L}^2(\Sigma)}}{\|\Phi^6_{\tau'}\|_{\mathrm{L}^2(\Sigma\times\Sigma)}} \leq \frac{\|G\|_{\mathrm{L}^2(\Sigma)}}{\|\Phi^6_\tau\|_{\mathrm{L}^2(\Sigma\times\Sigma)}} < \infty. \tag{146}$$

Using (145) we finally get the desired result $G \in V^6_{\tau'}(\Sigma)$. □

It is obvious that the multiresolution property for the scale spaces $V^5_\tau(\Sigma)$ for the non–spherical case can be proved in analogy to the considerations above.

6.6 A Tree Algorithm

Next we deal with some aspects of scientific computing (for a similar approach in spherical theory see [9]). We are interested in a *pyramid scheme* for the (approximate) recursive computation of the integrals $P^i_{\tau_j}(F)$ and $R^i_{\tau_j}(F)$ starting from an initial approximation of a given function $F \in \mathrm{L}^2(\Sigma)$. The tree algorithm (pyramid scheme) is based on the existence of a 'reproducing kernel function' on the regular surface Σ.
A pyramid scheme is a tree algorithm with the following ingredients. Starting from a sufficiently large $J \in \mathbb{N}$ such that for all $x \in \Sigma$

$$P^i_{\tau_J}(F) \simeq \sum_{k=1}^{N_J} a_k^{N_J} \Phi^i_{\tau_J}(x, y_k^{N_J}) \simeq \begin{cases} F(x) & , \quad i = 2,3,5,6 \\ 0 & , \quad i = 4,7 \\ P(0,0)F(x) & , \quad i = 1 \\ P_{|1}(0,0) & , \quad i = 8 \\ P_{|2}(0,0) & , \quad i = 9 \end{cases} \tag{147}$$

we want to calculate coefficients

$$a_{N_j} \in \mathbb{R}^{N_j}, a^{N_j} = \left(a_1^{N_j}, \ldots, a_{N_j}^{N_j}\right)^T, \quad j = J_0, \ldots, J,$$

such that the following statements hold true:

1. The vectors a^{N_j}, $j = J_0, \ldots, J-1$, are obtainable by recursion starting from the vector a^{N_J}.

2. For $j = J_0, \ldots, J$

$$P^i_{\tau_j}(F)(x) = \int_\Sigma \Phi^i_{\tau_j}(x,y)F(y)\ d\omega(y) \simeq \sum_{k=1}^{N_j} a_k^{N_j} \Phi^i_{\tau_j}\left(x, y_k^{N_j}\right) \quad .$$

For $j = J_0 + 1, \ldots, J$

$$R^i_{\tau_{j-1}}(F)(x) = \int_\Sigma \Psi^i_{\tau_{j-1}}(x,y)F(y)\ d\omega(y) \simeq \sum_{k=1}^{N_{j-1}} a_k^{N_{j-1}} \Psi^i_{\tau_{j-1}}\left(x, y_k^{N_{j-1}}\right) \quad .$$

(the symbol '$\simeq$' always means that the error is assumed to be negligible).

In the scheme we base the numerical integration on certain approximate formulae associated to known weights $w_k^{N_j} \in \mathbb{R}$ and prescribed knots $y_k^{N_j} \in \Sigma$, $j = J_0, \ldots, J$. This may be established, for example, by transforming the integrals over the regular surface Σ to integrals over the (unit) sphere Ω in case an explicit transformation $\Theta : \Omega \to \Sigma$ is given (see [4], [18]). Note that j denotes the scale of the discretized scaling function, N_j is the number of integration points to the accompanying scale j, and k denotes the index of the integration knot within the integration formulae under consideration, i.e.:

$$P^i_{\tau_j}(F)(x) \simeq \sum_{k=1}^{N_j} w_k^{N_j} F\left(y_k^{N_j}\right) \Phi^i_{\tau_j}\left(x, y_k^{N_j}\right), \tag{148}$$

$j = J_0, \ldots J$,

$$R^i_{\tau_{j-1}}(F)(x) \simeq \sum_{k=1}^{N_{j-1}} w_k^{N_{j-1}} F\left(y_k^{N_{j-1}}\right) \Psi^i_{\tau_{j-1}}\left(x, y_k^{N_{j-1}}\right), \tag{149}$$

$j = J_0 + 1, \ldots, J$.

The pyramid scheme – as every recursive implementation – is divided into two parts, the initial step and the recursion step, here called the pyramid step.

Initial Step. For a suitable large integer J, $P^i_{\tau_J}(x)$ is sufficiently close to the right hand side of (147) for all $x \in \Sigma$. Thus we simply get by (148)

$$a_k^{N_J} = w_k^{N_J} F(y_k^{N_J}), \quad k = 1, \ldots, N_J \quad . \tag{150}$$

Pyramid Step. The essential idea for the development of our recursive scheme is the existence of a (symmetric) kernel function $\Xi^i_j : \Sigma \times \Sigma \to \mathbb{R}$ such that

$$\Phi^i_{\tau_j}(x,y) \simeq \int_\Sigma \Phi^i_{\tau_j}(z,x)\Xi^i_j(y,z)\ d\omega(z) \tag{151}$$

and

$$\Xi^i_j(x,y) \simeq \int_\Sigma \Xi^i_j(z,x)\Xi^i_{j+1}(y,z)\ d\omega(z) \tag{152}$$

for $j = J_0, \ldots, J$.

Since our scaling functions are non–bandlimited, the scale spaces $V^i_{\tau_j}$ are infinite–dimensional. This leads us to choose the functions Ξ_j, for example, to be equal to

$$\Xi^i_j = \Phi^l_{\tau_{J+L}}, \quad j = J_0, \ldots, J;\ l \in \{2,3,5,6\}\ .$$

for suitable $L \in \mathbb{N}_0$. By virtue of the approximate integration rules on the sphere we thus get

$$\begin{aligned}
\int_\Sigma \Phi^i_{\tau_j}(\cdot, y) F(y)\ d\omega(y) &\simeq \int_\Sigma \Xi^i_j(y,z) \int_\Sigma \Phi^i_{\tau_j}(\cdot, z) F(y)\ d\omega(z)\ d\omega(y) \\
&\simeq \int_\Sigma \Phi^i_{\tau_j}(\cdot, z) \int_\Sigma \Xi^i_j(y,z) F(y)\ d\omega(y)\ d\omega(z) \\
&\simeq \sum_{k=1}^{N_j} a_k^{N_j} \Phi^i_{\tau_j}(\cdot, y_k^{N_j}) \qquad (154)
\end{aligned}$$

for $j = J_0, \ldots, J-1$, where we have set

$$a_k^{N_j} = w_k^{N_j} \int_\Sigma \Xi^i_j \left(y_k^{N_j}, y \right) F(y)\ d\omega(y) \qquad (155)$$

for $j = J_0, \ldots, J-1$ and $k = 1, \ldots, N_j$. Hence, in connection with (153), we find

$$\begin{aligned}
a_k^{N_j} &= w_k^{N_j} \int_\Sigma \Xi^i_j \left(y_k^{N_j}, y \right) F(y)\ d\omega(y) \qquad (156) \\
&\simeq w_k^{N_j} \int_\Sigma \int_\Sigma \Xi^i_{j+1}(z,y) \Xi^i_j(y_k^{N_j}, z)\ d\omega(z) F(y)\ d\omega(y) \\
&\simeq w_k^{N_j} \sum_{l=1}^{N_{j+1}} w_l^{N_{j+1}} \Xi^i_j \left(y_k^{N_j}, y_l^{N_{j+1}} \right) \int_\Sigma \Xi^i_{j+1} \left(y_l^{N_{j+1}}, y \right) F(y)\ d\omega(y) \\
&= w_k^{N_j} \sum_{l=1}^{N_{j+1}} w_l^{N_{j+1}} \Xi_j \left(y_k^{N_j}, y_l^{N_{j+1}} \right) a_l^{N_{j+1}} \ .
\end{aligned}$$

for $j = J-1, \ldots, J_0$ and $k = 1, \ldots, N_j$.

We see that the coefficients $a_k^{N_{J-1}}$ can be calculated recursively from $a_l^{N_J}$ for the initial level $J, a_k^{N_{J-2}}$ can be deduced from $a_l^{N_{J-1}}$, etc. Finally, we get as a reconstruction scheme

$$\begin{aligned}
P^i_{\tau_j}(F) &\simeq \sum_{k=1}^{N_j} a_k^{N_j} \Phi^i_{\tau_j} \left(\cdot, y_k^{N_j} \right), \quad j = J_0, \ldots, J, \qquad (157) \\
R^i_{\tau_{j-1}}(F) &\simeq \sum_{k=1}^{N_{j-1}} a_k^{N_{j-1}} \Psi^i_{\tau_{j-1}} \left(\cdot, y_k^{N_{j-1}} \right), \quad j = J_0+1, \ldots, J\ . \qquad (158)
\end{aligned}$$

Note that the coefficients a^{N_J} in the initial step do not depend on the choice of $\Xi^i_J = \Phi^l_{\tau_{J+L}}$. Furthermore, the functions Ξ^i_j, $j = J_0, \ldots, J_{-1}$, can be chosen independently of the scaling function $\{\Phi^i_{\tau_j}\}_{j \in \mathbb{Z}}$ used in (157) and (158).

Table 1: Pyramid Scheme (Tree Algorithm)

Initial step: For J sufficiently large

$$a_k^{N_J} = w_k^{N_J}\ F(y_k^{N_J}), \quad k = 1, \ldots, N_J$$

Pyramid step: For $j = J_0, \ldots, J-1$ and $k = 1, \ldots, N_j$

$$a_k^{N_j} = w_k^{N_j} \sum_{l=1}^{N_{j+1}} \Xi^i_j \left(y_k^{N_j}, y_l^{N_{j+1}} \right) a_l^{N_{j+1}}$$

In conclusion, the above considerations lead us to the following decomposition and reconstruction scheme:

$$\begin{array}{ccccccccccc} F & \to & a^{N_J} & \to & a^{N_{J-1}} & \to & \ldots & \to & a^{N_{J_0+1}} & \to & a^{N_{J_0}} \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ & & P^i_{\tau_J}(F) & & P^i_{\tau_{J-1}}(F) & & & & P^i_{\tau_{J_0+1}}(F) & & P^i_{\tau_{J_0}}(F) \end{array}$$

(decomposition scheme)

$$\begin{array}{ccccccccc} a^{N_{J_0}} & & & & a^{N_{J_0+1}} & & & & \\ \downarrow & & & & \downarrow & & & & \\ P^i_{\tau_{J_0}}(F) & & & & P^i_{\tau_{J_0+1}}(F) & & & & \\ & \searrow & & & & \searrow & & & \\ R^i_{\tau_{J_0}}(F) & \to & + & \to & R^i_{\tau_{J_0+1}} & \to & + & \ldots & \end{array}$$

$$\begin{array}{ccccccccc} & & & a^{N_{J-1}} & & & & \\ & & & \downarrow & & & & \\ & & & P^i_{\tau_{J-1}}(F) & & & & \\ & & & & \searrow & & & \\ \ldots & + & \to & R^i_{\tau_{J-1}}(F) & \to & + & \to & P_{\tau_J}(F) \end{array}$$

(reconstruction scheme).

The numerical effort of a pyramid step can drastically be reduced by use of a panel–clustering method (e.g. fast multipole procedures as developed by [10]). In doing so, the evaluations take advantage of the localizing structure of the kernels Ξ^i_j. Roughly spoken, the kernel is split into a near field and a far field component. The far field component is approximated by a certain expression obtaining the 'low frequency contributions'. For the points close to the evaluation position the evaluation uses the exact near field of the kernel. For the remaining points, the approximate far field contributions are put together.

The numerical results (based on the types $i = 5, 6$) presented in the Diploma thesis [18] illustrate the efficiency and economy in applications of our wavelet method for different types of regular surfaces (e.g. sphere, ellipsoid, Cassini's surfaces, etc). Figure 5 demonstrates the functionality of the multiscale analysis. The mechanism is, for example, as follows: To a scale–reconstruction at scale $J = 5$ the detail–structure at scale $J = 5$ has to be added to get the scale–reconstruction at scale $J = 6$. This can be done globally as shown in Figure 5 or locally as shown in Figure 4 without getting any oscillations because of the space–localizing properties of the scaling functions.
Finally three important applications of wavelet decomposition and reconstruction should be mentioned:

(i) The 'zoom–in' property allows a local high–scale reconstruction of fine structure based on global data. For the evaluation of a functional value under consideration, only wavelet coefficients close to the point have to be taken into account. This aspect of functional evaluation enables us to derive local features within a global model. This is demonstrated in Figure 4 by a reconstruction of the EGM96–geopotential model [16] from discrete data in local areas (for example, South–America).

(ii) Data compression techniques reduce storage requirements and speed up read or write operations. The loss of information emerged by data compression is shown in Figure 6.

(iii) In Figure 7 the detection of a high frequency phenomena is demonstrated. We added within the EGM96–model a mass point lying $63km$ under the (spherical) Earth's surface and at 80^o West and 30^o South to the EGM96–geopotential model. It is well known that phenomena with such short wavelengths cannot be detected with the spherical harmonic techniques known in the literature.

In conclusion, as mentioned in our Introduction, three esssential features are incorporated in this way of thinking about wavelets generated by layer potentials, namely the basis property, the zoom–in ability, and fast computation. In particular, these facts justify the characterization of our wavelets as 'building blocks' that enable fast decorrelation of data given on a regular surface.

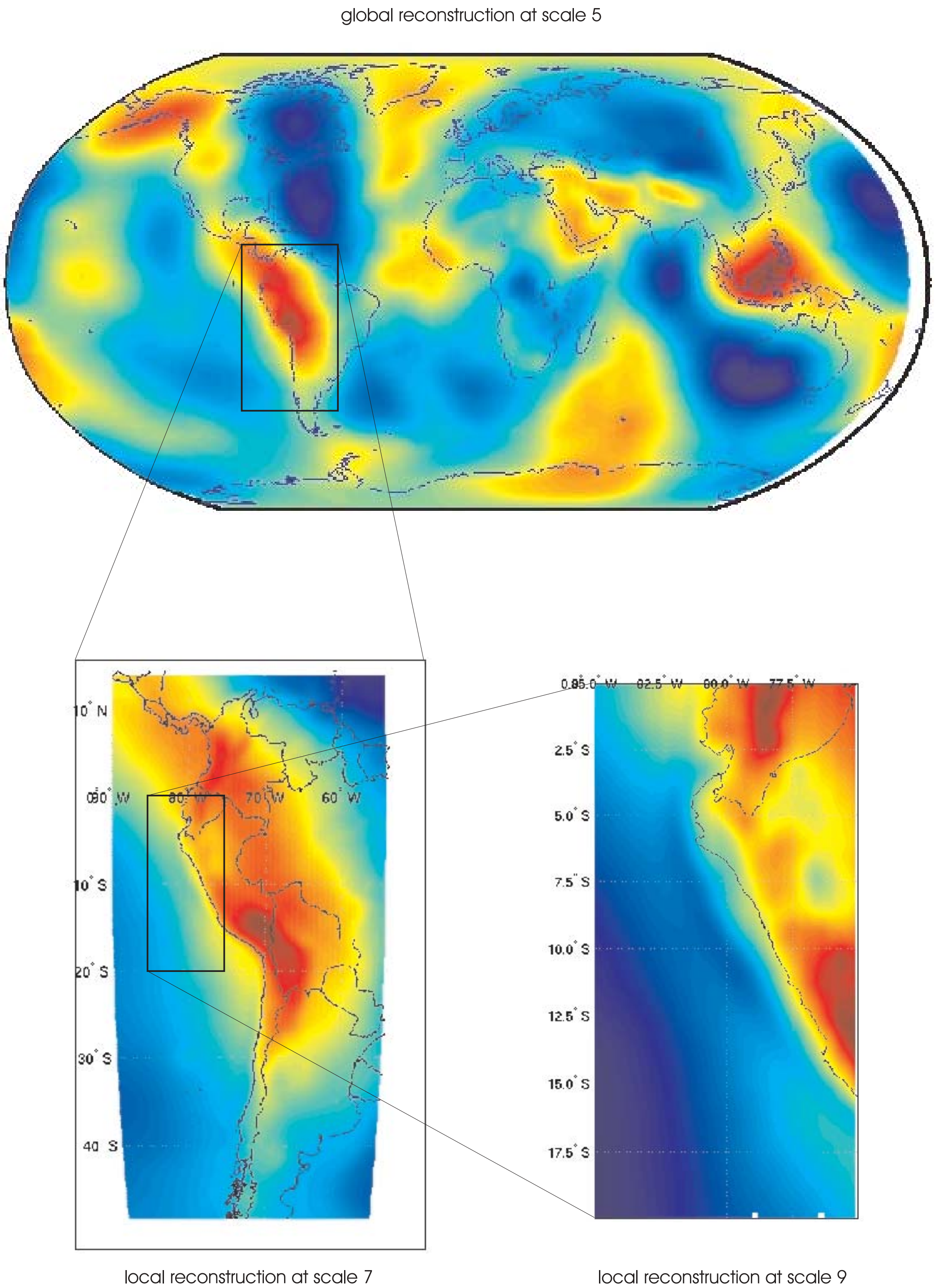
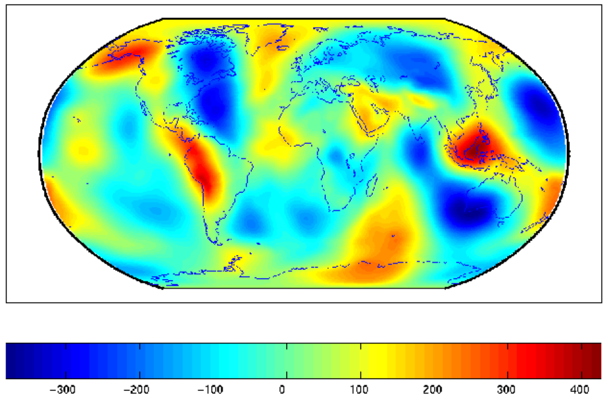
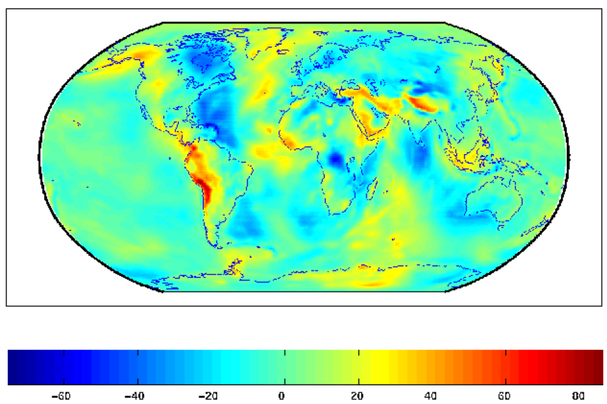


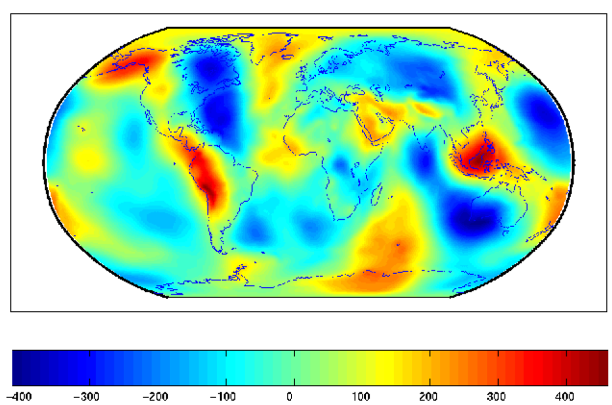
Figure 4: Illustration of the zoom–in property. In order to reconstruct a function on a local area, only data in a certain neighborhood of this area are used. Since global high–scale reconstruction of fine structure is very time–consuming, only the area of interest is reconstructed which can be done with a considerably fewer effort.



(a) global scale–reconstruction at scale $J = 5$

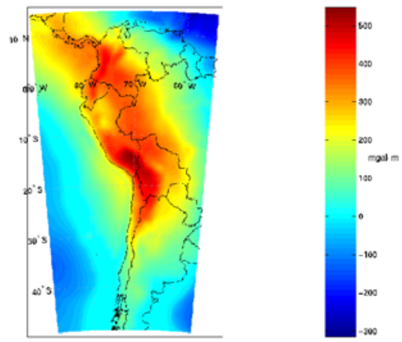


(b) global detail–reconstruction at scale $J = 5$

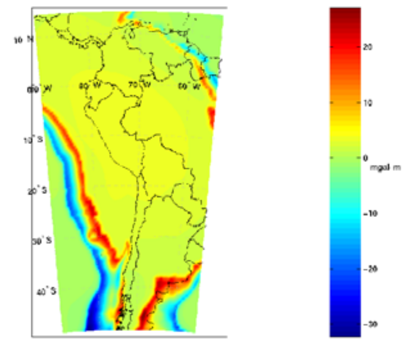


(c) global scale–reconstruction at scale $J = 6$

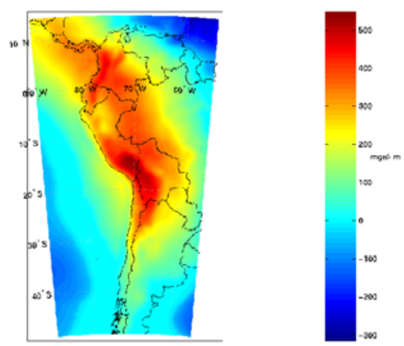
Figure 5: Scale–reconstruction at scale $J = 6$ (c) consists of detail–reconstruction at scale $J = 5$ (b) added to scale–reconstruction at scale $J = 5$ (a).



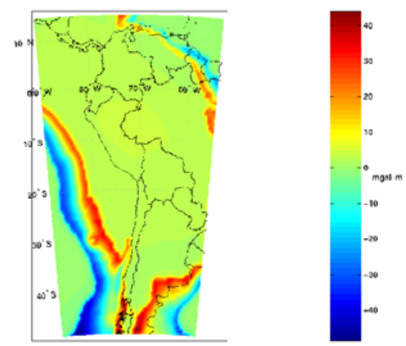
(a) reconstruction at scale $J = 8$ with compression rate of 19%



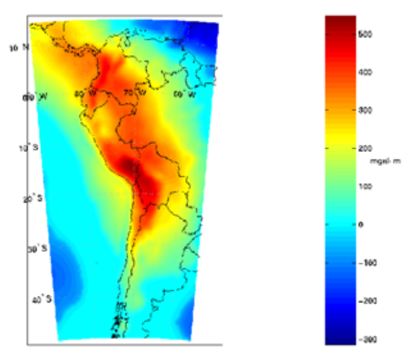
(b) error of reconstruction with compression rate of 19%



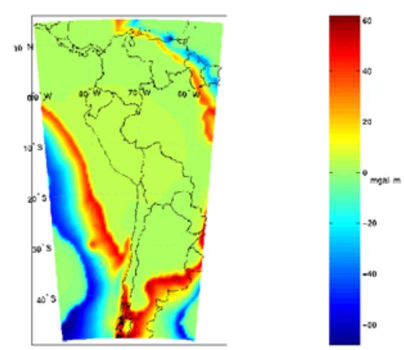
(c) reconstruction at scale $J = 8$ with compression rate of 25%



(d) error of reconstruction with compression rate of 25%

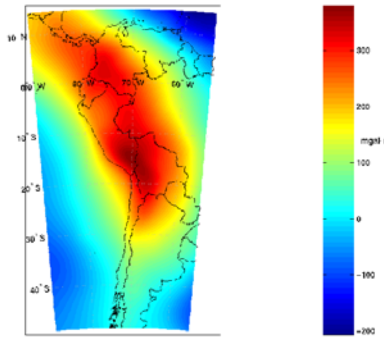


(e) reconstruction at scale $J = 8$ with compression rate of 34%

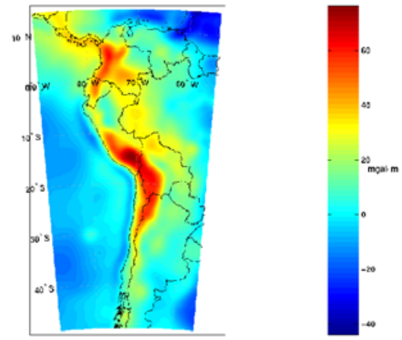


(f) error of reconstruction with compression rate of 34%

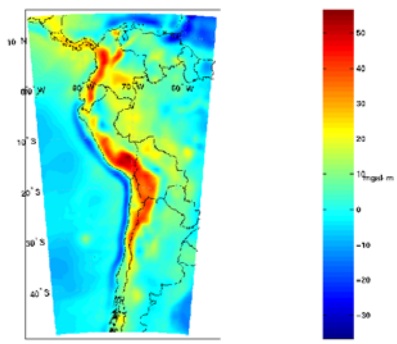
Figure 6: Local reconstruction with $\Sigma-$scaling function $\Phi^6_{\tau_j}$ at scale $j = 8$ with compressed rate of wavelet coefficients after setting a certain treshold. The compression rate indicates the percentage of neglected wavelet coefficients.



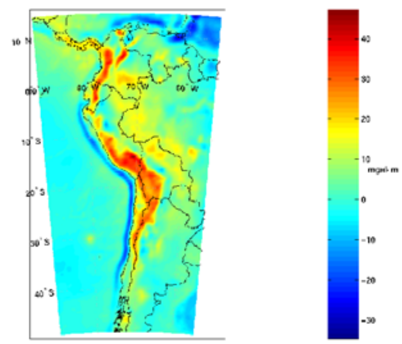
(a) local reconstruction with Σ−scaling function $\Phi^6_{\tau_j}$ at scale $j = 5$



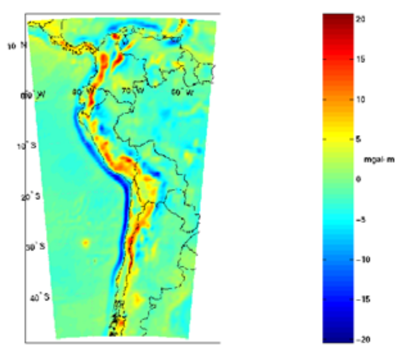
(b) local reconstruction with Σ−wavelet function $\Psi^6_{\tau_j}$ at scale $j = 5$



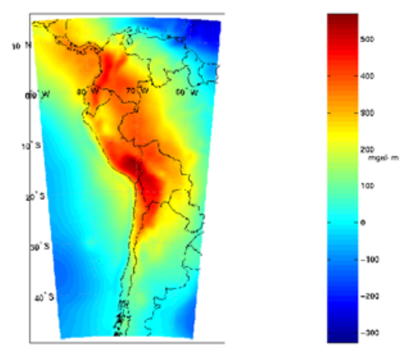
(c) local reconstruction with Σ−wavelet function $\Psi^6_{\tau_j}$ at scale $j = 6$



(d) local reconstruction with Σ−wavelet function $\Psi^6_{\tau_j}$ at scale $j = 7$



(e) local reconstruction with Σ−scaling function $\Psi^6_{\tau_j}$ at scale $j = 8$



(f) local reconstruction with Σ−scaling function $\Phi^6_{\tau_j}$ at scale $j = 5$

Figure 7: Detection of high frequency perturbation within a local area of the EGM96–geopotential model. The buried mass point at 80^o West, 30^o South is clearly detected, especially in the wavelet reconstruction at scale 8.

7 Multiscale Modelling of Boundary-Value Problems

The classical problem of solving a boundary-value problem for the Laplace equation $\Delta U = 0$ from given data on a regular boundary Σ arises in many applications (for example, geophysics, mechanics, electromagnetism, etc). Of particular importance is the Dirichlet (resp. Neumann) boundary-value problem, i.e., the determination of U from given potential values (resp. normal derivatives) on the boundary. Finding the solution in the exterior space of a regular boundary (such as e.g. sphere, ellipsoid, geoid, (actual) Earth's surface) is of importance in all Earth's sciences:

7.1 Formulation and Well–posedness

We begin with the formulation of the boundary–value problems.

Exterior Dirichlet Problem (EDP): Given $F \in \mathrm{C}^{(0)}(\Sigma)$, find a function $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ such that

$$U^+(x) = \lim_{\substack{\tau \to 0 \\ \tau > 0}} U(x + \tau\nu(x)) = F(x), \quad x \in \Sigma.$$

Exterior Neumann Problem (ENP): Given a function $F \in \mathrm{C}^{(0)}(\Sigma)$, find $U \in \mathrm{Pot}^{(1)}\ (\overline{\Sigma_{\mathrm{ext}}})$ such that

$$\frac{\partial U^+}{\partial \nu_\Sigma}(x) = \lim_{\substack{\tau \to 0 \\ \tau > 0}} \nu(x) \cdot (\nabla U)(x + \tau\nu(x)) = F(x), \quad x \in \Sigma.$$

Existence and Uniqueness: We recall the role of layer potentials in the aforementioned boundary-value problems:

(EDP) Let D^+ (more accurately, D^+_Σ) denote the following set:

$$D^+ = \{U^+ | U \in \mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})\}. \tag{159}$$

The solution of (EDP) is uniquely determined, hence, $\mathrm{C}^{(0)}(\Sigma) = D^+$. It can be formulated in terms of a potential of the form

$$U(x) = \int_\Sigma Q(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|}\ d\omega(y) + \frac{1}{|x|} \int_\Sigma Q(y)\ d\omega(y), \tag{160}$$

where $Q \in \mathcal{C}^{(0)}(\Sigma)$ satisfies the integral equation

$$F = U^+ = (2\pi I + P + P_{|2}(0,0))Q. \tag{161}$$

with

$$P(Q) : x \mapsto \frac{1}{|x|} \int_\Sigma Q(y)\ d\omega(y). \tag{162}$$

Setting
$$T = 2\pi I + P + P_{|2}(0,0) \tag{163}$$
we obtain
$$\mathrm{kern}(T^*) = \{0\}, \tag{164}$$
$$T(\mathrm{C}^{(0)}(\Sigma)) = D^+. \tag{165}$$
By completion,
$$\mathrm{L}^2(\Sigma) = \overline{D^+}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = \overline{\mathrm{C}^{(0)}(\Sigma)}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}}. \tag{166}$$
(ENP) Let N^+ (more accurately, N^+_Σ) denote the following set:
$$N^+ = \left\{ \frac{\partial U^+}{\partial \nu_\Sigma} \Big| \ U \in \mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}}) \right\}. \tag{167}$$
The solution of (*ENP*) is uniquely determined, hence, $\mathrm{C}^{(0)}(\Sigma) = N^+$. It can be formulated in terms of a single layer potential
$$U(x) = \int_\Sigma Q(y) \frac{1}{|x-y|} \ d\omega(y), \quad Q \in \mathrm{C}^{(0)}(\Sigma), \tag{168}$$
where Q satisfies the integral equations
$$F = \frac{\partial U^+}{\partial \nu_\Sigma} = (-2\pi I + P_{|1}(0,0))Q. \tag{169}$$
Setting
$$T = (-2\pi I + P_{|1}(0,0)) \tag{170}$$
we obtain
$$\mathrm{kern}\ (T^*) = \{0\}, \tag{171}$$
$$T(\mathrm{C}^{(0)}(\Sigma)) = N^+. \tag{172}$$
By completion,
$$\mathrm{L}^2(\Sigma) = \overline{N^+}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}}. \tag{173}$$
Analogous arguments, of course, hold for the inner boundary-value problems. The details are left to the reader. A more comprehensive treatment of classical potential theory may be found in standard textbooks, e.g. [11], [13], [17], [23].

7.2 Regularity Theorems

From the maximum/minimum principle of potential theory we already know that
$$\sup_{x \in \overline{\Sigma_{\mathrm{ext}}}} |U(x)| \le \sup_{x \in \Sigma} \left|U^+(x)\right| \tag{174}$$

holds for $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$. Moreover, from the theory of integral equations it can be easily detected (see e.g. [19]) that there exists a constant C (dependent on Σ) such that for $U \in \mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$

$$\sup_{x\in\overline{\Sigma_{\mathrm{ext}}}} |U(x)| \leq \mathrm{C} \sup_{x\in\Sigma} \left| \frac{\partial U^+}{\partial \nu}(x) \right| \tag{175}$$

In what follows we want to verify analogous *regularity theorems* in the $\mathrm{L}^2(\Sigma)$-context.

THEOREM 7.1. *Let U be of class* $\mathrm{Pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)$. *Then, for every (sufficiently small) $\rho > 0$, there exists a constant* $\mathrm{C}(=\mathrm{C}(k;K,\Sigma))$ *such that*

$$\sup_{x\in K} \left| \left(\nabla^{(k)} U\right)(x) \right| \leq C \left(\int_\Sigma \left| U^+(x)\right|^2 \; d\omega(x) \right)^{1/2} \tag{176}$$

for all $K \subset \Sigma_{\mathrm{ext}}$ with $\mathrm{dist}\,(\overline{K},\Sigma) \geq \rho$ *and for all $k \in \mathbb{N}_0$ (where $\nabla^{(0)}U = U$ and $\nabla^{(k)}U$ is the gradient of $\nabla^{(k-1)}U$).*

Proof. Recall that the exterior Dirichlet problem (*EDP*) can be solved by (160), (161). The operator T defined by (163) and its adjoint operator T^* with respect to $||\cdot||_{\mathrm{L}^2(\Sigma)}$ are bijective in the Banach space $(\mathrm{C}^{(0)}(\Sigma), ||\cdot||_{\mathrm{C}^{(0)}(\Sigma)})$ (see e.g. [19]). By virtue of the open mapping theorem (see e.g. [24]) the operators T and T^{-1} are linear and bounded with respect to $||\cdot||_{\mathrm{C}^{(0)}(\Sigma)}$. Furthermore, $(T^*)^{-1} = (T^{-1})^*$. Therefore, by virtue of the technique due to P. Lax (1954) (cf. Theorem 4.1), T and its inverse operator T^{-1} are bounded with respect to $||\cdot||_{\mathrm{L}^2(\Sigma)}$.
Now, for all sufficiently small values $\rho > 0$ and all points $x \in K \subset \Sigma_{\mathrm{int}}$ with $\mathrm{dist}(\overline{K},\Sigma) \geq \rho$, the Cauchy-Schwarz inequality gives

$$\begin{aligned} \left| \left(\nabla^{(k)} U\right)(x) \right| &= \left| \int_\Sigma Q(y) \left(\nabla_x^{(k)} \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} + \frac{1}{|x|} \right) d\omega(y) \right. \\ &\leq \left(\int_\Sigma \left| \nabla_x^{(k)} \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} + \frac{1}{|x|} \right|^2 d\omega(y) \right)^{\frac{1}{2}} \\ &\quad \left(\int_\Sigma |Q(y)|^2 \; d\omega(y) \right)^{\frac{1}{2}}. \end{aligned} \tag{177}$$

This shows us that

$$\sup_{x\in K} \left| \left(\nabla^{(k)} U\right)(x) \right| \leq D \left(\int_\Sigma |Q(y)|^2 \, d\omega(y) \right)^{\frac{1}{2}}, \tag{178}$$

where we have used the abbreviation

$$D = \sup_{x\in K} \left(\int_\Sigma \left| \nabla_x^{(k)} \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} + \frac{1}{|x|} \right|^2 d\omega(y) \right)^{\frac{1}{2}}. \tag{179}$$

However,

$$\sup_{x\in K} |\left(\nabla^{(k)}U\right)(x)| \leq D\ \left(\int_\Sigma \left|T^{-1}(F)(y)\right|^2\ d\omega(y)\right)^{\frac{1}{2}}. \tag{180}$$

Because of the boundedness of T^{-1} with respect to $||\cdot||_{\mathrm{L}^2(\Sigma)}$ this tells us with $\mathrm{C} = D\|T^{-1}\|_{\mathrm{L}^2(\Sigma)}$ that

$$\sup_{x\in K} |\left(\nabla^{(k)}U\right)(x)| \leq \mathrm{C}\ \left(\int_\Sigma |F(y)|^2\ d\omega(y)\right)^{\frac{1}{2}}. \tag{181}$$

Hence, the statement (Theorem 7.1) is true. □

An analogous argument yields the following theorem.

THEOREM 7.2. *Let U be of class $\mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$. Then, for every (sufficiently small) $\rho > 0$, there exists a constant $\mathrm{C}(= \mathrm{C}(k;K,\Sigma))$ such that*

$$\sup_{x\in K} \left|\left(\nabla^{(k)}U\right)(x)\right| \leq \mathrm{C}\left(\int_\Sigma \left|\frac{\partial U^+}{\partial\nu}(x)\right|^2\ d\omega(x)\right)^{1/2} \tag{182}$$

for all $K \subset \Sigma_{\mathrm{ext}}$ with dist $(\overline{K},\Sigma) \geq \rho > 0$ and for all $k \in \mathbb{N}_0$.

7.3 Solution by Outer Harmonics

Combining the L^2-closure (Corollary 5.3) for the system of outer harmonics and the regularity theorems (Theorem 7.1) we first obtain the following results.

THEOREM 7.3. *Let Σ be a regular surface satisfying the condition (10).*

(EDP) For given $F \in \mathrm{C}^{(0)}(\Sigma)$, let U be the potential of class $\mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ with $U^+ = F$. Then, for any given value $\varepsilon > 0$ and $K \subset \Sigma_{\mathrm{ext}}$ with $\mathrm{dist}(\overline{K},\Sigma) > 0$ there exist an integer m (dependent on ε) and a set of coefficients $a_{0,1},\dots,a_{m,1},\dots,a_{m,2m+1}$ such that

$$\left(\int_\Sigma \left|F(x) - \sum_{n=0}^{m}\sum_{j=1}^{2n+1} a_{n,j}H^\alpha_{-n-1,j}(x)\right|^2\ d\omega(x)\right)^{\frac{1}{2}} \leq \varepsilon$$

and

$$\sup_{x\in\overline{K}}\left|\left(\nabla^{(k)}U\right)(x) - \sum_{n=0}^{m}\sum_{j=1}^{2n+1} a_{n,j}\left(\nabla^{(k)}H^\alpha_{-n-1,j}\right)(x)\right| \leq \mathrm{C}\varepsilon$$

hold for all $k \in \mathbb{N}_0$.

(ENP) For given $F \in \mathrm{C}^{(0)}(\Sigma)$, let U satisfy $U \in \mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$, $\partial U^+/\partial\nu_\Sigma = F$. Then, for any given value $\varepsilon > 0$ and $K \subset \Sigma_{\mathrm{ext}}$ with $\mathrm{dist}(\overline{K},\Sigma) > 0$ there exist an integer m (dependent on ε) and a set of coefficients $a_{0,1},\ldots,a_{m,1},\ldots,a_{m,2m+1}$ such that

$$\left(\int_\Sigma \left| F(x) - \sum_{n=0}^{m}\sum_{j=1}^{2n+1} a_{n,j}\frac{\partial H^\alpha_{-n-1,j}}{\partial\nu}(x)\right|^2 d\omega(x)\right)^{\frac{1}{2}} \leq \varepsilon$$

and

$$\sup_{x\in\overline{K}}\left|\left(\nabla^{(k)}U\right)(x) - \sum_{n=0}^{m}\sum_{j=1}^{2n+1} a_{n,j}\left(\nabla^{(k)}H^\alpha_{-n-1,j}\right)(x)\right| \leq \mathrm{C}\varepsilon$$

hold for all $k \in \mathbb{N}_0$.

In other words, locally uniform approximation is guaranteed in terms of outer harmonics, i.e. the L^2- approximation in terms of outer harmonics on Σ implies the uniform approximation (in ordinary sense) on each subset K with positive distance of $\overline{K}$ to Σ.

Unfortunately, the theorems developed until now are non-constructive, since further information about the choice of m and the coefficients of the approximating linear combination is needed. In order to derive a constructive approximation theorem the system of potential values and normal derivatives, respectively, has to be orthonormalized on Σ. As result we obtain a *orthogonal Fourier approximation* that shows locally uniform approximation.

THEOREM 7.4. *Let Σ be a regular surface such that (10) holds true.*

(EDP) For given $F \in \mathrm{C}^{(0)}(\Sigma)$, let U satisfy $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}}), U^+ = F$. Corresponding to the countably infinite sequence $(H^\alpha_{-n-1,j})$ there exists a system $(H_{-n-1,j}(\Sigma;\cdot)) \in \mathrm{Pot}^{(0)}(\overline{A_{\mathrm{ext}}})$ such that $(H_{-n-1,j}(\Sigma;\cdot)|\Sigma)$ is orthonormal in the sense that

$$\int_\Sigma H_{-n-1,j}(\Sigma;y)H_{-l-1,k}(\Sigma;y)\ d\omega(y) = \delta_{nl}\ \delta_{jk}.$$

Consequently, U is representable in the form

$$U(x) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\left(\int_\Sigma F(y)H_{-n-1,j}(\Sigma;y)\ d\omega(y)\right) H_{-n-1,j}(\Sigma;x)$$

for all points $x \in K$ with $K \subset \Sigma_{\mathrm{ext}}$ and $\mathrm{dist}(\overline{K},\Sigma) > 0$. Moreover, for each $U^{(m)}$ given by

$$U^{(m)}(x) = \sum_{n=0}^{m}\sum_{j=1}^{2n+1}\left(\int_\Sigma F(y)H_{-n-1,j}(\Sigma;y)\ d\omega(y)\right) H_{-n-1,j}(\Sigma;x)$$

we have the estimate

$$\sup_{x\in\overline{K}} \left| \left(\nabla^{(k)} U\right)(x) - \left(\nabla^{(k)} U^{(m)}\right)(x) \right|$$

$$\leq \mathrm{C} \left(\int_\Sigma |F(y)|^2 \; d\omega(y) - \sum_{n=0}^{m} \sum_{j=1}^{2n+1} \left| \int_\Sigma F(y) H_{-n-1,j}(\Sigma; y) \; d\omega(y) \right|^2 \right)^{\frac{1}{2}} .$$

(ENP) For given $F \in \mathrm{C}^{(0)}(\Sigma)$*, let* U *satisfy* $U \in \mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$, $\frac{\partial U^+}{\partial \nu_\Sigma} = F$. *Corresponding to the countably infinite sequence* $(H^{\alpha}_{-n-1,j})$ *there exists a system* $(H_{-n-1,j}(\Sigma;\cdot)) \subset \mathrm{Pot}^{(0)}(\overline{A_{\mathrm{ext}}})$ *such that* $(\partial H_{-n-1,j}(\Sigma;\cdot)/\partial\nu_\Sigma)$ *is orthonormal in the sense that*

$$\int_\Sigma \frac{\partial H_{-n-1,j}(\Sigma; y)}{\partial \nu} \frac{\partial H_{-l-1,k}(\Sigma; y)}{\partial \nu} \; d\omega(y) = \delta_{nl} \; \delta_{jk}.$$

Consequently U *is representable in the form*

$$U(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\int_\Sigma F(y) \frac{\partial H_{-n-1,j}(\Sigma; y)}{\partial \nu} \; d\omega(y) \right) H_{-n-1,j}(\Sigma; x)$$

for all points $x \in K$ *with* $K \subset \Sigma_{\mathrm{ext}}$ *and* $\mathrm{dist}(\overline{K}, \Sigma) > 0$. *Moreover, for each* $U^{(m)}$ *given by*

$$U^{(m)}(x) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} \left(\int_\Sigma F(y) \frac{\partial H_{-n-1,j}(\Sigma; y)}{\partial \nu} \; d\omega(y) \right) H_{-n-1,j}(\Sigma; x)$$

we have the estimate

$$\sup_{x\in\overline{K}} \left| \left(\nabla^{(k)} U\right)(x) - \left(\nabla^{(k)} U^{(m)}\right)(x) \right|$$

$$\leq \mathrm{C} \left(\int_\Sigma |F(y)|^2 \; d\omega(y) - \sum_{n=0}^{m} \sum_{j=1}^{2n+1} \left| \int_\Sigma F(y) \frac{\partial H_{-n-1,j}(\Sigma; y)}{\partial \nu} \; d\omega(y) \right|^2 \right)^{\frac{1}{2}} .$$

Note that the orthonormalization procedure can be performed (e.g. by the well–known Gram–Schmidt orthonormalization process) once and for all when the regular surface Σ is specified.

Clearly, in the same way, the inner boundary-value problems can be formulated by generalized Fourier expansions (orthogonal expansions) in terms of inner harmonics. Furthermore, locally uniform approximation by ‘generalized Fourier expansions’ can be obtained not only for (the multipole system of inner/outer) harmonics, but also for the mass point and related kernel representations. The details are omitted.

7.4 Solution by Wavelets

(EDP) For given $F \in \mathrm{C}^{(0)}(\Sigma)$, the solution $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ with $U^+ = F$ of the exterior Dirichlet problem (EDP) can be written as layer potential (160), where the double layer $Q \in \mathcal{C}^{(0)}(\Sigma)$ satisfies the integral equation

$$2\pi Q(x) + \frac{1}{|x|}\int_\Sigma Q(y)\ d\omega(y) + \int_\Sigma \Phi^9_{\tau_L}(x,y)Q(y)\ d\omega(y) \simeq F(x) \tag{183}$$

for all $x \in \Sigma$ ($L \in \mathbb{N}$ sufficiently large). An approximation of scale J

$$\hat{P}^i_{\tau_J}(Q)(x) = \sum_{l=1}^{N_J} \hat{a}_l^{N_J} \Phi^i_{\tau_J}\left(x, \hat{y}_l^{N_J}\right), \quad x \in \Sigma \tag{184}$$

(with $i \in \{2,3,5,6\}, \hat{a}_l^{N_J} \in \mathbb{R}, \hat{y}_l^{N_J} \in \Sigma, l = 1,\ldots,N_J$ and $J, N_J \in \mathbb{N}$ sufficiently large) is deducable from (183) by solving a system of linear equations obtained by an appropriate approximation method such as collocation, Galerkin procedure, least squares approximation, etc .

(ENP) For given $F \in \mathrm{C}^{(0)}(\Sigma)$, the solution $U \in \mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$ with $\frac{\partial U^+}{\partial \nu_\Sigma} = F$ of the exterior Dirichlet problem (ENP) can be written as layer potential (168), where the single layer $Q \in \mathcal{C}^{(0)}(\Sigma)$ satisfies the integral equation

$$-2\pi Q(x) + \int_\Sigma \Phi^8_{\tau_L}(x,y)Q(y)\ d\omega(y) \simeq F(x) \tag{185}$$

for all $x \in \Sigma$ ($L \in \mathbb{N}$ sufficiently large). An approximation of scale J

$$\hat{P}^i_{\tau_J}(Q)(x) = \sum_{l=1}^{N_J} \hat{a}_l^{N_J} \Phi^i_{\tau_J}\left(x, \hat{y}_l^{N_J}\right), \quad x \in \Sigma \tag{186}$$

(with $i \in \{2,3,5,6\}$, $\hat{a}_l^{N_J} \in \mathbb{R}$, $\hat{y}_l^{N_J} \in \Sigma$, $l = 1,\ldots,N_J$ and $J, N_J \in \mathbb{N}$ sufficiently large) is deducable from (185) by solving a sytem of linear equations obtained by an appropriate approximation method such as collocation, Galerkin procedure, least squares approximation, etc .
For solving the linear systems fast multipole methods (FMM) are applicable (see e.g. [10]). The aforementioned observations concerning the exterior boundary value problems of potential theory lead us to tree algorithms with the following ingredients:
Starting from $\hat{a}^{N_J} \in \mathbb{R}^{N_J}, \hat{a}^{N_J} = \left(\hat{a}_1^{N_J}, \ldots, \hat{a}_{N_J}^{N_J}\right)^T$, the coefficients

$$\hat{a}^{N_j} \in \mathbb{R}^{N_j}, \hat{a}^{N_j} = \left(\hat{a}_1^{N_j}, \ldots, \hat{a}_{N_j}^{N_j}\right)^T, j = J_0, \ldots, J-1, \tag{187}$$

are determined such that the following rules hold true:

1. The vectors $\hat{a}^{N_j}, j = J_0, \ldots, J-1$ are given by recursion (see Section 6.6)

$$\hat{a}_k^{N_j} = w_k^{N_j} \sum_{l=1}^{N_{j+1}} \Xi_j^i \left(\hat{y}_k^{N_j}, y_l^{N_{j+1}}\right) \hat{a}_l^{N_{j+1}}. \tag{188}$$

2. For $j = J_0, \ldots J$

$$\hat{P}_{\tau_j}^i(Q)(x) \simeq \sum_{k=1}^{N_j} \hat{a}_k^{N_j} \Phi_{\tau_j}^i \left(x, \hat{y}_k^{N_j}\right), \quad x \in \Sigma \ . \tag{189}$$

For $j = J_0 + 1, \ldots J$

$$\hat{R}_{\tau_{j-1}}^i(Q)(x) \simeq \sum_{k=1}^{N_j} \hat{a}_k^{N_j} \Psi_{\tau_{j-1}}^i \left(x, \hat{y}_k^{N_j}\right), \quad x \in \Sigma \tag{190}$$

where

$$\hat{R}_{\tau_{j-1}}^i(Q)(x) = \hat{P}_{\tau_j}^i(Q)(x) - \hat{P}_{\tau_{j-1}}^i(Q)(x) \ . \tag{191}$$

THEOREM 7.5. *Let Σ be a regular surface such that* (10) *holds true.*
(EDP) For given $F \in \mathcal{C}^{(0)}(\Sigma)$, let U be the potential of class $\mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ *with $U^+ = F$. The function $F_J \in \mathcal{C}^{(0)}(\Sigma)$ given by*

$$\begin{aligned} F_J(x) & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (192) \\
= \ & 2\pi \sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \Phi_{\tau_{J_0}}^i \left(x, \hat{y}_l^{N_{J_0}}\right) + \frac{1}{|x|} \sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \int_\Sigma \Phi_{\tau_{J_0}}^i \left(y, \hat{y}_l^{N_{J_0}}\right) d\omega(y) \\
+ \ & \sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \int_\Sigma \Phi_{\tau_L}^9(x,y) \Phi_{\tau_{J_0}}^i \left(y, \hat{y}_l^{N_{J_0}}\right) d\omega(y) \\
+ \ & \sum_{j=J_0}^{J-1} \left(2\pi \sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \Psi_{\tau_j}^i \left(x, \hat{y}_l^{N_{j+1}}\right) + \frac{1}{|x|} \sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \int_\Sigma \Psi_{\tau_j}^i \left(y, \hat{y}_l^{N_{j+1}}\right) d\omega(y) \right. \\
& \left. + \sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \int_\Sigma \Phi_{\tau_L}^9(x,y) \Psi_{\tau_{j+1}}^i \left(y, \hat{y}_l^{N_{j+1}}\right) d\omega(y) \right), \end{aligned}$$

$x \in \Sigma$, represents a $J-$scale approximation of $F \in \mathcal{C}^{(0)}(\Sigma)$ in the $\|\cdot\|_{\mathrm{L}^2(\Sigma)}-$sense, while $U_J \in$ $\mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ *given by*

$$\begin{aligned} U_J(x) \ = \ & \sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \int_\Sigma \Phi_{\tau_{J_0}}^i \left(y, \hat{y}_l^{N_{J_0}}\right) \left(\frac{1}{|x|} + \frac{\partial}{\partial\nu(y)} \frac{1}{|x-y|} \right) d\omega(y) \\
+ \ & \sum_{J_0}^{J-1} \sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \int_\Sigma \Psi_{\tau_j}^i \left(y, \hat{y}_l^{N_{j+1}}\right) \left(\frac{1}{|x|} + \frac{\partial}{\partial\nu(y)} \frac{1}{|x-y|} \right) d\omega(y) \end{aligned}$$

represents a $J-$*scale approximation of* U *in the* $\|\cdot\|_{\mathcal{C}^{(0)}(\overline{K})}-$*sense for every* $K \subset \Sigma_{\text{ext}}$ *with* $\operatorname{dist}(\overline{K},\Sigma) > 0$. *Furthermore*

$$\sup_{x\in\overline{K}} \left|\nabla^{(k)}U(x) - \nabla^{(k)}U_J(x)\right| \leq C \left(\int_\Sigma |F(x) - F_J(x)|^2 \, d\omega(x)\right)^{1/2} \tag{193}$$

for all $k \in \mathbb{N}_0$.

(ENP) For given $F \in \mathcal{C}^{(0)}(\Sigma)$, *let* U *be the potential of class* $\operatorname{Pot}^{(1)}(\overline{\Sigma_{\text{ext}}})$ *with* $\frac{\partial U^+}{\partial \nu_\Sigma} = F$. *The function* $F_J \in \mathcal{C}^{(0)}(\Sigma)$ *given by*

$$\begin{aligned} F_J(x) & \qquad\qquad (194) \\ = \; & -2\pi \sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \Phi^i_{\tau_{J_0}}\left(x, \hat{y}_l^{N_{J_0}}\right) + \frac{1}{|x|}\sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \int_\Sigma \Phi^i_{\tau_{J_0}}\left(y, \hat{y}_l^{N_{J_0}}\right) d\omega(y) \\ + \; & \sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \int_\Sigma \Phi^8_{\tau_L}(x,y) \Phi^i_{\tau_{J_0}}\left(y, \hat{y}_l^{N_{J_0}}\right) d\omega(y) \\ + \; & \sum_{j=J_0}^{J-1} \left((-2\pi) \sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \Psi^i_{\tau_j}\left(x, \hat{y}_l^{N_{j+1}}\right) + \frac{1}{|x|}\sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \int_\Sigma \Psi^i_{\tau_j}\left(y, \hat{y}_l^{N_{j+1}}\right) d\omega(y) \right. \\ & \left. + \sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \int_\Sigma \Phi^8_{\tau_L}(x,y) \Psi^i_{\tau_{j+1}}\left(y, \hat{y}_l^{N_{j+1}}\right) d\omega(y) \right), \end{aligned}$$

$x \in \Sigma$, *represents a* $J-$*scale approximation of* $F \in \mathcal{C}^{(0)}(\Sigma)$ *in the* $\|\cdot\|_{\mathrm{L}^2(\Sigma)}-$*sense, while* $U_J \in \operatorname{Pot}^{(0)}(\overline{\Sigma_{\text{ext}}})$ *given by*

$$\begin{aligned} U_J(x) &= \sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \int_\Sigma \Phi^i_{\tau_{J_0}}\left(y, \hat{y}_l^{N_{J_0}}\right) \frac{1}{|x-y|} d\omega(y) \\ &+ \sum_{J_0}^{J-1} \sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \int_\Sigma \Psi^i_{\tau_j}\left(y, \hat{y}_l^{N_{j+1}}\right) \frac{1}{|x-y|} d\omega(y) \end{aligned}$$

represents a $J-$*scale approximation of* U *in the* $\|\cdot\|_{\mathcal{C}^{(0)}(\overline{K})}-$*sense for every* $K \subset \Sigma_{\text{ext}}$ *with* $\operatorname{dist}(\overline{K},\Sigma) > 0$. *Furthermore*

$$\sup_{x\in\overline{K}} \left|\nabla^{(k)}U(x) - \nabla^{(k)}U_J(x)\right| \leq C \left(\int_\Sigma |F(x) - F_J(x)|^2 \, d\omega(x)\right)^{1/2} \tag{195}$$

for all $k \in \mathbb{N}_0$.

In other words, the tree algorithm developed above uses an approximation method by solving a linear system for the initial step and integration rules with known weights and knots for the subsequent pyramid steps.

7.5 Concluding Remarks

The wavelet solution of boundary–value problems as presented here can be generalized in canonical way to boundary–value problems of elasticity and electromagnetic theory corresponding to regular surfaces. Furthermore, all considerations formulated here for the three–dimensional case may be generalized as well to higher dimensions.

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