Logic

(81-395)
4 + 2 SWS

13.45

Hauptstudium

Dozent: Prof. Schweigert

Inhalt: Axioms of set theory, ordinal and cardinal numbers, Boolean algebras, constraint satisfaction, syntax of sentential logic, thruth assignment, predicate logic, soundness an completeness theorems, introduction to model theory, non-classical logics: multiple-valued logic, fuzzy logic and rough sets, clones, undecidability, Gödel numbers and modal logic, theorems of Gödel and Löb.

We will emphasize the applications of logic:

1) Constraint satisfaction and combinatorial optimization.
2) DNA computing of the satisfaction problem.
3) Fuzzy logic and an application in marketing.

We will also collect some biographical data of the great mathematicians and make some remarks on the history of logic.

Leistungsnachweis: Übungsschein; 4 Credits.

Vorkenntnisse: Vordiplom.

Literatur: H. Enderton, A mathematical introduction to Logic, Academic Press 1972;
M.M. Richter, Logikkalküle, Teubner 1978;
M. Rautenberg, Klassische und nichtklassische Aussagenlogik, Vieweg 1978:
D. Schweigert, Logik, 2000, Semesterapparat.

Skript: See http://kluedo.ub.uni-kl.de/Mathematik/Metadaten/script_6.html.

Bemerkungen: The lecture will be given in English.

IV Predicate logic

V Nonclassical logics

VI Incompleteness
(Abridged version)

I Sets and numbers

§1 Some axioms of the set theory

For the convenience we use the language of the set theory in these formulas. (We will introduce ~~these~~ languages later on)

E1 Extensional axiom

If two sets have exactly the same members then hey are equal

$$\forall A \ \forall B \ [\forall x \ (x \in A \Leftrightarrow x \in B) \Rightarrow A = B)$$

We need the existence of some basic sets

E2 Empty set axiom

There is a set having no members

$$\exists B \ \forall x \quad x \notin B$$

E3 Pairing axiom

For any set u and v there is a set having as members just u and v

$$\forall u \ \forall v \ \exists B \ \forall x \ (x \in B \Leftrightarrow x = u \text{ or } x = v)$$

E4 Union axiom

For any set a there is a set whose member are those sets belonging either to a or b or both

$\forall a \forall b \exists B \forall x (x \in B \Leftrightarrow x \in a \text{ or } x \in b)$

(It can be extended for arbitrary unions)

E5 Power axiom

For any set a there is a set whose members are exactly the subsets of a

$\forall a \exists B \forall x (x \in B \Leftrightarrow x \subseteq a)$

(It is possible to avoid "$\subseteq$")

Notation 1.1 $\emptyset$ is the set having no members

We use some notations

pair set $\{u, v\} = \{x \mid x = u \text{ or } x = v\}$

union $a \cup b = \{x \mid x \in a \text{ or } x \in b\}$

power set $P(a) = \{x \mid x \subseteq a\}$

singleton $\{x\}$

$\emptyset = \{x \mid x \neq x\}$

E6 Subset axiom

For each formula α in language of the set theory not containing B the following is an axiom

$$\forall t_1 \ldots \forall t_k \forall c \exists B \forall x (x \in B \Leftrightarrow x \in c \text{ and } \alpha)$$

Example 1.2 One special case of the subset axioms is

$$\forall a \forall c \exists B \forall x (x \in B \Leftrightarrow x \in c \text{ and } x \in a)$$

This axiom asserts the existence of the set of intersection $c \cap a$

(The axiom E6 is often known as "Aussonderung" of Zermelo) (out seperated)

We can use the argument of Russell's paradox to show that the class V of all sets is not itself a set.

Thm 1.3 There is no set to which every set belongs

Proof. Let A be a set. We will construct a set not belonging to A. Let

$$B = \{x \in A \mid x \notin x\}$$

We claim that $B \notin A$. We have by construction of B

$$B \in B \iff B \in A \text{ and } B \notin B$$

If $B \in A$ then this reduces to

$$B \in B \iff B \notin B$$

which is impossible, since one side must be true and the other false. Hence $B \notin A$. $\square$

(It is not necessary to understand this proof.)

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§2 Natural numbers

In 1908 Zermelo and later on von Neumann sugested to introduce the natural numbers by sets:

$$0 = \emptyset$$

$$1 = \{0\} = \{\emptyset\}$$

$$2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$$

$$3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

Def. 2.3 For every set a its sucessor a^+ is defined by
$$a^+ = a \cup \{a\}$$

A set A is said to be inductive if $\emptyset \in A$ and it is closed under successors
$$(\forall a \in A)\, a^+ \in A$$

Illustration:
$$0 = \emptyset \;,\; 1 = \emptyset^+ \;,\; 2 = \emptyset^{++}$$

E7 Infinite axiom

There exists an inductive set
$$(\exists A)[\emptyset \in A \text{ and } (\forall a \in A)\, a^+ \in A]$$

Def. 2.4 A natural number is a set that belongs to every inductive set

Notation: $\omega = \bigcap \{A \mid A \text{ is inductive}\}$

Thm 2.5 There is a set whose members are exactly the natural numbers

Proof. Let A be an inductive set (By the infinite axioms it is possible to find such a set A)
By the subset axiom there is a set W such that for any x

$x \in W \Leftrightarrow x \in A$ and x belongs to every other inductive set

$\Leftrightarrow$ x belongs to every inductive set

$\square$

Induction principle for ω

Any inductive subset of ω coincides with ω

In 1889 Peano published an axiomatic approach to the natural numbers.
The formulation of the axioms is attributed to Dedekind

Def. 2.6 Let S be a function and A be a subset of domain S. A is said to be closed under S if whenever $x \in A$ then $S(x) \in A$

A Peano system is a triple $\langle N, S, e \rangle$ consisting of a set N, a function $S: N \longrightarrow N$ and a member $e \in N$ such that the following three conditions are met

i) $e \notin$ range of S

ii) S is bijective

iii) Any subset A of N that contains e and is closed under S equals N itself

Illustration

$$\overset{e}{\circ} \longrightarrow \overset{S(e)}{\circ} \longrightarrow \overset{S^2(e)}{\circ} \longrightarrow \overset{S^3(e)}{\circ} \cdots\cdots \circ \longrightarrow \cdots$$

Thm 2.7 Let σ be the restriction of the successor function to ω namely $\sigma = \{(n, n^+) \mid n \in N\}$

$\langle \omega, \sigma, 0 \rangle$ is a Peano system

Thm 2.8 Recursion thm on ω

Let A be a set, $a \in A$, and $F : A \to A$. Then there exists a unique function $h : \omega \to A$ such that

$$h(0) = a$$

and for every n in ω

$$h(n^+) = F(h(n))$$

Exercise: Prove this theorem!

§3 Well order sets

We use the concept of a partially ordered set (= poset)

Def. 3.1 Let $(A; \leq)$ be a partialley ordered set

((A binary relation of the set A is an order relation if it is reflexive, antisymmetric and transitive))

a is called a maximal element of $C \subseteq A$ if it holds

1) $a \in C$
2) $a \leq c$ implies $a = c$ for $c \in C$

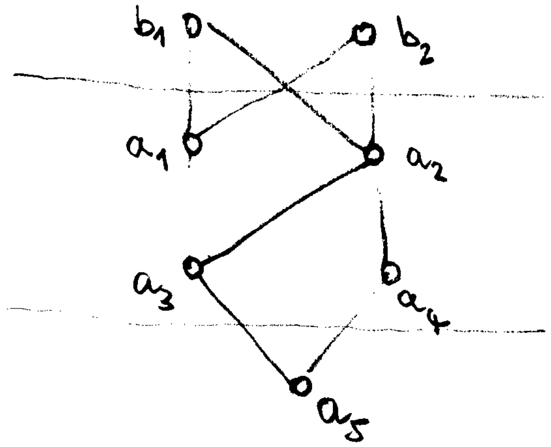
a is called an upper bound of $C \subseteq A$ if it holds

$$c \leq a \quad \text{for every } c \in C$$

a is called least upper bound or supremum of C if it holds

1) a is an upper bound of C
2) If b is an upper bound of C then $a \leq b$

Example



$A = \{a_1, \ldots, a_5, b_1, b_2\}$
$C = \{a_1, a_2, a_3, a_4\}$

a_1, a_2 are maximal element

a_3, a_4 are minimal element

b_1, b_2 are upper bounds

It exists no supremum of C

a_5 is the infimum of C

Def. 3.2 Two elements a, b is called comparable if $a \leq b$ or $b \leq a$.
andelse incomparable.
$(A; \leq)$ is linearly ordered or a chain if every two elements $a, b \in A$ are comparable

A linearly ordered set $(A; \leq)$ is well-ordered if every non-empty subset have aleast element (= infimum)

We like to illustrate of a well-order (wohl-ordnung)

$$t_0 < t_1 < \dots < t_\omega < t_{\omega+1} < \dots < t_{\omega\cdot 2} < t_{\omega\cdot 2+1} < \dots$$

Thm 3.3 Let $<$ be a linear ordering on A
Then it is a well ordering iffthere does not exist any fct $f: \omega \to A$ with $f(n^+) < f(n)$ for every $n \in \omega$

Proof. If there is a descending chain f then the range f is a nonempty subset of A. Clearly the range f has no least element. For each element $f(n)$ there is a smaller element $f(n^+)$ Hence the existence of descending chain implies that $<$ is not a well ordering

Conversively assume that $<$ is not a well ordering so that some non-empty $B \subseteq A$ lacks a least element. Then $(\forall x \in B)(\exists y \in B)\ y < x$. There are a descending chain $f : \omega \to B$ with $f(n^+) < f(n)$ for each n in ω □

If $<$ is some sort of ordering on A and $t \in A$ then the set

$$\text{seg}\, t = \{x \mid x < t\}$$

is called the initial segment up to t

Anfangsabschnitt

for instance

$$\text{seg}\, n = \{x \mid x \in n\} = n$$

34 Transfinite induction principle

Assume that $<$ is a well ordering on A. Assume that B is a subset of A with special property that for every t

$$\operatorname{seg} t \subseteq B \Rightarrow t \in B$$

Then B coincides with A

For example we are unable to prove that

$$\emptyset \cup P(\emptyset) \cup P(P(\emptyset)) \cup P(P(P(\emptyset))) \cup \dots$$

is a set. These deficiences will be eliminated by the axioms

E8 Replacement axioms

For every formula $\varphi(x,y)$ not containing the letter B the following is an axiom

$$\forall A\,[(\forall x \in A)\,\forall y_1 \forall y_2 (\varphi(x,y_1) \text{ and } \varphi(x,y_2))$$

$$\Rightarrow \exists B \forall y\,(y \in B \Leftrightarrow (\exists x \in A)\,\varphi(x,y))]$$

Notation $f : A \longrightarrow B$

domain $\mathrm{dom}(f) = A$ Definitionsbereich

range $\mathrm{ran}(f) = B$? Wertebereich ?

Example

If A is a set then $\{P(a) \mid a \in A\}$ is also a set. But now we have an easy proof of this fact. Take $\varphi(x,y)$ to be $y = P(x)$. That is, let each x nominate its own power set. Then the replacement tells us that the collection of all power set of members of A form a set.

§4 Ordinal numbers

Def 4.1 Assume that $<$ is a well ordering on A and take $\gamma(x,y)$ the formula $y = \text{range } x$.

The transfinite induction principle present us a unique fct E with domain A such that for any $t \in A$

$$E(t) = \{E(x) \mid x < t\}$$

The range E is called a ordinal number

Thm 4.1 The following are valid for any ordinal numbers α, β, γ

a) (transitive class) Any member of α is itself an ordinal number

a) (transitivity) $\alpha \in \beta$ and $\beta \in \gamma \Rightarrow \alpha \in \gamma$

b) (irreflexivity) $\alpha \notin \alpha$

c) (trichotomy) Exactly one of the alternative holds: $\alpha \in \beta$, $\alpha = \beta$, $\beta \in \alpha$

d) 0 is an ordinal number

e) If α is any ordinal number then α^+ is also an ordinal number

f) If A is any set of ordinals then $\bigcup A$ is also an ordinal number

Illustration:

The least ordinal number is 0

Next come 0^+, 0^{++}, 0^{+++}, ... the natural numbers

The least ordinal greater than these is the least upper bound of the set of natural numbers. This is ω.
(ω is an ordinal and the ordinals less than ω are exactly its members)
Then it begins $\omega^{+}, \omega^{++}, \omega^{+++}, \ldots$

All these are countable ordinals.

The least uncountable ordinal is the set of all smaller ordinals

But we have not shown that there exists uncoutables ordinals. We will correct that by

Eg Regularity axiom
Every non empty set A has a member m with $m \cap A = \emptyset$

(Remark: No set is member of itself)

Thm 4.2 There is no set to which every ordinal number belongs

The paradox of Russel shows the inconcistence of Cantor's use of the "set of all ordinals". Therefore the class of all ordinals is not a set but a class

§5 Cardinal numbers

Def 5.1 A set A is equinumerous to a set (written $A \approx B$) if there is a bijective function from A onto B

Examples 5.2

5.2.1 The set $\omega \times \omega$ is equinumerous to ω

$\omega \approx \omega \times \omega$

5.2.2 The set of natural numbers is equinumerous to $\mathbb{Q}$

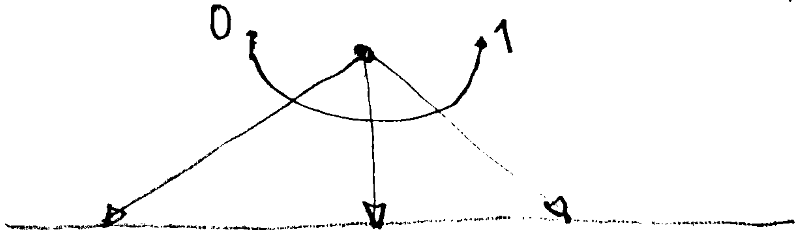
$\omega \approx \mathbb{Q}$

$$\begin{tikzcd}
5\ \frac{-2}{1} \arrow[d] & -\frac{1}{1}\ 4 \arrow[l] & \overset{0}{\frac{0}{1}} \arrow[r] & \frac{1}{1}\ 1 \arrow[d] \\
{} & 3\ -\frac{1}{2} \arrow[u] & \frac{0}{2} \arrow[l] & \frac{1}{2}\ 2 \arrow[l]
\end{tikzcd}$$

5.2.3 The open unit interval

$$(0,1) = \{x \mid x \in \mathbb{R},\ 0 < x < 1\}$$

is equinumerous to the set $\mathbb{R}$ of all reals



$$(0,1) \approx \mathbb{R}$$

5.2.4
Galileo remarked in 1638 that ω was equinumerous to the set $\{0,1,4,9,\ldots\}$ of the squares of natural numbers and found this to be a curious fact.

5.2.5 For any set A we have

$$P(A) \approx 2^A$$

Proof. For any subset B of A we consider the function $f_B : A \longrightarrow 2$

$$f_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & x \in A \setminus B \end{cases}$$

Then any function $g \in 2^A$ is the range H where H (B) is the characteristic fct of B

$$g = H(\{x \mid x \in A,\ g(x) = 1\})$$

Thm 5.3 (Cantor 1873)

a) The set ω is not equinumerous to the set $\mathbb{R}$ of reals

b) No set is equinumerous to its power ordered set

Def. 5.4 For any set A the cardinal number (card A) of A is the least ordinal number which is equinumerous to A

Notation 5.5

According to Cantor we have the name aleph

$$\operatorname{card} \omega = \aleph$$

We can expand the class of infinite cardinal numbers. We assume that we have already defined $\aleph_\beta$ for all β less than α. Then we can define

$\aleph_\alpha$ = the least infinite cardinal different from $\aleph_\beta$ for every β less than α

Such a cardinal must exist beause

$$\{\aleph_\beta \mid \beta \in \alpha\}$$

is merely a set where as the class of infinite cardinals is unbounded.

By transitive induction (34) one can prove that every infinite cardinal is of the form $\aleph_\alpha$ for some α

Remark. First-graders are not told about the recursion thm. Instead, if they want to add 2 and 3 they select two sets K and L with card K = 2 and card L = 3. Then they look at card(K ∪ L)
(We use greek letters)

Def. 5.6 Let $\varkappa$ and λ be any cardinal numbers

a) $\varkappa + \lambda = \operatorname{card}(K \cup L)$ where K and L are any disjoint sets of cardinality $\varkappa$ and λ

b) $\varkappa \cdot \lambda = \operatorname{card}(K \times L)$ where K and L are any sets of cardinality $\varkappa$ and λ respectively

c) $\varkappa^\lambda = \operatorname{card} {}^L K$

Def. 5.7 The statement $\omega^+ = 2^\omega$ is called the continuum hypothesis

The statement $\alpha^+ = 2^\alpha$ is called the general continuum hypothesis

In other words:

Every uncountable set of real numbers is equinumerous to the set of all real numbers?

Cantor conjectured that the continuum hypothesis was true.

See historical remarks (Seite 166)

§6 Axiom of choice

E 106.1 Axiom of choice

For any set A there is a function F (a choice function for A) such that the domain of F is a set of non-empty subsets of A and such that $F(B) \in B$ for any nonempty $B \subseteq A$

The analysis as well the algebra use the axiom of choice. This axiom is often hidden by harmless remark: "We choose a subset of a set with some properties."

Many statements of the axiom of choice are equivalent

$\boxed{\text{Axiom of choice}} \longleftrightarrow \boxed{\text{Thm of well ordering}} \longleftrightarrow \boxed{\text{Zorn's lemma}}$

6.2 Well-Ordering thm (of Zermelo)

For any set, there is a well ordering on A

One consequence of the thm of Zermelo is

6.3 Numeration Thm

Any set is equinumerous to some ordinal number

Remark

We have the asserting the existence of well ordering of the reals. But is impossible that there is a formula of the language of set theory that will explicitly define a well ordering on $\mathbb{R}$ (without the axiom of choice)

6.4 Zorn's lemma

Let $(A; \leq)$ be a nonempty poset in which every chain has an upper bound. Then $(A; \leq)$ has a maximal element

We will apply Zorn's lemma is the next chapter on Boolean algebras

§7 Remarks on history

7.1
Are there any sets with cardinality between

$\aleph_0$ and $2^{\aleph_0}$?

The set theory is created by Cantor 1845 - 1918
Cantor conjectured that the continuum hypothesis was true

In 1939 Gödel proved that on the bases of our axioms for the set theory (which we have assumed to be consistent (Widerspruchsfreiheit)) the continuum hypothesis could not be disproved

In 1963 Cohen showed that the continuum hypothesis could not be proved from our axioms either

7.2 The system of the axioms E1 - E9 of the set theory is also called Zermelo - Fraenkel - Set theory with together the axiom of choice E10. Some important results are of Tarski, Erdös, Hausdorf, J. von Neumann Kurotowski, König and naturally Cantor.

For the metamathematics one quotes Frege, Russel and Wittgenstein.

7.3 We like to collect some paradoxes

7.3.1 Epimenides: I am lying

The statement is false which stand in in the box

Is the set of all sets a set?

7.3.2 Zeno's paradoxes of motion (Zeno of Elea 490-430 B.C.)

Achilles can never overtake the turtle (tortoise)
Before he comes up to the point at which the turtle started the turtle will have got a little way on
This repeats ad infinitum.

7.3.3 The paradox of the baldness (of the Greeks)

„Baldness is a vague conception; some men are certainly bald, some are certainly not bald, while between them there are men of whom it is not true to say they must either be bald or not bald. The law of excluded middle is true when precise symbols are employed, but it is not true when symbols are vague, as, in fact, all symbols are." [Russell 1923, [redacted]]

LIST OF AXIOMS

Extensionality axiom

$$\forall A\ \forall B[\forall x(x \in A \quad \Leftrightarrow \quad x \in B) \quad \Rightarrow \quad A = B]$$

Empty set axiom

$$\exists B\ \forall x\ x \notin B$$

Pairing axiom

$$\forall u\ \forall v\ \exists B\ \forall x(x \in B \quad \Leftrightarrow \quad x = u \text{ or } x = v)$$

Union axiom

$$\forall A\ \exists B\ \forall x[x \in B \quad \Leftrightarrow \quad (\exists b \in A) x \in b]$$

Power set axiom

$$\forall a\ \exists B\ \forall x(x \in B \quad \Leftrightarrow \quad x \subseteq a)$$

Subset axioms For each formula φ not containing B, the following is an axiom:

$$\forall t_1 \cdots \forall t_k\, \forall c\ \exists B\ \forall x(x \in B \quad \Leftrightarrow \quad x \in c\ \&\ \varphi)$$

Infinity axiom

$$\exists A[\varnothing \in A\ \&\ (\forall a \in A)\ a^+ \in A]$$

Replacement axioms For any formula $\varphi(x, y)$ not containing the letter B, the following is an axiom:

$$\begin{aligned} \forall t_1 \cdots \forall t_k\, \forall A[(\forall x \in A)\forall y_1 \forall y_2(\varphi(x, y_1)\ \&\ \varphi(x, y_2) \quad &\Rightarrow \quad y_1 = y_2) \\ \Rightarrow \quad \exists B\ \forall y(y \in B \quad \Leftrightarrow \quad (\exists x \in A)\varphi(x, y))]& \end{aligned}$$

Regularity axiom

$$(\forall A \neq \varnothing)(\exists m \in A)\ m \cap A = \varnothing$$

Choice axiom

$$(\forall \text{ relation } R)(\exists \text{ function } F)(F \subseteq R\ \&\ \text{dom } F = \text{dom } R)$$

II Boolean algebra

§8 Relation system; partially ordered set (poset)

Def. 8.1 An ordered pair $\underset{\sim}{A} = (A; \underset{\sim}{r}_0, \dots, \underset{\sim}{r}_\gamma, \dots)$ consisting of a not-empty set A ((carrier)) and a set of predicates with the arity (place number) $n_0, \dots, n_\gamma, \dots \in \mathbb{N}$ is called relation system

To every n_i-place predicate $\underset{\sim}{r}_{i(A)}$ we assign a n_i-place relation $r_{i(A)}$ of A

The sequence $(n_0, \dots, n_\gamma, \dots)$ of places is called the type of A)

$T_\gamma(A)$ is ordinal number.

By abuse of the denotation we will use the predicate and its relation the same letter

(symbol $\longleftarrow\!\!-\!\!-\!\!\longrightarrow$ its realization)

(predicate $\longleftarrow\!\!-\!\!-\!\!\longrightarrow$ relation)

Moreover we will assume that the identity as a relation always exists

Example

$\underset{\sim}{A} = (A ; \varrho)$ is a partially ordered set (= poset) if it holds

1) $\forall x \ \varrho(x,x)$ reflexive ϱ

2) $\forall x \forall y \ ((\varrho(x,y) \overset{\text{and}}{\wedge} \varrho(y,x)) \rightarrow (x = y))$

antisymmetric

3) $\forall x \forall y \forall z \ ((\varrho(x,y) \wedge \varrho(y,z)) \rightarrow \varrho(x,y))$

transitive

Usual notation:

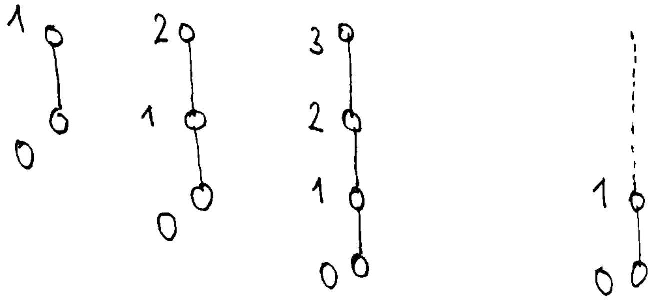
1) $x \leq x$

2) From $x \leq y$ and $y \leq z$ it follows $x \leq z$

3) From $x \leq y$ and $y \leq z$ it follows $x \leq z$

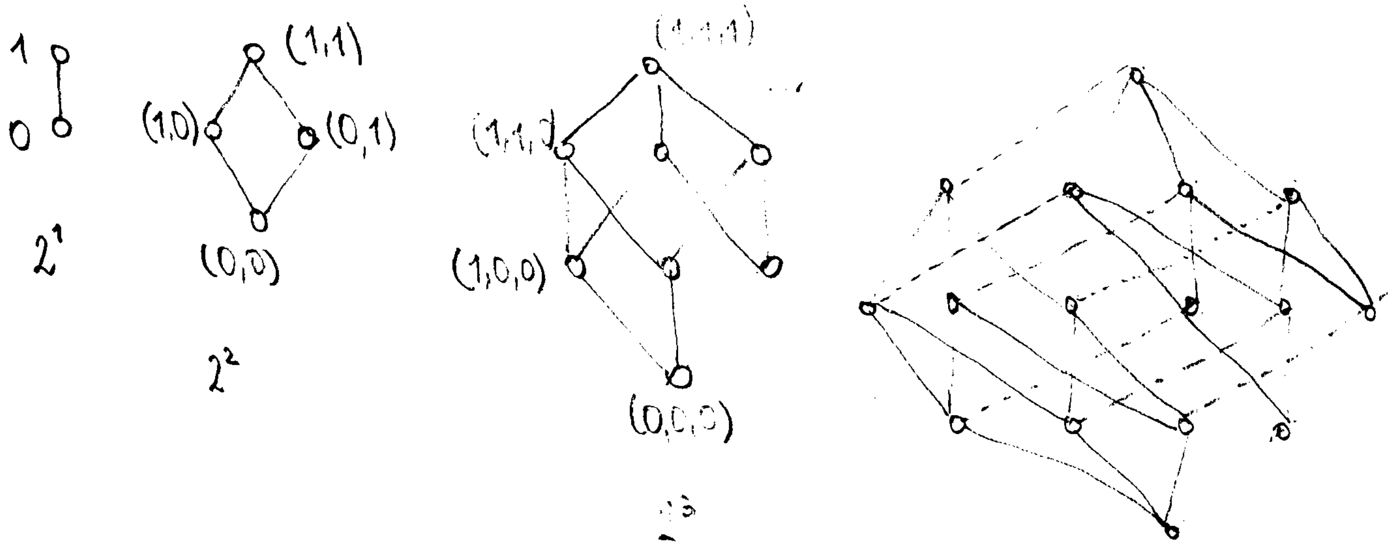
Presentation by a Hassediagram

1)



$(\mathbb{N}; \leq)$

2) Boolean algebra



2^1 2^2 2^3 2^4

Def. 8.2 A relation system $(B; r_0, \dots, r_\gamma, \dots)$ is called a subsystem of $(A; r_0, \dots, r_\gamma, \dots)$ if it holds

1) $B \subseteq A$ and $B \neq \emptyset$

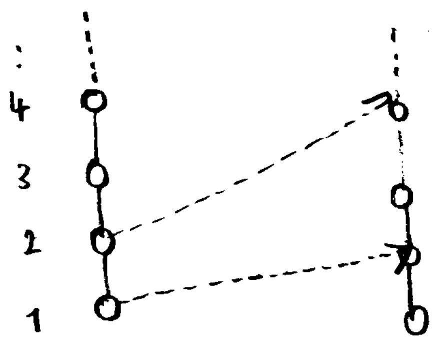
2) Every relation $r_{i(B)}$ is the restriction of the relation $r_{i(A)}$.

Example: $(2\mathbb{N}; \leq)$ is a subsystem of $(\mathbb{N}; \leq)$

Def. 8.3 A mapping $f: A \to B$ is called a homomorphism of the relation system $\underset{\sim}{A} = (A; r_0, \dots, r_\gamma, \dots)$ into the relation system $\underset{\sim}{B} = (B; r_0, \dots, r_\gamma, \dots)$ if it follows from $r_i(a_1, \dots, a_{n_i})$ to $r_i(f(a_1), \dots, f(a_{n_i}))$

Example

$f: \mathbb{N} \to 2\mathbb{N}$ defined by $f(n) = 2n$ is a homomorphism



Def. 8.4 Let $\{\underset{\sim}{A}_i \mid i \in I\}$ be a family of relation systems. Then the direct product $\underset{\sim}{A} = \prod_{i \in I} \underset{\sim}{A}_i$ is defined in the following.

The (carrier) set of $A = \{\prod_{i\in I} A_i \mid i \in I\}$ is the cartesian product.

The relation r_i is defined coordinate-wise

Let $a_1, \dots, a_{n_i} \in \prod_{i\in I} A_i$

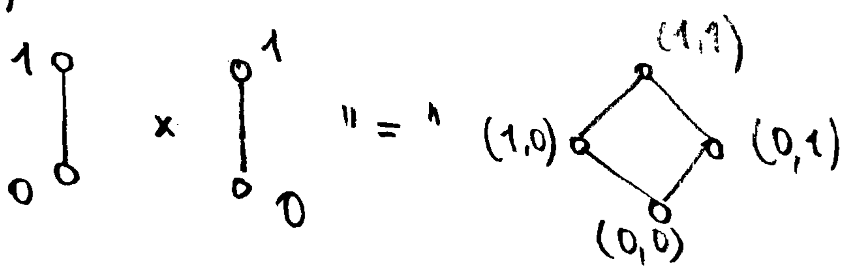
For all projections $e_i : \prod_{i\in I} A_i \longrightarrow A_i$ the relation $r_i(e_i(a_1), \dots, e_i(a_{n_i}))$ is fulfilled

Examples

1) $(\mathbb{N} \times \mathbb{N}; \leq)$

$(n_1, n_2) \leq (m_1, m_2)$ if and only if $n_1 \leq m_1$ and $n_2 \leq m_2$

2)



§ 9 Algebras; Boolean algebras

Def. 9.1 An ordered pair $\underset{\sim}{A} = (A; \underset{\sim}{f}_0, \dots, \underset{\sim}{f}_{\gamma}, \dots)$ which consists of a not empty (carrier) set A and a set of n_i-place operation symbols $\underset{\sim}{f}_{n_i}$ is called algebra

To every n_i-place operation symbol $\underset{\sim}{f}_{n_i}$ there is assigned a n_i-place operation $f_{n_i}(A)$

The sequence $(n_0, \dots, n_\gamma, \dots)$ of places (arities) is called the type of the algebra

By abuse of the denotation we use the same letter of operation symbol and operation

Symbol $\longleftrightarrow$ Realization

Def. 9.2 (and example)

The algebra $\underset{\sim}{B} = (B; \wedge, \vee, ', 0, 1)$ of the type $(2,2,1,0,0)$ is called Boolean algebra if the following holds.

1) Associativity

$x \wedge (y \wedge z) = (x \wedge y) \wedge z \qquad x \vee (y \vee z) = (x \vee y) \vee z$

2) Commutativity

$x \wedge y = y \wedge x \qquad x \vee y = y \vee x$

3) Idempotency

$x \wedge x = x \qquad x \vee x = x$

4) Adsorption

$x \wedge (y \vee x) = x \qquad x \vee (y \wedge z) = x$

lattice
~~Verband~~

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5) Distributivity

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

distributive latt

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6) De Morgan laws

$(x \wedge y)' = x' \vee y' \qquad (x \vee y)' = x' \wedge y'$

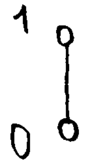
7) Complement

$x \wedge x' = 0 \qquad x \vee x' = 1$

8)

$x'' = x$

9.3 Examples for Boolean algebras

9.3.1 

$\underset{\sim}{B} = (\{0,1\}; \wedge, \vee, ', 0, 1)$

9.3.2 Algebra of the power set

Let X be a set (which is not-empty)

$(P(X); \cap, \cup, ', \emptyset, X$

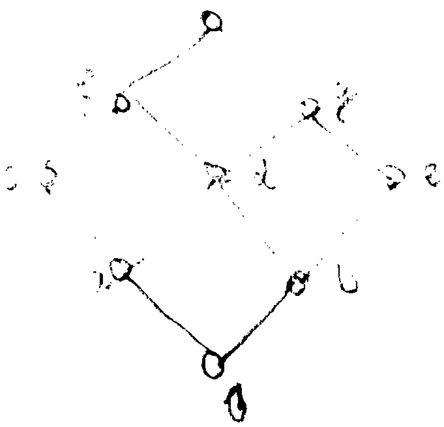
9.3.3 Let T be a topological space. Let X the family of all open-closed sets.

9.3.4 Probability

9.3 Examples of lattices

$\underset{\sim}{L} = (L; \wedge, \vee)$ is a lattice (Verband) if 1)–7) holds

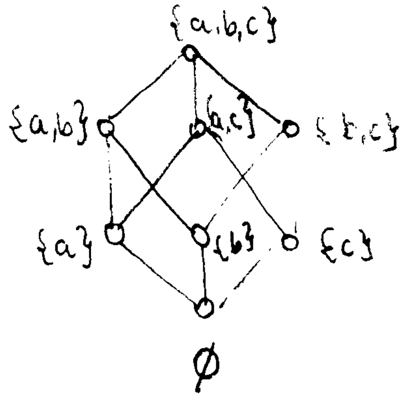
9.3.5



is a distributive lattice

9.4 Hasse diagrams of Boolean algebras (or [illegible])

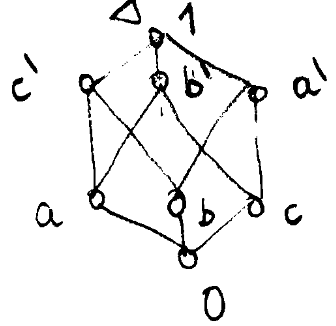
9.4.1



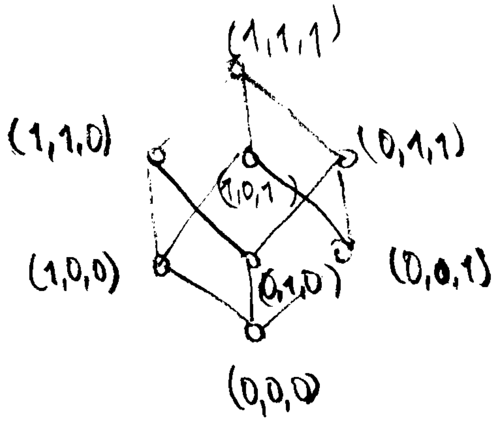
$X = \{a,b,c\}$

$(P(X); \cup, \cap, ', \emptyset, \{a,b,c\})$

9.4.2 duality



9.4.3 Switching algebra



§10 Structures

Def. 10.1 A structure $\underset{\sim}{A} = (A; \underset{\sim}{f_0}, \ldots, \underset{\sim}{f_\gamma}, \ldots; \underset{\sim}{r_0}, \ldots, \underset{\sim}{r_\beta}, \ldots)$ is a triple where it holds

1) A is a non-empty (carrier) set
2) $(A; \underset{\sim}{f_0}, \ldots, f_\gamma, \ldots)$ is an algebra
3) $(A; \underset{\sim}{r_0}, \ldots, r_\beta, \ldots)$ is a relation system

(γ, β are ordinal numbers)

Examples 10.2

10.2.1 On a Boolean algebra we can defined an order relation:

$$a \leq b \iff a \wedge b = a$$

$(B; \wedge, \vee, `, 0, 1; \leq)$ is a structure

10.2.2 The structure $(N; S, +, 0; <)$ is called Pressburger arithmetic

The concepts substructure
homomorphism
direct product
can be defined in the same way

Def. 10.3 Let algebras $\underset{\sim}{A}$, $\underset{\sim}{B}$ be of the same type.
A mapping $h: A \to B$ is called a homomorphism if it holds

$$h(f_{n_i(A)}(a_1, \dots, a_{n_i})) = f_{n_i(B)}(h(a_1), \dots, h(a_{n_i}))$$

Example

Let B_1, B_2 be Boolean algebras. $h: B_1 \to B_2$ is a homomorphism if it holds

$$h(x \wedge y) = h(x) \wedge h(y) \qquad h(0) = 0$$
$$h(x \vee y) = h(x) \vee h(y) \qquad h(1) = 1$$
$$h(x') = h(x)'$$

Def. 10.4 Let $\{A_i \mid i \in I\}$ be a family of algebras of the same type. The direct product $A = \prod_{i \in I} A_i$ is defined in the following

1) $\prod_{i \in I} A_i$ is (carrier) set A (it is the cartesian product)

2) The operation $f_{n_i(A)}$ is defined component-wise
Let $a_1, \dots, a_{n_i} \in A$. Then it holds

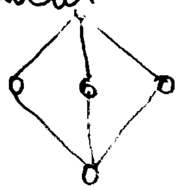
$$e_j(f_{n_i(A)}(a_1, \dots, a_{n_i})) = f_{n_i(A_j)}(e_j(a_1), \dots, e_j(a_{n_i}))$$

9.3.6

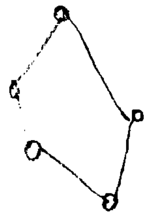
A lattice $(L; \wedge, \vee)$ is non-distributive

if it contains of isomorphic sublattices

modular



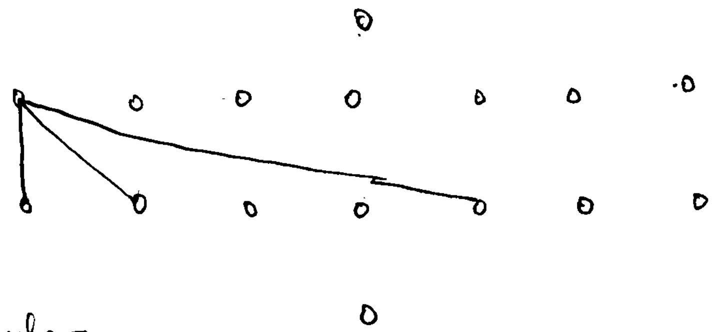
M_3



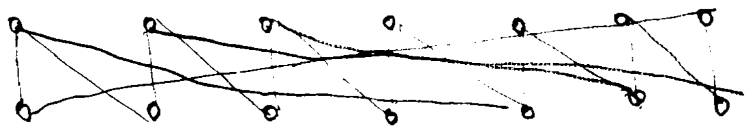
N_5

$K = GF(2)$

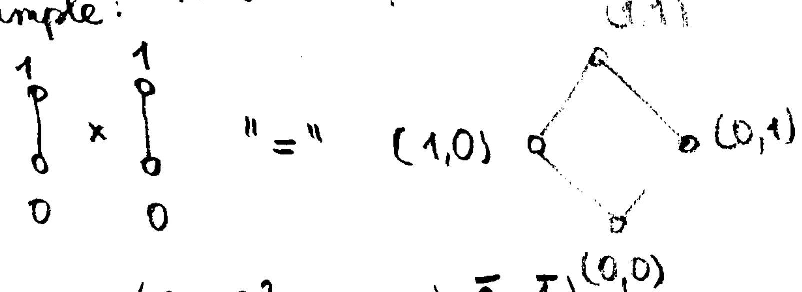
9.3.7 projective geometry | Vector space $K^3 = K \times K \times K$



modular lattice



Example: A direct product of Boolean algebras



We have $(\{0,1\}^2; \wedge, \vee, ', \bar{0}, \bar{1})$

$(a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge b_1, a_2 \wedge b_2)$

$(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee b_1, a_2 \vee b_2)$

$(a_1, a_2)' = (a_1', a_2')$

§11 Polynoms and terms

Def. 11.1 Let $\underset{\sim}{A} = (A; f_0, \ldots, \ldots)$ be an algebra of some type. The polynomial functions of $\underset{\sim}{A}$ consists of all n-place function which is defined recursively

i) the projection $e_i^n : A^n \longrightarrow A$ with $e_i^n(x_1, \ldots, x_n) = x_i$ are n-place polynomial fct

ii) the constant fct $c_a^n : A^n \longrightarrow A$ with $c_a^n(x_1, \ldots, x_n) = a$, $a \in A$, are n-place polynomial fct

iii) If f_m is a m-place operation of the algebra $\underset{\sim}{A} = (A; f_0, \ldots, f_n, \ldots)$ and $p_1, \ldots, p_m$ are n-place polynomial fct then $f_m(p_1(x_1, \ldots, x_n), \ldots p_m(x_1, \ldots$ is a polynomial fct.

One may apply (iii) only a finite number of times.

Examples 11.2

11.2.1 $\underset{\sim}{B} = (\mathbb{R}; +, -, 0, \cdot, 1)$

1-place polynomial fcts (normal form)

$$p(x) = a_k x^k + \cdots + a_1 x + a_0$$

2-place $p(x,y) = x^3 + xy + 1$

11.2.2 Group $\underset{\sim}{G} = (G; \cdot, e)$

example for an non-abelian group

$$p(x) = x^{-1} a x$$

example for abelian groups

$$p(x_1, x_2, \ldots, x_n) = a x_1^{k_1} \cdot \ldots x_n^{k_n}$$

11.2.3 Distributive lattices

Example: Let $a_1 \geq b_1$, $a_2 \geq b_2$

$[(a_1 \wedge x) \vee b_1] \wedge [(a_2 \wedge x) \vee b_2]$

$= \{[(a_1 \wedge a_2) \vee (b_1 \wedge a_2) \vee (a_1 \wedge b_2)] \wedge x\} \vee (b_1 \wedge b_2)$

$= [(a_1 \wedge a_2) \wedge x] \vee (b_1 \wedge b_2)$

Normal form

11.2.4 Boolean algebra

Example:

$(x_1 \vee x_2)' \vee (x_1' \wedge x_3) =$

$(x_1' \wedge x_2') \vee (x_1' \wedge x_3) =$

$[(x_1' \wedge x_2') \wedge (x_3 \vee x_3')] \vee [x_1' \wedge (x_2 \vee x_2') \wedge x_3] =$

$(x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2 \vee x_3)$

disjunctive normal form

Def. 11.3 The term fct of $\underset{\sim}{A}$ consists of all n-place fct which is defined by recursively

i) ... projection $e_i^n(x_1, \ldots, x_n) = x_i$

iii) ... $f_m(t_1(x_1), \ldots t_m(x_n))$

Algebra $\underset{\sim}{A} = (A; f_0, \ldots, f_\gamma, \ldots)$

Polynomial function

Definition by recursion

i) projection $e_i^n(x_1, \ldots, x_n) = x_i \quad \in P_n(A)$

ii) constant $c_a^n(x_1, \ldots, x_n) = a \quad \in P_n(A)$

iii) $f_m(p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n)) \in P_n(A)$

$P(A) = \bigcup_{n \in \mathbb{N}} P_n(A)$ is the set of all polynomial fct

Term fct : usually without constant

Example: $(\mathbb{R}; +, -, 0, \cdot, 1)$

$$p(x) = a_k x^k + \ldots + a_1 x + a_0$$

$$t(x) = x^k + \ldots + x$$

Distributive lattices

Example:

$$p(x, y) = [(x \wedge y) \vee a] \wedge y$$

$$= [x \wedge y \wedge y] \vee (a \wedge y) = (x \wedge y) \vee (a \wedge y)$$

Example.

$(\mathbb{R}; +, -, 0, \cdot, 1)$

$t(x) = x^k + x^{k-1} + \dots + x$ is a term fct

in a normal form.

In the difference the term function have no constants usually

Def. 11.4 We define words in X over the operation symbols $\Omega = \{f_0, \dots, f_\gamma, \dots\}$ "recursively

i) Words of rank 0 are $w_i = x_i$, $i = \{x_0, \dots, x_n, \dots\}$
(projection x_i)

ii) Words of rank $k+1$ is either a words of rank k or of the form $f_\gamma(w_1, \dots, w_n)$ where again ~~of~~ words $w_1, \dots, w_n$ of lower rank

Example

$$f_\gamma(f_1(x_1, x_2, x_3, \dots, x_{n_1}), x_2, \dots, x_{n_\gamma})$$

We can define the words together with constant fct symbols or without

Notation 11.5

A set N of words in symbols $x_1, \ldots, x_n, \ldots$ (and constant fct symbols) is a normal form system provided

a) for every representation $p = w(x_1, \ldots, x_n)$ can (and constant fct sym $w(a_1, \ldots, a_m, x_1, \ldots)$) find a word in N representing p in a finite number of steps

b) any different words of N represent different element of the algebra $\mathfrak{A}$

Thm 11.6 Let $(D; \wedge, \vee, 0, 1)$ be a bounded distributive lattice. The ~~set~~ N of all words $(a \wedge x) \vee b$, $a, b \in D$, $a \leq b$, is normal form system $D[x]$

Proof.

1) We show by induction on the minimal rank of $w(a_i, x)$ that, in finite numbers of steps for every presentation $p = w(a_i, x)$ an element $p \in D[x]$ we can find a word of N presenting p

Let $d \in D$ then $d = (d \wedge x) \vee d$ and $x = (1 \wedge x) \vee x$

holds for minimal rank 0

Supose that it holds for words of minimal rank $\leq m$

Every word of minimal rank $m+1$ has the form $w_1 \vee w_2$ or $w_1 \wedge w_2$ where w_1, w_2 are words of minimal rank $\leq m$

By induction we have $w_1 = (a_1 \wedge x) \vee b_1$ and $w_2 = (a_2 \wedge x) \vee b_2$. Therefore we have

$w_1 \vee w_2 = ((a_1 \vee a_2) \wedge x) \vee (b_1 \vee b_2)$ as $a_1 \vee a_2 \geq b_1 \wedge b_2$

In the other case we have

$w_1 \wedge w_2 = [(a_1 \wedge x) \vee b_1] \wedge [(a_2 \wedge x) \vee b_2]$

We use the distributivity

$$
\begin{aligned}
(a_1 \wedge x) \vee b_1 &= b_1 \vee (a_1 \wedge x) \\
&= (b_1 \vee a_1) \wedge (b_1 \vee x) \\
&= a_1 \wedge (b_1 \vee x)
\end{aligned}
$$

Now we get

$$w_1 \wedge w_2 = [a_1 \wedge (b_1 \vee x)] \wedge [a_2 \wedge (b_2 \vee x)]$$

$$= a_1 \wedge a_2 \wedge \underbrace{(b_1 \vee x) \wedge (b_2 \vee x)}_{\text{distributive law}}$$

$$= a_1 \wedge a_2 \wedge [x \vee (b_1 \wedge b_2)]$$

$$= [(a_1 \wedge a_2) \wedge x] \vee [(a_1 \wedge a_2) \wedge (b_1 \wedge b_2)] \text{ distributive}$$

$$= [(a_1 \wedge a_2) \wedge x] \vee (b_1 \wedge b_2)$$

because $a_1 \wedge a_2 \geq b_1 \wedge b_2$

2) We have to show that no two different words of N represent the same element of $D[x]$

Assume that $(a_1 \wedge x) \vee b_1 = (a_2 \wedge x) \vee b_2$

Now for $x = 0$ we get $b_1 = b_2$ and

for $x = 1$ we have $(a_1 \wedge 1) \vee b_1 = (a_2 \wedge 1) \vee b_2$

and therefore $a_1 = a_2$

□

Def. 11.7 In any poset with 0, two elements x, y are said to be disjoint when $x \wedge y = 0$

For the interval $[0, a']$ of a Boolean algebra we have

$$a \wedge x = 0 \Leftrightarrow x \leq a'$$

More generally, in any distributive lattice

$$a \wedge x = 0 \text{ and } x \wedge y = 0 \text{ implies } a \wedge (x \vee y) = 0$$

By the law of distributives lattice we have

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge b) = 0 \wedge 0 = 0$$

Thm 11.8 Let $p_1, \dots, p_m$ be pairwise disjoint elements of a distributive lattice L with a 0. Define the function $\theta : P(M) \longrightarrow L$ $M = \{1, \dots, m\}$ as a mapping which takes each subset $S \subset M$ into $\theta(S) = \bigvee_{i \in S} p_i$ and $\emptyset$ into 0

11.8.1 $\left(\bigvee_{i \in S} p_i\right) \vee \left(\bigvee_{j \in T} p_j\right) = \bigvee_{S \cup T} p_k$

11.8.2 $\left(\bigvee_{i \in S} p_i\right) \wedge \left(\bigvee_{j \in T} p_j\right) = \bigvee_{S \cap T} p_k$

Proof. By the commutative, associative, and idempotent laws it holds 11.8.1

Likewise by the general distributive law we have

$$\left(\bigvee_{i\in S} p_i\right) \wedge \left(\bigvee_{j\in T} p_j\right) = \bigvee_{S\times T} (p_i \wedge p_j)$$

Since $p_i \wedge p_j = 0$ is disjoint we have

$$\bigvee_{S\times T} (p_i \wedge p_j) = \bigvee_{S\cap T} p_k$$

□

Thm 11.9 Every Boolean term in $Y = \{y_1, \dots, y_n\}$ can be put into the disjunctive normal form (DNF)

$$(11.9) \quad \bigvee_S p(w) \quad \text{where } p(w) = \bigwedge_n z_i \quad z_i = \begin{cases} y_i & \text{if } w_i = 1 \\ y_i' & \text{if } w_i = 0 \end{cases}$$

in one and only one way

Proof. By thm 11.8, the set of all joins of the $p(w)$ is closed under $\wedge$ and $\vee$

Furthermore if S and S' are complementary in set of words then

$$\bigvee_{S} p(w) \vee \bigvee_{S'} p(w) = \bigvee_{W} p(w) = 1$$

Since

$$1 = 1 \wedge \dots \wedge 1 = (y_1 \vee y_1') \wedge \dots \wedge (y_n \vee y_n') = \bigvee_{W} p(w)$$

again by the (generalized) distributive law

Hence by the uniqueness of complements in a Boolean algebra

$\bigvee_{S'} p(w)$ is the complement of $\bigvee_{S} p(w)$

and the set of Boolean forms in disjoint canonical form is a Boolean subalgebra of a given Boolean algebra

It remains to show that <u>all</u> y_i are in set (11.9)

To show this note that

$$y_1 = y_1 \wedge 1 \wedge \dots \wedge 1 = y_1 \wedge (y_2 \vee y_2') \wedge \dots \wedge (y_n \vee y_n')$$

By distribution law we have

$$y_1 = \bigvee_{S(1)} p(w)$$

Similary

we have $y_i = \bigvee_{S(i)} p(w)$ □

It is less obvious that the 2^{2^n} terms present different term fct.

One can prove that the Boolean algebra $P(2^n) = B^{2^n}$ is generated by n elements $x_1, \dots, x_n$

§12 Filters on Boolean algebra

Def 12.1 Let $\mathcal{B} = (B; \wedge, \vee, ', 0, 1)$ be a Boolean algebra. A subset $F \subseteq B$ is called a filter if it holds

(i) $1 \in F$ (that means $F \neq \emptyset$)

(ii) If $x \in F$ and $x \leq y$ then $y \in F$

(iii) If $x \in F$ and $y \in F$ then $x \wedge y \in F$

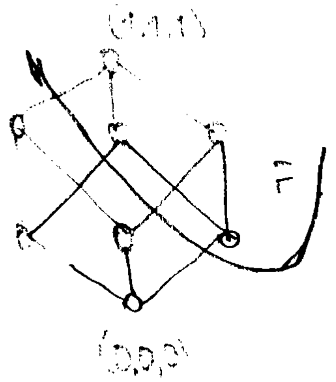
A filter F is an ultra filter if it holds

(iv) $0 \notin F$

(v) If $x \in \mathcal{B}$ then either $x \in F$ or $x' \in F$

Examples 12.2

12.2.1 $(\{0,1\}^3; \wedge, \vee, ', 0, 1)$



is an ultra filter

12.2.2 If $M \subseteq X$ then $\{A \mid A \in P(X), M \subseteq A\}$

is a filter of the Boolean algebra $(P(X); \cap, \cup, ', \emptyset, X)$

This filter is exactly an ultrafilter if M has only one element.

12.2.3 The set $\{A \mid A \in P(X), X \setminus A \text{ is finite}\}$

is a filter.

There are many ultrafilters which contains this filter

Remark 12.3

The dual concept of filter is the ideal, and ultrafilter is the prim ideal

The next thm we use the Lemma of Zorn also the axiom of choice

Thm 12.4 Every filter of a Boolean algebra is contained in an ultrafilter.

Proof. Let F be a filter in a Boolean algebra

Let $\underset{\sim}{F}$ be a set of all filters which contains F

As $F \in \underset{\sim}{F}$ there is $\underset{\sim}{F} \neq \emptyset$

$\underset{\sim}{F}$ is a partially ordered set in relation of containing "$\subseteq$"

We show: Every chain has an upper bound

Let $\underset{\sim}{D} = \{D_i \mid i \in I\}$ be a chain in $\underset{\sim}{F}$ and let

$D = \bigcup_{i \in I} D_i$.

If $x, y \in D$ then there exist $i, j \in I$ such that

$x \in D_i$, $y \in D_j$. As $\underset{\sim}{D}$ is a chain we have

either $D_i \subseteq D_j$ or $D_j \subseteq D_i$.

Let $D_i \subseteq D_j$. As D_j is a filter with $x, y \in D_j$

we have $x \wedge y \in D_j$ and therefore $x \wedge y \in D$.

As $x \in D$, $x \leq z$ we have that

$x \in D_i$ for some i and therefore D_i is a filter

It follows $z \in D_i$ and therefore $z \in D$

$0 \notin D_i$ for every $i \in I$ and therefore $0 \notin D$

Altogether we have that D is a filter

and $D \in \underset{\sim}{F}$

By the lemma of Zorn there exists a maximal element $\mathfrak{U}$ in $\underset{\sim}{F}$. It remains to show that $\mathfrak{U}$ is ultrafilter.

We assume that $x \in B$ and $x, x' \notin \mathfrak{U}$. We consider

$\mathfrak{U}' = \{y \mid \exists u \in \mathfrak{U} \ \ y \leq x \wedge u\}$. $\mathfrak{U}'$ is properly greater than $\mathfrak{U}$. As $\mathfrak{U}$ is maximal we have $\mathfrak{U}' \notin \underset{\sim}{F}$

Also we have $\mathfrak{U}' \notin \underset{\sim}{F}$ that means $0 \in \mathfrak{U}'$. Therefore there are $u \in \mathfrak{U}$ with $x \wedge u = 0$. But then we have

$$
\begin{aligned}
x' \wedge u &= (x' \wedge u) \vee 0 = (x' \wedge u) \vee (x \wedge u) \\
&= (x' \vee x) \wedge u = u \qquad \text{by distributivity}
\end{aligned}
$$

$\Rightarrow x' \in \mathfrak{U}$ ↯ $\square$

Def. 12.5 A subset A of elements of Boolean algebra is said to have the finite intersection property = FIP if the infinum of any finite subset of A is not equal to 0

Thm 12.6 Every subset A of elements of a Boolean algebra which have FIP is contained in an ultrafilter

Proof. Consider the sets

$$A^c = \{\inf X \mid X \text{ is a finite subset of } A\}$$

$$\overline{A} = \{x \mid x \in B,\ x \geq a \text{ for some } a \in A^c\}$$

We show:

$\overline{A}$ is a filter

If $x, y \in \overline{A}$ then there exist finite sets X, Y such that $x \geq \inf X$ and $y \geq \inf Y$

We have $\inf(X \cup Y) = \inf X \vee \inf Y$ $\leq x \vee y$. Therefore $x \wedge y \in \overline{A}$

It is clear that for $x \wedge \overline{A}$ and $x \leq z$ holds for $z \in \overline{A}$. Furthermore it is $0 \notin \overline{A}$ or else it would be $0 \in A^c$ and A has not the FIP. It is $A \subseteq \overline{A}$

By the ultrafilter theorem there exists an ultra filter U with $A \subseteq U$ $\square$

§13 Congruence relations

Def. 13.1 Let $\underset{\sim}{A} = (A; f_0, \dots, f_r, \dots)$ be an algebra. An equivalence relation ϱ is a congruence relation of $\underset{\sim}{A}$ if for every operation f_i holds:

If $(a_1, b_1) \in \varrho, \dots, (a_n, b_n) \in \varrho$

then $(f_i(a_1, \dots, a_n), f_i(b_1, \dots, b_n)) \in \varrho$

Examples

13.2.1 Let $(G; \cdot, e)$ be a group, N a normal subgroup of G

We define

$$(a,b) \in \varrho \iff aN = bN$$

ϱ is equivalence relation because =

i) Let $(a_1, b_1) \in \varrho$ and $(a_2, b_2) \in \varrho$

we have $a_1 N = b_1 N$, $a_2 N = b_2 N$

$$a_1 a_2 N = a_1 a_2 (a_2^{-1} N a_2) = a_1 N a_2$$

$$= b_1 N a_2 = b_1 (a_2 N a_2^{-1}) a_2 = b_1 a_2 N = b_1 b_2 N$$

$$\Rightarrow (a_1 \cdot a_2, b_1 \cdot b_2) \in \varrho$$

ii) $(a,b) \in \varrho \Rightarrow aN = bN \Rightarrow a^{-1} N = b^{-1} N$

$$\Rightarrow (a^{-1}, b^{-1}) \in \varrho$$

Example B 2.2 Let $(B; \wedge, \vee, `, 0, 1)$ be a Boolean algebra and let d be a fixed element of B.
We define:

$$(a,b) \in \rho \iff a \wedge d = b \wedge d$$

ρ is an equivalence relation because of "="

i) $(a_1, b_1) \in \rho$ and $(a_2, b_2) \in \rho$

$$\begin{aligned}(a_1 \wedge a_2) \wedge d &= (a_1 \wedge a_2) \wedge (d \wedge d) && \text{idempotency} \\ &= (a_1 \wedge d) \wedge (a_2 \wedge d_2) && \text{commutativity} \\ &= (b_1 \wedge d) \wedge (b_2 \wedge d) && \text{by hypothesis} \\ &= (b_1 \wedge b_2) \wedge d\end{aligned}$$

$\Rightarrow (a_1 \wedge a_2; b_1 \wedge b_2) \in \rho$

ii) $$\begin{aligned}(a_1 \vee a_2) \wedge d &= (a_1 \wedge d) \vee (a_2 \wedge d) && \text{distributivity} \\ &= (b_1 \wedge d) \vee (b_1 \wedge d) && \text{hypothesis} \\ &= (b_1 \vee b_2) \wedge d && \text{by distributivi}\end{aligned}$$

iii) $(a,b) \in \rho$

$\Rightarrow a \wedge d = b \wedge d$ $\Rightarrow (a \wedge d)` = (b \wedge d)`$

$\Rightarrow a` \vee d` = b` \vee d`$

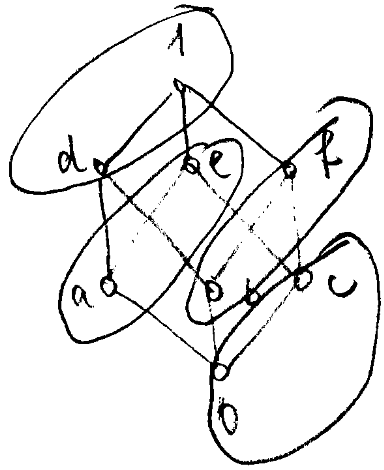
$\Rightarrow (a` \vee d`) \wedge d = (b` \vee d`) \wedge d$

$\Rightarrow (a` \wedge d) \vee (d \wedge d`) = (b` \wedge d) \vee (d` \wedge d)$

$\Rightarrow a` \wedge d = b` \wedge d$

$\Rightarrow (a`, b`) \in \rho$ $\square$

Example 13.2.3



$(x,y) \in \varrho \iff$

There exists 4 equivalence classes

$(e,a) \in \varrho$ because $e \wedge d = a = a \wedge d$

$(b,f) \in \varrho$ " $b \wedge d = b = f \wedge d$

Remark: In the general the equivalence classes have not equal many elements. But for the groups it holds the theorem of Lagrange $|aN| = |bN|$. It holds for Boolean algebra and rings

Def. 13.3 The algebra $A/_\varrho = (A/_\varrho ; f_0, \dots, f_\gamma, \dots)$ is called factor algebra. It is noted:

$$A/_\varrho = \{[a] \mid a \in A\}$$

$[a]$ is called an equivalence class and is defined by

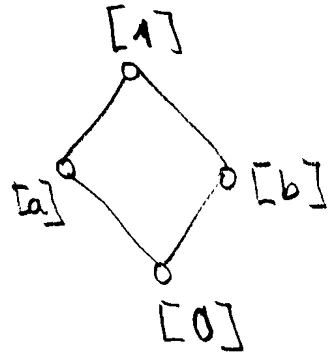
$$[a] = \{b \mid (a,b) \in \varrho, b \in A\}$$

$$f_i([a_1], \dots, [a_{n_i}]) := [f_i(a_1, \dots, a_{n_i})]$$

Continuation of 13.2.3

$[0] = \{0, c\}$
$[a] = \{a, e\}$
$[b] = \{b, f\}$
$[1] = \{d, 1\}$

$B/_\varrho$ "="



Factor algebra

$[a] \wedge [b] = [a \wedge b] = [0] \ldots\ldots$

Def. 13.4 The mapping $\mu : A \longrightarrow A/_\varrho$ of the algebra $\underset{\sim}{A}$ into the factor algebra is the kanonical homomorphism

Let $f : A \longrightarrow B$ be an homorphism from $\underset{\sim}{A}$ into $\underset{\sim}{B}$

then the relation $\ker f$ is called the Kernel

$$(a, b) \in \ker f \iff f(a) = f(b)$$

Thm 13.5 (Homomorphism Thm)

Let $f : A \longrightarrow B$ be a surjective homomorphism.

Then it holds
$$A/_{\ker f} \cong B$$

$$\left(\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\mu"'] & B \\ & A/_{\ker f} \arrow[u, "\cong"'] \end{tikzcd}\right)$$

§ 14 Tarski's Lemma

Prop. 14.1 Let F be a filter of Boolean algebra $\underset{\sim}{B}$

Then the relation $\eta = \eta(F)$ on $\underset{\sim}{B}$ is a congruence relation:

(*) $(x,y) \in \eta \iff$ There exists $f \in F$ with $x \wedge f = y \wedge f$

Proof

α) reflexive: Because $F \neq \emptyset$ there exists $f \in F$ with $x \wedge f = x \wedge f$. We have $(x,x) \in \eta$

β) antisymmetric:

Let $(x,y) \in \eta$. Then there exists $f \in F$ $x \wedge f = y \wedge f$.

It implies: there exists $f \in F$ $y \wedge f = x \wedge f$. $\Rightarrow (y,x) \in \eta$

γ) transitive

Let $(x,y) \in \eta$, $(y,z) \in \eta$. There exist $f \in F$ $x \wedge f = y \wedge f$ and $g \in F$ $y \wedge g = z \wedge g$

Now $f \wedge g \in F$ and

$$x \wedge (f \wedge g) = y \wedge (f \wedge g)$$
$$y \wedge (f \wedge g) = z \wedge (f \wedge g)$$

Therefore $x \wedge (f \wedge g) = z \wedge (f \wedge g) \Rightarrow (x,z) \in \eta$

η is an equivalence relation

i) $(x_1,y_1)\in\eta$, $(x_2,y_2)\in\eta$

$$\left.\begin{array}{ll}\exists f\in F & x_1\wedge f = y_1\wedge f\\ \exists g\in F & x_2\wedge g = y_2\wedge g\end{array}\right\} \Rightarrow \exists f\wedge g \quad x_1\wedge x_2\wedge(f\wedge g) = y_1\wedge y_2\wedge(f\wedge g)$$

ii) $(x_1,y_1)\in\eta$, $(x_2,y_2)\in\eta$

$$\exists f\in F \quad x_1\wedge f = y_1\wedge f \Rightarrow \exists f\wedge g\in F \quad x_1\wedge(f\wedge g) = y_1\wedge(f\wedge g)$$

$$\exists g\in F \quad x_2\wedge g = y_2\wedge g \Rightarrow \quad \dots$$

$$\Rightarrow \exists f\wedge g \quad [x_1\wedge(f\wedge g)]\vee[x_2\wedge(f\wedge g)] = [y_1\wedge f\wedge g]\vee[y_2\wedge f\wedge g]$$

$$\Rightarrow \exists f\wedge g \quad (x_1\vee x_2)\wedge(f\wedge g) = (y_1\vee y_2)\wedge(f\wedge g)$$

iii) $(x,y)\in\eta$

$$\Rightarrow \exists f\in F \quad x\wedge f = y\wedge f$$

We have also

$$f = f\wedge(x\vee x') = (f\wedge x)\vee(f\wedge x')$$
$$= (f\wedge y)\vee(f\wedge x')$$

Also
$$y'\wedge f = y'\wedge[(f\wedge x')\vee(f\wedge y)]$$
$$= [y'\wedge(f\wedge x')]\vee[y'\wedge y\wedge f]$$
$$= (y'\wedge x')\wedge f$$

Similarly $x'\wedge f = (y'\wedge x')\wedge f$ and $x'\wedge f = y'\wedge f$ $\square$

Prop. 14.2 Let F be a filter of a Boolean algebra $\underset{\sim}{B}$

For the canonical homorphism $h : B \to B/_{\eta(F)}$

$h(x) = 1$ it holds ifandonly if $x \in F$

Proof.

α) Let $d \in F$. Then $d \wedge d = 1 \wedge d$. Therefore $(d, 1) \in \eta(F)$

and $h(d) = h(1)$. For a homorphism we have $h(d) = 1$

β) Let $h(e) = 1$

Then wehave $h(e) = h(1)$ and $(e, 1) \in \eta(F)$

There exists a $f \in F$ such that $e \wedge f = 1 \wedge f$.

$e \wedge f = f \quad \Rightarrow e \geq f \quad , f \in F$

$\Rightarrow e \in F$ as F is a filter

Theorem 14.3 The following condition are equivalent

for a filter F on a Boolean algebra

α) $B/_{\eta(F)} \cong \{0, 1\}$

β) F is an ultrafilter

γ) If $x \wedge y \in F$ then either $x \in F$ or $y \in F$

(i.e F is a prim filter!)

Proof

$\alpha) \Rightarrow \beta)$

Let $x \notin F$. Then by prop. 14.2 we have $[x] \neq 1$

and by $B/_{\eta(F)} \cong \{0,1\}$ we have $[x] = 0$

Now $[x]' = [x'] = 1$. It follows $x' \in F$, then F is a ultrafilter

$\beta) \Rightarrow \gamma)$

Let $x \vee y \in F$ and assume that $x \notin F$, $y \notin F$

Then by $\beta)$ we have $x' \in F$ and $y' \in F$ and therefore $x' \wedge y' \in F$

and $(x \vee y)' \in F$. Contradiction: $(x \vee y) \in F$ ↯

$\gamma) \Rightarrow \beta)$

Because $x \vee x' = 1 \in F$ and because $\gamma)$ it follows

either $x \in F$ or $x' \in F$

$\beta) \Rightarrow \alpha)$

Let h be the canonical homomorphism from B

into $B/_{\eta(F)}$ and let be $h(x) \neq 1$

Now $x \notin F$ and therefore $x' \in F$

By Prop 14.2 we have $h(x') = 1$; $h(x)' = 1$

$\Rightarrow h(x)'' = 1' = 0$. Therefore $B/_{\eta(F)} \cong \{0,1\}$ ☒

Def. 14.4 Let $\{A_n \mid n \in I \subseteq \mathbb{N}\}$ be a countable family of subsets of a Boolean algebra which has infimas.

Let $a_n := \inf(A_n)$

A ultrafilter F of $\mathcal{B}$ is called infima preserving ($\wedge$-preserving) if it holds

$$h(a_n) = \inf\{h(a) \mid a \in A_n\}$$

where h is the canonical hom. from B into $B/\eta(F)$

Thm 14.5 (Tarski's Lemma)

For every element $x \in B$ $x \neq 0$ there exists an ultrafilter F with $x \in F$ which preserves the infimas

Proof.

We consider $\{A_n \mid n \in I \subseteq \mathbb{N}\}$ as defined as above.

- Furthermore we define recursively a sequence

$\{b_n \mid n \in I;\ I \subseteq \mathbb{N}\}$ such that

1) $b_n \in A_n$
2) $\{x, a_1 \vee b_1', \dots, a_n \vee b_n'\}$

(finite intersection property FIP)

- If $n = 0$ then we put $y := x$ and $\{x\}$ has the FIP because $x \neq 0$

Therefore we assume that we have found b_n for all $n < m$, m fixed chosen, $m \geq 1$

Put $y = x \wedge (a_1 \vee b_1') \wedge \dots \wedge (a_{m-1} \vee b_{m-1}')$

Which is unequal 0 by our assumption

- We assume $y \wedge (a_m \vee b') = 0$ for every $b \in A_m$

Now we have $y \wedge (a_m \vee b') = (y \wedge a_m) \vee (y \wedge b')$

i.e. $y \wedge a_m = 0$ and $y \wedge b = 0$ for all $b \in A_m$

therefore $y \leq b$ for all $b \in A_m$

Therefore $y \leq \inf(A_m) = a_m$

- Also we have $y \wedge a_m = y$ contradicting $y \wedge a_m = 0$

There is a $b_m \in A_m$ with

$$y \wedge (a_m \vee b_m) \neq 0$$

Consider the set $M = \{x, a_1 \vee b_1', \ldots, a_m \vee b_m', \ldots\}$

M has the FIP and by 12.4 M is contained in an Ultrafilter

Still to show: F preserve infimas

Let h be the canonical homomorphism from B into $B/\eta(F)$

As $a_n \vee b_n' \in F$ it holds $h(a_n \vee b_n') = 1$

by prop 14.2

i.e. $h(a_n) \vee h(b_n') = 1 \qquad \Rightarrow h(b_n) \leq h(a_n)$

therefore

$$\inf(\{h(b) \mid b \in A_n\}) \leq h(a_n)$$

As $a_n = \inf(A_n)$ and as $a_n \leq b$ for all $b \in A_n$

we have

$$h(a_n) \leq \inf(\{h(b) \mid b \in A_n\})$$

■

§ 15 Remarks

15.1 George Boole 1815-1864 (England)
Aside from his father's help and a few years of lotal schooles he was self-taught
He read the famous books on Newton, Laplace and Lagrange.
Boole contributed ~~an~~ original and remarkable method of logical inference, fully stated in Laws of Thought (1854).

15.2 Tarski 1902 - 1983 (Warsaw, Berkeley)
Tarski's Lemma is often formulated by the dual concept of the ultra filter and it is called prime ideal theorem (1952)

15.3 Topology

Let X be a topological space. For every point $x \in X$ there is a filter of x of the family $N(x)$ of neighborhood

Remark: A space X is exactly compact if every ultra filter of X converges to one point. In topology H. Cartan has indroduced the filters and ultrafilters (1937)

15.4 Epistemology

(episteme (knowledge) logos (reason))

Analytic philosophy was prevailing in the Anglo-American world in the 20th century. It has the origins of the symbolic logic (Frege, Peano, Schröder, Russels) and the empiricism

Wittgenstein published the Philosophical Invendtigations (1953) and On Certainty (1969)

III Propositional logic

§ 16 Language of the propositional logic

16.1 Notations

By an alphabet $V = \{A, B, C, \dots\}$ we understand a countable set of (propositional logical) variables. Moreover a set of operation symbols O are given:

$\neg$	(not)
$\wedge$	(and)
$\vee$	(or)
$\longrightarrow$	(if, then)
$\longleftrightarrow$	(iff, then)

We use also brackets (,) and the identity =

Def. 16.2 A word is a finite sequence of letters from $V \cup O \cup \{(,)\}$

Example $)A(\wedge B$

Def. 16.3 The set $\bar{A}$ of well-formed formulas is defined by recursion

i) Every (proposition) variable A is a well-formed formula

ii) If α and β well-formed formulas then $(\neg\alpha)$, $(\alpha\wedge\beta)$, $(\alpha\vee\beta)$, $(\alpha\rightarrow\beta)$ and $(\alpha\longleftrightarrow\beta)$ well-formed formulas

iii) No expression is well-formed unless it arise by the finite application of i) and ii)

The language of the proposition logic consists the words which are well-formed

Remark 16.4 The language of the proposition logic can be generated with an aid of a grammar.

$S\rightarrow A_i$, A_i (proposition) variables

$S\rightarrow(\neg S)$, $S\rightarrow(S\wedge S)$ and so on

These are the transformation rules.

Remark 16.5 We can also consider the set $\bar{A}$ as an algebra. $V = \{A_1, A_2, A_3, \ldots\}$ V the set of proposition variables is the generating system. The operations are also called formula-building operations

$f_\neg(\alpha) = (\neg\alpha)$ $\qquad$ $f_\to(\alpha,\beta) = \alpha \to \beta$

$f_\wedge(\alpha,\beta) = (\alpha \wedge \beta)$ $\qquad$ $f_\leftrightarrow(\alpha,\beta) = \alpha \leftrightarrow \beta$

$f_\vee(\alpha,\beta) = (\alpha \vee \beta)$

Remark. Two well-formed formulas are equal if they are identity as words

Thm 16.6 (Recursion theorem)

Let $\underset{\sim}{A} = (\bar{A}; f_\neg, f_\wedge, f_\vee, f_\to, f_\leftrightarrow)$ and $\underset{\sim}{B} = (B; g_\neg, g_\wedge, g_\vee, g_\to, g(\leftrightarrow))$ algebras of the same type $(1,2,2,2,2)$

Every mapping $h: V \longrightarrow B$ can be extended in only one form as a homomorphism $\underline{h}: A \to B$

Sketch of a proof (see my lecture page 28)

See Enderton page 27 respectively page 28

We define recursively by def. 16.3

$A_0 := V$

$A_n := \{f_\neg(\alpha), f_\wedge(\alpha,\beta) \dots f_\leftrightarrow(\alpha,\beta)\}$

Let $h_n : A_n \to B$ already be an extension of h on A_n

Now let $\gamma \in A_{n+1}$ and let $\gamma = f_d(\alpha, \beta)$ with $d = \{\wedge, \vee, \to, \leftrightarrow\}$

Then we define $h_{n+1}(\gamma) = f_d(h_n(\alpha), h_n(\beta))$

The extension is the only possibility to define h_{n+1} as a hommorphism

§ 17 Truth assignment

Notation 17.1 Let $B = \{0, 1\}$

0 is the truth value "false F "

1 is the truth value " true T "

A truth assignment h for a set V of variables is a function

$$h : V \longrightarrow B$$

assigning either T or F

Thm 17.2

The truth assignment h can be uniquely extended to the truth assignment $\bar{h} : \bar{A} \to B$ for $f_\neg, f_\wedge, f_\vee, f_\to, f_\leftrightarrow$

We define (17.2.1 – 17.2.5)

1) $\bar{h}(\neg\alpha) = \begin{cases} 1 & \text{if } \bar{h}(\alpha) = 1 \\ 0 & \text{else} \end{cases}$

2) $\bar{h}(\alpha \wedge \beta) = \begin{cases} 1 & \text{if } \bar{h}(\alpha) = 1 \text{ and } h(\beta) = 1 \\ 0 & \text{else} \end{cases}$

3) $\bar{h}(\alpha \vee \beta) = \begin{cases} 1 & \text{if } \bar{h}(\alpha) = 1 \text{ or } h(\beta) = 1 \text{ (or both)} \\ 0 & \text{else} \end{cases}$

4) $\bar{h}(\alpha \to \beta) = \begin{cases} 0 & \text{if } \bar{h}(\alpha) = 1 \text{ and } \bar{h}(\beta) = 0 \\ 1 & \text{else} \end{cases}$

5) $\bar{h}(\alpha \leftrightarrow \beta) = \begin{cases} 1 & \text{if } \bar{h}(\alpha) = \bar{h}(\beta) \\ 0 & \text{else} \end{cases}$

We have the table

α	β	$\neg\alpha$	$\alpha \wedge \beta$	$\alpha \vee \beta$	$\alpha \to \beta$	$\alpha \leftrightarrow \beta$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

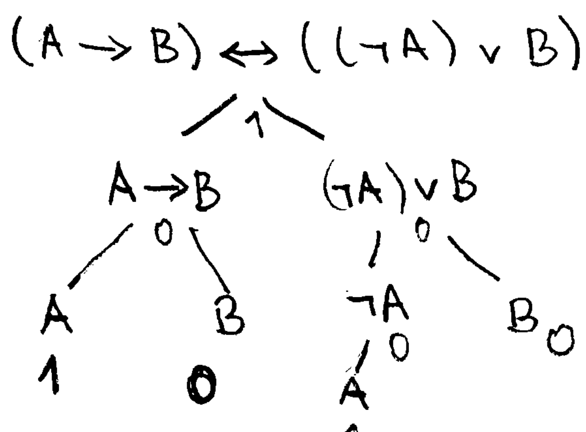
The proof of the thm 17.2 follows by the recursion thm

Example 17.3.1

$$\alpha \equiv (A \to B) \leftrightarrow ((\neg A) \vee B$$

Let $h(A) = 1$ and $h(B) = 0$ be a truth assignment

Then we have $\bar{h}(\alpha) = 1$



Example 17.3.2

For every possible truth assignment h of the variable A, B we get $\bar{h}(\alpha)=1$

Proof by the tables:

A	B	$(A \to B)$	$((\neg A) \vee B)$	$(A \to B) \Leftrightarrow ((\neg A) \vee B)$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
1	1	1	1	1

Notation 17.4

The set of well-formed formulas is denoted by $\bar{A}$

well-formed formulas:= wf formulas

Def. 17.5

A wf formula α is called valid or a tautology if $\bar{h}(\alpha)=1$ holds for every possible truth assignments h of every variables in α

We write:

$$\models \alpha$$

A wf formula α is satisfiable if there exists a truth assignment h such that $\bar{h}(\alpha) = 1$

The semantic implication

Def. 17.6 Let $\Sigma \subseteq \bar{A}$ and let $\varphi \in \bar{A}$

Σ semantically implies φ ($\Sigma \models \varphi$)
if and only if every truth assignment for the sentence in Σ and φ which satisfies every member of Σ also satisfies φ.

(If $\bar{h}(\psi) = 1$ for all $\psi \in \Sigma$ then $\bar{h}(\varphi) = 1$)

Examples

17.7.1
$\Sigma = \{(A \wedge B)$ semantically follows A

Proof.
Let $\bar{h}(A \wedge B) = 1$. By the definition 17.2.2
it follows $\bar{h}(A) = 1$ and $\bar{h}(B) = 1$
In any case it is $\bar{h}(A) = 1$, therefore $\Sigma \models \varphi$

17.7.2

$\Sigma = \{P, P \to Q\}$ semantically follows Q

Proof

Let $\bar{h}(P) = 1$ and let $\bar{h}(P \to Q)$.

From $\bar{h}(P \to Q)$ it follows that

$\bar{h}(Q) = 0$ if $\bar{h}(P) = 0$, else $\bar{h}(Q) = 1$

But as $\bar{h}(P) = 1$ it follows $\bar{h}(Q) = 1$ $\square$

§ 18 Syntax of the proposition logic

The syntax of the proposition logic consists of a set of wf formulas of the axioms and the derivation rules. There are many possibilities for the axioms and the rules.

We choose only one.

18.1. Axioms are wf formulas of the form

(A1) $\alpha \to \alpha$

(A2) $\alpha \to (\beta \to \alpha)$

A3 $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

~~A4~~ $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

A5 $\alpha \to (\alpha \vee \beta)$, $\beta \to (\alpha \vee \beta)$

A6 $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

A7 $(\alpha \wedge \beta) \to \alpha$, $(\alpha \wedge \beta) \to \beta$

A8 $(\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta))$

A9 $((\alpha \wedge \beta) \vee \gamma) \to ((\alpha \vee \gamma) \wedge (\beta \vee \gamma))$

$(\alpha \wedge \gamma) \vee (\beta \wedge \gamma) \to ((\alpha \wedge \beta) \vee \gamma$

A10 $((\alpha \vee \beta) \wedge \gamma) \to ((\alpha \wedge \gamma) \vee (\beta \wedge \gamma))$

$(\alpha \vee \gamma) \wedge (\beta \vee \gamma) \to ((\alpha \vee \beta) \wedge \gamma$

A11 $(\alpha \to \beta) \to ((\neg\beta) \to (\neg\alpha))$

A12 $(\alpha \wedge (\neg\alpha)) \to \beta$

A13 $\beta \to (\alpha \vee (\neg\alpha))$

II Rule

Modus ponens $\dfrac{\alpha, \alpha \to \beta}{\beta}$

The syntatic implication

Def. 18.2 A deduction of φ from $\Sigma \subseteq \bar{A}$ is a sequence $(\varphi_1, \dots, \varphi_n)$ of well-formed formulas with $\varphi_n \equiv \varphi$ such that for each $i \le n$ either

φ_i is an axiom or is in Σ or

for some k and l less than i, φ_i is obtained by modus ponens from φ_k and φ_l

We write $\Sigma \vdash \varphi$

φ is called derivable or a proof from Σ

If $\Sigma = \emptyset$ then we write $\vdash \varphi$

Examples 18.3

1) Let $\Sigma = (A \wedge B)$ and $\varphi = A$

Then $\Sigma \vdash \varphi$

Proof. Put $\varphi_1 \equiv (A \wedge B)$ and $\varphi_2 \equiv (A \wedge B) \to A$ is an axiom (A7) and $\varphi_3 = A$

For φ_3 we use modus ponens $\dfrac{(A \wedge B),\ (A \wedge B) \to A}{A}$

We have $\Sigma \vdash \varphi$

2) Let $\Sigma = \{P, P \to Q\}$ and $\varphi = Q$

Then $\Sigma \vdash \varphi$

Proof: Put $\varphi_1 = P$, $\varphi_2 = (P \to Q)$, $\varphi_3 = Q$

By the Modus ponens we have

$$\frac{P, \quad P \to Q}{Q}$$

Therefore $\Sigma \vdash \varphi$

Thm 18.4 (Correctness thm)

If $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$

Proof

One calculate the axioms with truth table method

A1 $\alpha \to \alpha$

α	$\alpha \to \alpha$
0	1
1	1

0	1	1
0	0	1
1	1	1
1	1	1

A2

α	β	$\beta \to \alpha$	$\alpha \to (\beta \to \alpha)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

The reader should check the Axioms A3 till A13 by table method

Let $h(\alpha) = 1$, $h(\alpha \to \beta) = 1$

Then $h(\beta) \neq 0$ by Def $h(\alpha \to \beta)$

Therefore we have $h(\beta) = 1$

and therefore the modus ponens is correct $\square$

18.5 Satisfiability Problem

Example:

Is the formula $\varphi \equiv (A \wedge B)$ satisfiable?

Is there are truth assignment h with $\bar{h}(\varphi) = 1$?

A	B	A ∧ B
0	0	0
0	1	0
1	0	0
1	1	1

You can decide after the 4th proposition

General: We have to make 2^n tests of n propositions

This "naive" algorithm is exponential !!!

18.6 Main problem

Does exists an algorithm which decide for n (proposion) variable in $f(n)$ steps this question where $f(n)$ is polynomial!

This question is open in mathematics and computer science:

$$P = NP \,?$$
$$P \neq NP \,?$$

complexity

§19 Completeness theorem

Our aim is to show

$$\text{If } \Sigma \models \varphi \quad \text{then } \Sigma \vdash \varphi$$

We consider the set $\bar{A}$ of well-formed formulas and the following relation $\approx$ (congruent)

$$\alpha \approx \beta \iff \Sigma \vdash (\alpha \to \beta) \text{ and } \Sigma \vdash (\beta \to \alpha)$$

Prop. 19.1 $\approx$ is an equivalence relation

Proof.

reflexive : $\alpha \approx \alpha$ because $\Sigma \vdash (\alpha \to \alpha)$ Axiom A1

symmetric : clear:

If $\alpha \approx \beta$ then $\Sigma \vdash (\alpha \to \beta)$ and $\Sigma \vdash (\beta \to \alpha)$ and then $\Sigma \vdash (\beta \to \alpha)$ and $\Sigma \vdash (\alpha \to \beta)$, also $\beta \approx \alpha$.

transitive : Let $\alpha \approx \beta$ and $\beta \approx \gamma$

Consider $\Sigma \vdash (\alpha \to \beta)$, $\Sigma \vdash (\beta \to \gamma)$ and A3

$$\frac{(\alpha \to \beta),\ (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))}{(\beta \to \gamma) \to (\alpha \to \gamma)} \quad \text{Modus ponens}$$

$$\frac{(\beta \to \gamma),\ (\beta \to \gamma) \to (\alpha \to \gamma)}{\alpha \to \gamma} \quad \text{Modus ponens}$$

Therefore we have

$\Sigma \vdash (\alpha \to \gamma)$

In the same way we get $\Sigma \vdash (\gamma \to \alpha)$

and therefore

$$\alpha \approx \beta$$

We define a relation $\leq$ by $[\alpha] \leq [\beta]$ on $\bar{A}/_{\approx}$ if and only if $\Sigma \vdash (\alpha \to \beta)$

Prop. 19.2 $(\bar{A}/_{\approx}; \leq)$ is a partially order

Proof.

$\leq$ is reflexive because $\Sigma \vdash (\alpha \to \alpha)$ A1

$\leq$ is antisymmetric beause the definition of "$\approx$"

$\leq$ is transitive as in the ~~proof~~ of 19.1.

If $\Sigma \vdash (\alpha \to \beta)$ and $\Sigma \vdash (\beta \to \gamma)$ then we get A3 and Modus ponens and we get

$\Sigma \vdash (\alpha \to \gamma)$

Prop. 19.3 $(\bar{A}/_{\approx}; \wedge, \vee, `, \bar{0}, \bar{1})$ is a Boolean algebra in regard of "$\leq$" if one defines

$$[\alpha] \wedge [\beta] := [\alpha \wedge \beta]$$

$$[\alpha] \vee [\beta] := [\alpha \vee \beta]$$

$$[\alpha]' := [\alpha']$$

Proof.

We show that $[\alpha \wedge \beta]$ is the infimum of $[\alpha]$ and $[\beta]$

Because A7 it follows $(\alpha \wedge \beta) \to \alpha$ and $(\alpha \wedge \beta) \to \beta$

Therefore $[\alpha \wedge \beta]$ is a lower bound of $[\alpha], [\beta]$

If $[\gamma]$ is another lower bound of $[\alpha], [\beta]$

then it holds $\Sigma \vdash (\gamma \to \alpha)$, $\Sigma \vdash (\gamma \to \beta)$

Furthermore:

A8 $(\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta)))$

By the help of the Modus ponens (MP)

$$\frac{(\gamma \to \alpha), (\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta)))}{(\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta))}$$

$$\frac{(\gamma \to \beta), (\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta))}{\gamma \to (\alpha \wedge \beta)}$$

Also $[\gamma] \leq [\alpha \wedge \beta]$

i.e. $[\alpha \wedge \beta]$ is the infimum of $[\alpha], [\beta]$

Similarly we show that $[\alpha \vee \beta]$ is the supremum of $[\alpha], [\beta]$. We need the axioms A5 and A6 and the MP (Modus ponens)

It follows that $(\bar{A}/_{\approx}\,;\wedge,\vee)$ is a lattice because any partially ordered set with infimum and supremum fulfills the axioms of a lattice

Obviously the axioms A9 and A10 is the distributivity

$\bar{0} := [\alpha \wedge (\neg\alpha)]$ and $\bar{1} := [\alpha \vee (\neg\alpha)]$

We show that $\bar{0} \leq [\beta]$

This is because of A12 namely the zero element $\bar{0}$ and A13 is the one element $\bar{1}$

It holds:

$$[\varphi] \vee [\neg\varphi] = [\varphi \vee (\neg\varphi)] = \bar{1}$$
$$[\varphi] \wedge [\neg\varphi] = [\varphi \wedge (\neg\varphi)] = \bar{0}$$

Therefore $(\bar{A}/_{\approx}\,;\wedge,\vee,\bar{0},\bar{1})$ is a distributive lattice where every element has a complement.

It follows:

$(\bar{A}/_{\approx}\,;\wedge,\vee,{}',\bar{0},\bar{1})$ is a Boolean algebra

$\square$

Thm 19.4 In the Boolean algebra

$$(\bar{A}/_{\approx}; \wedge, \vee, ', \bar{0}, \bar{1})$$

holds

$$[\alpha] = \bar{1} \text{ iff } \Sigma \vdash \alpha$$

That means: the derivable formulas are in the congruence class $\bar{1}$

Proof. Let $\Sigma \vdash \alpha$

By A2 $\alpha \to (\beta \to \alpha)$ it holds

If $\Sigma \vdash \alpha$ then $\dfrac{\alpha \quad \alpha \to (\beta \to \alpha)}{\beta \to \alpha}$

Therefore $\vdash (\beta \to \alpha)$ and also $[\beta] \leq [\alpha]$

for all $\beta \in \bar{A}$ Therefore $[\alpha] = \bar{1}$

On the other hand let $[\alpha] = \bar{1}$

Then $[\beta] \leq [\alpha]$ for all $\beta \in \bar{A}$

Also $\Sigma \vdash (\beta \to \alpha)$

If we choose β such that $\Sigma \vdash \beta$

then we have by the Modus ponens

$$\frac{\beta \quad \beta \to \alpha}{\alpha}$$

. Therefore $\Sigma \vdash \alpha$

Semantic

("meaning, interpretation")

$\Sigma \models \varphi$ iff every truth assignment in Σ and φ which satisfies Σ also satisfies φ

(If $\bar{h}(\psi)=1$ for all $\psi \in \Sigma$ then $\bar{h}(\varphi)=1$)

$$\Sigma \models \varphi$$

semantically follows

satisfy

is a model

truth table

Syntax

("building of form")

A deduction (or a proof) is a sequence $(\varphi_1, \dots, \varphi_n)$ such that either

φ_i is an axiom or

φ_i is in Σ or

φ_i is obtained by Modus ponens

$$\Sigma \vdash \varphi$$

syntactically follows

derivable

have axioms and rules

proof.

Correctness thm

If $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$

Completeness thm

If $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$

Notation. The algebra $\underset{\sim}{A} = (\bar{A}/_{\sim} ; \wedge, \vee, ', \bar{0}, \bar{1})$ is called the Lindenbaum algebra of the proposition logic.

Prop. 19.5 Let h be a homomorphism of the Lindenbaum algebra $\underset{\sim}{A}$ into the Boolean algebra $(\{0,1\}; \wedge, \vee, ', 0, 1)$. If f is defined in V by

$$f(A_i) = h([A_i])$$

then f is a truth assignment of $\bar{A}$ such that

$$f(\alpha) = h(\alpha)$$

The proof follows by the recursion thm 16.6

Thm 19.6 (Completeness theorem)

If $\Sigma \models \varphi$ then also $\Sigma \vdash \varphi$

Proof (indirect)

Let $\Sigma \models \varphi$ and φ be not derivable

Then we have in the Lindenbaum algebra $[\varphi] \neq \bar{1}$

and therefore $[\neg\varphi] \neq \bar{0}$

By Tarski's Lemma 14.5 there exists a ultrafilter F

which contains $[\neg\varphi]$

By the thm 14.3 we have $\bar{A}/_{\eta(F)} \cong \{0,1\}$

Let h the canonical homomorphism from

$\bar{A}$ into $\bar{A}/_{\eta(F)}$

As $[\neg\varphi] \in F$ then it holds $\bar{h}([\neg\varphi]) = 1$

For the truth assignment f induced ~~by~~ h

it holds that $f(\neg\varphi) = 1$

Therefore $f(\varphi) = 0$ and therefore $\Sigma \not\models \varphi$ $\lightning$

$\square$

Thm 19.7

$$\Sigma \models \varphi \text{ if and only if } \Sigma \vdash \varphi$$

The syntactical derivation $\vdash$ is equivalent

to the semantical implication

§20 Satisfiability and consistence

Thm 20.1 (Finiteness thm)

If $\Sigma \vdash \varphi$ then there exists a finite set $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \varphi$

Proof follows from the definition of the deduction

Thm 20.2 (Deduction thm)

Let Σ be a set of w.f. formulas

$\Sigma \cup \{\Psi\} \vdash \varphi$ if and only if $\Sigma \vdash (\Psi \to \varphi)$

Proof

I) Let $\Sigma \vdash (\Psi \to \varphi)$ and consider $\Sigma \cup \{\Psi\}$

Then $\Sigma \cup \{\Psi\} \vdash \Psi$ holds and then by Modus ponens

$$\frac{\Psi \,,\ \Psi \to \varphi}{\varphi}$$

Therefore $\Sigma \cup \{\Psi\} \vdash \varphi$

II) Let $\Sigma \cup \{\psi\} \vdash \varphi$

Then there exists a finite set of w.f. formulas $\{\psi_1, \dots, \psi_n\}$ with $\{\psi_1, \dots, \psi_n\} \cup \{\psi\} \vdash \varphi$

Let h be a truth assignment with $\bar{h}(\psi_1, \dots, \psi_n) = 1$

and $\bar{h}(\psi)$. By the correctness thm we have

$\bar{h}(\varphi) = 1$ (*)

We now form it

$$\alpha \equiv (\psi_1 \wedge \dots \wedge \psi_n) \to (\psi \to \varphi)$$

and we will prove that α is a tautology

$\psi_1 \wedge \dots \wedge \psi_n$	ψ	φ	$\psi \to \varphi$	α	
0	0	0	1	1	
0	0	1	1	1	
0	1	0	0	1	
⋮	⋮	⋮	⋮	⋮	
1	1	0	0	0	This case does not hold because (*)
1	1	1	1	1	

By the completeness thm it follows

$$\vdash (\psi_1 \wedge \dots \wedge \psi_n) \to (\psi \to \varphi)$$

or $\Sigma \vdash (\psi \to \varphi)$

$\square$

Def. 20.3 A set Σ of w.f. formulas is called consistent if φ and also $\neg\varphi$ can be derived from Σ

(Remark: Is the classical mathematik consistent? Is the set theory consistent?)

Lemma 20.4 Let Σ be a set of w.f. formulas. $\Sigma \cup \{\varphi\}$ is consistent if and only if $\Sigma \vdash (\neg\varphi)$ does not hold.

Proof.

I) If $\Sigma \vdash (\neg\varphi)$ then it holds also $\Sigma \cup \{\varphi\} \vdash (\neg\varphi)$

But it also holds $\Sigma \cup \{\varphi\} \vdash \varphi$

Therefore $\Sigma \cup \{\varphi\}$ is not consistent

II) Assume that $\Sigma \cup \{\varphi\}$ is not consistent

that means $\Sigma \cup \{\varphi\} \vdash \psi$ and $\Sigma \cup \{\varphi\} \vdash (\neg\psi)$

By the deduction thm we have

$$\Sigma \vdash (\varphi \to \psi) \text{ and } \Sigma \vdash (\varphi \to (\neg\psi))$$

Now $\alpha = (\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$

φ	ψ	$\varphi \to \psi$	$\varphi \to \neg\psi$	$(\varphi \to \neg\psi) \to \neg\varphi$	α
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	1	0	1
1	1	1	0	1	1

By the modus ponens

$$\frac{\varphi \to \psi, \quad \alpha}{(\varphi \to \neg\psi) \to \neg\varphi}$$

Again MP

$$\frac{\varphi \to (\neg\psi), \quad (\varphi \to \neg\psi) \to (\neg\varphi)}{\neg\varphi}$$

Therefore we have $\Sigma \vdash (\neg\varphi)$ $\square$

This theorem gives the following sequences

A w.f. formula φ is consistent if $\neg\varphi$ is not derivable.

By the completeness thm $\neg\varphi$ is not derivable if $\neg\varphi$ is not satisfiable.

Thm 20.5 A finite set Σ_0 of w.f. formulas is consistent if and only if Σ_0 is satisfiable.

§21 Compactness theorem

Thm 21.1 Let Σ be a set of of formulas.
Σ is satisfiable if and only if every finite subset of Σ is satisfiable.

Proof

I) If Σ is satisfiable then every finite subset of Σ is satisfiable.

II) Let every subset Σ_0 of Σ satisfiable.

We consider all finite subset Σ_0 of Σ with the property.

Every $\varphi \in \Sigma_0$ has less than n (proposition) variables $A_1, \ldots, A_n$, $n \in \mathbb{N}$ fixed !

All finite subsets Σ_0 are satisfiable by hypothesis

Therfore it holds the

Induction beginning: There is a truth assignment $h : \{A_i \mid i < n\} \longrightarrow \{0,1\}$ such that the truth assignment $\bar{h} : \Sigma_0 \to \{0,1\}$ is satisfied

Induction step

We show that h can be extended to $\{A_i \mid i \leq n\}$

Assume that it is not so, if we put $h(A_n) = 0$

Then there is a finite $\Sigma_1 \subseteq \Sigma$ such that $\bar{h}$ is not satisfied and have less than n variables.

Let $\Sigma_0 \subseteq \Sigma_1$ where Σ_0 with $n-1$ variables (which it also requires) By hypothesis Σ_1 must have be satisfied as finite set by $\bar{g}$. For Σ_0 it is $\bar{h} = \bar{g}$ and we conclude that $\bar{g}(A_n) = 1$ and we put $\bar{h}(A_n) = 1$

Let $\varphi \in \Sigma$. We choose n large enough such that all variables of φ are required in $\{A_i \mid i \leq n\}$

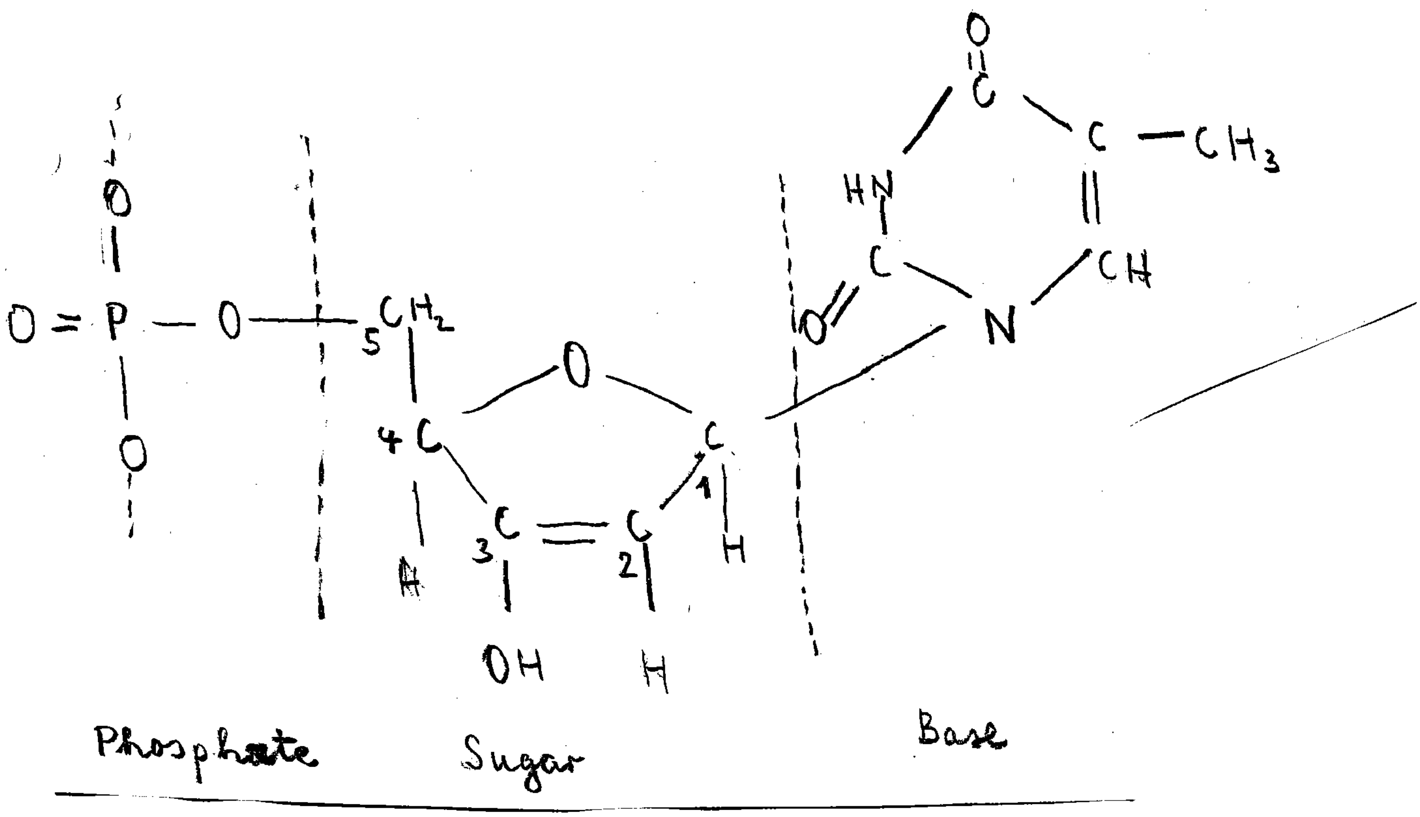
Also $\bar{h}(\varphi) = 1$ and then Σ is satisfied

§22 DNA Computing ((This a sketch))

Organic cells can be used in computers to replace digit switching circuits. The idea is that a "soup" = millions of DNA cells work as millions of "parallel" computers

22.1 DNA = Deoxyribo Nucleid Acid

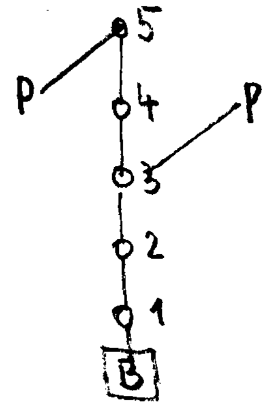
The chemical structure of a nucleotide



Oxygen | Sauerstoff O
Hydrogen | Wasserstoff H
Carbon | Kohlenstoff C

22.2

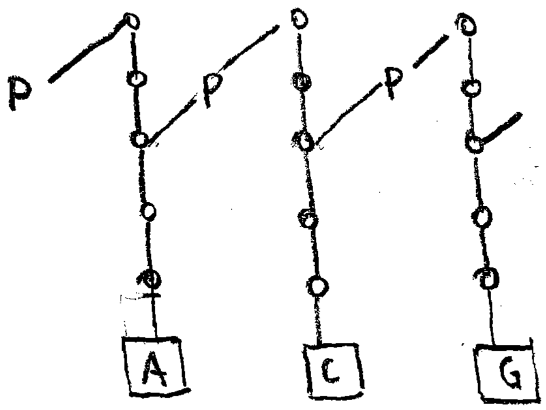
We will present this molecule in the form



P Phosphor

The 4 possible bases are A, T, C, G

P is a strong band (Band, Fessel)



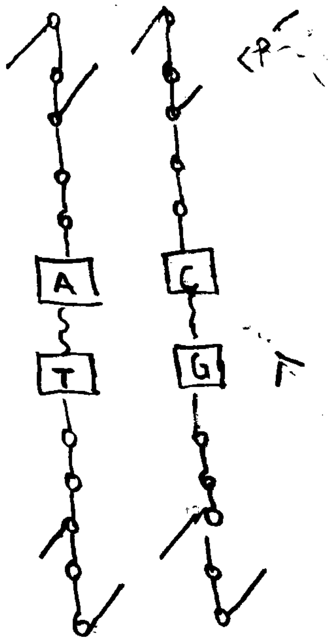
single stranded DNA

~~Many molecules build a double strand (Streifen)~~

This pairing principle is called the

Watson-Crick complementary of the famous double

helix structure



(The nucleotides link together in two different ways

1) phosphodiester bond

2) hydrogen bond

22.3 There are many operations on DNA molecules which we symbolize as strands. These operation can be applied in biochemical engineering.

Example

C G G A A A T
A C C T T T A

strand of the double helix

Operation 1 Lengthenging and shortening of the DNA

Operation 2 Cuting of the DNA

Operation 3 Linking (pasting) the DNA

Operation 4 Multiplying the DNA

and many others

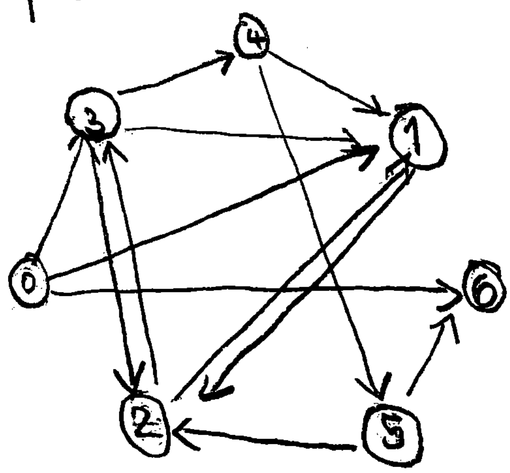
22.B HPP = Hamiltonian Path Problem

The HPP consists to deciding whether or not an arbitrarily given digraph has a hamiltonian path.

G digraph, input v_{in}, output v_{out}

A path from v_{in} to v_{out} is called hamiltonian if it involves every vertex exactly once

Example 22.4



The path 0 1 2 3 4 5 6 is hamiltonian

HPP is NP-complete.

Adleman's solution is based on the following nondeterministic algorithm for solving HPP

22.4 Adleman's algorithm

Input : A directed graph G with n vertices among which are designated vertices v_{in} and v_{out}

Step 1 Generate paths in G randomly in large quantities

Step 2 Reject all path that do not begin with v_{in} and end in v_{out}

Step 3 Reject all path that do not involve exactly n vertices

Step 4 For each of the n vertices v reject all paths that do not involve v

Out put: Yes if any path remains

No otherwise.

22 C Satisfiability problem

Repetition 22.6

A w.f. formula α is satisfiable if it assumes the truth-value 1 for at least one truth assignment. Clearly α is not satisfiable exactly in the case if its negation $\neg\alpha$ is a tautology

Lipton's DNA based solution of the satisfiability problem is an exhaustive search made by the massive parallelism of the DNA.

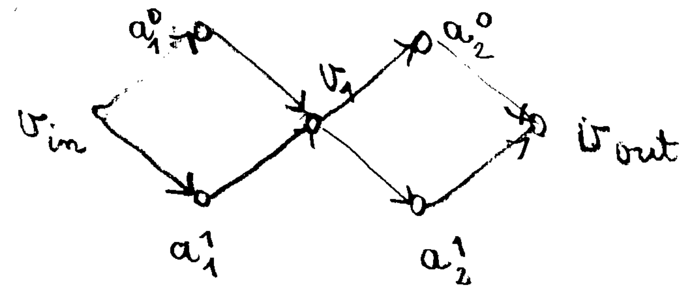
We begin with a graphical description of the truth assignments

Example 22.7

$$\beta \equiv (x_1 \vee x_2) \vee (\neg x_1 \vee \neg x_2)$$

We draw 2^k paths from v_{in} to v_{out} $(2^k = 4)$

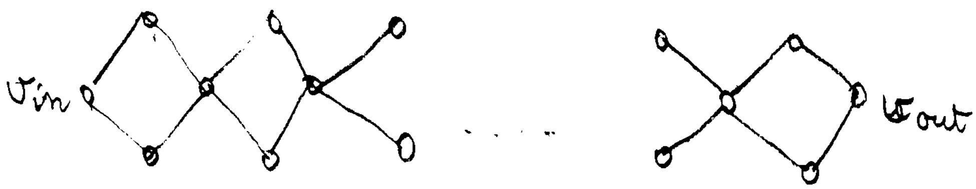
The paths and the truth assignments for the variables $x_1, x_2, \ldots, x_k$ $(k = 2)$ have a correspondence



general
$v_{in}, v_1, v_2 \ldots v_k, v_{out}$

Each of the 4 paths through this graph corresponds to one of the 4 truth assignment for the variables x_1 and x_2

In general we have:



Now we have the same way as in Adleman's experiment

For each vertex and edge in the graph we create a "soup" which contain a DNA double strand encoding of truth assignments for k variables.

§23 Remarks

1. On the history

The idea of the Lindenbaum algebra was ariside in Lindenbaum (1935) and also in Tarski

The first proof of the completeness thm was made by E. Post (1921) and independently Łukasiewicz (1921)

Our proof of the completeness thm is essentially made by Helen Rasiowa and Sikorski (and Tarski) (1951)

2. Noun expression

We make a difference in

a) proposition logic

b) the logic of the noun expressions

In the proposition logic there are variables A, B, C, ...

Examples of the logic of the noun expressions are

"Human are mortal"

"Greeks are men und women"

The logic of the nouns consists of simple sentences which can be divided small parts. The category of name (Socrates, Father of the ~~Socrates~~) and the category of the commoun nouns, also predecates, verbs, for example (are, mortal) and expressions of verbs: (is a philosopher)

3. Intuitionism.

Exercise: Find two irrational number a, b such that a^b is rational.

Solution: Either is $\sqrt{2}^{\sqrt{2}}$ rationl and then $a = \sqrt{2}$, $b = \sqrt{2}$ or $\sqrt{2}^{\sqrt{2}}$ is irrational, then choose $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{2} = 2$

The intuitions don't agree this solution. They do not accept the law of excluded middle (A 13) $\beta \rightarrow (\alpha \vee (\neg\alpha))$ Only by the constructuction as a proof is accepted,

4. The discussion of the A2

(A2) $\alpha \rightarrow (\beta \rightarrow \alpha)$ entailment

Some philosophers require that a implication $\alpha \rightarrow \beta$ should be a meaningful statement combining verbal and nominal elements and not only a formal implication.

- Relevance Logic - http://plato.stanford.edu/entries/logic-relevance/
  Article in the Stanford Encyclopaedia of Philosophy, by Edwin Mares.
- Conditionals and Counterfactuals - http://www.c-parr.freeserve.co.uk/hcp/if.htm
  Essay by Hector Parr, considering some philosophical difficulties with `if'.
- Paradoxes of Material Implication - http://www.earlham.edu/~peters/courses/log/mat-imp.htm
  Lecture notes by Peter Suber.
- Faithful and Fruitful Logic - http://www.bu.edu/wcp/Papers/Logi/LogiHowe.htm
  Article by John Howes, presented at the 20th World Congress of Philosophy.
- Nomic Dependencies & Contrary-to-fact Conditionals - http://www.lawrence.edu/fac/boardmaw/contr_cond_nomic.html
  Essay by William S. Boardman discussing Fred Dretske's measles examples in his book `Knowledge and the Flow of Information'.
- Peirce and Philo - http://www.clas.ufl.edu/users/jzeman/peirce_and_philo.htm
  Discusses Charles Peirce's account of conditionals, hypotheticals, and what he takes from Cicero's account of the debate between Philo and Diodorus.
- Conditional Sentences - http://www.icsi.berkeley.edu/~kay/bcg/lec07.html
  Lecture notes on the analysis of condition sentences by Paul Kay.

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5. The discussion of the A11

$(\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \neg\alpha)$

In constructive mathematics (Brouwer, Heiting (Intuitionism), Erett Bishop) it is distincted between the construction of a mathematical object and an indirect proof of its existence. In the constructive mathematics it is universally rejected.

the indirect proofs and the reductio ad absurdum.

(Indirect proof: $\dfrac{\alpha \to \beta,\ \neg\beta}{\neg\alpha}$)

6. Topology

The theorem of compactness is equivalent of

$\{0,1\}^V$

$\{0,1\}$ is the discrete topology

$\{0,1\}^V$ is the product topology and is compact.

(In graph theory there is the compactness theorem in form of Hall's theorem)

Mariage thm)

7. Aristotle's logical works, which have been grouped together under the name of the *Organon*, are concerned with two major problems, the technique of proof and the principles of proof.

The broad sequence of subjects followed in the first three books of the *Organon* pursues the problems involved in such a construction. The first treatise, the *Categories*, is concerned with *simple*, uncombined terms treated under ten most universal heads or categories. The second treatise, *On interpretation*, is concerned with *pairs* of terms combined in propositions and expressive of truths and falsities conceived by the mind. The third treatise, the *Prior analytics*, is concerned with inference or, since all perfect inference may be stated as a syllogism or a series of syllogisms, with combinations of *three* terms in an argument.

In the last three treatises of the *Organon*, he distinguished three kinds of syllogisms according to their principles or premisses, and that differentiation of three modes of argumentation sets his doctrine, even when it seems derivative, apart from the philosophy of Plato; for the dialectical method, which was the one scientific and philosophic method according to Plato, becomes according to Aristotle a second-best method distinct from the method of science. The conditions relevant to the selection of true first principles, particularly of definitions, determined to the nature of things are treated in the *Posterior analytics*: syllogisms based on such premisses are scientific and demonstrative. The conditions relevant to the selection or rejection of principles which express only opinions, whether generally accepted or stamped with the authority of experts, are treated in the *Topics*: syllogisms based on such premisses are dialectical and probable. The analysis of arguments dependent on opinions which seem to be generally accepted but are not, or which consist, not in reasoning, but in apparent reasoning from generally accepted opinions or from opinions apparently so accepted, as well as the methods for refuting such arguments, is taken up in *On sophistical refutations*; and as Aristotle remarks, such fallacious arguments may be traced for the most part to ambiguities of language.

IV Predicate logic

§ 24 First-Order language

24.1 For the language of the predicate logic we use the following alphabet Δ which is divided into groups

I Logical symbols

1. Variables $x_1, x_2, x_3, \dots$
2. sentential connective symbols: $\neg, \wedge, \vee, \rightarrow$
3. brackets $($, $)$ (eventually $\equiv$ identity)

II Parameters

1. Quantifier symbols $\forall x$
2. Predicate symbols
   There are of possibly empty set of n-place predicate symbols for every $n \in \mathbb{N}$
3. Constant symbols
   A possibly empty set of constant symbols. We will conceive it as nullary predicate symbols

4. Function symbols

There are a possibly empty set of n-place function symbols.

Example 24.2 The language of the elementary number theory

predicate symbol	$\leq$	
constant	0	
functions	S	successor fct.
	$+$	$S(x) = x+1$
	$\cdot$	
	E	
	$(=)$	

Example 24.3 Set theory

predicate	$\in$	2-places
constant	$\emptyset$	
no fct		
identity	$=$	

Words are finite sequences of letters of the alphabet for example: $\in \forall x + y$. But only those words are interesting which are well formed.

1. Terms

We define a n-place term-building operation F_{f_i} for every n-place function symbols f_i by:

Let $\varepsilon_1, \dots, \varepsilon_n$ be words then

$$F_{f_i}(\varepsilon_1, \dots, \varepsilon_n) := f_i(\varepsilon_1, \dots, \varepsilon_n)$$

Def. 24.4 The set of terms is the set of words which can be built by the constants symbols and variable with the aid of the term building operations

Example 24.5 Language of number theory

Terms are $+xy$ in other way: $x+y$

$SSSS0$ " 4

Polish notation

(If there are no fct symbols then the variables (and eventually constants) are terms)

2. Atomic formulas.

Def. 24.6 An atomic formula is a word of the form $P(t_1, \dots, t_n)$

where P is a n-place predicate symbols and $t_1, \dots, t_n$ are terms

Example 1 $\leq 0 S x$ another writing $0 \leq S x$

Example 2

$\in x y$ or $x \in y$

$\in \emptyset x$ or $\emptyset \in x$

3. Formulas

Formula-building operation are

1) $f_{\neg}(\alpha) := (\neg \alpha)$ 2) $f_{\rightarrow}(\alpha, \beta) = (\alpha \rightarrow \beta)$

3) $f_{i\forall}(\alpha) := \forall x_i \alpha$

Def. 24.7 The set of formulas is the set of words which can be build of the atomic formulas with the aid of the formula building operations

Example 1

$\forall x \forall y \leq x + xy$ or usually: $\forall x \forall y \; x \leq x+y$

Example 2

$(\forall x \, (x \in y)) \rightarrow (x \in z)$

4. Free variables

Def. 24.8 We define recursively for every formula α which it means that a variable x is free in α.

1) If α atomic then x occurse free in α iff x is in α

2) x is free in $(\neg \alpha)$ iff x is free in α

3) x is free in $(\alpha \rightarrow \beta)$ iff x is free in α or in β

4) x is free in $\forall y \, \alpha$ iff x is free in α and $x \neq y$

Example

$\forall x (x \in y)$ is bounded and therefore <u>not</u> free

$\forall y (x \in y)$ is free

5. Notations

$(\alpha \wedge \beta)$ is a shortening notation for $((\neg\alpha)\to\beta)$

$(\alpha \vee \beta)$ abbreviates $(\neg(\alpha \to (\neg\beta)))$

$(\alpha \leftrightarrow \beta)$ abbreviates $(\alpha\to\beta)\wedge(\beta\to\alpha)$

$\exists x\, \alpha$ abbreviates $(\neg\forall x(\neg\alpha))$

The language with alphabet Δ we denote with

$$L(\Delta)$$

§25 Structures and interpretations

A structure $\tilde{A}$ for a language $L(\Delta)$ of first order consists of a (carrier) set A and relations.

The set A of the structure $\tilde{A}$ is assigned by the quantifier symbol $\forall$

Every n-place predicate symbol P is assigned to a n-place relation P_A

Every constant symbol c is assigned to 0-place function symbol c_A

Every n-place function symbol f_i is assigned to n-place function f_i, $n \geq 1$

$\tilde{A}$ is an interpretation for $L(\Delta)$

Example.
Let $L(\Delta)$ be the language of the elementary number theory. Then the $L(\Delta)$ is assigned to the structure $(\mathbb{N}; 0, S, +, \cdot, \exp; \leq)$ with the type $(0,1,2,2,2;2)$

Thm 25.1 (Recursion thm)
Let $L(\Delta)$ be a language and $\tilde{A}$ be a structure belonging to $L(\Delta)$
Every mapping $h: X \longrightarrow A$ can be extended in exactly one way to $\bar{h}: T \longrightarrow A$ where T is the set of terms of L such that

(i) $\bar{h}(x) = h(x)$ for all $x \in X$

(ii) $\bar{h}(f_i(t_1, \ldots, t_n)) = f_i(\bar{h}(t_1), \ldots, \bar{h}(t_n))$

for all function symbols (also constant symbols) and all terms $t_1, \ldots, t_n \in T$

§ 26 Semantic

Let Δ be an alphabet and $L(\Delta)$ be a language. As a truth assignment of the variable we understand a belonging structure. $\tilde{A} = (A; f_1, \dots, f_\gamma, \dots; r_1, \dots, r_\beta, \dots)$ together with a mapping $h : X \longrightarrow A$.

By the recursion thm there exists exactly an extension $\bar{h} : T \longrightarrow A$, T set of terms.

The concept of satisfiability

Notation. We say that $\tilde{A}$ satisfiy the (well-formed) formula φ with h

$$\models_{\tilde{A}} \varphi [h]$$

We define $\models_{\tilde{A}} \varphi [h]$ recursively by building the formula

1. If φ is a term then $\models_{\tilde{A}} \varphi [h]$ holds iff $\bar{h}(\varphi) \in A$

It holds for every variable $\bar{h}(x) = h(x)$ and for every function symbos

$$\bar{h}(f_i(t_1,\dots,t_n)) = f_i(\bar{h}(t_1),\dots,\bar{h}(t_n))$$

2. If φ is an atomic formula then

a) equality $t_1 \approx t_2$

$\models_{\mathfrak{A}} (t_1 \approx t_2)[h]$ iff then $\bar{h}(t_1) = \bar{h}(t_2)$

b) P is a n-place predicate symbol and $P_{\mathfrak{A}}$ is the belonging relation of $\mathfrak{A}$

$\models_{\mathfrak{A}} P t_1 \dots t_n [h]$ iff then $(\bar{h}(t_1),\dots,\bar{h}(t_n)) \in P_{\mathfrak{A}}$

3. Other formulas

a) For atomic formulas see 2.

b) $\models_{\mathfrak{A}} \neg\varphi[h]$ iff $\not\models_{\mathfrak{A}} \varphi[h]$

c) $\models_{\mathfrak{A}} (\varphi \to \psi)[h]$ iff either $\not\models_{\mathfrak{A}} \varphi[h]$ or $\models_{\mathfrak{A}} \psi[h]$ or both

In other words: If $\mathfrak{A}$ satisfies the formula φ by h then $\mathfrak{A}$ satisfies the formula ψ by h

d) $\models_{\underset{\sim}{A}} \forall x \, \varphi \, [h]$ iff then it holds for every

$a \in A$

$$\models_{\underset{\sim}{A}} \varphi \, [h(x|a)]$$

Here is the function $h(x|a)$ the same function as h with the exception

$$h(x|a)(y) = \begin{cases} h(y) & \text{if } y \neq x \\ a & \text{if } y = x \end{cases}$$

Semantical implication

Def. 26.1 Let Γ be a set of formulas and φ a formula of $L(\Delta)$.

Γ implies semantically to φ

$$\Gamma \models \varphi$$

if and only if then for every structure $\underset{\sim}{A}$ for $L(\Delta)$ and every function $h: X \rightarrow A$ it holds

If $\underset{\sim}{A}$ satisfies every formula $\psi \in \Gamma$ with h then $\underset{\sim}{A}$ satisfies the formula φ with h

"From $\models_{\mathfrak{A}} \psi[h]$ for all $\psi \in \Gamma$ it follows $\models_{\mathfrak{A}} \varphi[h]$"

Notation. If $\Gamma = \emptyset$ then we write

$$\models \varphi$$

and called φ as a valid formula

Example 26.2

Let $L(\Delta)$ be given. The parameter of Δ consists

1) $\forall$
2) P 2-place predicate symbol
3) f 1-place function symbol
4) f_0 0-place function symbol = constant symbol

We consider $\mathfrak{A} = (\mathbb{N}; \leq, S, 0)$ as a structure

Consider the formula $\varphi \equiv P f_0 f x_i$

We show $\models_{\mathfrak{A}} \varphi[h]$

Let $X = \{x_1, \ldots, x_n, \ldots\}$ the set of variables

$h: X \longrightarrow \mathbb{N}$, $h(x_i) = i-1$.

Then it holds $\bar{h}(\varphi) = (0 \leq i-1)$ (exactly: $(\leq 0 \, S \, h(x_i))$)

Therefore $\models_{\mathfrak{A}} \varphi[h]$

We have to consider arbitary $\tilde{A}$ and arbitary h
Therefore the following theorem is important

Thm 26.3 If h_1, h_2 are truth assignment of the variables of X in A and h_1, h_2 agree on all free variables of φ then it holds

$$\models_{\tilde{A}} \varphi[h_1] \quad \text{iff} \quad \models_{\tilde{A}} \varphi[h_2]$$

Proof. Let φ be a atomic formula $Pt_1 \ldots t_n$. Then every occuring variable is free. By the recursion theorem we have $\bar{h}_1(t_i) = \bar{h}_2(t_i)$ and also $P\bar{h}_1(t_1) \cdots \bar{h}_1(t_n)$
$= P\bar{h}_2(t_1) \cdots \bar{h}_2(t_n)$ and also $\bar{h}_1(\varphi) = \bar{h}_2(\varphi)$

If φ is the form $\neg\alpha$ or $\alpha \to \beta$ then it follows $\bar{h}_1(\varphi) = \bar{h}_2(\varphi)$ by the definition of the free variables

If $\varphi = \forall x \psi$ then the variables are free in φ which are free in ψ with out the exception of x.

Therefore $h_1(x|a)$ and $h_2(x|a)$ agree for all $a \in A$
Therefore $\forall x \psi$ is satisfied with h_1 exactly iff $\forall x \psi$ is satisfied with h_2

$\square$

Def. 26.4 A formula σ is a sentence if σ has no free variables

Example. Language of group theory

$$\sigma \equiv \forall x \forall y \forall z \; x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Corollary 26.5 For a sentence it holds:
either $\breve{A}$ satisfies the sentence σ for every $h : X \to A$
or $\breve{A}$ satisfies σ with no h

Def. 26.6
A sentence σ is true in $\breve{A}$ $\quad (\models_{\breve{A}} \sigma)$
if $\breve{A}$ satisfies the sentence α with some h.

If σ is not true in $\breve{A}$ then σ is called false in $\breve{A}$

$\breve{A}$ is a model of Σ of a set of sentences
if $\breve{A}$ is a model for every sentence $\sigma \in \Sigma$

Corollary 26.7 Let Σ, σ be sentences.

$\Sigma \models \sigma$ holds if and only if every model of Σ is also a model of σ

Example 26.7 $\sigma \equiv (\forall x \varphi \to \exists x \varphi)$

Statement: σ is a valid formula; i.e $\models \sigma$

To show: If $\models_{\mathfrak{A}} \forall x \varphi [h]$ then $\models_{\mathfrak{A}} \exists x \varphi$

Let also $\models_{\mathfrak{A}} \forall x \varphi [h]$

then it holds $\models_{\mathfrak{A}} \varphi [h(x|a)]$ for an arbitrary $a \in A$

Therefore $\not\models_{\mathfrak{A}} \neg \varphi [h(x|a)]$

and therefore $\not\models_{\mathfrak{A}} \forall x \neg \varphi [h(x|a)]$

$\models_{\mathfrak{A}} \neg (\forall x \neg \varphi [h(x|a)]$

$\models_{\mathfrak{A}} \exists x \varphi [h]$

Example 26.8 Let $\Gamma = \{\alpha, \beta, \gamma\}$ and ε

$\alpha \equiv \forall x \forall y \forall z \;\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$\beta \equiv \forall x \;\; x \cdot e = x$
$\gamma \equiv \forall x \exists y \;\; x \cdot y = e$
$\varepsilon \equiv \;\; e \cdot e = e$

The class of models of Γ is the class of groups

Statement: $\Gamma \models \varepsilon$

We show that $\models_{\mathfrak{A}} \varepsilon [h]$

Now by β we have $\models_{\mathfrak{A}} \forall x \;\; x \cdot e = x \;\; [h]$

$\models_{\mathfrak{A}} x \cdot e = x \, [h(x|a)$

$\models_{\mathfrak{A}} x \cdot e = x \; [h(x|e)]$ therefore $e \cdot e = e$

Building the formulas of the language of predicate logic

1. Terms

Functions $f(x_1, \dots, x_n)$

Examples

$x + y$

2. Atomic formulas

Predicates $P t_1 \cdots t_n$

$x \leq y$

$x \leq x + y$

3. Formulas

1) $\neg \alpha$

2) $\alpha \to \beta$

3) $\forall x_i\, \alpha$

$\neg (x \leq x + y)$

$(x \leq y) \to (x \leq x + y)$

$\forall x \quad x \leq 2 \cdot x$

4. Free variables

$\forall x\ \alpha^x_t$

bounded

$\forall x \quad x \leq 2 \cdot x$

Valid formulas of the language of predicate logic

For all generalization $\forall x_1 \cdots \forall x_n\, \alpha$

1. Tautologies (Axioms A1 – A13)
2. $\forall x \alpha \to \alpha^x_t$ — t is substitutable for x in α
3. $\forall x (\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta)$
4. $\alpha \to \forall x \alpha$ where x is not free

If the language contains an equality

5. $x \approx x$
6. $x \approx y \to (\alpha \to \alpha')$

Rule: Modus ponens

$$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

§27 Syntax

Def. 27.1 A formula φ is a generalization of the formula ψ if it holds for a n $\varphi \equiv \forall x_1 \dots \forall x_n \psi$

For $n=0$ we have $\varphi \equiv \psi$ also a generalization

I Logical axioms

Axioms are all generalization of formulas

1. Tautology
2. $\forall x \alpha \rightarrow \alpha^x_t$ where t can be substituted for x in α
3. $\forall x (\alpha \rightarrow \beta) \rightarrow (\forall x \alpha \rightarrow \forall x \beta)$
4. $\alpha \longrightarrow \forall x \alpha$ where x is not free in α

In the case that the language $L(\Delta)$ has an equality $\approx$ then we add

5. $x \approx x$
6. $x \approx y \rightarrow (\alpha \rightarrow \alpha')$ where α is atomic and α' araised from α in which x is substituted in one or more places by y

II Rule

Only one rule is the modus ponens

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Def. 27.2

A deduction of φ from Γ is a finite sequence

$(\varphi_0, \dots, \varphi_n)$ with $\varphi_n \equiv \varphi$ such that either

i) $\varphi_i \in \Gamma$ or

ii) φ_i is a (logical) axiom

iii) for some j and k less than i, φ_i is obtained by modus ponens from φ_j and φ_k

$$\Gamma \vdash \varphi$$

We say: φ can be proved from Γ

We have to diskuss the different groups of axioms and we begin with the substitution

Substitution (Group 2 of axioms)

$$\forall x\, \alpha \to \alpha^x_t \qquad t \text{ substitutable for } x \text{ in } \alpha$$

We describe α^x_t :

α^x_t is the formula which you get from α

if you put the term t for x in α.

1. Atomic formula

t can also substituted for x in α

Example. $\alpha \equiv Pxy$

$\alpha^x_t \equiv Pty$

2. $(\neg\alpha)^x_t = (\neg\alpha^x_t)$

3. $(\alpha \to \beta)^x_t = (\alpha^x_t \to \beta^x_t)$

4. $(\forall y \alpha)^x_t = \begin{cases} \forall y \alpha & \text{if } x = y \\ \forall y \alpha^x_t & \text{if } x \neq y \end{cases}$

it means: x can be substituted by t

whenever it is free

Example. $\alpha \equiv (Pxy \to \forall x\, Pyx)$

$\alpha^x_t \equiv (Pty \to \forall x\, Pyx)$

Tautologies (Group 1 of axioms)

In this group there are the formulas which arise the tautologies of the proposition logic. Every proposition variables can be substituded by a formula.

Example
$$\forall x P x y \rightarrow (f_i x y \rightarrow \forall x P x y)$$

because A2 $\alpha \rightarrow (\beta \rightarrow \alpha)$

Naturally you have all generalizations

Example
$$\forall x \forall y (\forall x P x y \rightarrow (f_i x y \longrightarrow \forall x P x y)$$

is an axiom of the predicate logic.

§28 Correctness

By the definition 27.2 it follows

Theorem 28.1 (Finiteness thm)

If $\Sigma \vdash \varphi$ then there are a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \varphi$

Notations 28.2

If $\Sigma = \emptyset$ then we write $\vdash \varphi$ and we say "φ is provable" or "φ is derivable"

Let φ be a formula and let $x_1, ---, x_n$ be all free variables in φ. Then the formula $\forall x_1 \forall x_2 \dots \forall x_n \varphi$ is called the complete generalization of φ

$\forall x_1 \dots \forall x_n \varphi$ is a sentence

Thm 28.3 A formula φ is **provable** if and only if the complete generalization of φ is provable.

Proof

(I) Let φ be provable. Then there a finite sequence $(\varphi_1, \dots, \varphi_n)$; $\varphi_n \equiv \varphi$ with φ_i as axiom or with φ_k and $\varphi_k \to \varphi_i$, $k \leq i$ such that $\dfrac{\varphi_k,\ \varphi_k \to \varphi_i}{\varphi_i}$

For every axiom φ_i you can put the generalization $\forall x_1 \dots \forall x_n \varphi_i$ Then you have $\forall x_1 \dots \forall x_n \varphi_k$ and $\forall x_1 \dots \forall x_n (\varphi_k \to \varphi_i)$. Because the group 3 of axioms we have $\forall x_1 \dots \forall x_n (\varphi_k \to \varphi_i) \to (\forall x_1 \dots \forall x_n \varphi_k \to \forall x_1 \dots \forall x_n \varphi_i)$. By the modus ponens we have $\forall x_1 \dots \forall x_n \varphi_k \to \forall x_1 \dots \forall x_n \varphi_i$

(II) Let $\forall x_1 \dots \forall x_n \varphi$ be provable.

By the group 2 of axioms we can substitute the term $t = x_1$ in $\varphi^{x_1}_{x_1}$, i.e

$\forall x_1 \dots \forall x_n \varphi \longrightarrow \forall x_2 \forall x_3 \dots \forall x_n \varphi^{x_1}_{x_1}$

By modus ponens it follows that $\forall x_2 \dots \forall x_n \varphi$ is provable. By n steps it follows that φ is provable $\square$

Thm 28.4 (Correctness thm)

If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

(If φ is provable from Γ then φ follows semantically from Γ)

((Idea: Modus ponens preserves semantical implication ||

Proof.

We assume that we have already shown:

(*) Every logical axiom is valid

Now let φ provable from Γ

1) Let φ be an axiom

Then the statement follows by (*)

It holds $\models \varphi$ and also it holds $\Gamma \models \varphi$

2) $\varphi \in \Gamma$; then it holds naturally $\Gamma \models \varphi$

3) φ is obtained by modus ponens from ψ and $\psi \to \varphi$. Then it holds $\Gamma \models \psi$ and $\Gamma \models \psi \to \varphi$ Let $\tilde{A}$ be a structure and h be an assignment. Then it holds

$\Gamma \models_{\tilde{A}} \psi[h]$ and $\Gamma \models (\psi \to \varphi)[h]$

The latter holds by the definition of the satis fiability with h. If $\models_{\mathfrak{A}} \psi[h]$ then also $\models_{\mathfrak{A}} \varphi[h]$ Therefore we have $\Gamma \models \varphi[h]$ and therefore $\Gamma \models \varphi$

It remains to prove (*)

Group 1 of axioms

Here we have to consider only the tautologies which correspond the axioms A1 - A13
(The other tautologies follow with the aid of the modus ponens)

A1 $(\varphi \rightarrow \varphi)$

If $\models_{\mathfrak{A}} \varphi[h]$ then also $\models_{\mathfrak{A}} \varphi[h]$

A2 $\varphi \rightarrow (\psi \rightarrow \varphi)$

If $\models_{\mathfrak{A}} \varphi$ then we have by the definition of the satisfiability that

$$\models_{\mathfrak{A}} (\psi \rightarrow \varphi)[h]$$

The other axioms A3 - A13 is left by the reader

Group 2 of axioms

$\forall x \alpha \rightarrow \alpha^x_t$

Let t be a substitution of x in α and let

be $\models_{\mathcal{A}} \forall x \alpha [h]$

To show: $\models_{\mathcal{A}} \alpha^x_t [h]$

We know that the every $a \in A$ holds

$$\models_{\mathcal{A}} \alpha [h(x|a)]$$

Now choose $a = \bar{h}(t)$

$$\models_{\mathcal{A}} \alpha [h(x|\bar{h}(t))]$$

Proof by induction of the building of formula α

(1) α is an atomic formula

$$\bar{h}(\alpha^x_t) = \bar{h}(x|\bar{h}(t))(\alpha^x_t) = h(x|\bar{h}(t))(\alpha)$$

Therefore $\models_{\mathcal{A}} \alpha^x_t [h]$

(2) α is of the form $\alpha \equiv (\neg\beta)$ or $\alpha \equiv (\gamma \rightarrow \beta)$

By induction we have $\bar{h}(\alpha^x_t) =$

$\bar{h}(x|\bar{h}(t))\, \alpha^x_t = h(x|\bar{h}(t))\, \alpha$

(3) $\alpha \equiv \forall y \varphi$

(3a) x is not free in φ

Then $h(x)$ and $h(x|\bar{h}(t))$ agree on all variables.

(3b) x is free in φ

As t is substitutable for x in φ it follows that y is not in φ

We have $\bar{h}(t) = \bar{h}(y|a)(t)$ for some $a \in A$

As $x \neq y$ we have $\alpha^x_t = \forall y \varphi^x_t$

$\models_{\mathfrak{A}} \alpha^x_t [h]$ if and only if $\models_{\mathfrak{A}} \varphi^x_t [h(y|a)]$

(Induction: φ is ~~s~~maller than α) iff $\models_{\mathfrak{A}} \varphi [h(y|a)(x|\bar{h}(t))]$

iff $\models_{\mathfrak{A}} \varphi [h(x|\bar{h}(t))]$

The rest of the group of axioms are correct $\square$

Notations 28.5

Γ is called consistent (or without contradictions) if there exists a formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$

Γ is called satisfiability if there is a structure $\mathfrak{A}$ and a (truth) assignment h such that $\mathfrak{A}$ satisfies every formula of Γ by h

Remark. Is the mathematics (for example the number theory $(\mathbb{N}; 0, S, +, -, \cdot, \exp, \leq)$) consistent?

Thm 28.6 If Γ is satisfiable then Γ is also consistent.

Proof.
We assume that for a formula φ it holds $\Gamma \vdash \varphi$ and $\Gamma \vdash (\neg\varphi)$

By the correctness thm it follows $\Gamma \models \varphi$ and $\Gamma \models (\neg\varphi)$. As Γ is satisfiable then there is a structure $\mathfrak{A}$ and an assignment h which satisfies every $\psi \in \Gamma$. Therefore we have $\models_{\mathfrak{A}} \psi\,[h]$ for every $\psi \in \Gamma$. By the definition there are a $\models_{\mathfrak{A}} \varphi\,[h]$ and $\models_{\mathfrak{A}} \neg\varphi\,[h]$

But $\models_{\mathfrak{A}} (\neg\varphi)\,[h]$ if and only if $\not\models \varphi\,[h]$ ↯ □

Remark. The thm 28.6 is equivalent to the correctness thm

§29 Metatheorems

(Metatheorems = theorems transcending the predicate logic)

Thm 29.1 (Generalization thm)

If $\Gamma \vdash \varphi$ and x is bounded in every formula of Γ then $\Gamma \vdash \forall x\, \varphi$

Proof. Let x be bounded under every formula of Γ

The statement is shown by the recursion of φ

1) φ is an (logical) axiom

Then $\forall x\, \varphi$ is also a logical axiom

2) $\varphi \in \Gamma$

Then x is only bounded in Γ. By the group 4 of axioms it holds

$$\varphi \to \forall x \varphi$$

By modus ponens

$$\frac{\varphi \quad \varphi \to \forall x \varphi}{\forall x \varphi}$$

3) φ follows by the modus ponens from ψ and $\psi \to \varphi$

By induction we have $\quad \forall x \psi \,,\ \forall x(\psi \to \varphi)$

by group 3 of axioms $\quad \forall x(\psi \to \varphi) \to (\forall x \psi \to \forall \varphi)$

and by modus ponens $\quad \dfrac{\forall x(\psi \to \varphi) \quad \forall x(\psi \to \varphi) \to (\forall x \psi \to \forall x \varphi)}{\forall x \psi \to \forall x \varphi}$

And modus ponens $\quad \dfrac{\forall x \psi \quad \forall x \psi \to \forall x \varphi}{\forall x \varphi}$

$\square$

Lemma 29.2 If $\Gamma \to \alpha_1, \dots, \Gamma \to \alpha_n$ and $\alpha_1 \to \alpha_2 \to \dots \to \alpha_n \to \beta$ is a tautology then it holds $\Gamma \vdash \beta$

Proof. Apply n-times of modus ponens

Thm 29.3 (Deduction thm)

$\Gamma \cup \{\gamma\} \vdash \varphi$ if and only if then $\Gamma \vdash (\gamma \to \varphi)$

Proof.

(A) Let $\Gamma \vdash (\gamma \to \varphi)$.

Obviously $\Gamma \cup \{\gamma\} \vdash \gamma$ and $\Gamma \cup \{\gamma\} \vdash (\gamma \to \varphi)$

By the modus ponens we have $\Gamma \cup \{\gamma\} \vdash \varphi$

(B) Let $\Gamma \cup \{\gamma\} \vdash \varphi$

1) If $\gamma \equiv \varphi$ then it holds by A1 $\gamma \to \gamma$

and therefore $\Gamma \vdash (\gamma \to \varphi)$

2) If $\varphi \in \Gamma$ or φ is an (logical) axiom

then $\Gamma \vdash \varphi$. Because A2 $\varphi \to (\gamma \to \varphi)$

and by the modus ponens $\Gamma \vdash (\gamma \to \varphi)$

3) In the case that φ is arised by the

modus ponens we have ψ and $\psi \to \varphi$. By induction

hypothesis, $\Gamma \vdash (\gamma \to \psi)$ and $\Gamma \vdash (\gamma \to (\psi \to \varphi))$

Now by A4 $(\gamma \to (\psi \to \varphi)) \to ((\gamma \to \psi) \to (\gamma \to \varphi))$

By modus ponens $\dfrac{(\gamma \to (\psi \to \varphi)),\ A4}{(\gamma \to \psi) \to (\gamma \to \varphi)}$

Onceagain modus ponens $\dfrac{(\gamma \to \psi) \quad (\gamma \to \psi) \to (\gamma \to \varphi)}{\gamma \to \varphi}$

Therefore $\Gamma \vdash (\gamma \to \varphi)$ $\square$

Thm 29.3 (Contraposition)

$\Gamma \cup \{\varphi\} \vdash (\neg \psi)$ if and only if $\Gamma \cup \{\psi\} \vdash (\neg \varphi)$

Proof. Let $\Gamma \cup \{\varphi\} \vdash (\neg \psi)$

By the deduction thm $\Gamma \vdash (\varphi \to (\neg \psi))$

By axiom A11 $(\varphi \to (\neg \psi)) \to (\psi \to (\neg \varphi))$

Modus ponens $\Gamma \vdash (\psi \to \neg \varphi)$

Again deduction thm $\Gamma \cup \{\psi\} \vdash (\neg \varphi)$

The other direction of the proof holds because symmetry $\square$

Thm 29.4 (reductio ad absurdum := reduction by absurd)

If $\Gamma \cup \{\varphi\}$ is inconsistent then $\Gamma \vdash (\neg \varphi)$

Proof.

Assume that $\Gamma \cup \{\varphi\} \to \beta$ and $\Gamma \cup \{\varphi\} \to \neg \beta$

Then by the deduction thm

$\Gamma \vdash (\varphi \to \beta)$ and $\Gamma \vdash (\varphi \to \neg \beta)$

By the axiom A8

$$(\varphi \to \beta) \to ((\varphi \to \neg\beta) \to (\varphi \to (\beta \wedge \neg\beta))$$

By the modus ponens : 2x times

$$\Gamma \vdash (\beta \wedge \neg\beta)$$

By the axiom A 12

$$(\beta \wedge \neg\beta) \to (\neg\varphi)$$

Again modus ponens

$$\Gamma \vdash (\neg\varphi)$$

~~Again 2-times modus ponens~~

$$\Gamma \vdash (\varphi \to (\beta \wedge \neg\beta)$$

Axiom A 11

$$(\varphi \to (\beta \wedge \neg\beta)) \to (\neg(\beta \wedge \neg\beta) \to \neg\varphi)$$

Modus ponens

$$\Gamma \vdash (\neg(\beta \wedge \neg\beta) \to \neg\varphi)$$

Now $\neg(\beta \wedge \neg\beta)$ is an tautology beause it is equivalent to $\beta \vee \neg\beta$. Therefore it always holds $\Gamma \vdash \neg(\beta \wedge \neg\beta)$

By modus ponens we have $\Gamma \vdash (\neg\varphi)$ $\square$

Example 29.5 $\quad \vdash \forall x \varphi \to \exists x \varphi$

to show $\forall x \varphi \vdash \exists x \varphi$ — deduction thm

to show $\forall x \varphi \vdash \neg(\forall x \neg\varphi)$ — definition of $\exists$

to show $\forall x \neg\varphi \vdash \neg\forall x \varphi$ — contraposition

to show $\{\forall x \neg\varphi, \neg\forall x \varphi\}$ is inconsistent and therefore reductio ad absurdum.

Proof. $\{\forall x \neg\varphi, \forall x \varphi\}$

Group 2 of axioms $\quad \forall x \neg\varphi \to \neg\varphi^x_x$

Modus ponens $\quad \dfrac{\forall x \neg\varphi, \ \forall x \neg\varphi \to \neg\varphi^x_x}{\neg\varphi^x_x}$

Therefore $\Gamma \vdash \neg\varphi^x_x$ and ~~there~~for $\Gamma \vdash \neg\varphi$

On the other hand $\forall x \varphi \to \varphi^x_x$

Modus ponens $\quad \dfrac{\forall x \varphi, \ \forall x \varphi \to \varphi^x_x}{\varphi^x_x}$

Therefore $\Gamma \vdash \varphi$ and Γ is inconsistent $\square$

§ 30 Completeness

30.1 Construction of the Lindenbaum algebra for the predicate logic

Let Γ be a set of formulas. Let F be the set of all formulas over the alphabet Δ

We define an equivalence relation $\approx$ on F

$\varphi \approx \Psi$ if and only if $\Gamma \vdash (\varphi \to \Psi)$ and $\Gamma \vdash (\Psi \to \varphi)$

We consider the faktor set $F/_{\approx}$ and we define an order relation by

$$[\varphi] \leq [\Psi] \text{ if and only if } \Gamma \vdash (\varphi \to \Psi)$$

In the same way of the proposition logic we show that $(F/_{\approx} ; \leq)$ is a partially ordered set which corespond the Boolean algebra. This Boolean algebra is the Lindenbaum algebra which we consider.

Notations 30.2

$X = \{x_1, x_2, x_3, \ldots\}$ is the set of variables.

Let φ be a formula.

Let j the larghest index in which the variables of φ come along

In the case that x_i is bounded then we rename

x_i into the variable x_{j+i}

$\varphi\begin{pmatrix} x_{k_1} & \ldots & x_{k_n} \\ x_{p_1} & & x_{p_n} \end{pmatrix}$ is the formula which arise

from φ after the all bounded variables are renamed

and all free variables $x_{k_1}, \ldots, x_{k_n}$ are substituted

by $x_{p_1}, \ldots, x_{p_n}$

Lemma 30.3 For every formula φ holds

$$[\forall x_k \varphi] = \inf\left\{\left[\varphi\begin{matrix} x_k \\ x_p \end{matrix}\right] \;\middle|\; p = 1, 2, 3, \ldots\right\}$$

Proof. By the group 2 of axioms we have

$$\forall x_k \varphi \longrightarrow \varphi \begin{matrix} x_k \\ x_p \end{matrix}$$

This holds for $p = 1, 2, 3, \ldots$

Therefore $[\forall x_k \varphi] \leq [\varphi\, {}^{x_k}_{x_p}]$ for $p = 1, 2, 3, \ldots$

Also $\quad [\forall x_k \varphi] \leq \inf\{[\varphi\, {}^{x_k}_{x_p}] \mid p = 1, 2, 3, \ldots\}$

<u>Assumption</u>: ψ is a lower bound of $\{[\varphi\, {}^{x_k}_{x_p}] \mid p = 1, \ldots\}$

<u>Show</u>: $[\psi] \leq [\forall x_k \varphi]$

The assumption follows $[\psi] \leq [\varphi\, {}^{x_k}_{x_p}]$ for $p = 1, \ldots$

Also $\Gamma \vdash (\psi \rightarrow \varphi\, {}^{x_k}_{x_p})$ for $p = 1, \ldots$

We apply the generalization thm

$$\Gamma \vdash (\psi \rightarrow \forall x_p \varphi\, {}^{x_k}_{x_p})$$

By the group 2 of axioms we put

$$\forall x_p \varphi\, {}^{x_k}_{x_p} \rightarrow \varphi\, {}^{x_p}_{x_p} \equiv p$$

Once more the generalization thm

$$\Gamma \vdash (\forall x_p \varphi\, {}^{x_k}_{x_p} \rightarrow \forall x_k \varphi)$$

By the axiom A3 and the modus ponens

$$\Gamma \vdash (\psi \rightarrow \forall x_k \varphi)$$

Therefore $[\psi] \leq [\forall x_k \varphi]$ $\square$

Building the formulas of the language of predicate logic

1. Terms

	Examples
Functions $f(x_1, \ldots, x_n)$	$x+y$

2. Atomic formulas

Predicates $P t_1 \cdots t_n$	$x \leq y$
	$x \leq x+y$

3. Formulas

1) $\neg\alpha$	$\neg(x \leq x+y)$
2) $\alpha \to \beta$	$(x \leq y) \to (x \leq x+y)$
3) $\forall x_i \, \alpha$	$\forall x \;\; x \leq 2 \cdot x$

4. Free variables

	bounded
$\forall x \; \alpha_t^x$	$\forall x \;\; x \leq 2 \cdot x$

Valid formulas of the language of predicate logic

For all generalization $\forall x_1 \cdots \forall x_n \, \alpha$

1. Tautologies (Axioms A1 – A13)
2. $\forall x \alpha \to \alpha_t^x$ $\quad$ t is substitutable for x in α
3. $\forall x (\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta)$
4. $\alpha \to \forall x \alpha$ $\quad$ where x is not free

If the language contains an equality

5. $x \approx x$
6. $x \approx y \to (\alpha \to \alpha')$

Rule: Modus ponens

$$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

Thm 30.4 (Completeness thm)

If $\Sigma \models \varphi$ then it holds $\Sigma \vdash \varphi$

Proof. Let $\Sigma \models \varphi$

We assume that $\Sigma \nvdash \varphi$

As $\Sigma \nvdash \varphi$ it holds in Lindenbaum algebra

$$[\varphi] \neq 1$$

and therefore

$$[\neg\varphi] \neq 0$$

By the Lemma 30.3 it holds for every formula ψ in $L(\Delta)$

$$[\forall x_k \psi] = \inf\{[\psi\tfrac{x_k}{x_p}] \mid p = 1,2,3,\dots\}$$

By definition 14.4 and Tarski's lemma there exists an ultrafilter D in the Lindenbaum algebra which contains $[\neg\varphi]$ and preserves the infimas

i.e $\quad [\forall x_k \psi] \in D \Leftrightarrow [\psi\tfrac{x_k}{x_p}] \in D$ for all p

We define the structure $\hat{A}$ of $L(\Delta)$ in the following. Let X be the set of the variables of $L(\Delta)$

$$x_i \sim x_j \Leftrightarrow [x_i = x_j] \in D$$

Furthermore we define

$$[x_i]_\sim = \{x_j \mid x_j \sim x_i,\ x_j \in X\}$$

and
$$[X]_\sim = \{[x_i]_\sim \mid i = 1,2,3,\ldots\}$$

$A = [X]_\sim$ is the (carrier) set for the structure $\tilde{A}$

— For every n we define the n-place function fi on $A = [X]_\sim$ by

$$fi([y_1], \ldots, [y_{n_i}]) = [f_i(y_1, \ldots, y_{n_i})]$$

where fi corresponds to the function symbol $f_i \in \Delta$

— For every n we define the n-place relation r_i on $A = [X]_\sim$ by

$$([y_1], \ldots, [y_n]) \in r_i \iff [P_i(y_1, \ldots, y_n)] \in D$$

(If $z_1 \sim y_1, \ldots, z_n \sim y_n$ then we remark

$$[P_i(y_1, \ldots, y_n)] \in D \iff [P_i(z_1, \ldots, z_n)] \in D$$

We get the structure

$$A = ([X]_\sim;\ fi, (i \in I);\ r_j, (j \in J)\}$$

We show

For every formula $\varphi(x_0, \ldots, x_n)$ which are free variables $x_0, \ldots, x_n$, it holds

(*) $\models_{\tilde{A}} \varphi [y_0, \ldots, y_n] \Leftrightarrow [\varphi^{x_0 \ldots x_n}_{y_0 \ldots y_n}] \in D$

(that means: $h(x_0) = y_0, \ldots, h(x_n) = y_n$)

The proof of (*) can be find by the recursion of φ

Atomic formulas

1) $\varphi \equiv (x_i = x_j)$

By the def. of $[X]_\sim$ it follows that (*) holds

2) $\varphi \equiv P_j(y_1, \ldots, y_n)$

By the def. of r_i it follows that (*) holds

3) $\varphi \equiv (\neg \varphi)$

4) $\varphi \equiv (\psi_1 \rightarrow \psi_2)$

In this cases it follows by the definition of satisfiability, inccording induction that (*) holds

5) $\varphi \equiv \forall x_k \psi$

$\models_{\tilde{A}} \forall x_k \psi$ holds if and only if $\models_{\tilde{A}} \psi [h(x_k|a)]$ for all $a \in A$

Let $a = x_p$

By induction hypothesis we may apply (*)

$$\models_{\tilde{A}} \psi[y_0, \ldots, y_p, \ldots, y_n] \iff [\psi \, {}^{x_0 \ldots x_k \ldots x_n}_{y_0 \ldots x_p \ldots y_n}] \in D$$

for every x_p

As the ultrafilter preserves infima we have

$$[\forall x_k \, \psi \, {}^{x_0 \ldots x_k \ldots x_n}_{y_0 \ldots y_p \ldots y_n}] \in D$$

Therefore (*) holds.

We are ready because $\tilde{A}$ is a structure with

$$\models_{\tilde{A}} [\neg \varphi]$$

which contradicts $\Sigma \models \varphi$ ∎

Conclusions: Thm 30.5

Every consistent set of formula is satisfiable

You can show that the thm 30.5 and 30.4 are equivalent.

§31 Logic on the power set

31A Power-$\subseteq$ sets

We consider the binary relation "$\subseteq$" on a set E

Def. 31.1 Let $\vec{E} = (E; \subseteq)$ be a finite set E with the relation "contains$\subseteq$"

$\tilde{P}(E) = (P(E); \subseteq_p)$ is a power-contained set of $\tilde{E}$ if the relation $\subseteq_p$ is defined on the power set $P(E)$ in the following way.

For any subsets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ of $P(E)$ with $n \leq m$ we have

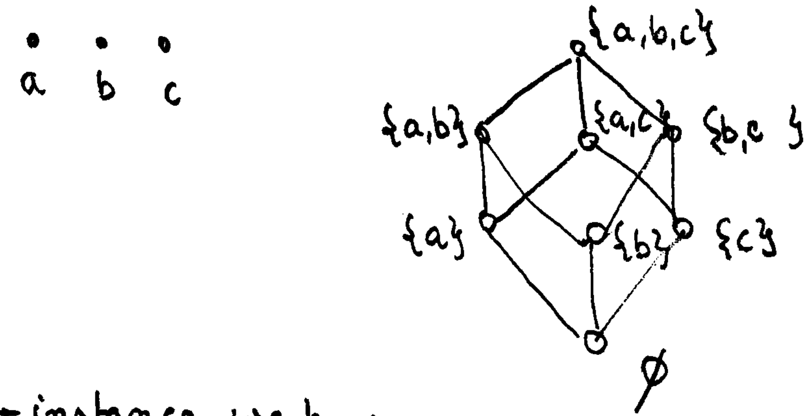
$$\{a_1, \dots, a_n\} \subseteq_p \{b_1, \dots, b_m\}$$

if and only if there exists an injective mapping

$$\pi : \{a_1, \dots, a_n\} \to \{b_1, \dots, b_m\}$$

such that $a_i \subseteq \pi(a_i)$ for $i = 1, \dots, n$

Example 31.2 We consider the set $\{a,b,c\}$



For instance we have

$\{a\} \subseteq_{pr} \{a,c\}$ because there exists an injective map $\pi : \{a\} \to \{a,c\}$ which preserves the relation "contains" namely $a \to a$

With this relation "contains" we have the Boolean algebra without the complement $(P(E);\cap,\cup,\emptyset,E)$

We will generalize for all relation but because of our comfort we confine us to the binary relation.

Def. 31.3 Let $\tilde{E} = (E; \varrho)$ be a binary relation. $\tilde{P}(E) = (P(E); \varrho_p)$ is called the power-ϱ set if the relation ϱ_p is defined on the power set $P(E)$ in the following way. For any subsets $\{a_i\}$ and $\{b_j\}$ of $P(E)$ there is the relation

$$\{a_i \mid i \in I\} \, \varrho_p \, \{b_j \mid j \in J\}$$

if and only if there exists an injective mapping

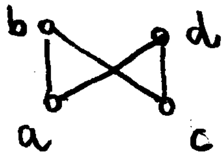
$$\pi : \{a_i \mid i \in I\} \longrightarrow \{b_j \mid j \in J\}$$

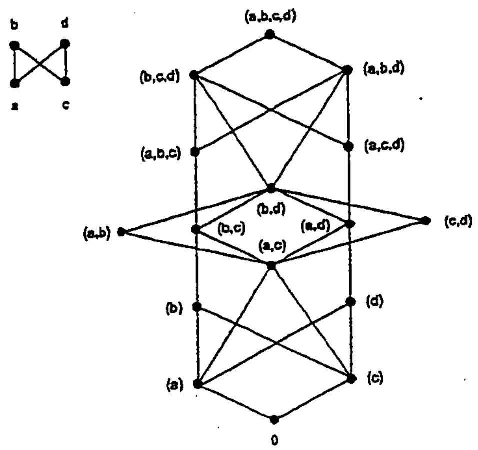
such that

$$a_i \varrho b_j$$

for some $i \in I$ and $j \in J$ and $\{\pi(a_i) \mid i \in I\} \subseteq \{b_j \mid j \in J\}$

Example 31.4 We consider the order relation $\tilde{E} = (E; \leq)$ with $E = \{a, b, c, d\}$ and the poset





We have for instance $\{a, b\} \leq_p \{b, c, d\}$ because there exists an injective map $\pi : \{a, b\} \to \{b, c, d\}$ which preserves the order namely $a \to d$, $b \to b$.

Remark. The poset $D12$ is not a lattice.

30B On the theorem of Cantor-Bernstein

Theorem 30.5 The relation $\leq$ and $\leq^{-1}$ is antisymmetric

Proof. We consider the following subsets of $P(E)$

$$\{a_i \mid i \in I\} \leq \{b_j \mid j \in J\}$$

$$\{b_j \mid j \in J\} \leq^{-1} \{a_i \mid i \in I\}$$

where

By definition there are two injective mappings

$$\pi : \{a_i \mid i \in I\} \to \{b_j \mid j \in J\}$$

$$\varphi : \{b_j \mid j \in J\} \to \{a_i \mid i \in I\}$$

We define recursively $C_0 = \{a_i | i \in I\}$ \range (φ) and $C_{n+1} := \varphi(\pi(C_n))$
Using the theorem of Cantor-Bernstein we are able to define a bijection by

$$\phi(x) := \begin{cases} \pi(x) \text{ if } x \in C_n \text{ for some } n \\ \varphi^{-1}(x) \text{ else} \end{cases}$$

Hence the considered sets $\{a_i | i \in I\}$ and $\{b_j | j \in J\}$ are equivalent and we get

$$\{a_i | i \in I\} =_p \{b_j | j \in J\}$$

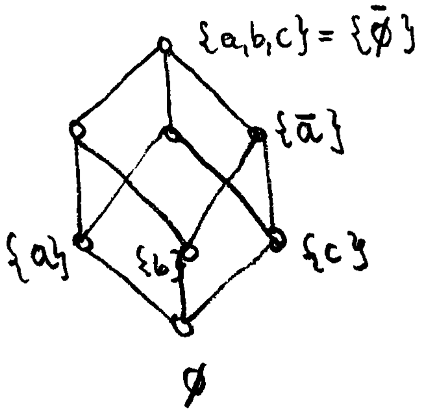
as required.

31C Complement

Once again we consider the relation "contains" and also negation.

Let $\tilde{E} = (E; \subseteq, \notin)$ and the $P(\tilde{E}) = (E; \subseteq_p, \notin_p)$

We consider the example on the elements a, b, c.



?

31 D Algebra of formulas

Def. 31.6 Let $\bar{A}$ be the set of formulas and its powerset $P(\bar{A})$. Σ be a set of formula $\subseteq \bar{A}$.

The relation "implies" "$\to$" on $\bar{A}$ is defined by

$$\alpha \leq \beta \iff \Sigma \vdash (\alpha \to \beta)$$

The relation "power implication" "$\to_p$" on the powerset $P(\bar{A})$ is defined

$$\{\alpha_1, \dots, \alpha_n\} \to_p \{\beta_1, \dots, \beta_m\}$$

as the same conditions in 30.1

The properties are

1) $\{\alpha_1, \dots, \alpha_n\} \to_p \{\beta_1, \dots, \beta_m\} \to_p \{\gamma_1, \dots, \gamma_p\}$

$\Rightarrow \{\alpha_1, \dots, \alpha_n\} \to_p \{\gamma_1, \dots, \gamma_p\}$

2) $\{\alpha_1, \dots, \alpha_n\} \to_p \bar{A}$

and so on....

§32 Remarks

1. The difference of the semantic and the syntax has been emphasized by Hilbert and his school. Gödel has proved the completeness thm 1930 for the predicate logic.
Various proofs have been given later on, for instance by Henkin 1949
Our proof using Boolean algebra is due to Helen Rasiowa and Sikorski 1951
The recursive definition of the satisfaction is by Tarski 1935

(1980?)

2. David Hilbert
(1862 – 1943)

Between 1900 and 1928, the German mathematician David Hilbert proposed that each mathematical theory T, such as geometry or number theory, should be given firm logical foundations: a theorem of T should be deducible from an agreed set of axioms (basic assumptions in T) by a finite number of applications of the rules of logic, called a proof. This programme, called formalism, aimed to establish the consistency and completeness of each theory T, and to decide algorithmically whether any given statement is a theorem of T, thus reducing mathematics to a mechanical process.

At the turn of the century Hilbert proposed 23 mathematical problems for investigation. Most have since been solved. He also tried to establish the underlying consistency of all mathematics, an effort that was eventually proved impossible by the logician Kurt Gödel in 1931.

3. Kurt Gödel
(1906 – 1978)
(Brünn/Mähren) Brno Moravia

University of Vienna 1924
Teacher of Gödel was Hans Hahn which is famous for instance of the thm of Hahn-Banach

Gödel was in touch with the Vienna Circle
Topics are rational thinking, checking the science, searching structure of science.
Gödel was in touch with Karl Menger

(Geometry, Lattice theory) Menger father was a famous professor in economy of the Austrian Empire

Gödel has read the book Hilbert, Ackermann: Grundzüge der theoretischen Logik and visit a lecture of Carnap on logic.

1930 PhD was the complete theorem of the predicate logic.

1931 Incompleteness (Habilitation)

a) Every formal system $\mathfrak{S}$ which contains at least part of arithmethics is incomplete (for instance $(\mathbb{N}; 0, S, +, \cdot; \leq)$ is incomplete)

b) The statement C is not proovable that $\mathfrak{S}$ is consistent

"If the mathematics is consistent then the consistente of the mathematics cannot be proved."

The incomplete thm was a "disaster" for the program of Hilbert

Tarski was a guest of Menger's colloquium

Cantor: Is the continuum $\aleph$ the smallest cardinality $> \aleph_0$ (countable)?

A famous result of Gödel is:

The set theory Zermelo-Fraenkel ZF together with the continuum hypothesis and axiom of choice is consistent if ZF is consistent.

Important methods have been developed by his work; for instance the theory of the recursive functions.

1940 Princeton

Friendship with Albert Einstein and Oskar Morgenstern

Karl Menger, Olga Tausky

Papers of science and also philosophy.

1978 Death

IV Non-classical logics

§ 33 Fuzzy sets

33 A Membership function

Remark 33.1

Let E be a universe (:= fundamental set) and A be a subset of E

$$A \subseteq E$$

If x is an element of A then we may write using the symbol $\in$

$$x \in A$$

But we may also write using the membership function

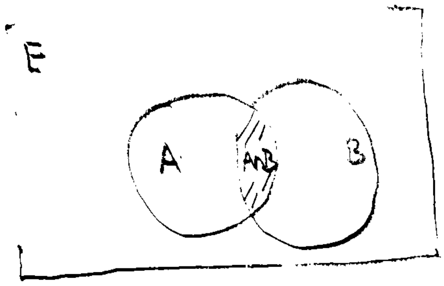
$$\mu_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

In this way we can describe the set theoretical operations

Notation 33.2

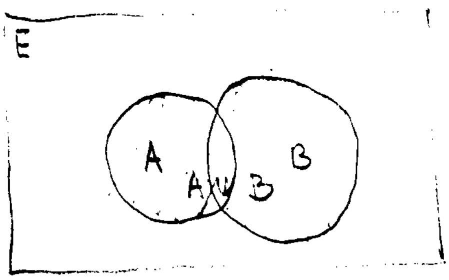
32.2.1 If $\mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ and $\mu_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$

then $\mu_{A \cap B}(x) = \begin{cases} 1 & x \in A \cap B \\ 0 & x \notin A \cap B \end{cases}$



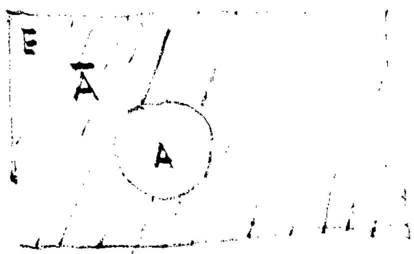
332.2 If $\mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ and $\mu_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$

then $\mu_{A \cup B}(x) = \begin{cases} 1 & x \in A \cup B \\ 0 & x \notin A \cup B \end{cases}$



332.3 If $\mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

then $\mu_{\bar{A}}(x) = \begin{cases} 1 & x \in \bar{A} \\ 0 & x \notin \bar{A} \end{cases}$ ($\bar{A}$ complement,



Boolean algebra of [illegible]

$(P(E);\ \cap, \cup, {}^{-}, \emptyset, E)$

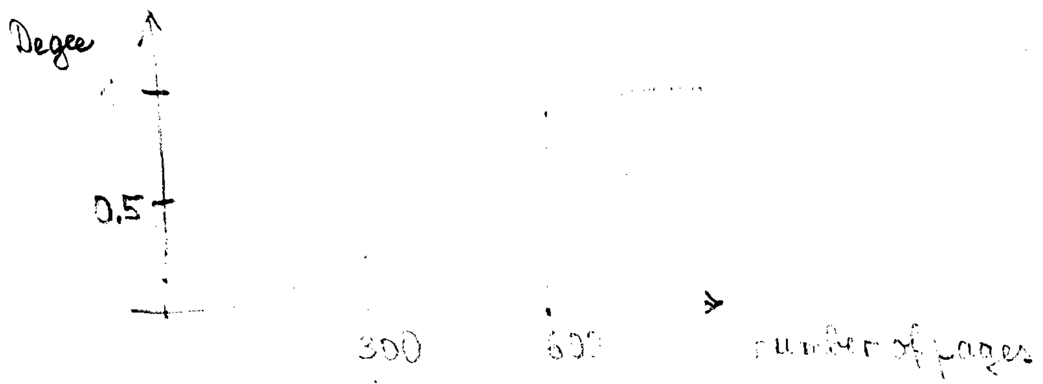
33 B Fuzzy sets

Remark 333 We use the membership function in order to describe the fuzzy sets

fuzzy := not exactly

crisp := distinct, exactly

Example. A fuzzy set of big books



Def. 334 Let E be a universe and μ be a function from E into the unit interval $[0,1]$

$$\mu : E \longrightarrow [0,1]$$

The set A of all pairs $(x, \mu(x))$ is called a fuzzy set over E

μ is called membership function (Zugehörigkeitsfunkt)

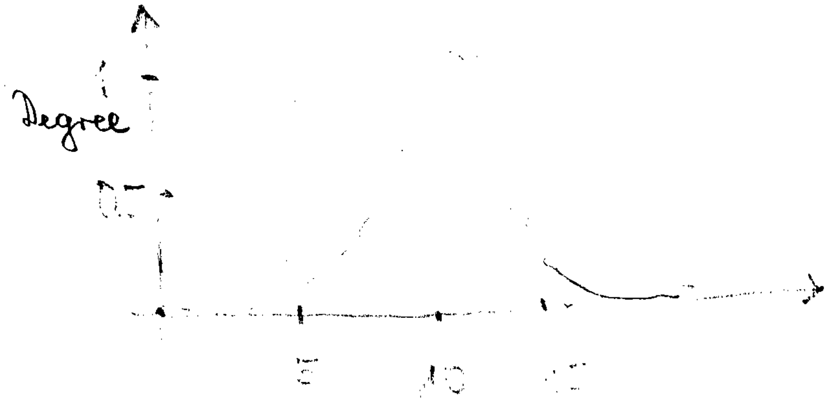
μ is also called the degree of membership

(also degree of truth)

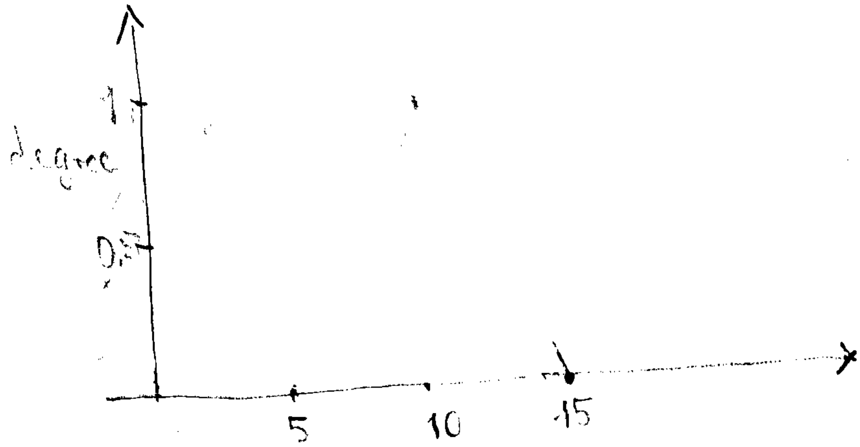
Remark. If an order relation on E already exists which corresponds to [illegible] of the fuzzy set then we can define a member function.

Example 32.5

33.5.1 A fuzzy set of the real number 10



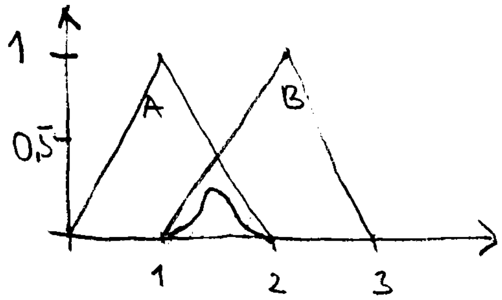
33.5.2 In optimization we often study fuzzy set in triangle form



The form of a fuzzy set [illegible] determined by norms for instance t and s-norm

Example 33.6.1

Join of two fuzzy sets A, B



For the join we like to give two definitions

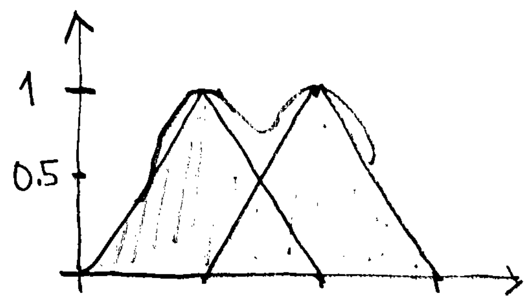
1) $A \cup B := \min(\mu_A, \mu_B)$

2) $A \cup B := alg_t(\mu_A, \mu_B)$

where $alg_t(x,y) := x \cdot y$ (t-Norm)

Which of the definitions you choose depends from the linguistical interpretation

33.6.2



1) $A \cap B := \max(\mu_A, \mu_B)$

2) $A \cap B := alg_{cs}(\mu_A, \mu_B)$

where $alg_s(x,y) = x + y - x \cdot y$

33C α - cuts

The description of fuzzy logic is not easy. Therefore we like to decompose the fuzzy sets into families of crisp sets

We can determine the threshold (Schwellwert, Niveau) value to discribe the crisp sets

Def. 33.7 Let $A = (x, \mu_A(x))$ a fuzzy set over the universe E and let $\alpha \in [0,1]$. Then the crisp set

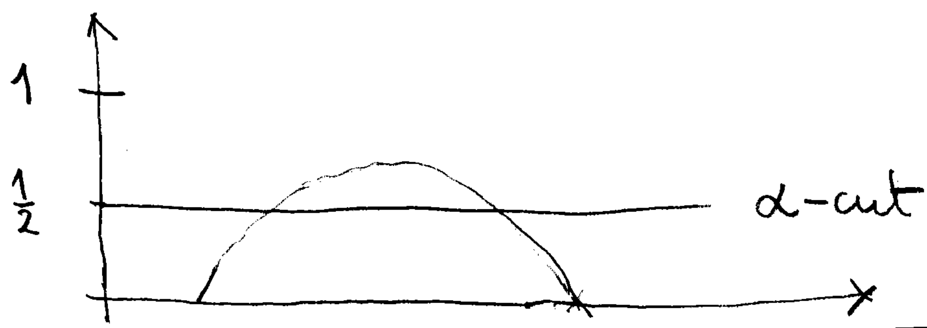
$$A_\alpha := \{x \mid \mu_A(x) \geq \alpha\}$$

is called a α-cut

We use the definition of the strict α-cut

Where $$A_\alpha^* := \{x \mid \mu_A(x) > \alpha\}$$

Example



Notation 33.8 There are special α-cuts namely

$$\mathrm{core}(\mu_A) := \{x \mid \mu_A(x) = 1\} \qquad \text{core} = \text{Kern}$$

$$\mathrm{support}(\mu_A) := \{x \mid \mu_A(x) > 0\}$$

We can also use the following notation for fuzzy sets.

Notation 33.9

For a finite fuzzy set A over E we put

$$A = \frac{\mu_A(x_1)}{x_1} + \dots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i}$$

For a infinite fuzzy set A over E we put

$$A = \int_E \frac{\mu_A(x)}{x}$$

Remarks. Zadeh (Berkley, Department of electronics) has introduced the fuzzy sets (1965)

Some similar ideas before him are

Łukasiewicz (School of Warsaw) (1921) (3-valued logic) and

Heisenberg (Unschärfe relation)

33 D Operations of fuzzy sets

Def. 33.10 Let E be a universe and let A, B two fuzzy sets of E

$$A = B \iff \mu_A(x) = \mu_B(x)$$

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x)$$

Def. 32.11 Let E be a universe. The fuzzy sets A, B is given by the membership function μ_A, μ_B

It holds
$$\mu_{A \cap B}(x) := \min(\mu_A(x), \mu_B(x))$$
$$\mu_{A \cup B}(x) := \max(\mu_A(x), \mu_B(x))$$
$$\mu_{\bar{A}}(x) := 1 - \mu_A(x)$$

Of course the propositions of classical logic does not hold. But some axioms holds:

33.12 Neutral elements:

$$\min(\mu_A(x), 1) = \mu_A(x) \implies A \cap E = A$$

$$\max(\mu_A(x), 0) = \mu_A(x) \implies A \cup \emptyset = A$$

Commutativity

$$\min(\mu_A(x), \mu_B(x)) = \min(\mu_B(x), \mu_A(x)) \Rightarrow A \cap B = B \cap A$$
$$\max(\mu_A(x), \mu_B(x)) = \max(\mu_B(x), \mu_A(x)) \Rightarrow A \cup B = B \cup A$$

Associativity

$$\min(\min(\mu_A(x), \mu_B(x)) \mu_C(x)) = \min(\mu_A(x), \min(\mu_B(x), \mu_C(x))$$

$\Rightarrow (A \cap B) \cap C = A \cap (B \cap$ ......

max .....

Monotony

$\mu_A(x) \leq \mu_C(x) \wedge \mu_B(x) \leq \mu_D(x) \Rightarrow$

$$\min(\mu_A(x), \mu_B(x)) \leq \min(\mu_C(x), \mu_D(x))$$

that means $A \subseteq C \wedge B \subseteq D \Rightarrow (A \cap B) \subseteq (C \cap D$

Idempotency

$$\min(\mu_A(x), \mu_A(x)) = \mu_A(x) \Rightarrow A \cap A = A$$

$\max(\mu_A$ .... $\rightarrow$

Distributivity

$$\min(\mu_A(x), \max(\mu_B(x), \mu_C(x)) =$$
$$\max(\min(\mu_A(x), \mu_B(x)), \min(\mu_A(x), \mu_C(x))$$

$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and dual

Adsorption

$\min(\mu_A(x), \max(\mu_A(x), \mu_B(x))) = \mu_A(x) \Rightarrow A \cap (A \cup B) = A$

$\max(\ldots\ldots \ \ldots\ldots$

De Morgan

$1 - \min(\mu_A(x), \mu_B(x)) = \max(1-\mu_A(x), 1-\mu_B(x)) \Rightarrow \overline{A \cap B} = \overline{A} \cup \overline{B}$

$1 - \max \ldots\ldots$

33 E Fuzzy algebras

Def 33 13 A MV-algebra is an algebra $\mathcal{A} = (A; \oplus, \odot, -, 0, 1)$ where A is a non-empty set and where the operations fulfill the following axioms

(MV stands for many-valued)

B1 $x \oplus y = y \oplus x$ $\quad x \odot y = y \odot x$

B2 $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ $\quad x \odot (y \odot z) = (x \odot y) \odot z$

B3 $x \oplus \bar{x} = 1$ $\quad x \odot \bar{x} = 0$

B4 $x \oplus 1 = 1$ $\quad x \odot 0 = 0$

B5 $\quad x \oplus 0 = x \qquad x \odot 1 = x$

B6 $\quad \overline{x \oplus y} = \bar{x} \odot \bar{y} \qquad \overline{x \odot y} = \bar{x} \oplus \bar{y}$

B7 $\quad \bar{\bar{x}} = x$

B8 $\quad \bar{0} = 1$

We use the following derived formulas

$$x \veebar y := (x \odot y) \oplus y \qquad x \veebar y := (x \oplus y) \odot y$$

B9 $\quad x \veebar y = y \veebar x \qquad x \barwedge y = y \barwedge x$

B10 $\quad x \veebar (y \veebar z) = (x \veebar y) \veebar z$

$\qquad x \barwedge (y \barwedge z) = (x \barwedge y) \barwedge z$

B11 $\quad x \oplus (y \barwedge z) = (x \oplus y) \barwedge (x \oplus z)$

dual $\quad x \odot (y \veebar z) = (x \odot y) \veebar (x \odot z)$

Remark 33,14

The following MV-algebras are the most important from the theoretical and practical view

$$A_M = \{0, \tfrac{1}{M}, \tfrac{2}{M}, \dots, \tfrac{M-1}{M}, 1\} \qquad \text{(carrier set)}$$

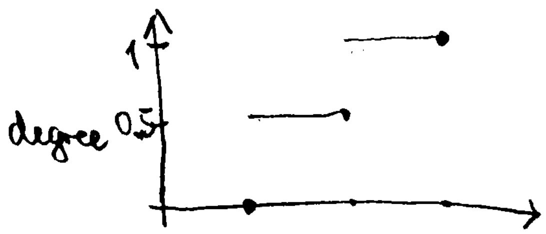
(operations)

$$\frac{p}{M} \oplus \frac{q}{M} := \min\{1, \frac{p+q}{M}\}$$

$$\frac{p}{M} \odot \frac{q}{M} := \min\{0, \frac{p+q-M}{M}\}$$

$$\overline{\frac{p}{M}} = \frac{M-p}{M}$$

Example $A_2 = \{0, \frac{1}{2}, 1\}$



33 F The extension principle

Example

The age of a animal is estimated by roughly 3 years. How old is the animal later in 5 years?

If we suppose that

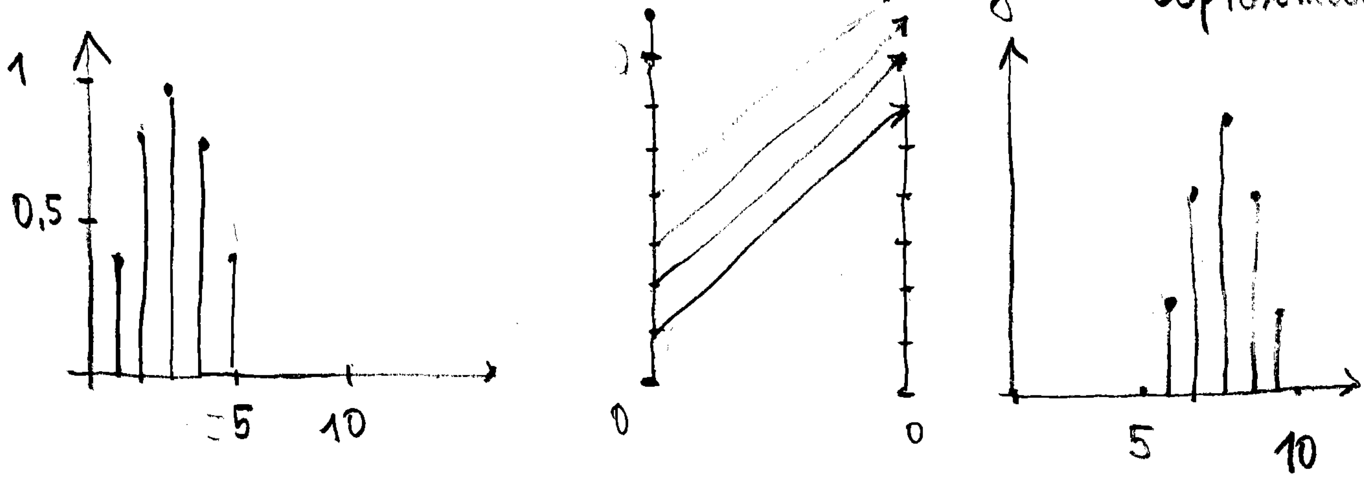
$$\text{"aproximated 3 years"} = \left\{ \frac{0.4}{1} + \frac{0.8}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.4}{5} \right\}$$

we can calculate the fuzzy sum by the extension principle

$$5 + \left\{ \frac{0.4}{1} + \frac{0.8}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.4}{5} \right\}$$

$$= \left\{ \frac{0.4}{1+5} + \frac{0.8}{2+5} + \cdots \right\}$$

The fuzzy set $B = \left\{ \frac{0.4}{6} + \frac{0.8}{7} + \frac{1}{8} + \cdots \right\}$ can also be interpreted as the " linguistic aproximation



((The mapping of the extension principle is not always injetive))

---

§ 34 Fuzzy relations

33 A Composition of relation

Def. 34.1 Let $X \times Y$ be a universe and let μ_R be binary fct $\mu_R : X \times Y \to [0,1]$

The set $R = \{((x,y), \mu_R(x,y)) \mid x \in X, y \in Y\}$ is called a binary fuzzy relation

---

Notation 34.2 We use also the following fuzzy relation by $\{x_1, \dots, x_n\} \times \{y_1, \dots, y_m\}$

in the finite case

$$R = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\mu_R(x_i, y_j)}{(x_i, y_j)}$$

and the infinite case

$$R = \int_E \frac{\mu_R(x_i, y_j)}{(x_i, y_j)}$$

We are interested for fuzzy reflexive relations

Def. 34.3 A fuzzy relation $R \subseteq X \times X$ is called reflexive if the membership fct μ_R holds

$$\mu_R(x,x) = 1 \quad \text{for every } x \in X$$

If the main diagonal of a matrix consists of all ones then we recognize the reflexivity

Def 34.4 A fuzzy relation $R \subset X \times X$ is called symmetric if the membership fct μ_R is symmetric

$$\mu_R(x,y) = \mu_R(y,x) \quad \text{for all } x,y \in X$$

Def 34.5 Let R_1 and R_2 be finite, binary fuzzy relation on the sets X, Y, Z

$R_1 = R_1(X,Y) \subseteq X \times Y$, $R_2 = R_2(Y,Z) \subseteq Y \times Z$

The composition $R_2 * R_1(X,Z)$ of relation is given by the pairs (x,z) with the membership

$$\mu_{R_2} * \mu_{R_1}(x,z) = \max_{y \in Y} \left(\min\left(\mu_{R_1}(x,y), \mu_{R_2}(y,z)\right)\right)$$

Example 3.46 Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$, $Z = \{z_1, z_2, z_3\}$

We consider the following scheme of matrices

R_2

	z_1	z_2	z_3
y_1	0	0.6	0.2
y_2	0.5	0.1	0.8

R_1

	y_1	y_2
x_1	0.5	0.3
x_2	1	0
x_3	0.2	1

$R_2 * R_1$

	z_1	z_2	z_3
x_1	0.3	?	?
x_2	?	?	?
x_3	?	?	?

$$\mu_{R_2 * R_1}(x_1, z_1) = \max(\min(\mu_{R_1}(x_1, y)\, \mu_{R_2}(y, z_1))$$

$$= \max(\min(0.5; 0), \min(0.3; 0.5))$$

$$= \max(0; 0.3)$$

$$= 0.3$$

Remark. This composition of relation is only possible for infinite fuzzy relations if there exists all supremas.

Def 3.4.6 A (finite) fuzzy relation $R \subseteq X \times X$ is called transitive if its composition of R by itself

$$R * R \subseteq R$$

i.e. if the membership fct μ_R fulfill the condition

$$\mu_{R*R}(x,y) \leq \mu_R(x,y) \quad \text{for every } x,y \in X$$

Def 3.4.7 A binary fuzzy relation is called a fuzzy order relation if the following holds

i) reflexive $\mu_R(x,x) = 1$ for every $x \in X$

ii) antisymmetric
$\mu_R(x,y) > 0$ and $\mu_R(y,x) > 0$
implies $x = y$

iii) transitive

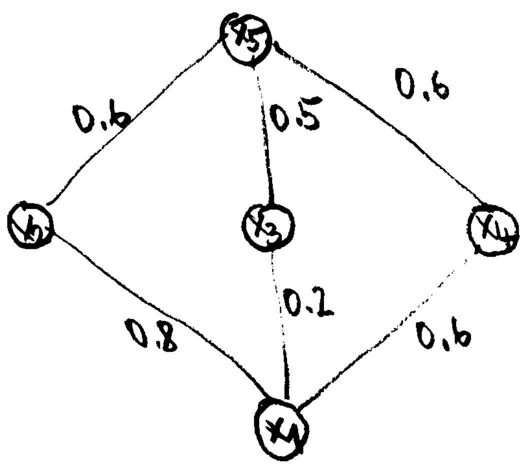
$$\mu_{R*R}(x,y) \leq \mu_R(x,y) \quad \text{for all } x,y \in X$$

Def. 34.8 A order $(A; \leq)$ is called a fuzzy poset if the fuzzy relation fulfill the conditions i, ii, iii.

ii is also called perfect antisymmetric

Example 34.9

	x_1	x_2	x_3	x_4	x_5
x_1	1.0	0.8	0.2	0.6	0.6
x_2	0.0	1.0	0.0	0.0	0.6
x_3	0.0	0.0	1.0	0.0	0.5
x_4	0.0	0.0	0.0	1.0	0.6
x_5	0.0	0.0	0.0	0.0	1.0



Def. 34.10 A binary fuzzy relation with the properties reflexive, symmetric and transitive is called a fuzzy similarity

Example 34.11

	a	b	c	d	e	f	g
a	1	0.8	0	0.4	0	0	0
b	0.8	1	0	0.4	0	0	0
c	0	0	1	0	1	0.9	0.5
d				1	0	0	0
e					1	0.9	0.5
f						1	0.5
g							1

The reader recognizes the reflexivity (~~all~~ 1 in the diagonal), the symmetry; the transitivity has to be sheck! ($R * R \subseteq R$)

For the crisp equivalence relation one find the decomposition of the classes. On the other hand we form relations

§35 Approximate reasoning

35A Possibility theory

In the center of the possibility theory is the concept of the possibility distribution

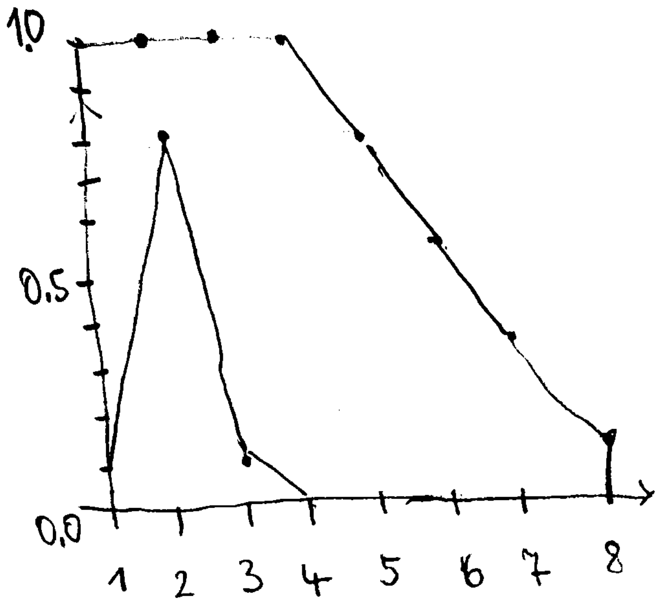
Example. Consider a numerical age, say $u = 28$, is approximately 0.7. The compatibility of a value of u with young becomes converted into the possibility of that value of u given "John is young" [Zadeh 1978]

(We write X instead of E the universe)

Def. 35.1 If A is a fuzzy set which the values of a fuzzy variable is bounded under $\mu_A : \Omega \to [0,1]$ there is defined a possibility distribution such that for $\omega \in \Omega$

$$\pi_X(\omega) = \mu_A(\omega) \qquad \text{holds}$$

Example. Possibility distribution:
"Hans ate X eggs for breakfast"



$\Omega = \{1,2,3,4,5,6,7,8\}$

ω	$\Pi_X(\omega)$	$P_X(\omega)$	
1	1	0.1	
2	1	0.8	
3	1	0.1	
4	1	0	
5	0.8	0	
6	0.6	0	
7	0.4	0	
8	0.2	0	

B Possibility measures

Def. 35.2 The probability measure is a fct $P : \mathcal{P}(\Omega) \to [0,1]$ which fulfill the axioms of Kolmogorow.

1) $P(A) \geq 0$
2) $P(\Omega) = 1$
3) $(A \cap B = \emptyset) \Rightarrow (P(A \cup B) = P(A) + P(B))$

Def 35.3 The fuzzy measure is a fct $g : \mathcal{P}(\Omega) \to [0,1]$ which fulfill the following axioms

1) $g(\emptyset) = 0$
2) $g(\Omega) = 1$
3) $(A \subseteq B) \Rightarrow g(A) \leq g(B)$

More over for infinite universes Ω it holds

4) $\lim\limits_{n \to \infty} g(A_n) = g(\lim\limits_{n \to \infty} A_n)$

for all monotone sequences $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ of sets $A_i \subseteq \mathcal{P}(\Omega)$

It follows

$$g(A \cup B) \geq \max\{g(A), g(B)\}$$

$$g(A \cap B) \leq \min\{g(A), g(B)\}$$

If we consider the limits of $g(A \cap B)$ and $g(A \cup B)$ you get possibility measure

Def. 35.4. Let A be a fuzzy set over Ω and Π_X a possibility distribution over Ω. The possibility measure $\Pi(A)$ is defined by

$$\Pi(A) = \sup_{\omega \in \Omega} \min\{\mu_A(\omega), \Pi_X(\omega)\}$$

(In the case that A is a crisp set we write

$$\Pi(A) = \sup_{\omega \in \Omega} \{\Pi_X(\omega) \mid \omega \in \Omega\}$$

35 C Rules of approximate reasoning

An example of a fuzzy propositions

Premise : This tomato is very red

Implication : If a tomato is red then the tomato is ripe

Conclusion : This tomato is very ripe

Zadeh has introduced some forms on implication.
We use the simple one.

Def. 35.5
Implication :

If $x = A$ then $y = B \rightarrow \Pi_{(x,y)}$

where $\Pi_{(x,y)} = \max(\min(\mu_A(x), \mu_B(y)), \min(1-\mu_A(x),$

Zadeh has also give two suggestions for the "general" modus ponens. The general form which we will use delivers the same conclusions if it applies the

the usual (crisp) sets. But is more general because it works by fuzzy sets and fuzzy implications

Def. 35.6 Generalized Modus ponens

$$\begin{array}{l} p: \ x = A' \\ q: \ \text{if } x = A \text{ then } y = B \to \Pi_{(x,y)} \\ \hline r: \ y = B' \end{array}$$

A', B' are fuzzy propositions, q is a fuzzy implication

Universität Kaiserslautern
Fachbereich Mathematik
Prof. Dr. D. Schweigert

Wintersemester 2001/02
14.01.02

Tutorial
"Logic"
Page 9

Deadline: Friday, 25.01.2002, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:
A modal logic in the logical symbols $\neg, \wedge, \vee, \rightarrow, \Box$ contains
1) all axioms of the proposition logic and $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$
2) is closed under the rule Modus ponens MP$\frac{\alpha\alpha\rightarrow\beta}{\beta}$.
3) is closed under the rule Modus necesse MN$\frac{\alpha}{\Box\alpha}$.
We introduce an additional axiom $G : \Box(\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha$ and call it modal logic of the class G (G :=Gödel). A derivation in the modal logic G we denote

$$\vdash^G$$

Prove the following theorem of Löb:
If $\vdash^G (\Box\alpha \rightarrow \alpha)$ then it holds $\vdash^G \alpha$.

Exercise 2:
Heating rule by the generalized modus ponens.
Rule: If the temperature of the room is normal then the ventil should be half open. The fuzzy values are defined in the following:

"normal temperature" = normal

"half open ventil" = half open

"about 20 degree Celsius" = about 20

Datas are given by

temperature	=	[15 16 17....25]
ventil	=	[0 0.5 1....5]
normal	=	[0 0 0.3 0.7 1 1 1 0.7 0.3 0 0]
halfopen	=	[0 0 0 0.3 0.7 1 0.7 0.3 0 0 0]
about 20	=	[0 0 0 0.2 0.7 1 0.7 0.2 0 0 0]

How can you put the ventil such that the temperature is about 20 degree ?

$R_{\text{normal}\rightarrow\text{halboffen}} = I_{LU}(\text{normal, halboffen})$

$$= \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1\\ 0.7&0.7&0.7&1&1&1&1&1&0.7&0.7&0.7\\ 0.3&0.3&0.3&0.6&1&1&1&0.6&0.3&0.3&0.3\\ 0&0&0&0.3&0.7&1&0.7&0.3&0&0&0\\ 0&0&0&0.3&0.7&1&0.7&0.3&0&0&0\\ 0&0&0&0.3&0.7&1&0.7&0.3&0&0&0\\ 0.3&0.3&0.3&0.6&1&1&1&0.6&0.3&0.3&0.3\\ 0.7&0.7&0.7&1&1&1&1&1&0.7&0.7&0.7\\ 1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix}$$

etwa 20 $\circ$ I_{LU}(normal, halboffen)

$$[0\ 0\ 0\ 0.2\ 0.7\ 1\ 0.7\ 0.2\ 0\ 0\ 0] \circ \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1\\ 0.7&0.7&0.7&1&1&1&1&1&0.7&0.7&0.7\\ 0.3&0.3&0.3&0.6&1&1&1&0.6&0.3&0.3&0.3\\ 0&0&0&0.3&0.7&1&0.7&0.3&0&0&0\\ 0&0&0&0.3&0.7&1&0.7&0.3&0&0&0\\ 0&0&0&0.3&0.7&1&0.7&0.3&0&0&0\\ 0.3&0.3&0.3&0.6&1&1&1&0.6&0.3&0.3&0.3\\ 0.7&0.7&0.7&1&1&1&1&1&0.7&0.7&0.7\\ 1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix}$$

$= [0.2\ 0.2\ 0.2\ 0.3\ 0.7 \ldots$

ARISTOTLE VS. THE BUDDHA

Everything must either be or not be, whether in the present or in the future.

ARISTOTLE
DE INTERPRETATIONE

I have not explained that the world is eternal or not eternal. I have not explained that the world is finite or infinite.

THE BUDDHA
MAJJHIMA-NIKAYA

The fundamental idea of Buddhism is to pass beyond the world of opposites, a world built up by intellectual distinctions and emotional defilements.

D. T. SUZUKI
THE ESSENCE OF BUDDHISM

Fuzzy logic is a concept derived from the branch of mathematical theory of fuzzy sets. Unlike the basic Aristotelian theory that recognizes statements as only "true" or "false," or "1" or "0" as represented in digital computers, fuzzy logic is capable of expressing linguistic terms such as "maybe false" or "sort of true." In general, fuzzy logic, when applied to computers, allows them to emulate the human reasoning process, quantify imprecise information, make decisions based on vague and incomplete data, yet by applying a "defuzzification" process, arrive at definite conclusions.

FUZZY LOGIC: A KEY TECHNOLOGY FOR FUTURE COMPETITIVENESS
U.S. DEPARTMENT OF COMMERCE, NOVEMBER 1991

Syllogism

A syllogism consists of three parts

major premise
minor premise

conclusion

Example (Barbara)

M is P	all men are mortal
S is M	Socrates is a man
S is P	Socretes is mortal

Fuzzy Syllogism

is a scheme in which both premise and the conclusion are fuzzy and contains fuzzy quantors.

Example (intersection - product - syllogism)

Q_1 : A is B	$Q_1 = 0.8$: most students are not married 80%
Q_2 : (A and B) is C	$Q_2 = 0.6$: most students are male (60%)
Q_3 : A is (B and C)	$Q_3 = 0.48$: most st. are unmarried and male. 48%

$Q_3 = Q_1 \cdot Q_2$ if monotone
or else $\geq Q_1 \cdot Q_2$

Q_1, Q_2, Q_3 are fuzzy quantors and A, B, C are fuzzy predicates which are connected.

Example 35.7

If the temperature is normal then the (heating) ventilatorshould be half open

— Temperature = [15, 16, ---- , 25] Celsius

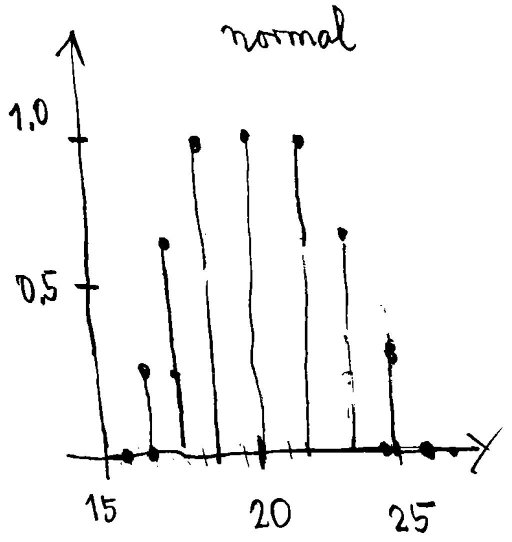
discrete values

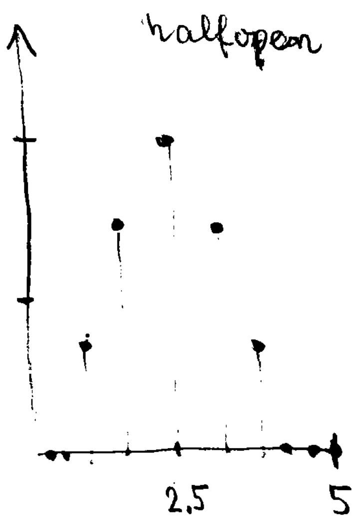
— Ventilation = [0, 0.5, 1, 1.5, ---- 5]

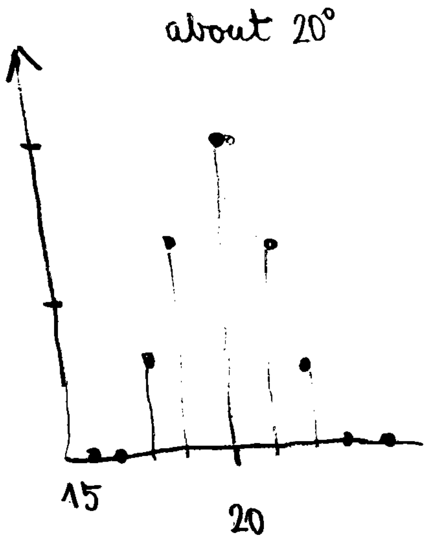
normal = [0, 0, 0.3, 0.7, 1, 1, 1, 0.7, 0.3, 0, 0]

half open = [0, 0, 0, 0.3, 0.7, 1, 0.7, 0.3, 0, 0, 0] fuzzy va

— about 20° = [0, 0, 0, 0.2, 0.7, 1, 0.7, 0.2, 0, 0, 0]







We get the position of the ventilator of the fuzzy input value " about 20° " with the implication $\Pi_{(x,y)}$

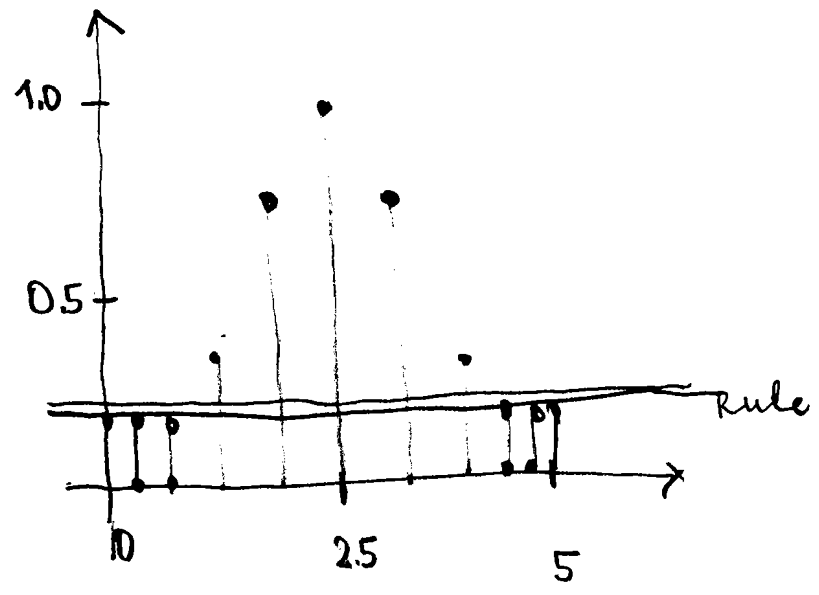
$\Pi_{(x,y)}$: normal, half open

" about 20° "

$$[0,0,0,0.2,0.7,1,0.7,0.2,0,0,0] \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0.7 & 0.7 & 0.7 & 1 & 1 & 1 & 1 & 1 & 0.7 & 0.7 & 0.7 \\ 0.3 & 0.3 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0.3 & 0.7 & 1 & 0.7 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 1 & 0.7 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 1 & 0.7 & 0.2 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0.3 & 0.3 \\ 0.7 & 0.7 & 0.7 & 1 & 1 & 1 & 1 & 1 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$= [0.2, 0.2, 0.2\ 0.3\ 0.7\ 1\ 0.7\ 0.3\ 0.2\ 0.2\ 0.2$

Conclusion : ((Rule of heating)) Position of the ventilator



§36 Rough sets

In 1981 Pawlak has developed another concept of fuzziness. At first we use the concept of an equivalence relation.

A binary relation R on X is an equivalence relation which is reflexive, symmetric and transitive. We use the notation

$$x \sim y \iff (x,y) \in R$$

For a set of elements we write

$$[z]_R = \{x \mid x \in X,\ x \sim z\}$$

which is equivalent for $z \in X$

The factor set we write

$$X/_R := \{[z]_R \mid z \in X\}$$

Example of the number theory:

x is congruent to y $\iff$ mod 3 $\iff$ $x - y \equiv 0$

$R = \{(x,y) \mid x,y \in \mathbb{N},\ x \equiv y \bmod 3\}$

Notation 36.1 Let E be fundamental set (an universe) and R be equivalence relation on E.
The tupel $A = (E; R)$ is called an approximation space which is generated by the equalence classes of R and is named as elementary sets.
Every union of elementary sets is called as composed sets of E

Def. 36.2 Let $A = (E; R)$ be an approximation space. Then the following holds:

$$\overline{A}(X) := \{ z \mid z \in E,\ [z]_R \cap X \neq \emptyset \}$$

$$\underline{A}(X) := \{ z \mid z \in E,\ [z]_R \subseteq X \}$$

$$B_A(X) := \overline{A}(X) \setminus \underline{A}(X)$$

$\overline{A}(X)$ is the upper, and $\underline{A}(X)$ is the lower approximation, and $B_A(X)$ is the boundary
The pair $(\overline{A}(X), \underline{A}(X))$ is called a rough set

36.3 Properties of rough sets

B1 $\underline{A}(X) \subseteq X \subseteq \overline{A}(X)$

B2 $\underline{A}(E) = \overline{A}(E) = E$

B3 $\underline{A}(\emptyset) = \overline{A}(\emptyset) = \emptyset$

B4 $\overline{A}(X \cup Y) = \overline{A}(X) \cup \overline{A}(Y)$

B5 $\underline{A}(X \cup Y) \supseteq \underline{A}(X) \cup \underline{A}(Y)$

B6 $\overline{A}(X \cap Y) \subseteq \overline{A}(X) \cap \overline{A}(Y)$

B7 $\underline{A}(X \cap Y) = \underline{A}(X) \cap \underline{A}(Y)$

B8 $\overline{A}(-X) = -\underline{A}(X)$

B9 $\underline{A}(-X) = -\overline{A}(X)$

Def 36.4 Let $A = (E, R)$ be an approximation space, $X \subseteq E$ non-empty subset of E and $(\overline{A}(X), \underline{A}(X))$ be a rough set

The Δ-accuracy measure is

$$\Delta_R(X) = \frac{|\underline{A}(X)|}{|\overline{A}(X)|}$$

The limit of the quotient exist always

$$0 \leq \Delta_R(X) \leq 1$$

Notation 36.5

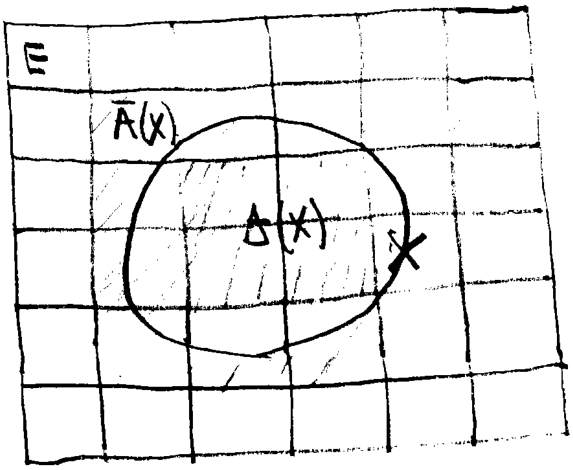
$$x \underline{\in}_R X \iff x \in \underline{A}(X) \qquad \text{lower}$$

$$x \overline{\in}_R X \iff x \in \overline{A}(X) \qquad \text{upper}$$

We say $\overline{\in}$ x possibly belongs to X

$\underline{\in}$ x surely belongs to X

Example 36.7



The upper approximation is the join of the lower approximation with together the boundary

Therefore:
The smallest composed set of A which contains X is the upper approximation

The largest composed set of A which contains X is the lower approximation.

Example 36.8

$K = (E; R)$

$$E = \{x_1, x_2, x_3, \dots, x_8\}$$

Equivalence classes:

$$E_1 = \{x_1, x_4, x_8\}$$

$$E_2 = \{x_2, x_5, x_7\}$$

$$E_3 = \{x_3\}$$

$$E_4 = \{x_6\}$$

It is $E/R = \{E_1, E_2, E_3, E_4\}$

Let $X_1 = \{x_1, x_4, x_7\}$ and $X_2 = \{x_2, x_8\}$

$$\underline{A}(X_1 \cup X_2) = E_1$$

$$\underline{A}(X_1) = \emptyset \quad \underline{A}(X_2) = \emptyset$$

Hence $\underline{A}(X_1 \cup X_2) \supsetneq \underline{A}(X_1) \cup \underline{A}(X_2)$ ✓

§ 37 Remarks

Fuzzy logic   Zadeh 1965 Berkeley

Rough sets   Pawlak 1982 Warsaw

Many valued logic

Łukasiewicz 1920

Work of Post (Boolean algebra and Post classes)
1921 and detailed version 1941

Switching algebra

Russian School
Jablonski has characterized the functional

completeness for the three valued logic 1958

Rosenberg   n-valued logic
functional completeness 1970

Modal Logic
Aristoteles has described the concepts "Necessarity"

and "Possibility" with together its negation
In the mediaval age it was discussed the

works of Aristoteles (in the way of Islam)

For instance Euklidi's element was brought
from the Okzident and translated from
the arabian by Adelhard of Bath (1120)
(clothed as a mohammedian student in Cordoba)

Ockham introduced more modalities like
opinion, thout, ...

School of Port-Royal (Paris 1662)
Discussion of the concepts.

Gottfried Wilhelm Leibniz 1651-1716
Possibilible worlds

Kurt Gödel 1906 - 1978
The modality of provability

VI Incompleteness
(Abridged version)

§ 38 Predicate logic of the second order

38.1 Language of the second order

The language of the predicatelogic is called a language of the first order L_I

The language L_{II} of second order contains the language of the first order L_I. The alphabet of L_{II} contains additionally countable precate variables for every $n \geq 1$ (relation variables)

$$X_1^n, X_2^n, X_3^n, \ldots$$

We write only $X, Y, Z, \ldots$ if the arity of $X, Y, Z, \ldots$ is known.

There are following formula-building operations

38.1.1 If X is n-place predicate variable and if $t_1, \ldots, t_n$ terms then $X(t_1, \ldots, t_n)$ is a formula of the second order

38.1.2 If φ is a formula of second order and X is a predicate variable then $\forall X \varphi$ is a formula of the second order

We will usea also function variables. These can be considered as predicate variables as every function presents nothing else a relation We write $F, G, H, \ldots$ for the function variables Naturally it holds 38.1.1 and 38.1.2 for such a language with $F, G, H, \ldots$

Example 38.2
We consider the following axiom system of the natural numbers (by Peano) in a language with unary functions symbols S and constant symbol 0

38.2.1 $\quad \forall x\,(S(x) \neq 0)$

38.2.2 $\quad \forall x \forall y\,(S(x) = S(y) \to x = y)$

38.2.3 $\quad \forall Y\,(Y(0) \wedge \forall x\,(Y(x) \to Y(S(x)) \to \forall x\, Y(x))$

This formula 38.2.3 is the induction principle in the second order. Y is an unary predicate variable

38.3 Models for a language of the second order

A (truth) assignment of the second order in a structure $\tilde{A}$ is a mapping which every variable x_i maps an element of A and every predicate variables X_i^n maps every n-place relation of $\tilde{A}$

If $\tilde{A}$ is a structure, h a (truth) assignment of second order in $\tilde{A}$ then we put

38.3.1 $\quad \tilde{A} \models X(t_1, \dots, t_n) \Leftrightarrow (t_1, \dots, t_n) \in h(X)$

i.e. if $(t_1, \dots, t_n)$ is in the relation $h(X)$

38.3.2 $\quad \tilde{A} \models \forall X \varphi$ if and only if every assignment h of X satisfies the formula φ

(i.e. every n-place relation of A satisfies the formula φ)

We consider the axiom system of natural numbers (Peano)

Every model is unique till up isomorphism

Such an axiom system is catecorical
This property have been proved by Dedekind

Thm 38.4 Every model of 38.2.1 - 38.2.3 (Peano) is uniquely determined till up isomorphism

Proof. Let $(A; f, a)$ be a model of 38.2.1 - 38.2.2
We consider $\grave{A} = \{a, f(a), f(f(a)), \dots\}$

By 38.2.1 and 38.2.2 it follows that $(\grave{A}; f/_{\grave{A}}, a)$ is isomorphic to $(\mathbb{N}; S, 0)$

We denote: $\mathcal{Y}(x)$ is the one-place relation R on A with $x \in R$ if and only if $x \in A$ and $x \in \grave{A}$

$\mathcal{Y}(0)$ holds because $a \in A \cap \grave{A}$

If $\mathcal{Y}(x)$ hold then it follows $\mathcal{Y}(S(x))$ because $x \in A \cap \grave{A}$ and $S(x) \in A \cap \grave{A}$

By the axiom 38.2.3 it follows that $\forall x\, \mathcal{Y}(x)$
i.e. $A \subseteq \grave{A}$

Obviously we have $A' \subseteq A$ and so we have $A = A'$

Remark 38.5

In the second order language we have all n-place relations (for $n \geq 1$). Therefore we can ~~formd~~ all concept like subset Therefore one part of set theory is atomatically built

Remark 38.6

In general mathematics is built up by the set theory and the predicate logic of the first order

In both ways you cannot avoid the difficulties which results the Gödel's incompleteness theorem and other problems.

/

§ 39 Recursive functions

Def. 39.1 The Peano arithmetic PA consists of the following axioms of the formal number theory in the language of first order.
((Parameters are $S, \tilde{0}, +, \cdot$ as well as primitive recursive functions))

I $\quad \tilde{0} \neq Sx$

$Sx = Sy \rightarrow x = y$

(*) $p_i^k(x_1, \dots, x_k) = x_i$

(**) $c_n^k(x_1, \dots, x_k) = \tilde{n}$

II $\quad x < z \leftrightarrow \exists x\,(x + z = y)$

III $\quad x + \tilde{0} = x \qquad x + Sy = S(x+y)$

$x \cdot \tilde{0} = \tilde{0} \qquad x \cdot Sy = x \cdot y + y$

(***) f is defined by primitive recursion of g, h if it holds

i) $f(0, x_1, \dots, x_k) = g(x_1, \dots, x_k)$

ii) $f(Sx, x_1, \dots, x_k) = h(f(x, x_1, \dots, x_n), x_1, \dots, x_k)$

f belongs to PA if g, h belong to PA

(***) f is defined by composition of $g, h_1, \ldots, h_n$ if it holds

$$f(x_1, \ldots, x_k) = g(h_1(x_1, \ldots, x_k), \ldots, h_k(x_1, \ldots, x_k))$$

f belongs to PA if $g, h_1, \ldots, h_k$ belong to PA

IV $\varphi(\tilde{0}) \wedge \forall x (\varphi(x) \to \varphi(Sx)) \to \forall x\, \varphi(x)$

Def. 39.2 The set of primitive recursive functions is the smallest set of functions which contain the function (*) and (**) and which is closed under primitive recursion (***) and composition (****)

Example 39.3 The addition can be defined by

$x + 0 = x$

$x + (y + 1) = (x + y) + 1$

It is a primitive recursive function because

$+(0, x) = p_2^2(0, x)$

$+(y+1, x) = S(p_1^3(+(y, x), x, y)$

Remark

The concept of recursive function is equivalent to the concept of the Turing machine

A Turing machine is an abstract represention of a computer

Alan Turing has given a mathematical precise definition of algorithm.

§ 40 Gödel numbers and modal logic

40.1 Coding by Gödel numbers

Every symbol of the alphabet of the language of the number theory can be correspond to numbers. These numbers is called Gödelnumbers

(i) $0 \longmapsto 1$ $\qquad S \longmapsto 9$

$\vee \longmapsto 3$ $\qquad + \longmapsto 11$

$\neg \longmapsto 5$ $\qquad \cdot \longmapsto 13$

$\exists \longmapsto 7$ $\qquad = \longmapsto 15$

and

$x_i \longmapsto 2_i$ $\qquad i = 1, 2, 3, \ldots$

(ii) The Gödel numbers of the terms is defined in following

$\ulcorner x_i \urcorner = \langle 2i \rangle$ $\qquad \ulcorner \bar{0} \urcorner = \langle 1 \rangle$

$\ulcorner St \urcorner = \langle 9, \ulcorner t \urcorner \rangle$ $\qquad \ulcorner t + s \urcorner = \langle 11, \ulcorner t \urcorner, \ulcorner s \urcorner \rangle$

$\ulcorner t \cdot s \urcorner = \langle 13, \ulcorner t \urcorner, \ulcorner s \urcorner \rangle$

(iii) The Gödel numbers of the formulas are defined as following

$$\ulcorner (s = t) \urcorner = \langle 15, \ulcorner s \urcorner, \ulcorner t \urcorner \rangle \qquad \ulcorner (\varphi \vee \psi) \urcorner = \langle 3, \ulcorner \varphi \urcorner, \ulcorner \psi \urcorner \rangle$$

$$\ulcorner \neg \varphi \urcorner = \langle 5, \ulcorner \varphi \urcorner \rangle \qquad \ulcorner \exists x_i \varphi \urcorner = \langle 7, \ulcorner x_i \urcorner, \ulcorner \varphi \urcorner \rangle$$

Resumé 40.2

The terms t or the formulas φ are corresponding to the Gödel numbers $\ulcorner t \urcorner$ or respectivelly $\ulcorner \varphi \urcorner$. By the syntax we can simulated by the primitive recursive functions

Example

a) $neg \ulcorner \varphi \urcorner = \ulcorner \neg \varphi \urcorner$

b) $sub(\ulcorner \varphi x \urcorner, \ulcorner n \urcorner) = \ulcorner \varphi n \urcorner$

substitution of n in x of the formula φx

The derivation (= proofs) can be considered as finite sequences of formulas.

The relation "x is coded by the derivation of the formula which is coded by y" is primitive recursive.

We write

(40.2.) $\quad \mathrm{Prov}(x,y) : p(x,y) = \tilde{0}$

and say

Prov(x,y) is true if and only if
$x = (\ulcorner \varphi_1 \urcorner, \ldots, \ulcorner \varphi_n \urcorner)$ and $y = (\ulcorner \varphi_n \urcorner)$
and $(\varphi_1, \ldots, \varphi_n)$ is a (formal) proof of φ_n

Notation 40.3

For $\exists x\, \mathrm{Prov}(x,y)$ we write

$$\Box y$$

Remark 40.4

If $f(x_1, \ldots, x_n)$ is a primitive recursive function and if $(m_1, \ldots, m_k)$ is given then $f(m_1, \ldots, m_k)$ is really given in finite steps

(40.4) If $f(m_1, \ldots, m_k) = n$ then
$$PA \vdash f(\tilde{m}_1, \ldots, \tilde{m}_k) = \tilde{n}$$

If we consider (40.4) we recognize that induction is used in order to define f.

If f is defined by primitive recursion we need an additional induction for the "recursion variable". As PA contain the induction scheme IV it holds

(40.5) $\mathrm{PA} \vdash \forall x_1 \cdots \forall x_k \forall y \,(f(x_1, \dots, x_k) = y$

$$\to \Box\,(\mathrm{sub}(\ulcorner f(x_1, \dots, x_k) = y \urcorner, \ulcorner x_1 \urcorner, \dots \ulcorner x_k \urcorner, \ulcorner y \urcorner, x_1 \dots x_k,$$

§41 Incompleteness theorems

Thm 41.1 (Conditions of Löb)

For all propositions φ, ψ it holds

41.1.1 If $PA \vdash \varphi$ then also $PA \vdash \Box \ulcorner \varphi \urcorner$

41.1.2 $PA \vdash (\Box \ulcorner \varphi \urcorner \wedge \Box (\ulcorner \varphi \to \psi \urcorner) \to \Box \ulcorner \psi \urcorner)$

41.1.3 $PA \vdash (\Box \ulcorner \varphi \urcorner \to \Box (\Box \ulcorner \varphi \urcorner))$

The condition 41.1.3 is the self reference

Proof 41.1.1 If $PA \vdash \varphi$ implies that there exist a derivation $(\varphi_1, ..., \varphi_{n-1}, \varphi)$ of φ. Therefore (40.2) we have

$$P((\ulcorner \varphi_1 \urcorner, ..., \ulcorner \varphi_{n-1} \urcorner), \ulcorner \varphi \urcorner, \ulcorner \varphi \urcorner) = 0$$

If we use (40.3) we have

$$PA \vdash Prov((\ulcorner \varphi_1 \urcorner, ..., \ulcorner \varphi_{n-1} \urcorner, \ulcorner \varphi \urcorner), \ulcorner \varphi \urcorner)$$

$$PA \vdash \exists x \, Prov(x, \ulcorner \varphi \urcorner)$$

$$PA \vdash \Box \ulcorner \varphi \urcorner$$

The other condition 41.1.2, 41.1.3 can be proved in similar way Δ

Remark. The condition 41.1.1 – 41.1.3 are a key for $\square X$

Thm 41.2 (Diagonalization lemma). We assume that in the formula ψ only one variable x is free. Then there exists a proposition φ such that

$$PA \vdash (\varphi \leftrightarrow \psi(\ulcorner \varphi \urcorner))$$

Thm 41.3 (Incompleteness thms of Gödel)

Let $PA \vdash \varphi \leftrightarrow \neg\square(\ulcorner \varphi \urcorner)$

Then it holds

(i) $PA \nvdash \varphi$

(ii) $PA \nvdash \neg\varphi$

Sketch:

(i) We assume that $PA \vdash \varphi$. Then by Löb 40.1.1

$$PA \vdash \square(\ulcorner \varphi \urcorner)$$

By the hypothesis follows

$$PA \vdash (\neg\varphi)$$

This is a contradiction of the concidence of PA

(ii) We assume $PA \vdash (\neg\varphi)$

By the hypothesis there is

$$PA \vdash \Box(\ulcorner\varphi\urcorner)$$

As in PA only true theorems are provable

then φ must be true and derivable

$$PA \vdash \varphi$$

Remark

The diagonalization lemma is the condition

that the $PA \vdash (\varphi \leftrightarrow \neg\Box(\ulcorner\varphi\urcorner))$ holds

That means that there exist a φ which is satisfiable

Therefore the incompleteness thm of Gödel holds.

§ 42 Remarks

Gödel's idea was the following self reference

Epimenides: " I am lying

respectivelly

Gödel: " What I say, is not provable"

1950 Leon Henkin put the question:

As we know the Gödel's proposition on the own not provability is not provable

What holds now?

" What I say is provable"

1954 Martin Löb answered this question by

yes

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05.11.01

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Monday 19.11.

Deadline: ~~Friday, 16.11.~~01, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:

a)The following in equality holds for every lattice

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$$

b) Consider the following identities and inequality

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$

(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(iii) $(x \vee y) \wedge z \leq x \vee (y \wedge z)$

Then $(i), (ii)$ and (iii) are equivalent in any lattice.

Exercise 2:

a) For every Boolean algebra B the relation $\leq$ is defined by

$$x \leq y \Leftrightarrow x \wedge y = x$$

Prove that the relation "$\leq$" is a partially-ordered set (=poset).
b) Prove $x \wedge b' = 0 \Rightarrow x \leq y$.

Exercice 3:

The following conditions are equivalent on a filter F of an Boolean algebra **B**

a) $B/_F \simeq 2$.

b) F is an ultrafilter.

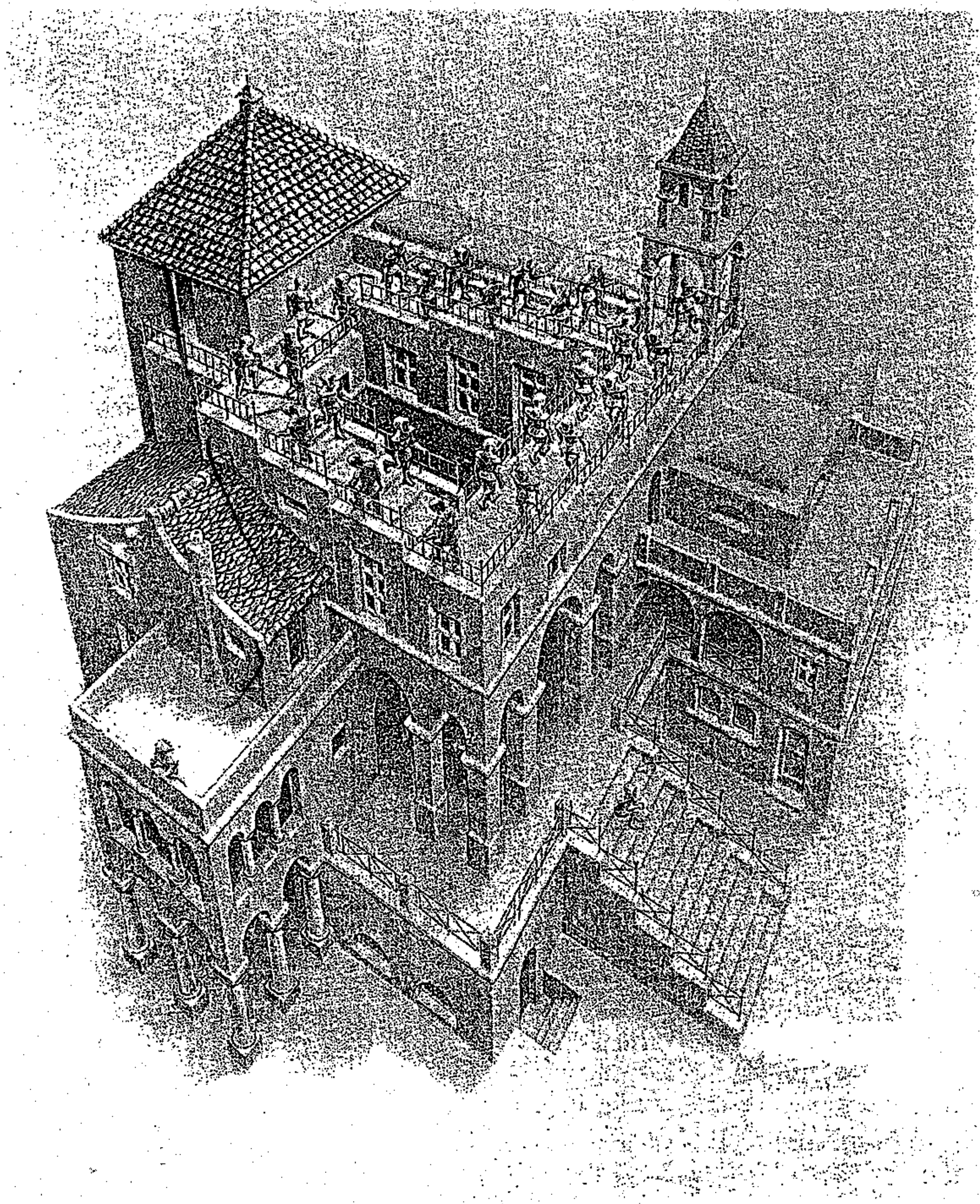
c) F is a primfilter.

A filter F of an Boolean algebra B is a primfilter if for all $x, y \in B$ holds:

If $x \wedge y \in F$ then either $x \in F$ or $y \in F$).

Exercise 4:

a) Let A be a countable set. Then the set of all finite sequences of members of A is also countable.
b) What holds for the set of all countable sequences ?



Treppauf, Treppab, *von M. C. Escher (Lithographie, 1960).*

"I know what you're thinking about" said Tweedledum:
"but it is not so, nowhow"
"Contrariwise" continued Tweedledee
"if it was so, it might be;
and if it were so, it would be;
but as it is not, it ain't.
That is Logic"

Lewis Caroll

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Winters[redacted]ter 2001
15.11.01

Tutorial
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Page 2

Deadline: Friday, 23.11.01, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:

A pseudo Boolean function is a function $f : B_2^n \longrightarrow \mathbb{Z}$ where B_2 is the 2-valued Boolean algebra and $\mathbb{Z}$ the ring of integers $(\mathbb{Z}, +, -, 0, \cdot, 1)$.

Let $\sum\limits_{I \in \Omega} c_i \prod\limits_{i \in I} x_i$ be the polynomial normal form of a pseudo-Boolean term where Ω is a collection of subsets of $\{1, ...n\}$ and the c_I are coefficients in $\mathbb{Z}\backslash\{0\}$. The pseudo-Boolean term t is linear if $|I| \leq 1$ for all $I \in \Omega$.

A vector $(x_1^*, ..., x_n^*) \in B_2^n$ is a minimum of the pseudo-Boolean function $f(x_1, ..., x_n)$ if

$$f(x_1^*, ..., x_n^*) \leq f(x_1, ..., x_n)$$

for any $(x_1, ..., x_n)$ in B_2^n.

a) Determine the minimum of the function

$$f(x_1, ..., x_7) = 2 + 3x_1 - 2x_2 + 5x_3 + 2x_6 - x_7$$

p_4 and p_5 are arbitrary parameters.

b) Give an algorithm which determine the minimas

Exercise 2:

Consider a pseudo Boolean function. Determine the minimum of the following program

$$\min 2 + 3x_1 - 2x_2 - 5x_3 + 2x_4 + 4x_6$$

with the constraints

$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 + 4x_4 + 2x_5 - x_6 &\leq 2 \\ 4x_1 + 2x_2 + x_3 + 8x_4 - x_5 - 3x_6 &\leq 4 \end{aligned}$$

Exercice 3:

Let D be a set of divisors of a number $d \in \mathbb{N}$. Then there is a lattice $\mathrm{D} = (D; \wedge, \vee)$ if one defines

$a \wedge b :=$ least common multiple of a, b $(\mathrm{lcm}(a, b))$.

$a \vee b :=$ greatest common divisor of a, b $(\gcd(a, b))$

a) Draw the Hasse diagram for $d = 36$.

b) Prove that D is distributive.

Exercise 4:

The algebra $\mathbf{B} = (B; \wedge, \vee, ', 0, 1, \Box)$ is called modal algebra of type $(2, 2, 1, 0, 0, 1)$

1) $(B; \wedge, \vee, ', 0, 1, \cancel{\Box})$ is a Boolean algebra.

2) $\Box$ is an unary operation such that

$$\Box(1) = 1$$
$$\Box(x \wedge y) = \Box(x) \wedge \Box(y)$$

A modal algebra $\mathbf{B}$ is of the class $S4$ if it holds

$$\Box(x) \leq x$$
$$\Box(x) \leq \Box(\Box(x))$$

We denote: $\Diamond(x) =' (\Box('(x)))$.

Give a normal form system for the modal algebra.

Remark.

The operator $\Box$ plays a pure formal role and have no meaning.

One can give a meaning on the sake of the understanding.

Once an choose

i) $\Box(\alpha)$: it is necessary that α holds

ii) $\Box(\alpha)$: α holds of all possible worlds (Leibniz)

iii) $\Box(\alpha)$: is is provable that α holds (Gödel)

In our case (iii) would be the best.

"Die Mathematiker sind eine Art Franzosen; redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es sobald ganz etwas anderes."

(J.W. von Goethe)

Die Welt in der wir leben, ist ja nicht die einzig mögliche; sonst wäre sie wiederum nicht nur möglich und wirklich, sondern absolut notwendig. Wohl aber ist sie wegen der Güte Gottes es hypothesi notwendig, und zwar die beste aller möglichen oder bestmögliche.

(Leibniz)

Pangloss disait quelquefois à Candide:

Tous les événements sont enchainés dans le meilleur des mondes possibles: car enfin si vous n'aviez pas été chassé d'un beau chateau à grands coups de pied dans le derrière pour l'amour de maidemoiselle Cunégonde, si vous n'aviez pas été mis à l'Inquisition, si vous n'aviez pas couru l'Amérique à pied si vous n'aviez pas perdu tous vous moutons du bon pays d'Eldorado, vous ne mangeriez pas ici des cédrats confits et des pistaches. –

Cela est bien dit, répondit Candide, mais il faut cultiver notre jardin.

(Voltaire, Candide)

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20.11.01

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Page 3

Deadline: Friday, ~~22~~ 30.11.01, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:

1a) Present all congruence relations of the Boolean algebra $(\{0,1\}^3; \wedge, \vee, ', 0, 1)$ (ideals, filters)

1b) Present all congruence relations of the distributive lattice $(D_3; \wedge, \vee)$ with the elements $0, a, 1$.

1c) Present all congruence relations of the ring

$$\mathbf{Z} = (\mathbb{Z}/_{4\mathbb{Z}}; +, -, 0, \cdot, 1)$$

with $\mathbb{Z}/_{4\mathbb{Z}} = \{0,1,2,3\}$ (ideals, filters)

Exercise 2:
Let $\mathbf{B} = (B; \wedge, \vee, ', 0, 1)$ be the Boolean algebra with $B = \{0, 1\}$.

For every function $f : B^n \to B$ there exists a term function $t : B^n \to B$ of the Boolean algebra $\mathbf{B}$ such that $f = t$.

Hint: Thm 15B Enderton/respectively: Thm 11.9

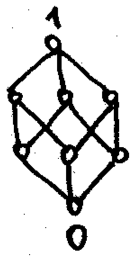
Exercice 3:
Consider $\mathbb{Z}/_{p\mathbb{Z}} = \{0, 1, ..., p-1\}$ with p a prime number and consider the ring $Z = \mathbb{Z}/_{p\mathbb{Z}}$ namely $\mathbf{Z} = (Z; +, -, 0, \cdot, 1)$. For every function $f : Z^n \to Z$ of the ring $\mathbf{Z}$ there exists a polynomial function $pf : Z^n \to Z$ of the ring $\mathbf{Z}$ such that $f = pf$

(Lagrange interpolation)

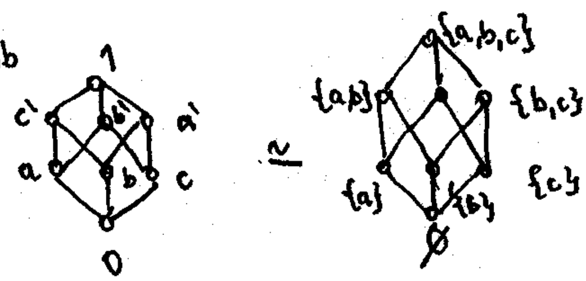
Exercise 4:

4a) Show: Not every Boolean algebra is isomorphic to the power set Boolean algebra.

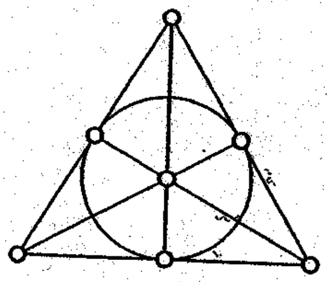
4b) Prove: Every Boolean algebra is isomorphic to a subalgebra of a power set Boolean algebra.

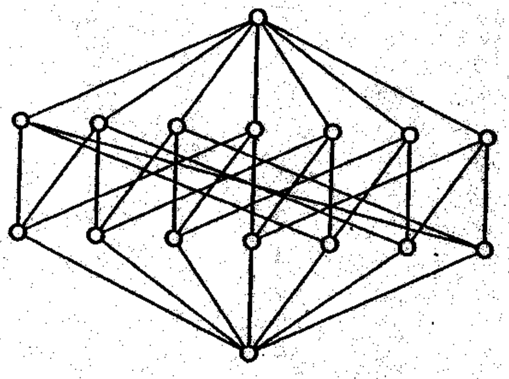
Ex 1a


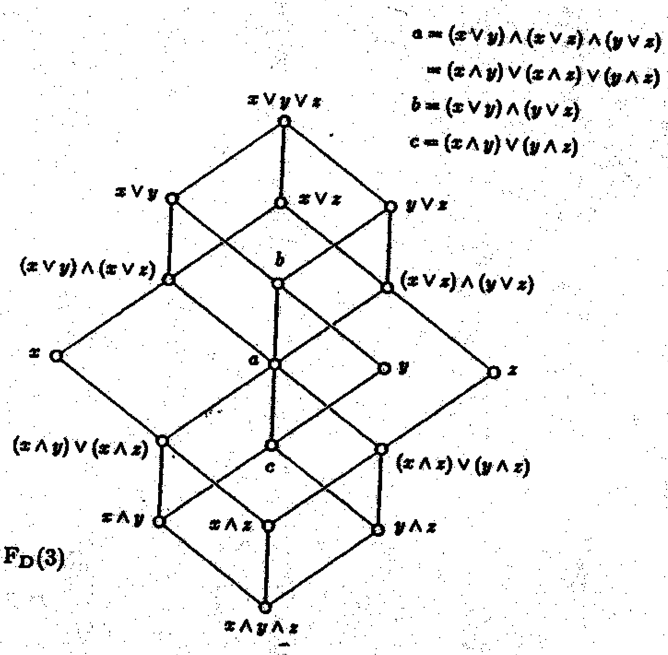
Ex 1b


Ex 4a,b


Power set Boolean algebra: $(P(X); \cap, \cup, ', \emptyset, X)$







$F_D(3)$

Das Ganze ist gleich der Summe seiner Teile.
EUKLID, Elemente

Das Ganze ist mehr als die Summe seiner Teile.
MAX WERTHEIMER, Productive Thinking

"A book should have either intelligibility or correctness; to combine the two is impossible" – BERTRAND RUSSELL [1901].

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Deadline: Friday, 07.12.01, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:
Show that the formulas are tantologies:

$$((\neg(A \to B)) \leftrightarrow (A \wedge (\neg B)))$$
$$((\neg(A \wedge B)) \leftrightarrow ((\neg A) \wedge (\neg B)))$$

Exercise 2:
Show or contradict for a set $\sum$ of formulas and formulas α und β.

(a) If either $\sum \models \alpha$ or $\sum \models \beta$ then it holds $\sum \models (\alpha \vee \beta)$.

(b) If $\sum \models (\alpha \vee \beta)$ then it holds either $\sum \models \alpha$ or $\sum \models \beta$.

Exercise 3:
Show $\{((\neg A_1) \to (\neg A_2)); ((\neg A_1) \to A_2)\} \models A_1$.

Exercise 4:
The algebra $\mathbf{B} = (B; \wedge, \vee, ', 0, 1, \Box)$ is a modal algebra if

1) $\mathbf{B} = (B; \wedge, \vee, ', 0, 1,)$ is a Boolean algebra

2) $\Box$ is an unary operation such that

$$\Box(1) = 1$$
$$\Box(x \wedge y) = \Box(x) \wedge \Box(y)$$

The algebra is called of the class M if $\Box(x) \leq x$

4a) Let $\mathbf{B}$ be finite and is of the class M. Every congruence of B corresponds to a fixed point and vice versa.

4b) Give two examples of simple modal algebras.

Remark.
$\Box$ is called necessary operator, $\Diamond$ is called possibility operator. $\Diamond(\varphi) := \neg\Box\neg(\varphi)$.

An algebra A is simple if the algebra hat only trivial congruences, that means all relation (A, A) and the diagonal $\{(x, x) \mid x \in A\}$.

Validity in modal logic.

The task of defining validity for **modal** wffs is complicated by the fact that, even if the truth values of all of the variables in a wff are given, it is not obvious how one should set about calculating the truth value of the whole wff. Nevertheless, a number of definitions of validity applicable to **modal** wffs have been given, each of which turns out to match some axiomatic **modal** system, in the sense that it brings out as valid those wffs, and no others, that are theorems of that system. Most, if not all, of these accounts of validity can be thought of as variant ways of giving formal precision to the idea that necessity is truth in every "possible world" or "conceivable state of affairs." The simplest such definition is this: Let a model be constructed by first assuming a (finite or infinite) set W of "worlds." In each world, independently of all the others, let each propositional variable then be assigned either the value 1 or the value 0. In each world the values of truth functions are calculated in the usual way from the values of their arguments in that world. In each world, however, $L\alpha$ is to have the value 1 if α has the value 1 not only in that world but in every other world in W as well and is otherwise to have the value 0; and in each world $M\alpha$ is to have the value 1 if α has value 1 either in that world or in some other world in W and is otherwise to have the value 0. These rules enable one to calculate a value (1 or 0) in any world in W for any given wff, once the values of the variables in each world in W are specified. A model is defined as consisting of a set of worlds together with a value assignment of the kind just described. A wff is valid if and only if it has the value 1 in every world in every model.

Aristoteles 384 – 322 A.C.

In the logic he has developed rules which the hypothesis leads to correct conclusions

Example. All men are mortal. All greeks are human.

It results to the valid conclusion. All greeks are mortal.

Syllogism ("put it together") consists of three sentences which the general into the special concludes.

Book of Aristotle "Analytica Priora"

Example Modus "Barbara"

$$\begin{array}{c} S \rightarrow M \\ M \rightarrow P \\ \hline S \rightarrow P \end{array}$$

There are four figures Aristotle "schemata"

$$\begin{array}{cc} M & P \\ S & M \\ \hline S & P \end{array} \qquad \begin{array}{cc} P & M \\ S & M \\ \hline S & P \end{array} \qquad \begin{array}{cc} M & P \\ M & S \\ \hline S & P \end{array} \qquad \begin{array}{cc} P & M \\ M & S \\ \hline S & P \end{array}$$

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05.12.01

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Deadline: Friday, 14.12.01, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:
Consider:
$S \equiv$ the stocks are falling
$P \equiv$ the interest rate is climbing
$U \equiv$ the most people are unhappy
1) If the interest rate is climbing then the stocks are falling.
2) If the stocks are falling then the most people are unhappy.
3) The interest rate is climbing.
Prove that the most people are unhappy because 1) $\wedge$ 2) $\wedge$ 3)

Exercise 2:
α) Show that neither of the following two formulas tautologically implies the other:

$$(A \leftrightarrow (B \leftrightarrow C)),$$
$$((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C)))).$$

β) Is $(((P \rightarrow Q) \rightarrow P) \rightarrow P)$ is tautology?

Exercise 3:
Define a *literal* to be a wff which is either a sentence symbol or the negation of a sentence symbol. An *implicant* of φ is a conjunction α of literals (using distinct sentence symbols) such that $a \models \varphi$. We showed in 11.9 that any satisfiable wff φ is tautologically equivalent to a disjunction $a_1 \vee ... \vee a_n$, where each a_i is an implicant of φ. An implicant α of φ is *prime* iff it ceases to be an implicant upon the deletion of any of its literals. Any disjunction of implicants equivalent to φ clearly must, if it is to be of minimum length, consist only of prime implicants.
(a) Find all prime implicants of

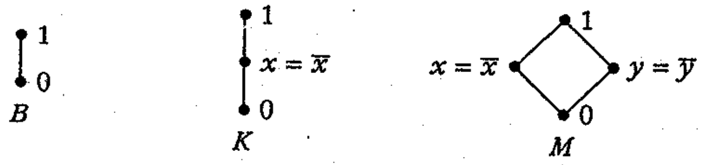
$$(A \rightarrow B) \wedge (\neg A \rightarrow C).$$

(b) Which disjunctions of prime implicants enjoy the property of being tautologically equivalent to the formula in part (a) ?

Exercise 4:
A distributive Ockham algebra is an algebra $(L; \wedge, \vee, f, 0, 1)$ of the type $(2, 2, 1, 0, 0)$ which is a bounded distributive lattice $(L; \wedge, \vee, 0, 1)$ and which it holds for an unary operation $f : x \longrightarrow f(x)$.

$$f(0) = 1, f(1) = 0, f(x \wedge y) = f(x) \vee f(y), f(x \vee y) = f(x) \wedge f(y)$$

Prove: A distributive Ockham algebra is simple if it is either B or K or M :



Ockham, William of (c. 1285-c. 1349)

Ockham was born in Surrey, England. He entered the Franciscan order and studied and taught at the University of Oxford from 1309 to 1319. Denounced by Pope John XXII for dangerous teachings, he was held in house detention from 1324 to 1328 at the papal palace in Avignon, France, while the orthodoxy of his writings was examined. Siding with the Franciscan general against the pope in a dispute over Franciscan poverty, Ockham fled to Munich in 1328 to seek the protection of Louis IV, Holy Roman Emperor, who had rejected papal authority over political matters. Excommunicated by the pope, Ockham wrote against the papacy and defended the emperor until the latter's death in 1347. The philosopher died in Munich, apparently of the plague, while seeking reconciliation with Pope Clement VI.

Ockham won fame as a rigorous logician who used logic to show that many beliefs of Christian philosophers (for example, that God is one, omnipotent, creator of all things; and that the human soul is immortal) could not be proved by philosophical or natural reason but only by divine revelation. His name is applied to the principle of economy in formal logic, known as *Ockham's razor,* which states that entities are not to be multiplied without necessity.

However abstract and impersonal the style of **Ockham**'s writings may be, they reveal at least two aspects of **Ockham**'s intellectual and spiritual attitude: he was a theologian-logician (*theologicus logicus* is Luther's term). On the one hand, with his passion for logic he insisted on evaluations that are severely rational, on distinctions between the necessary and the incidental and differentiation between evidence and degrees of probability--an insistence that places great trust in man's natural reason and his human nature. On the other hand, as a theologian he referred to the primary importance of the God of the creed whose omnipotence determines the gratuitous salvation of men; God's saving action consists of giving without any obligation and is already profusely demonstrated in the creation of nature. The medieval rule of economy, that "plurality should not be assumed without necessity," has come to be known as "**Ockham**'s razor"; the principle was used by **Ockham** to eliminate many entities that had been devised, especially by the scholastic philosophers, to explain reality.

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Fachbereich Mathematik
Prof. Dr. D. Schweigert

Wintersemester 2001
12.12.01

Tutorial
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Deadline: Friday, 21.12.01, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:
For all well-formed formulas φ, ψ holds:

$$\dashv ((\neg\varphi) \to (\varphi \to \psi))$$

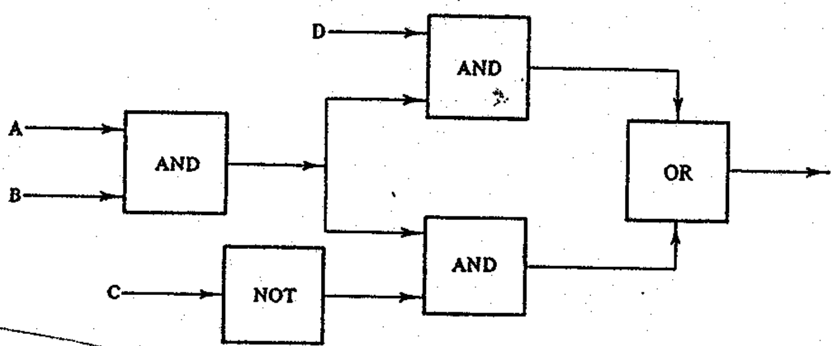
Exercise 2:
Show or contradict for a set $\sum$ of formulas and for the formulas α and β :

(a) If it holds either $\sum \models \alpha$ or $\sum \models \beta$ then it holds $\sum \models (\alpha \vee \beta)$.

(b) If it holds $\sum \models (\alpha \vee \beta)$ then we have either $\sum \models \alpha$ or $\sum \models \beta$.

Exercise 3:
Show $\{((\neg A_1) \to (\neg A_2); ((\neg A_1) \to A_2)\} \models A_1$.

Exercise 4:
While every function $f : \{0,1\}^n \to \{0,1\}$ for any $n \geq 1$ can be realized as a well-formed formula with the symbols $\{\neg, \wedge, \vee\}$ we say that the algebra $(\{0,1\}; \wedge, \vee, ')$ is term complete. Show that the algebra $(\{0,1\};\ \neg, \to)$ is term complete (see: Enderton, page 49).



Circuit for ((A ∧ B) ∧ D) ∨ ((A ∧ B) ∧ ¬C).

Llull, Ramon, also called RAYMOND LULLY (b. *c.* 1235, Majorca—d. 1316, probably Tunis), Catalan mystic and poet whose writings helped to develop the Romance Catalan language and widely influenced Neoplatonic mysticism throughout medieval and 17th-century Europe. He is best known in the history of ideas as the inventor of an "art of finding truth" (*ars inveniendi veritatis*) that was primarily intended to support the Catholic faith in missionary work but was also designed to unify all branches of knowledge.

Reared at the royal court of Majorca, Llull developed characteristics of a troubadour in his chivalrous upbringing. A manual of chivalry he wrote appeared in a 15th-century English version edited by William Caxton, the first English printer. From the large Moorish population in Majorca he acquired a knowledge of Arabic, which he used in some of his writings. His milieu also aroused in him an interest in Islāmic Ṣūfī mysticism and the Eastern contemplative spirit.

Having married, Llull at about the age of 30 experienced mystical visions of Christ on the Cross, after which he abandoned courtly life and devoted himself to missionary work. Influenced by the pacifist spirituality of Francis of Assisi, he traveled throughout North Africa and Asia Minor attempting to convert Muslims to Christianity.

About 1272, after another mystical experience on Majorca's Mount Randa in which Llull related seeing the whole universe reflecting the divine attributes, he conceived of reducing all knowledge to first principles and determining their convergent point of unity. Borrowing certain tenets from the 11th-century Scholastic theologian Anselm of Canterbury, he wrote his principal work; this is collectively known as the *Ars magna* (1305–08; "The Great Art") and includes the treatises *Arbor scientiae* ("The Tree of Knowledge") and *Liber de ascensu et descensu intellectus* ("The Book of the Ascent and Descent of the Intellect"). Llull attempted to place Christian apologetics on the level of rational discussion, mainly to meet the needs of disputation with the Muslims. Llull used logic and complex mechanical techniques (the *Ars magna*) involving symbolic notation and combinatory diagrams to relate all forms of knowledge, including theology, philosophy, and the natural sciences as analogues of one another and as manifestations of the godhead in the universe. Llull thus used original logical methods in an attempt to prove the dogmas of Christian theology. The *Ars magna*'s apologetic applications receded into the background after Llull's death, and it was as a universal system and compendium of knowledge that the *Ars* remained influential until long after the Renaissance.

Llull devoted his life to the spread of his *Ars* and attempted to interest rulers and popes in his projects. King James II of Aragon was persuaded to establish a school at Majorca for the study of Oriental languages so that the *Ars* could be disseminated throughout the Islāmic world.

According to legend, Llull was martyred by stoning at Bougie, North Africa. Charges of confusing faith with reason led to the condemnation of Llull's teaching by Pope Gregory XI in 1376. In the 19th century, however, the Roman Catholic Church showed more sympathetic interest and approved of his veneration. Current interest centres on his mystical writings, particularly the *Llibre d'amic e amat* (*The Book of the Lover and the Beloved*). In Catalan culture his allegorical novels *Blanquerna* (*c.* 1284) and *Felix* (*c.* 1288) enjoy wide popularity. Llull's works in Catalan were critically edited by M. Obrador, S. Galmes, *et al.*, 21 vol. (1905–52).

Llull and the ladders of his "Art," miniature from Thomas Le Myesier's "Breviculum," 14th century; in the Badische Landesbibliothek, Karlsruhe, Ger.

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Deadline: Friday, 11.01.2002, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:
1a) Show that $\{\Gamma, \alpha\} \models \varphi$ if and only if $\Gamma \models (\alpha \to \varphi)$
1b) Show that $\varphi \models \psi$ and $\psi \models \varphi$ if and only if $\models (\varphi \leftrightarrow \psi)$

Exercise 2:
Show that $\{\forall x(\alpha \to \beta), \forall x \alpha\} \models \forall x \beta$

Exercise 3:
Assume that the language has equality and the parameters $\forall, 0, 1, +, \cdot$. Fields can be regarded as structures for this language.

3a) Show that the class of all fields is an elementary class (EC).
3b) Show that the class of fields of characteristic zero is not an elementary class.

Exercise 4:
The modal algebra $\mathbf{B} = (B; \wedge, \vee, ', 0, 1, \Box)$ is of the class $S4$ if

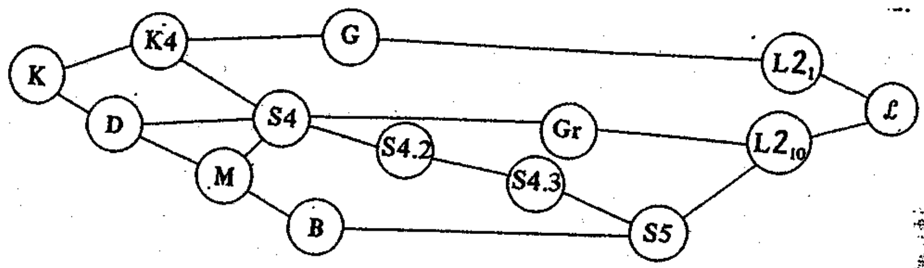
$$\Box x \leq x$$
$$\Box x \leq \Box^2 x$$

4a) Describe the simple algebras of this class $S4$.
4b) Characterise the polynomially complete algebras of the class $S4$.
(An algebra $\mathbf{A} = (A, \Omega)$ is polynomially complete if the algebra $\mathbf{B} = (A; \Omega \cup \{a \,|\, a \in A\})$ is term complete for all constants $a \in A$).
($\Box^2 x = \Box\Box x$)

The modal logic have the rules MP (Modus ponens) and MN (Modus necess[illegible]) $\dfrac{p}{\Box p}$

($\Box\vec{0}$): $\Box(p \to q) \to \Box p \to \Box q$
($\Box$d): $\Box p \to \Diamond p$
* ($\Box$r): $\Box p \to p$
($\Box$s): $p \to \Box\Diamond p$
* ($\Box$t): $\Box p \to \Box\Box p$

($\Box\ell$v): $\Diamond\Box p \to \Box\Diamond p$; ($\Box$e): $\Box\Diamond p \to \Diamond\Box p$
($\Box\ell$x): $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(\Diamond p \wedge q)$
($\Box$0iτ): $\Box(\Box p \to p) \to \Box p$
($\Box$0rτ): $\Box(\Box(p \to \Box p) \to p) \to p$
($\Box$t^m): $\bigwedge_{i<m} \Box^i p \to \Box^{m+1} p$



Abelard, Peter (1079-c. 1142), French philosopher and theologian, whose fame as a teacher made him one of the most celebrated figures of the 12th century. Born in Le Pallet, Brittany, Abelard left home to study at Loches with the French nominalist philosopher Roscelin and later in Paris with the French realist philosopher William of Champeaux. Critical of his masters, Abelard began to teach at Melun, at Corbeil, and, in 1108, at Paris. He soon gained fame throughout Europe as a teacher and as an original thinker. In 1117 he became tutor to Héloïse, the niece of Fulbert, a canon of the Cathedral of Notre-Dame in Paris.

Héloïse and Abelard fell in love, and she gave birth to a son whom they named Astrolabe. At Abelard's insistence they were married secretly; he persuaded Héloïse to take holy vows at the Benedictine Abbey of Saint-Argenteuil. Her uncle Fulbert, at first enraged by the relationship between the two and later somewhat placated by their marriage, finally decided, however, that Abelard had abandoned Héloïse at the abbey and had him castrated. The couple then separated: Héloïse joined an order of nuns, while Abelard retired to a religious retreat, the Abbey of Saint-Denis-en-France, in Paris.

Abelard's first published work, a treatise on the Trinity (1121), was condemned and ordered to be burnt by a Roman Catholic council that met at Soissons in the same year. Forced by criticism to leave Saint-Denis-en-France, Abelard founded a chapel and oratory, called the Paraclete, at Nogent-sur-Seine. In 1125 he was elected abbot of the monastery at Saint-Gildas-de-Rhuis, where he wrote his autobiographical *Historia Calamitatum* (History of Misfortunes, 1132). At this time the famous exchange of letters with Héloïse began, letters that have become classics of romantic correspondence. In 1140 St Bernard of Clairvaux, an eminent French ecclesiastic who thought Abelard's influence dangerous, prevailed upon a Roman Catholic council in session at Sens, and upon Pope Innocent II, to condemn Abelard for his sceptical, rationalistic writings and teaching. On his way to Rome to appeal against the condemnation, Abelard accepted the hospitality of Peter the Venerable, abbot of the Abbey of Cluny, remaining there for many months. Abelard died at a Clunist priory near Chalon-sur-Saône. His body was taken to the Paraclete; when Héloïse died in 1164 she was buried beside him. In 1817 both bodies were moved to a single tomb in the cemetery of Père Lachaise in Paris.

The romantic appeal of the life of Abelard often overshadows the importance of his thought. He was, however, one of the leading thinkers of the Middle Ages. In the emphasis he placed on dialectical discussion, Abelard followed the 9th-century philosopher and theologian John Scotus Erigena, and he foreshadowed the Italian Scholastic philosopher Thomas Aquinas. Abelard's important dialectical thesis that truth must be attained by carefully weighing all sides of any issue is presented in *Sic et Non* (Thus and Otherwise, c. 1123). He also foreshadowed the later theological reliance on the works of Aristotle, rather than on those of Plato.

Abelard reacted strongly against the theories of extreme realism, denying that universals have an independent existence outside the mind. According to Abelard, "universal" is a functional word expressing the combined image of that word's common associations within the mind. This position is not nominalism, because Abelard adds that the associations from which the image is formed and to which a universal name is given have a certain likeness, or common nature. His theory is a definite step towards the moderate realism of Aquinas, but it lacks an explanation of how ideas are formed. In the development of ethics, Abelard's great contribution was to maintain that an act is to be judged by the intention of the doer in doing it.

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Fachbereich Mathematik
Prof. Dr. D. Schweigert

Wintersemester 2001/02
07.01.02

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Deadline: Friday, 18.01.2002, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:
Show that $\{\forall x(\alpha \to \beta), \forall x\alpha\} \vdash \forall x\beta$ and compare $\{\forall x(\alpha \to \beta), \forall x\alpha\} \models \forall x\beta$.

Exercise 2:
2a) Show $\vdash \forall x \forall y((x = y) \to (y = x))$
2b) To which axiom groups, if any, do each of the following formulas belong?
α) $\forall y[\forall x(Px \to Px) \to (Pc \to Pc)]$
β) $\forall x \exists y Pxy \to \exists y Pyy$

Exercise 3:
3a) Show that if $\vdash \alpha \to \beta$ then $\vdash \forall x\alpha \to \forall x\beta$
3b) Show that it is not in general true that

$$a \to \beta \models \forall x\alpha \to \forall x\beta$$

Exercise 4:
A correlation lattice $\widetilde{L} = (L; \wedge, \vee, \sigma, \sigma^{-1}, 0, 1)$ is a bounded lattice endowed with two unary operations such that
1) $\sigma^{-1}(\sigma(x)) = \sigma(\sigma^{-1}(x)) = x$
2) $\sigma(x \wedge y) = \sigma(x) \vee \sigma(y)$
3) $\sigma(x \vee \beta) = \sigma(x) \wedge \sigma(\beta)$
4) $\sigma(0) = 1, \sigma(1) = 0$

Describe the simple, Boolean, correlation lattices.

'Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.'
R. Dedekind: *Was sind und was sollen die Zahlen?*

Vienna Circle (German, Wiener Kreis), group of philosophers and scientists who met periodically for discussions in Vienna, Austria, during the 1920s and 1930s, and proposed controversial conception of scientific philosophy. Initiated by the mathematician Hans Hahn and centred around the philosopher Moritz Schlick, it also included Rudolf Carnap, Herbert Feigl, Philipp Frank, Viktor Kraft, Otto Neurath, and Friedrich Waismann, and counted Kurt Gödel, Karl Menger, and Edgar Zilsel among its associates. The Circle's activities were confined to private meetings until 1929, when they began publishing several series of monographs and collaborated with the Berlin Society of Empirical Philosophy (which included Hans Reichenbach and C. G. Hempel) in organizing international conferences and editing the journal *Erkenntnis* ("knowledge"). The death and dispersion in exile of key members from 1934 onwards did not mean the extinction of Vienna Circle philosophy, however. Through the incessant revision and refinement of earlier theses by emigré members and collaborators, so-called logical positivism strongly influenced the development of analytic philosophy—occasionally suffering outright distortions of its original ideas.

The Circle's influences were Ernst Mach, Jules Henri Poincaré, Pierre Duhem, and Albert Einstein concerning empirical science, and Gottlob Frege, David Hilbert, Bertrand Russell, and early theories of Ludwig Wittgenstein on formal science; it was opposed to neo-Kantianisms and German and Catholic idealisms. Most strikingly, the Circle rejected the need for metaphysics and for an epistemology that bestowed justification on scientific knowledge claims from beyond science itself.

An empiricist meaning criterion such as the "verification principle", which required that statements be either empirically verifiable or analytical, would, they argued, exhibit the knowledge claim of science and simultaneously eliminate metaphysics as meaningless. (Unconditional norms were also robbed of their cognitive base.) Yet, critics asked, what is the status of such a criterion? Carnap's suggestion that it represents not a discovery but a convention, a proposa for future scientific language use, deserves to be taken seriously, for it amplifies the Circle's "linguistic turn" according to which philosophy is concerned with ways of representing, rather than the nature of the represented. Since a satisfactory formalization of the meaning criterion was never achieved, however, one must note that the Vienna Circle was neither a monolithic, formalist, nor a necessarily reductionist philosophical movement. In its time and place, it was a minority voice; the socio-political dimension of its theories—stressed by Neurath—as a renewal of enlightenment thought against the then-rising tide of fascism is gaining recognition. After the celebrated "death"of reductionist logical positivism in the 1960s, the historical Vienna Circle is re-emerging, revealing strands of reasoning still significant for post-positivisr

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Wintersemester 2001/02
14.01.02

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Deadline: Friday, 25.01.2002, 10:00 hrs , Mailbox of Dipl. Math. R. Lenz

Exercise 1:
A modal logic in the logical symbols $\neg, \wedge, \vee, \to, \Box$ contains
1) all axioms of the proposition logic and $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
2) is closed under the rule Modus ponens MP$\frac{\alpha\,\alpha\to\beta}{\beta}$.
3) is closed under the rule Modus necesse MN$\frac{\alpha}{\Box\alpha}$.
We introduce an additional axiom $G : \Box(\Box\alpha \to \alpha) \to \Box\alpha$ and call it modal logic of the class G (G :=Gödel). A derivation in the modal logic G we denote

$$\vdash^G$$

Prove the following theorem of Löb:
If $\vdash^G (\Box\alpha \to \alpha)$ then it holds $\vdash^G \alpha$.

Exercise 2:
Heating rule by the generalized modus ponens.
Rule: If the temperature of the room is normal then the ventil should be half open. The fuzzy values are defined in the following:

"normal temperature" = normal

"half open ventil" = half open

"about 20 degree Celsius" = about 20

Datas are given by

temperature	= [15 16 17....25]
ventil	= [0 0.5 1....5]
normal	= [0 0 0.3 0.7 1 1 1 0.7 0.3 0 0]
halfopen	= [0 0 0 0.3 0.7 1 0.7 0.3 0 0 0]
about 20	= [0 0 0 0.2 0.7 1 0.7 0.2 0 0 0]

How can you put the ventil such that the temperature is about 20 degree ?

$R_{\text{normal} \to \text{halboffen}} = I_{LU}(\text{normal, halboffen})$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0.7 & 0.7 & 0.7 & 1 & 1 & 1 & 1 & 1 & 0.7 & 0.7 & 0.7 \\ 0.3 & 0.3 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0.3 & 0.7 & 1 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 1 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 1 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0.3 & 0.3 \\ 0.7 & 0.7 & 0.7 & 1 & 1 & 1 & 1 & 1 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

etwa 20 $\circ$ $I_{LU}(\text{normal, halboffen})$

$$\begin{bmatrix} 0 & 0 & 0 & 0.2 & 0.7 & 1 & 0.7 & 0.2 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0.7 & 0.7 & 0.7 & 1 & 1 & 1 & 1 & 1 & 0.7 & 0.7 & 0.7 \\ 0.3 & 0.3 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0.3 & 0.7 & 1 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 1 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 1 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0.3 & 0.3 \\ 0.7 & 0.7 & 0.7 & 1 & 1 & 1 & 1 & 1 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$= [0.2\ 0.2\ 0.2\ 0.3\ 0.7$ [illegible]

ARISTOTLE VS. THE BUDDHA

Everything must either be or not be, whether in the present or in the future.

ARISTOTLE
DE INTERPRETATIONE

I have not explained that the world is eternal or not eternal. I have not explained that the world is finite or infinite.

THE BUDDHA
MAJJHIMA-NIKAYA

The fundamental idea of Buddhism is to pass beyond the world of opposites, a world built up by intellectual distinctions and emotional defilements.

D. T. SUZUKI
THE ESSENCE OF BUDDHISM

Fuzzy logic is a concept derived from the branch of mathematical theory of fuzzy sets. Unlike the basic Aristotelian theory that recognizes statements as only "true" or "false," or "1" or "0" as represented in digital computers, fuzzy logic is capable of expressing linguistic terms such as "maybe false" or "sort of true." In general, fuzzy logic, when applied to computers, allows them to emulate the human reasoning process, quantify imprecise information, make decisions based on vague and incomplete data, yet by applying a "defuzzification" process, arrive at definite conclusions.

FUZZY LOGIC: A KEY TECHNOLOGY FOR FUTURE COMPETITIVENESS
U.S. DEPARTMENT OF COMMERCE, NOVEMBER 1991

Syllogism

A syllogism consists of three parts

$$\frac{\begin{array}{l}\text{major premise}\\ \text{minor premise}\end{array}}{\text{conclusion}}$$

Example (Barbara)

M is P	all men are mortal
S is M	Socrates is a man
S is P	Socretes is mortal

Fuzzy Syllogism

is a scheme in which both premise and the conclusion are fuzzy and contains fuzzy quantors.

Example (intersection - product - syllogism)

Q_1 : A is B	$Q_1 = 0.8$: most students are not married 80%
Q_2 : (A and B) is C	$Q_2 = 0.6$: most students are male (60%)
Q_3 : A is (B and C)	$Q_3 = 0.48$: most st. are unmarried and male. 48%

$Q_3 = Q_1 \cdot Q_2$ if monotone
or else $\geq Q_1 \cdot Q_2$

Q_1, Q_2, Q_3 are fuzzy quantors and A, B, C are fuzzy predicates which are connected.