A limiter based on kinetic theory *

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Abstract

In the present paper the low Mach number limit of kinetic equations is used to develop a discretization for the incompressible Euler equation. The kinetic equation is discretized with a first and second order discretization in space. The discretized equation is then considered in the limit of low Mach and Knudsen number which gives rise to an interesting limiter for the convective part in the incompressible Euler equation. Numerical experiments are shown comparing different approaches.

Keywords. kinetic equations, asymptotic analysis, low Mach number limit, second order upwind discretization, slope limiter, incompressible Euler equation

1 Introduction

Kinetic equations or discrete velocity models of kinetic equations yield in the limit of small Knudsen and Mach numbers an approximation of macroscopic equations like the incompressible Euler or Navier Stokes equation. Hence, discretizations of kinetic models can be used in combination with the limiting procedures to develop discretizations for the corresponding macroscopic limit equation. For examples of this approach, we refer to [16, 15, 4, 5, 11] for kinetic schemes, [7, 2, 9, 6] for Lattice-Boltzmann methods, and [10, 13] for relaxation schemes for diffusive limits.

In the present paper we start by recalling the scaling of kinetic equations which leads to the incompressible Euler equation. Then, a natural discretization of the kinetic equation is used to obtain in the limit a second order slope limiting procedure for the convective term of the Euler equation. This slope limiter has interesting algebraic properties like reflection and rotation invariance.

The paper is organized as follows: section 2 contains a short description of the results of the asymptotic analysis leading from kinetic equations to the incompressible Euler equation. In section 3 the asymptotic procedure is performed for the discretized kinetic equations and a general limit discretization for the

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incompressible Euler equation is derived. In section 4 we concentrate on the derivation of the discretization of the convective part. A first and second order upwind discretization for the limit equation are presented. Whereas the first order discretization is standard, the second order discretization includes a multidimensional slope limiting procedure which is analyzed further in section 5. Finally, the new slope limiter is tested in several examples.

2 Kinetic equations and the incompressible Euler equation

The incompressible Euler equation

$$\partial_t u + u \cdot \nabla u + \nabla_x p = 0, \qquad \operatorname{div}_x u = 0 \tag{2.1}$$

can be formally obtained as scaling limit of a Boltzmann type kinetic equation (see [1, 3, 17])

$$\partial_t f + v \cdot \nabla_x f = J(f). \tag{2.2}$$

Here, $f = f(x, v, t)$ is a phase space density which we consider, for simplicity, in the two dimensional case $x = (x_1, x_2) \in \mathbb{R}^2$, $v = (v_1, v_2) \in \mathbb{R}^2$. We will not specify the complete structure of the collision operator $J(f)$. Only those properties which are important in the Euler limit will be listed below.
We follow the approach in [1] and use the scaled kinetic equation

$$\partial_t f + \frac{1}{\epsilon} v \cdot \nabla_x f = \frac{1}{\epsilon^{q+1}} J(f) \tag{2.3}$$

with $q > 1$. Furthermore, we assume that f is only a small perturbation of the Maxwellian velocity distribution M

$$M(v) = \frac{1}{2\pi} \exp\left(-\frac{|v|^2}{2}\right), \qquad v \in \mathbb{R}^2.$$

Our precise assumption on the structure of f is

$$f = M(1 + \epsilon g_\epsilon). \tag{2.4}$$

In the next section, the formal asymptotic analysis is carried out in a slightly more general situation where (2.3) is modified by adding a diffusive term $D_h(v) f$ and replacing ∇_x with an approximation ∇_x^h.
Let us now list some properties of J which will be needed for the analysis: if (2.4) is inserted into (2.3) we need a Taylor expansion of $J(M + \epsilon M g_\epsilon)$. We have

$$\frac{1}{M} J(M + \epsilon M g_\epsilon) = \epsilon L g_\epsilon + \frac{1}{2}\epsilon^2 Q(g_\epsilon, g_\epsilon) + \epsilon^3 R(g_\epsilon) \tag{2.5}$$

where L involves the first and Q the second Frechet derivative of J at the point M (see [1] for details). The exact structure of the remainder R is not relevant in the limit. Note that the zero order term in (2.5) drops out because of the assumed equilibrium condition

$$J(M) = 0. \tag{2.6}$$

Another important assumption is that the collision invariants of J are the functions $1, v_1, v_2$ (for simplicity, we consider isothermal flows and suppress the energy equation with corresponding collision invariant $|v|^2$), which means in terms of the weighted $\mathbb{L}^2$ scalar product $\langle g, h\rangle = \int_{\mathbb{R}^2} ghM\, dv$

$$\left\langle \frac{1}{M} J(f), \psi \right\rangle = 0, \qquad \psi \in \{1, v_1, v_2\}. \tag{2.7}$$

Note that (2.7) implies together with (2.5) that also

$$\langle Lg_\epsilon, \psi\rangle = \langle Q(g_\epsilon, g_\epsilon), \psi\rangle = \langle R(g_\epsilon), \psi\rangle = 0 \tag{2.8}$$

for all collision invariants ψ. Important assumptions on the operator L are

1) L is selfadjoint with respect to $\langle \cdot, \cdot \rangle$.

2) L satisfies a Fredholm alternative with a three dimensional kernel spanned by the collision invariants.

Finally, we need a property of Q which is a direct consequence of the relation

$$Q(h, h) = -Lh^2 \qquad \text{for } h = \alpha + \beta \cdot v \tag{2.9}$$

(see [1] for the derivation). Using the fact that $1, v_1, v_2$ are in the kernel of L, we conclude

$$-Q(h, h) = \beta_i \beta_j L(v_i v_j). \tag{2.10}$$

For convenience, we list some moments of the standard Maxwellian which will be frequently used later

$$\begin{aligned} \langle 1, 1\rangle = 1, \qquad \langle 1, v_i\rangle = 0, \qquad \langle v_i, v_j\rangle = \delta_{ij} \\ \langle v_i v_j, v_k\rangle = 0, \qquad \langle v_i v_j, v_k v_l\rangle = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}. \end{aligned} \tag{2.11}$$

3 The discretized kinetic equation and derivation of macroscopic discretization

We start with the kinetic equation (2.2) which is discretized using the method of lines

$$\partial_t f + v \cdot \nabla_x^h f - D_h(v) f = J(f).$$

In this section, we only assume that $\nabla_x^h = (\partial_{x_1}^h, \partial_{x2}^h)$ and $D_h(v)$ are linear operators, that ∇_x^h is independent of v, and that the components of ∇_x^h commute. In the next section we choose $\partial_{x_i}^h$ as central difference approximations and $D_h(v)$ as numerical viscosity term.

As in the previous section, we introduce a scaled version of the kinetic equation (with $q > 1$)

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x^h f - D_h(v) f = \frac{1}{\varepsilon^{q+1}} J(f).$$

With the expansion (2.4) and (2.5), we then get

$$\partial_t g_\epsilon + \frac{1}{\varepsilon} v \cdot \nabla_x^h g_\epsilon - D_h(v) g_\epsilon = \frac{1}{\varepsilon^{q+1}} L g_\epsilon + \frac{1}{2\varepsilon^q} Q(g_\epsilon, g_\epsilon) + \frac{1}{\varepsilon^{q-1}} R(g_\epsilon). \tag{3.1}$$

In our formal analysis, we will assume that g_ϵ converges in a suitable sense for $\epsilon \to 0$ to some function g_0 and that all relevant operations behave continuously for that particular sequence. Moreover, terms which are formally of order ϵ are assumed to vanish in the limit (see [1] for a more detailed investigation). Upon multiplying (3.1) with ϵ^{q+1} we thus find that $Lg_0 = \lim_{\epsilon \to 0} L g_\epsilon = 0$ which shows that $g_0 \in \ker L$, i.e.

$$g_0(v) = \rho + u \cdot v, \qquad \rho \in \mathbb{R},\ u \in \mathbb{R}^2 \tag{3.2}$$

with parameters ρ, u which are yet undetermined. In order to get more information about these parameters, we consider the mass and momentum conservation equation related to (3.1) which are obtained by multiplying (3.1) with M and Mv and integrating over v

$$\partial_t \langle g_\epsilon, 1 \rangle + \frac{1}{\epsilon} \left\langle v \cdot \nabla_x^h g_\epsilon, 1 \right\rangle - \langle D_h(v) g_\epsilon, 1 \rangle = 0, \tag{3.3}$$

$$\partial_t \langle g_\epsilon, v \rangle + \frac{1}{\epsilon} \left\langle v \cdot \nabla_x^h g_\epsilon, v \right\rangle - \langle D_h(v) g_\epsilon, v \rangle = 0. \tag{3.4}$$

Multiplying the equations by ϵ and letting ϵ tend to zero, we conclude

$$\left\langle v \cdot \nabla_x^h g_0, 1 \right\rangle = 0, \qquad \left\langle v \cdot \nabla_x^h g_0, v_i \right\rangle = 0. \tag{3.5}$$

Using (3.2), the first condition can be reformulated

$$\begin{aligned} 0 = \partial_{x_i}^h \langle g_0, v_i \rangle &= \partial_{x_i}^h \langle \rho + v_j u_j, v_i \rangle \\ &= \langle 1, v_i \rangle \partial_{x_i}^h \rho + \langle v_j, v_i \rangle \partial_{x_i}^h u_j = \partial_{x_i}^h u_i =: \operatorname{div}_x^h u \end{aligned}$$

where we have used moment relations from (2.11). Similarly, we find

$$0 = \partial_{x_j}^h \langle v_j g_0, v_i \rangle = \langle v_j, v_i \rangle \partial_{x_i}^h \rho + \langle v_j v_k, v_i \rangle \partial_{x_i}^h u_k = \partial_{x_i}^h \rho$$

so that (3.5) implies

$$\operatorname{div}_x^h u = 0, \qquad \nabla_x^h \rho = 0 \tag{3.6}$$

which has to be satisfied by the parameters ρ, u in (3.2). Since ρ is essentially determined by the condition $\nabla_x^h \rho = 0$, it remains to find the time evolution of u. First, we rewrite the second term in (3.4) as

$$\frac{1}{\epsilon} \left\langle v \cdot \nabla_x^h g_\epsilon, v_j \right\rangle = \frac{1}{\epsilon} \partial_{x_i}^h \langle g_\epsilon, v_i v_j \rangle = \frac{1}{\epsilon} \partial_{x_i}^h \langle g_\epsilon, v_i v_j - \delta_{ij} \rangle + \frac{1}{\epsilon} \partial_{x_j}^h \langle g_\epsilon, 1 \rangle .$$

Since $\partial_{x_j}^h \langle g_\epsilon, 1\rangle \to \partial_{x_j}^h \rho = 0$ for $\epsilon \to 0$, we can assume that $\partial_{x_j}^h \langle g_\epsilon, 1\rangle / \epsilon$ converges to $\partial_{x_j}^h \rho_1$ if we think of an expansion $g_\epsilon = g_0 + \epsilon g_1 + \ldots$ with corresponding $\langle g_\epsilon, 1\rangle = \rho + \epsilon\rho_1 + \ldots$. Secondly, the function $v_i v_j - \delta_{ij}$ is orthogonal to the kernel of L (which is easily checked with relations 2.11). Hence, $v_i v_j - \delta_{ij} = LL^{-1}(v_i v_j - \delta_{ij})$ and

$$\frac{1}{\epsilon}\langle g_\epsilon, v_i v_j - \delta_{ij}\rangle = \frac{1}{\epsilon}\left\langle L g_\epsilon, L^{-1}(v_i v_j - \delta_{ij})\right\rangle.$$

Using (3.1) after multiplication with ϵ^q, we find

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle g_\epsilon, v_i v_j - \delta_{ij}\rangle = -\frac{1}{2}\left\langle Q(g,g), L^{-1}(v_i v_j - \delta_{ij})\right\rangle$$

so that (2.10) implies

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle g_\epsilon, v_i v_j - \delta_{ij}\rangle &= \frac{u_k u_l}{2}\left\langle L(v_k v_l), L^{-1}(v_i v_j - \delta_{ij})\right\rangle \\ &= \frac{u_k u_l}{2}\langle v_k v_l, v_i v_j - \delta_{ij}\rangle.\end{aligned}$$

In view of (2.11), we conclude

$$\lim_{\epsilon\to 0}\partial_{x_i}^h \frac{1}{\epsilon}\langle g_\epsilon, v_i v_j\rangle = \partial_{x_j}^h \rho_1 + \partial_{x_i}^h (u_i u_j).$$

Hence, in the limit $\epsilon \to 0$, equation (3.4) turns into

$$\partial_t u_j + \partial_{x_i}^h (u_i u_j) - \langle D_h(v) g_0, v_j\rangle + \partial_{x_j}^h \rho_1 = 0, \qquad \operatorname{div}_x^h u = 0. \tag{3.7}$$

Note that (3.7) reduces to the incompressible Euler equation (2.1) if we choose $\nabla_x^h = \nabla_x$ and $D_h(v) = 0$. Obviously, ρ_1 takes the role of the pressure and $\partial_{x_i}^h(u_i u_j) - \langle D_h(v) g_0, v_j\rangle$ gives a discretization of the convective terms.

4 First and second order upwind schemes

To find expressions for ∇_x^h and the numerical viscosity $D_h(v)$ we consider the linear transport part of the kinetic equation in two dimensions:

$$v\cdot\nabla_x f = v_1 \partial_{x_1} f + v_2 \partial_{x_2} f. \tag{4.1}$$

A *first order* discretization is given by

$$v_1 \partial_{x_1}^h f + v_2 \partial_{x_2}^h f - \frac{c_1 h}{2}\partial_{x_1}^{2,h} f - \frac{c_2 h}{2}\partial_{x_2}^{2,h} f \tag{4.2}$$

with positive constants c_1, c_2 and

$$(\partial_{x_1}^h f)_{ij} = \frac{1}{2h}(f_{i+1j} - f_{i-1j}), \qquad (\partial_{x_1}^{2,h} f)_{ij} = \frac{1}{h^2}(f_{i+1j} - 2f_{ij} + f_{i-1j}),$$
$$(\partial_{x_2}^h f)_{ij} = \frac{1}{2h}(f_{ij+1} - f_{ij-1}), \qquad (\partial_{x_2}^{2,h} f)_{ij} = \frac{1}{h^2}(f_{ij+1} - 2f_{ij} + f_{ij-1}).$$

In view of (4.2), we define

$$D_h(v)f = \left(c_1\frac{h}{2}\partial^{2,h}_{x_1}f + c_2\frac{h}{2}\partial^{2,h}_{x_2}f\right)$$

and obtain

$$D_h(v)g_0 = c_1\partial^{2,h}_{x_1}\rho\frac{h}{2} + c_2\partial^{2,h}_{x_2}\rho\frac{h}{2} + c_1\partial^{2,h}_{x_1}u_i\frac{hv_i}{2} + c_2\partial^{2,h}_{x_2}u_i\frac{hv_i}{2}$$

which yields with (2.11) the required expressions in (3.7)

$$\langle D_h(v)g_0, v\rangle = \frac{h}{2}\left(c_1\partial^{2,h}_{x_1}u + c_2\partial^{2,h}_{x_2}u\right).$$

We may choose c_1 and c_2 constant proportional to the maximal flow velocity:

$$c_1 = \max_{ij}\{|2(u_1)_{ij}|\}, \quad c_2 = \max_{ij}\{|2(u_2)_{ij}|\}.$$

Alternatively the local flow velocity can be used:

$$c_1(i,j) = \max\{|2(u_1)_{i+1j}|, |2(u_1)_{i-1j}|\}, \quad c_2(i,j) = \max\{|2(u_2)_{ij+1}|, |2(u_2)_{ij-1}|\}$$

giving rise to the expression

$$\langle D_h(v)g_0, v\rangle_{ij} = \frac{h}{2}\left(c_1(i,j)(\partial^{2,h}_{x_1}u)_{ij} + c_2(i,j)(\partial^{2,h}_{x_2}u)_{ij}\right) \tag{4.3}$$

Note that (4.3) has the usual form of the numerical viscosity related to an upwind discretization of $\operatorname{div} u \otimes u$ which is first order accurate.
A *second order* discretization of (4.1) is obtained by slope limiting

$$\begin{aligned}(v\cdot\nabla^h_x f)_{ij} - \Big[&\frac{c_1(i,j)}{2h}\Big((1-\varphi_{i+\frac{1}{2}j})\Delta_{i+\frac{1}{2}j}f - (1-\varphi_{i-\frac{1}{2}j})\Delta_{i-\frac{1}{2}j}f\Big)\\ &+\frac{c_2(i,j)}{2h}\Big((1-\varphi_{ij+\frac{1}{2}})\Delta_{ij+\frac{1}{2}}f - (1-\varphi_{ij-\frac{1}{2}})\Delta_{ij-\frac{1}{2}}f\Big)\Big]\end{aligned} \tag{4.4}$$

where ∇^h_x are again central differences, the f increments are defined by

$$\Delta_{i+\frac{1}{2}j}f = f_{i+1j} - f_{ij}, \qquad \Delta_{ij+\frac{1}{2}}f = f_{ij+1} - f_{ij},$$

and

$$\begin{aligned}\varphi_{i+\frac{1}{2}j} &= \varphi(r_{i+\frac{1}{2}j}), \qquad & r_{i+\frac{1}{2}j} &= \Delta_{i-\frac{1}{2}j}f/\Delta_{i+\frac{1}{2}j}f,\\ \varphi_{ij+\frac{1}{2}} &= \varphi(r_{ij+\frac{1}{2}}), \qquad & r_{ij+\frac{1}{2}} &= \Delta_{ij-\frac{1}{2}}f/\Delta_{ij+\frac{1}{2}}f,\end{aligned}$$

with $\varphi(r) = \max\{0, \min\{r, 1\}\}$ being the minmod limiter. Using the definition of φ, one can write expressions like $(1-\varphi_{i+\frac{1}{2}j})\Delta_{i+\frac{1}{2}j}f$ as $\phi(\Delta_{i-\frac{1}{2}j}f, \Delta_{i+\frac{1}{2}j}f)$ where ϕ is a continuous, piecewise linear function on $\mathbb{R}^2$ defined according to figure 1. Extracting the viscosity term in (4.4), we get

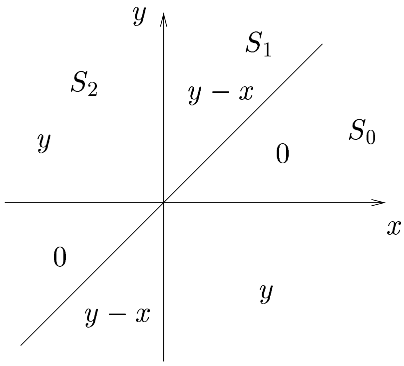


Figure 1: Piecewise linear definition of $\phi(x,y)$ in the sets S_0, S_1, S_2

$$
\begin{aligned}
D_h(v)f_{ij} = {} & \frac{c_1(i,j)}{2h}\Big(\phi(\Delta_{i-\frac{1}{2}j}f, \Delta_{i+\frac{1}{2}j}f) - \phi(\Delta_{i-\frac{3}{2}j}f, \Delta_{i-\frac{1}{2}j}f)\Big) \\
& + \frac{c_2(i,j)}{2h}\Big(\phi(\Delta_{ij-\frac{1}{2}}f, \Delta_{ij+\frac{1}{2}}f) - \phi(\Delta_{ij-\frac{3}{2}}f, \Delta_{ij-\frac{1}{2}}f)\Big)\Big]
\end{aligned}
$$

In order to calculate $\langle D_h(v)g_0, v_j\rangle$, we note that

$$\Delta_{i+\frac{1}{2}j}g_0 = (\Delta_{i+\frac{1}{2}j}u)\cdot v$$

because ρ satisfies $\nabla_x^h \rho = 0$. Hence, a typical term appearing in $\langle D_h(v)g_0, v\rangle$ has the form

$$L(\delta_1, \delta_2) = \langle \phi(\delta_1 \cdot v, \delta_2 \cdot v), v\rangle \qquad \delta_1, \delta_2 \in \mathbb{R}^2. \tag{4.5}$$

and we find the numerical viscosity for the second order discretization

$$
\begin{aligned}
\langle D_h(v)g_0, v\rangle_{ij} = {} & \frac{c_1(i,j)}{2h}\Big(L(\Delta_{i-\frac{1}{2}j}u, \Delta_{i+\frac{1}{2}j}u) - L(\Delta_{i-\frac{3}{2}j}u, \Delta_{i-\frac{1}{2}j}u)\Big) \\
& + \frac{c_2(i,j)}{2h}\Big(L(\Delta_{ij-\frac{1}{2}}u, \Delta_{ij+\frac{1}{2}}u) - L(\Delta_{ij-\frac{3}{2}}u, \Delta_{ij-\frac{1}{2}}u)\Big)\Big].
\end{aligned}
$$

We remark that row-wise application of the minmod limiter in the discretization of $\operatorname{div} u \otimes u$ gives rise to a numerical viscosity of the same form with L replaced by $\hat{L}$, where

$$\hat{L}(\delta_1, \delta_2) = \begin{pmatrix} \phi(\delta_{11}, \delta_{21}) \\ \phi(\delta_{12}, \delta_{22}) \end{pmatrix}, \qquad \delta_i = \begin{pmatrix} \delta_{i1} \\ \delta_{i2} \end{pmatrix}. \tag{4.6}$$

In the next section, we study properties of the function L and compare it to the row-wise minmod limiter $\hat{L}$.

5 The kinetic limiter

Introducing the linear map

$$T = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}$$

we can rewrite (4.5) as $L(\delta_1,\delta_2) = \langle\phi(Tv), v\rangle$. Note that ϕ is linear in each of the convex sets S_0, S_1, S_2 (see figure 1). With the unit vectors $e_1 = (1,0)$, $e_2 = (0,1)$ and

$$S(a,b) = S^+(a,b) \cup S^-(a,b), \qquad S^\pm(a,b) = \{\pm(\lambda_1 a + \lambda_2 b) : \lambda_1, \lambda_2 \geq 0\},$$

we can describe these sets as

$$S_0 = S(e_1, e_1 + e_2), \quad S_1 = (e_1 + e_2, e_2), \quad S_2 = S(e_2, -e_1).$$

Assuming that T is invertible, we conclude that $\phi \circ T$ is linear on each of the sets $\hat{S}_i = T^{-1} S_i$. Since $S(a,b) = cS(a,b)$ for all $c \neq 0$, we have $\hat{S}_i = \hat{T}(S_i)$ where

$$\hat{T} = (\det T) T^{-1} = \begin{pmatrix} \delta_{22} & -\delta_{12} \\ -\delta_{21} & \delta_{11} \end{pmatrix} = \begin{pmatrix} -\delta_2^\perp & \delta_1^\perp \end{pmatrix}.$$

Hence,

$$\hat{S}_0 = S(-\delta_2^\perp, \delta_1^\perp - \delta_2^\perp), \quad \hat{S}_1 = S(\delta_1^\perp - \delta_2^\perp, \delta_1^\perp), \quad \hat{S}_2 = S(\delta_1^\perp, \delta_2^\perp).$$

Taking into account that ϕ vanishes on S_0, we find

$$\langle\phi(Tv), v\rangle = \int_{\hat{S}_1} (\delta_2 \cdot v - \delta_1 \cdot v) v M(v)\, dv + \int_{\hat{S}_2} (\delta_2 \cdot v) v M(v)\, dv$$

or with the help of the matrix valued function

$$I(a,b) = \int_{S(a,b)} v \otimes v M(v)\, dv \tag{5.7}$$

that

$$L(\delta_1,\delta_2) = I(\delta_1^\perp - \delta_2^\perp, \delta_1^\perp)(\delta_2 - \delta_1) + I(\delta_1^\perp, \delta_2^\perp)\delta_2. \tag{5.8}$$

Next, we derive an explicit formula for the function I. Using the symmetry of $M(v)$ and $v \otimes v$, we find

$$I(a,b) = 2\int_{S^+(a,b)} v \otimes v M(v)\, dv.$$

To parameterize the cone $S^+(a,b)$ which has some opening angle $0 < \beta < \pi$ around the ray in direction α (see figure 2), we go over to polar coordinates and find

$$I(a,b) = 2\int_{\alpha-\beta/2}^{\alpha+\beta/2} \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix} d\psi \int_0^\infty \frac{r^2}{2\pi} e^{-\frac{r^2}{2}}\, r\, dr.$$

After some straight forward calculations we get

$$I(a,b) = \frac{1}{\pi}\left(\beta + \sin\beta \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}\right).$$

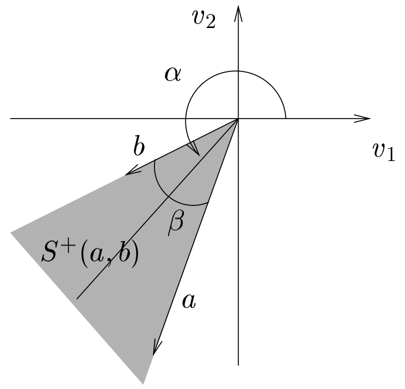


Figure 2: Angles α, β characterizing the cone $S^+(a,b)$

In terms of a, b, the angles α and β are given by

$$\alpha = \sphericalangle\left(\frac{a}{|a|} + \frac{b}{|b|}, e_1\right), \qquad \beta = \sphericalangle(a,b).$$

We remark that an efficient implementation of (5.8) requires the evaluation of scalar products and square roots and only one arccos call per evaluation of I to find the angle β.
Up to now, we have assumed that the arguments δ_1, δ_2 of L are linearly independent. If this is not the case, one can either slightly modify δ_1, δ_2 in order to make them independent (note that L is continuous), or one can use the relation

$$L(\gamma_1 e, \gamma_2 e) = \phi(\gamma_1, \gamma_2)e, \qquad \gamma_1, \gamma_2 \in \mathbb{R},\ e \in \mathbb{R}^2. \tag{5.9}$$

To prove (5.9), we go back to (4.5) which yields together with the homogeneity of ϕ and (2.11)

$$L(\gamma_1 e, \gamma_2 e) = \langle \phi(\gamma_1, \gamma_2)(e \cdot v), v\rangle = \phi(\gamma_1, \gamma_2)\,\langle 1, v \otimes v\rangle\, e = \phi(\gamma_1, \gamma_2)e.$$

Note that we also have

$$\hat{L}(\gamma_1 e, \gamma_2 e) = \begin{pmatrix} \phi(\gamma_1 e_1, \gamma_2 e_1) \\ \phi(\gamma_1 e_2, \gamma_2 e_2) \end{pmatrix} = \phi(\gamma_1, \gamma_2)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = L(\gamma_1 e, \gamma_2 e)$$

so that the standard minmod limiter $\hat{L}$ coincides with L in the case of linearly dependent arguments. We summarize our results in the following Lemma.

Lemma 5.1 *Let $\delta_1, \delta_2 \in \mathbb{R}^2$. If δ_1, δ_2 are linearly dependent, i.e. $\delta_i = \gamma_i e$ for some $e \in \mathbb{R}^2$, $\gamma_i \in \mathbb{R}$, then $L(\gamma_1 e, \gamma_2 e) = \phi(\gamma_1, \gamma_2)e$. If δ_1, δ_2 are independent, then*

$$L(\delta_1, \delta_2) = I(\delta_1^\perp - \delta_2^\perp, \delta_1^\perp)(\delta_2 - \delta_1) + I(\delta_1^\perp, \delta_2^\perp)\delta_2$$

where $(e_1, e_2)^\perp = (e_2, -e_1)$,

$$I(a,b) = \int_{S(a,b)} v \otimes v M(v)\, dv, \qquad a, b \in \mathbb{R}^2,$$

and $S(a,b) = \{\lambda_1 a + \lambda_2 b : \lambda_1, \lambda_2 \in \mathbb{R}\}$.

Using the representation of L, we can deduce the following symmetry properties:

Proposition 5.2 *Let $\delta_1, \delta_2 \in \mathbb{R}^2$. Then, for any $\lambda \in \mathbb{R}$, we have $L(\lambda\delta_1, \lambda\delta_2) = \lambda L(\delta_1, \delta_2)$. If $B \in \mathbb{R}^{2\times 2}$ is any rotation or reflection matrix, then $L(B\delta_1, B\delta_2) = BL(\delta_1, \delta_2)$. If $\delta_1 = \delta_2$ then $L(\delta_1, \delta_2) = 0$ (no limiting).*

Proof: Using (4.5) and the homogeneity of ϕ, the relation $L(\lambda\delta_1, \lambda\delta_2) = \lambda L(\delta_1, \delta_2)$ follows at once. Also, in the case of linearly dependent vectors δ_1, δ_2, we get

$$L(B(\gamma_1 e), B(\gamma_2 e)) = \phi(\gamma_1, \gamma_2)Be = BL(\gamma_1 e, \gamma_2 e)$$

which is even true for any matrix $B \in \mathbb{R}^{2\times 2}$. Let us therefore assume that δ_1, δ_2 are linearly independent. Since $B^T = B^{-1}$ for reflections and rotations, we conclude that also $B\delta_i$ are independent and we can use relation (5.8). we first investigate the function I. Applying the change of variables $v = B^{-1}w = B^T w$ in (5.7), we have in view of $|\det B| = 1$

$$I(a,b) = \int_{S(a,b)} v \otimes v M(v)\, dv = \int_{BS(a,b)} (B^T w) \otimes (B^T w) M(B^T w)\, dw.$$

Since B is an isometry, we find $|B^t w| = |w|$ so that $M(B^T w) = B(w)$. Due to the definition of $S(a,b)$ we also get $BS(a,b) = S(Ba, Bb)$ and finally

$$(B^T w) \otimes (B^T w) e = \left((B^T w)\cdot e\right) B^T w = B^{-1}\left(w \cdot (Be)\right) w = B^{-1}(w \otimes w)e$$

implies together with the other remarks that

$$BI(a,b) = I(Ba, Bb)B. \tag{5.10}$$

Writing the $\perp$-operation in terms of a matrix A

$$e^\perp = Ae = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e,$$

it is easy to check that A commutes with all rotation matrices (in fact, A itself is a rotation matrix and rotations commute in 2D). For a reflection matrix B, we find, on the other hand, $AB = -BA$. Using the relation $S(-a,-b) = S(a,b)$, we thus have

$$I((Ba)^\perp, (Bb)^\perp) = I(Ba^\perp, Bb^\perp). \tag{5.11}$$

Combining (5.10) and (5.11), we conclude $L(B\delta_1, B\delta_2) = BL(\delta_1, \delta_2)$. Finally, in the case $\delta_1 = \delta_2$) we have $L(\delta_1, \delta_2) = \phi(1,1)\delta_1 = 0$ because $\phi(1,1) = 0$. ■

We conclude with the remark that the row-wise minmod limiter $\hat{L}$ defined in (4.6) also has the homogeneity property and coincides with L for linearly dependent arguments. The rotation and reflection invariance, however, is not shared by $\hat{L}$ as is easily checked with the rotation and reflection matrices

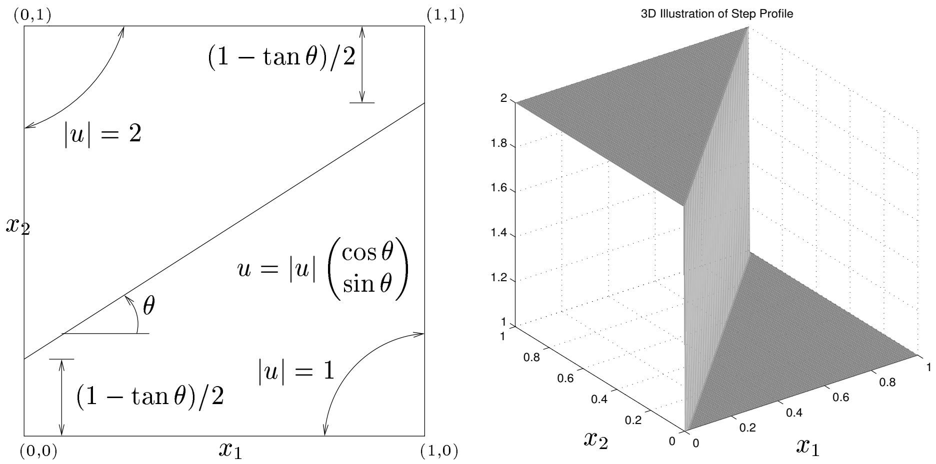
$$B_1 = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}, \qquad B_2 = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

6 Numerical results

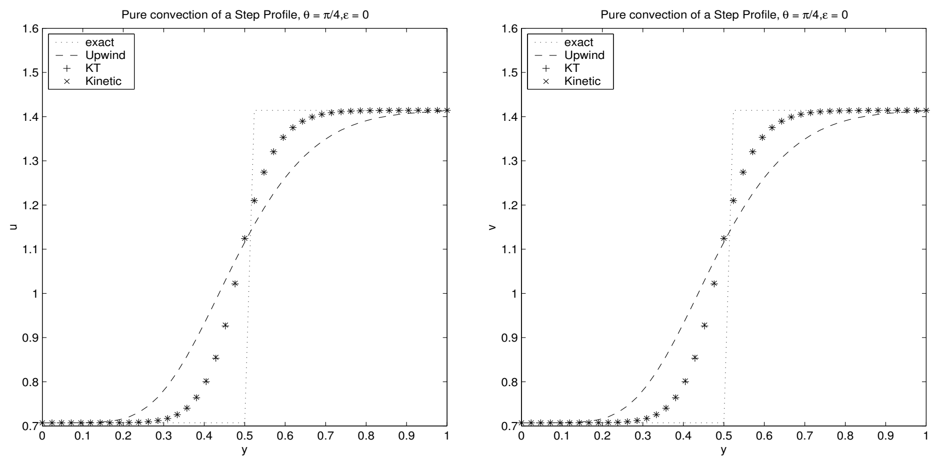
Since the new feature in the discretization of the incompressible Euler equation is the limiter L for the convective term, we separate effects and just study the behavior of the nonlinear part. As first test case, we take the stationary equation

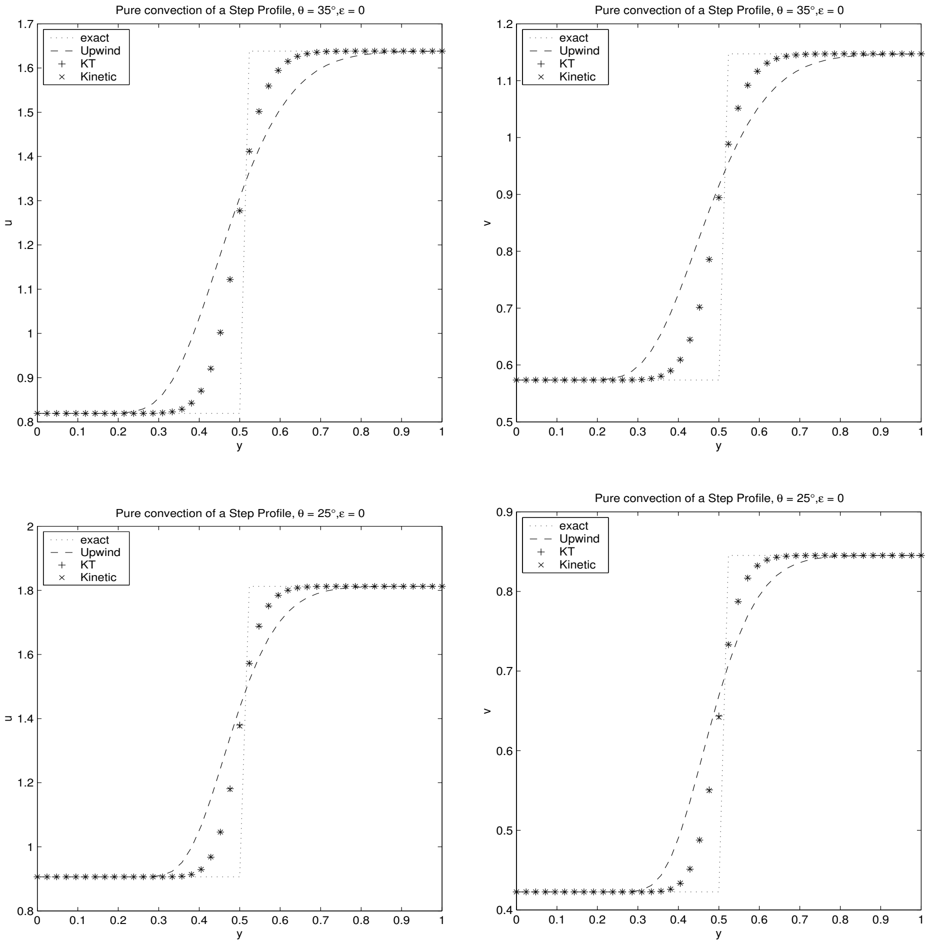
$$\nabla_x(u \otimes u) = 0 \quad + \text{boundary conditions}$$

and compare the scheme based on L with the second order scheme described in [14]. The test example is taken from [8]: Consider (x_1, x_2) in $[0,1]^2$. The domain of computation is divided into two subdomains which give a step profile as sketched below.



Boundary values are chosen according to the respective domain. The solution domain is discretized using a 41×41 regular mesh for different flow angles θ. The resulting nonlinear system is solved by a GMRES-based solver described by Kelley [12]. The results are plotted in the following figures. They show the computed profile at the line $x_1 = \frac{1}{2}$ for both the velocity components u_1 and u_2. In the figures we make a comparison of the first order kinetic method (which is equivalent to the usual upwind method), the second order approach by Kurganov and Tadmor [14] and the kinetic minmod approach developed here. We always use the local flow velocity to determine c_1 and c_2.





Obviously, all methods coincide within plotting accuracy. The reason for this behavior is probably related to the fact that the solution is constant in large subregions so that velocities u in neighboring cells are linearly dependent. As we have seen in the previous section, at least the kinetic and the row-wise minmod approach coincide in this situation.
In our second example, we therefore study a problem with a solution that exhibits a more complicated structure. We consider the Cauchy problem for the pressure-less Euler equation in the unit square with periodic boundary conditions

$$\partial_t u + \operatorname{div}_x u \otimes u = 0, \qquad u(0,x) = u_0(x).$$

The initial condition is piecewise constant with velocity directions as shown in figure 3. The speed is zero in the corner cells, $\sqrt{2}$ in the center, and one in the remaining cells. Because of the strong diagonal movement, the solution develops a jet-like structure (see figure 4 for isolines of $|u|$ at time $t = 0.5$ obtained on a fine grid). For a 50×50 grid, we compare the standard minmod limiter with the new kinetic limiter by considering the solution $|u|$ along several y-sections. Note that, due to symmetry of the solution, similar results are obtained for x-sections. Both algorithms are used with a second order Runge Kutta time

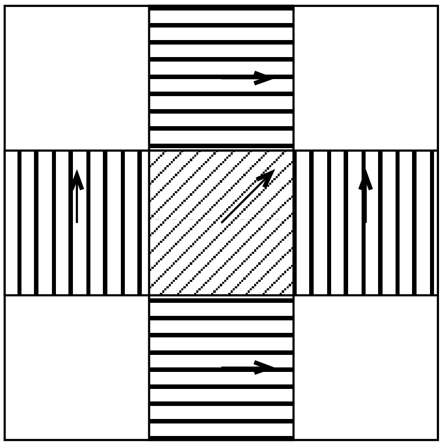


Figure 3: Initial condition

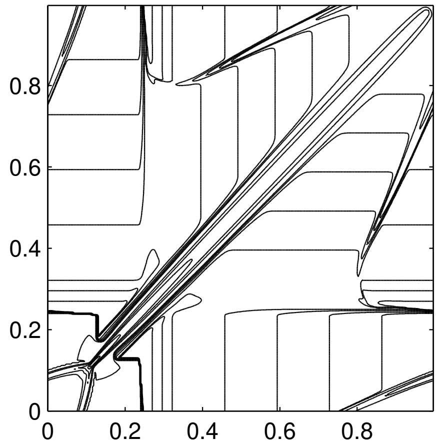


Figure 4: Solution at $t = 0.5$

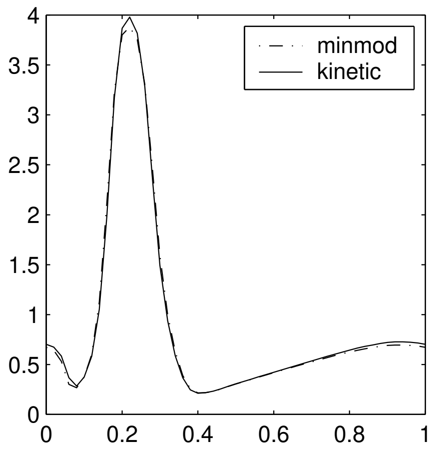


Figure 5: $|u|$ at $y = 0.12$

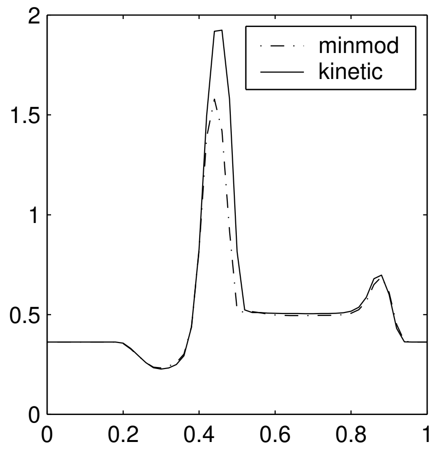


Figure 6: $|u|$ at $y = 0.52$

discretization and a CFL number 0.5 based on the maximal velocity. Because of its complexity, the kinetic limiter requires more computational time than the minmod approach. However, the factor two for the increase in runtime is still moderate. In many of the y-sections, the two results are almost identical (see figure 5 and 7) but in general, the kinetic limiter (solid line without symbols) yields higher extrema. In figure 8, the ratio of the maxima of $|u|$ in each y-section is shown. Typically, the maxima obtained with the kinetic limiter exceed those of the minmod limiter by 2%. In other words, the kinetic limiter better works out the details of the solution. One can also see from the results that, at certain points, the kinetic limiter is considerably less diffusive than the minmod approach (see figure 6).

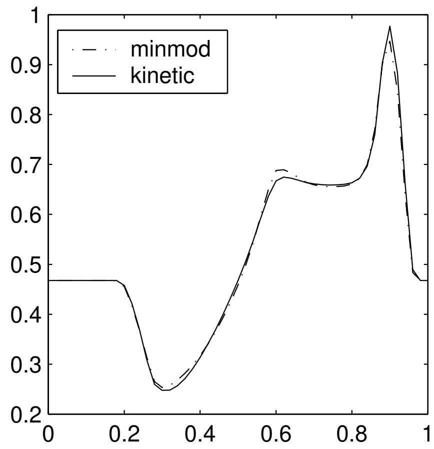


Figure 7: $|u|$ at $y = 0.6$

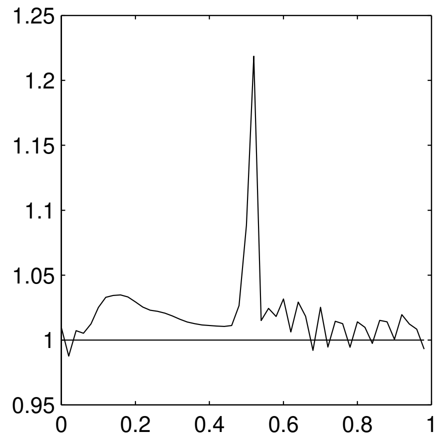


Figure 8: Ratio of maxima

Conclusion

Starting from special discretizations of the Boltzmann equation we have obtained corresponding discretizations of the incompressible Euler equation in the limit of low Mach and Knudsen numbers. In particular, a second order slope limiting approach on the kinetic level gives rise to a limiter for the nonlinear term in the Euler equation which exhibits an interesting structure. From simple numerical tests one can conclude that the new scheme yields results which are comparable or better than those obtained by other second order methods. A test of the new limiter in more complicated flow situations with strongly locally varying velocity fields is in preparation.

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