Rigorous Navier-Stokes Limit of the Lattice Boltzmann Equation

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Abstract

In this article, we rigorously investigate the diffusive limit of a velocity-discrete Boltzmann equation which is used in the lattice Boltzmann method (LBM) to construct approximate solutions of the incompressible Navier-Stokes equation. Our results apply to LBM collision operators with multiple collision frequencies (generalized lattice Boltzmann) which include the widely used BGK operators.

Keywords. lattice Boltzmann method, Navier Stokes equation, multiple collision frequencies, diffusive scaling, stability, incompressible limit
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1 Introduction

In this article, we are concerned with a velocity-discrete Boltzmann equation in the diffusive scaling

$$\frac{\partial f_i}{\partial t} + \frac{1}{\epsilon}\boldsymbol{c}_i \cdot \nabla f_i = \frac{1}{\epsilon^2} J_i(f), \qquad i = 0, \ldots, N \tag{1}$$

which arises in connection with a numerical method for the incompressible Navier-Stokes equation, the so-called lattice Boltzmann method (LBM) [3, 5]. The system (1) describes the evolution of a hypothetical gas or liquid in which the atoms can only travel with velocities from the discrete set $\mathcal{V} = \{\boldsymbol{c}_0, \ldots, \boldsymbol{c}_N\}$. The particle densities f_i specify how many particles have the velocity $\boldsymbol{c}_i \in \mathcal{V}$ at time $t \in [0, T]$ and position $\boldsymbol{x} \in \Omega$. While the left-hand side in (1) describes the transport of the particles, the right-hand side models interaction of the particles by collisions.

Before we specify details of the structure of $\mathcal{V}$ and J, let us briefly mention how the lattice Boltzmann method is related to (1) (for more details, see [11, 12]). Integrating (1) along characteristics, we find

$$f_i(t + \Delta t, \boldsymbol{x} + \boldsymbol{c}_i \Delta t/\epsilon) = f_i(t, \boldsymbol{x}) + \int_0^{\Delta t} \frac{1}{\epsilon^2} J_i(f)(t + s, \boldsymbol{x} + \boldsymbol{c}_i s/\epsilon)\, ds.$$

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Setting $\epsilon = \Delta x, \Delta t = \epsilon \Delta x$, and approximating the integral by the simple rectangle rule with evaluation at the left point of the interval, we obtain

$$f_i(t + \Delta t, \boldsymbol{x} + \boldsymbol{c}_i \Delta x) = f_i(t, \boldsymbol{x}) + J_i(f)(t, \boldsymbol{x}), \tag{2}$$

which is exactly the lattice Boltzmann evolution. If the discrete velocity set $\mathcal{V}$ is chosen in such a way that the set of all integer linear combinations forms a regular lattice $\mathcal{X} = \{\sum_i n_i \boldsymbol{c}_i : n_i \in \mathbb{Z}\}$, then (1) is already completely discretized if $\boldsymbol{x}$ is restricted to $\Delta x \, \mathcal{X}$. Under suitable conditions on the initial values for (1), it turns out that the average $\boldsymbol{u} = \sum_i \boldsymbol{c}_i f_i / \epsilon$ is an approximate solution of the incompressible Navier-Stokes equation

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \nu \Delta \boldsymbol{u}, \qquad \boldsymbol{u}|_{t=0} = \bar{\boldsymbol{u}}. \tag{3}$$

This relation is usually justified by carrying out a formal Chapman-Enskog expansion in ϵ (see for example [1, 8]).

In this article, our aim is to give a *rigorous* justification of the relation between the continuous version (1) of the lattice Boltzmann equation and the Navier-Stokes equation (3). Our analysis will be in the spirit of [19, 6, 2, 7] but it differs from these results because we concentrate on the collision operators which are used in the lattice Boltzmann methods. Our result is also different from those in [20, 16, 22] because our limit system consists of incompressible Navier-Stokes equations.

To fix ideas, we will work in a specific two-dimensional situation but the ideas can be transfered to other models and three dimensions. The spatial domain Ω will be the unit torus (i.e. the unit square with periodic boundary conditions) and the velocity set is chosen as in the D2Q9 model (nine velocities in two space dimensions – see Fig. 1) where $\mathcal{V} = \{\boldsymbol{c}_0, \ldots, \boldsymbol{c}_8\}$ with $\boldsymbol{c}_0 = \boldsymbol{0}$ and

$$\begin{array}{llll} \boldsymbol{c}_1 = \binom{1}{0} & \boldsymbol{c}_2 = \binom{0}{1} & \boldsymbol{c}_3 = \binom{-1}{0} & \boldsymbol{c}_4 = \binom{0}{-1} \\ \boldsymbol{c}_5 = \binom{1}{1} & \boldsymbol{c}_6 = \binom{-1}{1} & \boldsymbol{c}_7 = \binom{-1}{-1} & \boldsymbol{c}_8 = \binom{1}{-1} \end{array}$$

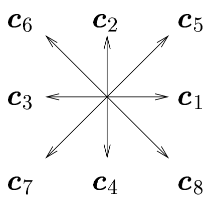

Figure 1: Discrete velocities in the D2Q9 model

To simplify notation, we introduce the Euclidean vector space $\mathcal{F}$ of real valued functions $f : \mathcal{V} \to \mathbb{R}$ with the canonical scalar product

$$\langle f, g \rangle = \sum_{\boldsymbol{v} \in \mathcal{V}} f(\boldsymbol{v}) g(\boldsymbol{v}), \qquad f, g \in \mathcal{F}.$$

With the multiplication operators $V_1, V_2 : \mathcal{F} \to \mathcal{F}$ defined by $(V_i f)(\boldsymbol{v}) = v_i f(\boldsymbol{v})$, we can form the vector $\boldsymbol{V} = (V_1, V_2)^T$ and rewrite (1) as equation for $f(t, \boldsymbol{x}, \boldsymbol{c}_i) = f_i(t, \boldsymbol{x})$:

$$\frac{\partial f}{\partial t} + \frac{1}{\epsilon} \boldsymbol{V} \cdot \nabla f = \frac{1}{\epsilon^2} J(f). \tag{4}$$

The collision operator $J : \mathcal{F} \to \mathcal{F}$ is chosen as

$$J(f) = \mathcal{A}(f^{eq}(f) - f)$$

where $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ is a linear mapping (with properties specified in section 2) and $f^{eq} : \mathcal{F} \to \mathcal{F}$ is the so-called equilibrium distribution which we choose as in [8]. It depends linearly on the average density $\rho = \langle 1, f \rangle$ and quadratically on the average momentum $\bar{\rho}\boldsymbol{u} = \langle 1, \boldsymbol{V} f \rangle$

$$f^{eq}(f) = F^{eq}_{\bar{\rho}}(\langle 1, f \rangle, \langle 1, \boldsymbol{V} f \rangle)$$

with

$$F^{eq}_{\bar{\rho}}(\rho, \boldsymbol{u}; \boldsymbol{v}) = \big(\rho + 3\bar{\rho}\boldsymbol{u} \cdot \boldsymbol{v} + \bar{\rho}(3\boldsymbol{u} \cdot \boldsymbol{v})^2/2 - 3\bar{\rho}|\boldsymbol{u}|^2/2\big) f^*(\boldsymbol{v}).$$

The function $f^* = F^{eq}_{\bar{\rho}}(1, \mathbf{0})$ is defined by

$$f^*(\mathbf{c}_i) = \begin{cases} 4/9, & i = 0 \\ 1/9, & i = 1, 2, 3, 4 \\ 1/36, & i = 5, 6, 7, 8 \end{cases}$$

In the following, we set $\bar{\rho} = 1$. As initial data, we choose a small perturbation of the function f^*

$$f|_{t=0} = F^{eq}_1(1, \epsilon \bar{\boldsymbol{u}}) + O(\epsilon^2) \tag{5}$$

where $\bar{\boldsymbol{u}} : \Omega \to \mathbb{R}^2$ is a smooth, divergence-free velocity field. The $O(\epsilon^2)$-term will be specified later to avoid initial-layers. We remark that in lattice Boltzmann simulations, the initial value typically lacks this correction, i.e. $f|_{t=0} = F^{eq}_1(1, \epsilon \bar{\boldsymbol{u}})$ but we postpone a careful investigation of the resulting initial-layer effects to some future work.
In the following, we are going to show that the solution f^ϵ of the initial value problem (4), (5) has the property that $\boldsymbol{u}_\epsilon/\epsilon = \langle 1, f^\epsilon \rangle / \epsilon$ differs only by terms of order ϵ^2 from the solution $\boldsymbol{u}$ of the incompressible Navier-Stokes equation (3) with a viscosity parameter ν being related to an eigenvalue of the operator $\mathcal{A}$.

2 Collision operator and stability structure

The idea to use collision operators of relaxation type $J(f) = \mathcal{A}(f^{eq}(f) - f)$ in the lattice Boltzmann approach goes back to [9, 17]. In the multiple-relaxation-time (or generalized) lattice Boltzmann model [10], the operator $\mathcal{A}$ is set up by specifying an orthonormal basis of $\mathcal{F}$ and assuming that $\mathcal{A}$ diagonalizes in this basis. This approach has shown to be more flexible and stable [15] than the widely used BGK collision operator, where $\mathcal{A}$ is a multiple of the identity [18, 4]. In the following, we will adopt the multiple-relaxation-time model where the linear operator $\mathcal{A}$ is essentially determined by certain algebraic properties (which reflect physical symmetries and conservation): $\mathcal{A}$ should be symmetric and positive semidefinite, it should commute with 90° rotations and reflections, and the kernel should be generated by the functions $p_1(\boldsymbol{v}) = 1$, $p_2(\boldsymbol{v}) = v_1$, $p_3(\boldsymbol{v}) = v_2$. Introducing rotation and reflection by the operators $(Rf)(\boldsymbol{v}) = f(-v_2, v_1)$, $(Sf)(\boldsymbol{v}) = f(-v_1, v_2)$, these properties can be written as

i. $\langle \mathcal{A}f, g\rangle = \langle f, \mathcal{A}g\rangle$

ii. $\mathcal{A}R = R\mathcal{A}$, $\mathcal{A}S = S\mathcal{A}$

iii. $\mathcal{A}$ is positive semidefinite

iv. $\{1, v_1, v_2\}$ generates the kernel of $\mathcal{A}$

It is possible to completely characterize operators $\mathcal{A}$ which satisfy conditions (i) to (iv). This investigation is easily carried out in an orthonormal basis $\{q_1, \ldots, q_9\}$ of $\mathcal{F}$ related to eigenvectors of the rotation operator R and the reflection operator S (up to the signs of q_6, q_7, q_8 this is the basis used in [10]):

$$q_1(\boldsymbol{v}) = \frac{1}{3},$$
$$q_2(\boldsymbol{v}) = \frac{v_1}{\sqrt{6}}, \qquad q_3(\boldsymbol{v}) = \frac{v_2}{\sqrt{6}},$$
$$q_4(\boldsymbol{v}) = \frac{v_1 v_2}{2}, \qquad q_5(\boldsymbol{v}) = \frac{v_1^2 - v_2^2}{2}, \qquad q_6(\boldsymbol{v}) = \frac{3v_1^2 + 3v_2^2 - 4}{6},$$
$$q_7(\boldsymbol{v}) = \frac{3v_1 v_2^2 - 2v_1}{2\sqrt{3}}, \qquad q_8(\boldsymbol{v}) = \frac{3v_1^2 v_2 - 2v_2}{2\sqrt{3}},$$
$$q_9(\boldsymbol{v}) = \frac{3}{2}v_1^2 v_2^2 - v_1^2 - v_2^2 + \frac{2}{3}.$$

Introducing the orthogonal projectors $(Q_i f) = \langle f, q_i\rangle\, q_i$, we can write (the majority of) all operators satisfying (i) to (iv) in the form

$$\mathcal{A} = \sum_{i=1}^{9} \lambda_i Q_i, \qquad \lambda_1, \lambda_2, \lambda_3 = 0, \quad \lambda_7 = \lambda_8, \quad \lambda_4, \ldots, \lambda_9 > 0. \tag{6}$$

For later use, we introduce the projection $Q = \sum_{i=1}^{3} Q_i$ onto the kernel of $\mathcal{A}$. Note that

$$Qf = 0 \qquad \Longleftrightarrow \qquad \langle 1, f\rangle = 0, \quad \langle 1, \boldsymbol{V}f\rangle = \boldsymbol{0}. \tag{7}$$

Since the remaining eigenvalues are strictly positive, we can define a pseudo inverse of $\mathcal{A}$ using $P = I - Q$

$$\mathcal{A}^{\dagger} := \left(\mathcal{A}|_{P(\mathcal{F})}\right)^{-1} P.$$

Note that $\mathcal{A}^\dagger : \mathcal{F} \to \mathcal{F}$ satisfies

$$Q\mathcal{A}^\dagger = \mathcal{A}^\dagger Q = 0, \qquad P\mathcal{A}^\dagger = \mathcal{A}^\dagger P = \mathcal{A}^\dagger, \qquad \mathcal{A}\mathcal{A}^\dagger = P = \mathcal{A}^\dagger\mathcal{A}. \tag{8}$$

Another important property of $P, Q, \mathcal{A}$, and $\mathcal{A}^\dagger$ is the preservation of even/odd symmetry. The reason is that all these operators commute with double rotations R^2 – the building block for the odd/even projections

$$S_e = \frac{1}{2}(I + R^2), \qquad S_o = \frac{1}{2}(I - R^2)$$

(note that $(S_e f)(\boldsymbol{v}) = (f(\boldsymbol{v}) + f(-\boldsymbol{v}))/2$, and $(S_o f)(\boldsymbol{v}) = (f(\boldsymbol{v}) - f(-\boldsymbol{v}))/2$). For example, we have

$$S_e\mathcal{A} = (\mathcal{A} + R^2\mathcal{A})/2 = \mathcal{A}(I + R^2)/2 = \mathcal{A}S_e$$

so that $\mathcal{A}$ applied to some even function f (i.e. $f = S_e f$) is again even: $S_e\mathcal{A}f = \mathcal{A}S_e f = \mathcal{A}f$. Similarly, one shows that $P, Q, \mathcal{A}$, and $\mathcal{A}^\dagger$ commute with S_e, S_o and thus preserve odd/even symmetry.
Next, we consider the equilibrium distribution $f^{eq}(f) = F_1^{eq}(\langle 1, f\rangle, \langle 1, \boldsymbol{V}f\rangle)$ which can be split into a linear and a quadratic part

$$f^{eq}(f) = f^{eq}_{lin}(f) + f^{eq}_{quad}(f, f) \tag{9}$$

where

$$f^{eq}_{lin}(f) = F^{eq}_{lin}(\langle 1, f\rangle, \langle 1, \boldsymbol{V}f\rangle), \qquad f^{eq}_{quad}(f, g) = F^{eq}_{quad}(\langle 1, \boldsymbol{V}f\rangle, \langle 1, \boldsymbol{V}g\rangle)$$

and

$$\begin{aligned} F^{eq}_{lin}(\rho, \boldsymbol{u}) &= (\rho + 3\boldsymbol{u}\cdot\boldsymbol{V})f^*, \\ F^{eq}_{quad}(\boldsymbol{u}, \boldsymbol{w}) &= ((3\boldsymbol{u}\cdot\boldsymbol{V})(3\boldsymbol{w}\cdot\boldsymbol{V})/2 - 3\boldsymbol{u}\cdot\boldsymbol{w}/2)f^*. \end{aligned}$$

The functions $F^{eq}_{lin}, F^{eq}_{quad}$ are constructed in such a way that

$$\left\langle 1, F^{eq}_{lin}(\rho, \boldsymbol{u})\right\rangle = \rho, \quad \left\langle 1, \boldsymbol{V}F^{eq}_{lin}(\rho, \boldsymbol{u})\right\rangle = \boldsymbol{u}, \quad \left\langle 1, V_iV_jF^{eq}_{lin}(\rho, \boldsymbol{u})\right\rangle = \frac{\rho}{3}\delta_{ij} \tag{10}$$

and

$$\left\langle 1, (1, \boldsymbol{V})F^{eq}_{quad}(\boldsymbol{u}, \boldsymbol{w})\right\rangle = (0, \boldsymbol{0}), \quad \left\langle 1, V_iV_jF^{eq}_{quad}(\boldsymbol{u}, \boldsymbol{w})\right\rangle = \frac{u_iw_j + u_jw_i}{2}. \tag{11}$$

Inspecting the definition of F^{eq}_{quad}, it easily follows that

$$f^{eq}_{quad}(f, g) = S_e f^{eq}_{quad}(S_o f, S_o g), \qquad f^{eq}_{quad}(f, 0) = 0. \tag{12}$$

Note that, in view of (7) and (10), we have $Q(f^{eq}(f) - f) = 0$ so that for any $\tau > 0$, $\mathcal{A}(f^{eq}(f) - f) = (Q/\tau + \mathcal{A})(f^{eq}(f) - f)$. Hence, if we choose $\lambda_4 = \cdots = \lambda_9 = 1/\tau$, we obtain the so-called BGK collision operator $J(f) = (f^{eq}(f) - f)/\tau$ which is frequently used in LBM [18, 4, 5] and which will be covered by our considerations.

Introducing the linear part of the collision operator

$$J_0 = \mathcal{A}(f^{eq}_{lin} - I), \tag{13}$$

we can write

$$J(f) = J_0 f + \mathcal{A} f^{eq}_{quad}(f, f). \tag{14}$$

We remark that the directional derivative of J at the point $f \in \mathcal{F}$ in direction $h \in \mathcal{F}$ is given by

$$DJ(f)h = J_0 h + 2\mathcal{A} f^{eq}_{quad}(f, h). \tag{15}$$

Finally, let us concentrate on the stability structure of equation (2). We observe that (4) is a symmetric hyperbolic system. However, for stability reasons, we require also some symmetry and definiteness properties of the right-hand side. To achieve this, we need the following result.

Lemma 2.1
Let B_0 be the positive definite multiplication operator $B_0 f = f/f^*$ and let $\mathcal{A}$ be of the form (6) with $\lambda_6 = \lambda_9$. Then there exist linear operators $P_k : \mathcal{F} \to \mathcal{F}$, $k = 1, \ldots, 9$ with adjoints P_k^* such that

$$P_i^* P_k = \delta_{ik} P_k^* P_k, \qquad B_0 = \sum_{k=1}^{9} P_k^* P_k, \qquad B_0 J_0 = -\sum_{k=1}^{9} \lambda_k P_k^* P_k. \tag{16}$$

and

$$P_i J_0 = 0, \quad P_i J(f) = 0, \qquad i = 1, 2, 3. \tag{17}$$

Proof: We first show that $B_0 J_0$ is a symmetric operator. To see this, we introduce the orthogonal subspaces

$$U_4 = \text{span}\{q_4\},\ U_5 = \text{span}\{q_5\},\ U_6 = \text{span}\{q_1, q_6, q_9\},\ U_7 = \text{span}\{q_2, q_3, q_7, q_8\},$$

and note that $3f^* = q_1 - q_6 + q_9/2 \in U_6$, $V_i f^* \in U_7$, and $B_0 U_k \subset U_k$. Moreover, because of the assumption $\lambda_6 = \lambda_9$, we have $\mathcal{A} U_k \subset U_k$. Using a subscript k on $f \in \mathcal{F}$ to denote the projection onto U_k, we find

$$f^{eq}(f) - f = -f_4 - f_5 - (f_6 - \langle 1, f_6 \rangle f_6^*) - (f_7 - 3\langle 1, \boldsymbol{V} f_7 \rangle \cdot (\boldsymbol{V} f^*)_7).$$

Since

$$\langle f_6 - \langle 1, f_6 \rangle f_6^*, q_1 \rangle = 0, \qquad \langle f_7 - 3\langle 1, \boldsymbol{V} f_7 \rangle \cdot (\boldsymbol{V} f^*)_7, q_i \rangle = 0, \quad i = 2, 3$$

we conclude

$$J_0 f = -\lambda_4 f_4 - \lambda_5 f_5 - \lambda_6 (f_6 - \langle 1, f_6 \rangle f^*) - \lambda_7 (f_7 - 3\langle 1, \boldsymbol{V} f_7 \rangle \cdot \boldsymbol{V} f^*). \tag{18}$$

To show symmetry of $B_0 J_0$ we recall that U_k are invariant, orthogonal subspaces of $B_0 J_0$ so that

$$\langle B_0 J_0 f, g\rangle = \sum_{k=4}^{7} \langle B_0 J_0 f_k, g_k\rangle .$$

Thus, it remains to show that $\langle B_0 J_0 f_k, g_k\rangle$ are symmetric expressions in g_k and f_k. For $k = 4, 5$ this follows immediately from

$$\langle B_0 J_0 f_k, g_k\rangle = -\lambda_k \langle B_0 f_k, g_k\rangle = -\lambda_k \left\langle B_0^{\frac{1}{2}} f_k, B_0^{\frac{1}{2}} g_k \right\rangle, \quad k = 4, 5.$$

For the other subspaces, we have with $B_0 f^* = 1$

$$\langle B_0 J_0 f_6, g_6\rangle = -\lambda_6 \left\langle B_0^{\frac{1}{2}} f_6, B_0^{\frac{1}{2}} g_6 \right\rangle + \lambda_6 \langle 1, f_6\rangle \langle 1, g_6\rangle ,$$

and

$$\langle B_0 J_0 f_7, g_7\rangle = -\lambda_7 \left\langle B_0^{\frac{1}{2}} f_7, B_0^{\frac{1}{2}} g_7 \right\rangle + \lambda_7 \langle 1, \boldsymbol{V} f_7\rangle \cdot \langle 1, \boldsymbol{V} g_7\rangle .$$

We remark that the eigenvectors and eigenvalues of J_0 can easily be read off from (18). Obvious eigenvectors are $q_4, \ldots, q_9$ with eigenvalues $-\lambda_4, \ldots, -\lambda_9$. The remaining three eigenvectors belong to the eigenvalue zero: f^*, $V_1 f^*$, $V_2 f^*$. Using the fact that

$$B_0^{\frac{1}{2}} J_0 B_0^{-\frac{1}{2}} = B_0^{-\frac{1}{2}} B_0 J_0 B_0^{-\frac{1}{2}}$$

is symmetric and has the same eigenvalues $-\lambda_k$ as the operator J_0, we can find a basis of orthonormal eigenvectors r_k. Using the corresponding orthogonal projectors $R_k f = \langle f, r_k\rangle r_k$, we have $\sum_{k=1}^{9} R_k = I$, and

$$B_0^{\frac{1}{2}} J_0 B_0^{-\frac{1}{2}} = -\sum_{k=1}^{9} \lambda_k R_k.$$

Defining $P_k = R_k B_0^{\frac{1}{2}}$, relations (16) follow immediately. Finally, to show (17), we observe for $i = 1, 2, 3$

$$P_i J_0 = R_i B_0^{\frac{1}{2}} J_0 B_0^{-\frac{1}{2}} B_0^{\frac{1}{2}} = -\sum_{k=1}^{9} \lambda_k R_i R_k B_0^{\frac{1}{2}} = -\lambda_i R_i B_0^{\frac{1}{2}} = 0.$$

In view of (14)

$$P_i J(f) = P_i J_0 f + P_i \mathcal{A} f^{eq}_{quad}(f, f) = P_i \mathcal{A} f^{eq}_{quad}(f, f) = R_i B_0^{\frac{1}{2}} \mathcal{A} f^{eq}_{quad}(f, f).$$

Since r_i is, for $i = 1, 2, 3$, in the kernel of $J_0 B^{-\frac{1}{2}}$, we have with suitable coefficients $\alpha_i, \boldsymbol{\beta}_i$

$$r_i = B_0^{\frac{1}{2}} (\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) f^* = B_0^{-\frac{1}{2}} (\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) 1.$$

so that with $g = f^{eq}_{quad}(f, f)$ and the structure of the kernel of $\mathcal{A}$

$$\begin{aligned} P_i J(f) &= \left\langle B_0^{\frac{1}{2}} \mathcal{A} g, B_0^{-\frac{1}{2}} (\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) 1 \right\rangle r_i \\ &= \langle (\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) 1, \mathcal{A} g \rangle = \langle \mathcal{A}(\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) 1, g \rangle = 0. \end{aligned}$$

■

3 Formal asymptotic expansion

In this section, we generally assume that $\mathcal{A}$ is of the form (6) with $\lambda_4 = \lambda_5 = 1/(3\nu) > 0$ for some $\nu > 0$. To investigate the asymptotic behavior of initial value problems for (4) in the limit $\epsilon \to 0$, we introduce a regular expansion $f_\epsilon \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \ldots$ with $f_0 = f^*$. Plugging the expansion into (4) and setting $f_m = 0$ for $m < 0$, we obtain in order ϵ^k, $k \geq -1$

$$\frac{\partial f_k}{\partial t} + \boldsymbol{V} \cdot \nabla f_{k+1} - \mathcal{A} \left(f^{eq}_{lin}(f_{k+2}) - f_{k+2} + \sum_{p+q=k+2} f^{eq}_{quad}(f_p, f_q) \right) = 0, \quad (19)$$

from which we can determine the expansion coefficients f_i. First, we note that in view of (7), (10), and (11),

$$Q(f^{eq}_{lin}(f_{k+2}) - f_{k+2}) = 0, \qquad Q f^{eq}_{quad}(f_n, f_m) = 0.$$

Hence,

$$P(f^{eq}_{lin}(f_{k+2}) - f_{k+2}) = f^{eq}_{lin}(f_{k+2}) - f_{k+2}, \qquad P f^{eq}_{quad}(f_n, f_m) = f^{eq}_{quad}(f_n, f_m),$$

so that an application of the pseudo inverse $\mathcal{A}^\dagger$ to (19), yields in view of (8) for any $k \in \mathbb{Z}$

$$f_k = f^{eq}_{lin}(f_k) + \sum_{p+q=k} f^{eq}_{quad}(f_p, f_q) - \mathcal{A}^\dagger \left(\frac{\partial f_{k-2}}{\partial t} + \boldsymbol{V} \cdot \nabla f_{k-1} \right). \quad (20)$$

We remark that (20) does not specify f_k completely since f_k also appears on the right-hand side as argument of f^{eq}_{lin}. Due to the structure of f^{eq}_{lin}, we can also say that (20) determines f_k up to the moments $\rho_k = \langle 1, f_k \rangle$ and $\boldsymbol{u}_k = \langle 1, \boldsymbol{V} f_k \rangle$. To fix these remaining degrees of freedom, we apply Q to (19) which yields $Q(\partial_t f_k + \boldsymbol{V} \cdot \nabla f_{k+1}) = 0$. In view of (7), we can express this equation also in terms of the moments $\rho_k, \boldsymbol{u}_k$

$$\frac{\partial \rho_k}{\partial t} + \operatorname{div} \boldsymbol{u}_{k+1} = 0, \quad (21)$$

$$\frac{\partial \boldsymbol{u}_k}{\partial t} + \operatorname{div} \langle 1, \boldsymbol{V} \otimes \boldsymbol{V} f_{k+1} \rangle = \boldsymbol{0}. \quad (22)$$

Here, the symmetric tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ is defined as the matrix with components $(a_i b_j + a_j b_i)/2$ and the divergence is applied row-wise.
In order to carry out the expansion, the following result is crucial.

Lemma 3.1
Assume $\rho_{2m-1}|_{t=0} = 0, \boldsymbol{u}_{2m}|_{t=0} = \boldsymbol{0}$ for $m = 1, 2, \cdots$. Then the expansion coefficients satisfy $S_e f_{2m} = f_{2m}$, and $S_o f_{2m-1} = f_{2m-1}$, i.e. f_{2m} are even functions and f_{2m-1} are odd functions for all $m \in \mathbb{Z}$. The moments ρ_{2m} and $\boldsymbol{u}_{2m-1}$ are solutions of the equation

$$\frac{\partial \boldsymbol{u}_{2m-1}}{\partial t} + c_{2m-1} \operatorname{div} \boldsymbol{u}_{2m-1} \otimes \boldsymbol{u}_1 + \frac{1}{3} \nabla \rho_{2m} = \nu \Delta \boldsymbol{u}_{2m-1} + \boldsymbol{G}_{2m-1} \tag{23}$$

with the divergence condition $\operatorname{div} \boldsymbol{u}_{2m-1} = -\partial_t \rho_{2m-2}$. In the case $m = 1$, we have $c_1 = 1$ and $\boldsymbol{G}_1 = \boldsymbol{0}$. Otherwise, $c_{2m-1} = 2$ and the source terms $\boldsymbol{G}_{2m-1}$, $m > 1$, depend only on derivatives of ρ_k, $\boldsymbol{u}_k$ with $k < 2m - 1$.

Proof: We prove the symmetry result by induction over m, where we add the additional statement $\rho_{2m+1} = 0$ (which follows from $S_o f_{2m-1} = f_{2m-1}$ once the proof is carried out and therefore does need to be stated in the lemma). The induction base $m = 0$ is quite simple because $f_{-1} = 0$ is odd and $f_0 = f^*$ is even. To show that $\rho_1 = 0$, we first exploit relation (20). Taking $k = 1$ and keeping in mind that $f_m = 0$ for $m < 0$ as well as (12) with $f_0 = f^* = S_e f^*$, we conclude $f_1 = f^{eq}_{lin}(f_1)$. In view of (10) and the fact that $\boldsymbol{u}_0 = \langle 1, \boldsymbol{V} f^* \rangle = \boldsymbol{0}$, (22) yields $\nabla \rho_1 = \boldsymbol{0}$ so that ρ_1 is independent of $\boldsymbol{x}$. Integrating (21) over the unit torus Ω, we thus get with the help of the divergence theorem

$$|\Omega| \frac{d\rho_1}{dt} = -\int_\Omega \operatorname{div} \boldsymbol{u}_2 \, d\boldsymbol{x} = 0.$$

Since $\rho_1 = 0$ initially, we conclude that $\rho_1 = 0$ for all $t \geq 0$ which completes the base of induction.
The induction step starts with the observation that f_{2m+1} is odd. This follows from (20) with $k = 2m + 1$ because all terms on the right-hand side are odd functions: $f^{eq}_{lin}(f_{2m+1})$ is odd since $\rho_{2m+1} = 0$ by induction assumption; all quadratic terms $f^{eq}_{quad}(f_p, f_q)$ vanish in view of (12) because if $p + q = 2m + 1$ is odd, either p or q has to be even so that $S_o f_q = 0$ or $S_o f_p = 0$; since f_{2m-1} is odd, the same holds for $\partial_t f_{2m-1}$ and the even symmetry of f_{2m} leads to odd symmetry of $\boldsymbol{V} \cdot \nabla f_{2m}$. The fact that $\mathcal{A}^\dagger$ preserves the symmetry thus shows that f_{2m+1} is odd. Using similar arguments in the case $k = 2m + 2$ (note that $f^{eq}_{quad}(f_p, f_q)$ is even according to (12)), we find that f_{2m+2} is even if and only if $\boldsymbol{u}_{2m+2} = \boldsymbol{0}$. Thus, to finish the induction proof, it remains to show that $\boldsymbol{u}_{2m+2} = \boldsymbol{0}$ and $\rho_{2m+3} = 0$.
Equation (20) with $k = 2m + 3$ and the fact that f_{2n} are even for $n \leq m$ imply

$$f_{2m+3} = f^{eq}_{lin}(f_{2m+3}) + 2 f^{eq}_{quad}(f_{2m+2}, f_1) - \mathcal{A}^\dagger \left(\frac{\partial f_{2m+1}}{\partial t} + \boldsymbol{V} \cdot \nabla f_{2m+2} \right).$$

In view of (22), we multiply this equation with $v_i v_j$ and apply $\langle 1, \cdot \rangle$. Using (10), (11), and summation convention for the repeated indices k, l, we obtain

$$\begin{aligned} \langle 1, V_i V_j f_{2m+3} \rangle = & \frac{\rho_{2m+3}}{3} \delta_{ij} + (\boldsymbol{u}_{2m+2})_i (\boldsymbol{u}_1)_j + (\boldsymbol{u}_{2m+2})_j (\boldsymbol{u}_1)_i \\ & - \frac{\partial (\boldsymbol{u}_{2m+2})_k}{\partial x_l} \left\langle 1, V_i V_j \mathcal{A}^\dagger (3 V_k V_l f^*) \right\rangle. \end{aligned}$$

By direct calculation, one finds

$$\left\langle \mathcal{A}^\dagger V_1^2 1, 3V_1^2 f^* \right\rangle = \frac{1}{3\lambda_5} = \nu, \qquad \left\langle \mathcal{A}^\dagger V_1^2 1, 3V_2^2 f^* \right\rangle = -\frac{1}{3\lambda_5} = -\nu,$$
$$\left\langle \mathcal{A}^\dagger V_2^2 1, 3V_1^2 f^* \right\rangle = -\frac{1}{3\lambda_5} = -\nu, \qquad \left\langle \mathcal{A}^\dagger V_2^2 1, 3V_2^2 f^* \right\rangle = \frac{1}{3\lambda_5} = \nu,$$
$$\left\langle \mathcal{A}^\dagger V_1 V_2 1, 3V_1 V_2 f^* \right\rangle = \frac{1}{3\lambda_4} = \nu,$$

and $\left\langle \mathcal{A}^\dagger V_i V_j 1, 3V_k V_l f^* \right\rangle = 0$ for all other choices of i, j, k and l. We conclude

$$\operatorname{div} \langle 1, \boldsymbol{V} \otimes \boldsymbol{V} f_{2m+3} \rangle = \frac{1}{3} \nabla \rho_{2m+3} + 2 \operatorname{div} \boldsymbol{u}_{2m+2} \otimes \boldsymbol{u}_1 - \nu \Delta \boldsymbol{u}_{2m+2}.$$

Using $\rho_{2m+1} = 0$ from the previous step, we find with (21), (20) that ρ_{2m+3}, $\boldsymbol{u}_{2m+2}$ are obtained as solutions of the Oseen problem

$$\operatorname{div} \boldsymbol{u}_{2m+2} = 0, \qquad \frac{\partial \boldsymbol{u}_{2m+2}}{\partial t} + 2 \operatorname{div} \boldsymbol{u}_{2m+2} \otimes \boldsymbol{u}_1 + \frac{1}{3} \nabla \rho_{2m+3} = \nu \Delta \boldsymbol{u}_{2m+2}.$$

Since the initial data $\rho_{2m+3}|_{t=0}$ and $\boldsymbol{u}_{2m+2}|_{t=0}$ are assumed to be zero, we conclude that $\rho_{2m+3} = 0$, $\boldsymbol{u}_{2m+2} = \boldsymbol{0}$ is the unique solution of this problem (see Lemma 5.1). This completes the induction.
Finally, let us derive the equation satisfied by $\boldsymbol{u}_{2m-1}$ and ρ_{2m}. The divergence condition is an immediate consequence of (21). To evaluate (22), we note that with (20) applied to $k = 2m$ and $k = 2m - 1$

$$f_{2m} = \rho_{2m} f^* + c_{2m-1} f^{eq}_{quad}(f_{2m-1}, f_1) - \frac{\partial (\boldsymbol{u}_{2m-1})_k}{\partial x_l} \mathcal{A}^\dagger (3V_k V_l f^*) + g_{2m-1}$$

where we have collected all terms involving f_k with $k < 2m - 1$ in

$$\begin{aligned} g_{2m-1} = & \sum_{k=2}^{2m-2} f^{eq}_{quad}(f_k, f_{2m-k}) \\ & - \mathcal{A}^\dagger \left(\frac{\partial f_{2m-2}}{\partial t} - \boldsymbol{V} \cdot \nabla \mathcal{A}^\dagger \left(\frac{\partial f_{2m-3}}{\partial t} - \boldsymbol{V} \cdot \nabla f_{2m-2} \right) \right). \end{aligned}$$

Introducing the field $\boldsymbol{G}_{2m-1} = \operatorname{div} \langle 1, \boldsymbol{V} \otimes \boldsymbol{V} g_{2m-1} \rangle$, the result follows from (22). ∎

To determine $\boldsymbol{u}_{2m-1}$ for $m \geq 1$ from (23), appropriate initial conditions are needed. In view of (5), we take

$$\boldsymbol{u}_1(0, \boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}), \qquad \boldsymbol{u}_{2m-1}(0, \boldsymbol{x}) = \boldsymbol{0}, \quad m = 2, 3, \cdots. \tag{24}$$

To determine ρ_{2m} for $m \geq 1$ from (23) and thus the expansion, we recall (21) and the periodicity of the data and impose

$$\int_\Omega \rho_{2m}(t, \boldsymbol{x})\, d\boldsymbol{x} = 0. \tag{25}$$

In addition, we remark that (23) is essentially an Oseen problem for the modified velocity field

$$\tilde{\boldsymbol{u}}_{2m-1} = \boldsymbol{u}_{2m-1} - \nabla\Phi, \qquad \Delta\Phi = -\partial_t \rho_{2m-2}$$

which satisfies the incompressibility condition $\operatorname{div} \tilde{\boldsymbol{u}}_{2m-1} = 0$.
By the above formal process, the expansion f_ϵ can be constructed completely. However, we do not know how to show the convergence of the expansion. Instead, we are interested in truncated expansions of the form

$$f_\epsilon^r = f^* + \epsilon f_1 + \cdots + \epsilon^{r+1} f_{r+1}$$

with r a positive integer and f_k defined by (20) (setting $f_0 = f^*$, $f_p = 0$ for $p < 0$), where the moments $(\boldsymbol{u}_k, \rho_{k+1})$ are either set to zero (if k is even) or taken as solution of equation (23) with (24)-(25) (see the appendix for existence and uniqueness results). Inserting the truncated expansion into (4), we find in order ϵ^k the left-hand side of (19) where now $f_p = 0$ for $p > r+1$. By construction of f_k, this expression vanishes exactly as long as $k + 2 \leq r + 1$. Thus, f_ϵ^r satisfies

$$\frac{\partial f_\epsilon^r}{\partial t} + \frac{1}{\epsilon} \boldsymbol{V} \cdot \nabla f_\epsilon^r - \frac{1}{\epsilon^2} J(f_\epsilon^r) = \epsilon^r \hat{R}_r \tag{26}$$

with

$$\begin{aligned} \hat{R}_r = {} & \frac{\partial f_r}{\partial t} + \boldsymbol{V} \cdot \nabla f_{r+1} - \sum_{p+q=r+2} \mathcal{A} f_{quad}^{eq}(f_p, f_q) \\ & + \epsilon \left(\frac{\partial f_{r+1}}{\partial t} - \sum_{p+q=r+3} \mathcal{A} f_{quad}^{eq}(f_p, f_q) \right) - \sum_{k=r+2}^{2r} \epsilon^{k-r} \sum_{p+q=k+2} \mathcal{A} f_{quad}^{eq}(f_p, f_q). \end{aligned}$$

The averages $\boldsymbol{u}_\epsilon^r = \langle 1, \boldsymbol{V} f_\epsilon^r \rangle$ and $\rho_\epsilon^r = \langle 1, f_\epsilon^r \rangle$ have expressions (because of even/odd symmetry of the coefficients f_k)

$$\boldsymbol{u}_\epsilon^r = \epsilon \boldsymbol{u}_1 + \epsilon^3 \boldsymbol{u}_3 + \ldots, \qquad \rho_\epsilon^r = 1 + \epsilon^2 \rho_2 + \epsilon^4 \rho_4 + \ldots.$$

Note that, in view of (23) with $m = 1$, $(\boldsymbol{u}_1, \rho_2/3)$ is the solution of the Navier-Stokes equation (3).
To investigate the regularity of the truncated expansion, we introduce some notation related to the Sobolev spaces H^s with s a non-negative integer. $L^2 = H^0$ is the space of square integrable ($\mathcal{F}$- or $\mathcal{F}^2$-valued) functions on the unit torus Ω. Its norm is denoted by $\|\cdot\|$. For $s > 0$, H^s is defined as the space of functions which are in L^2 together with their distributional $\boldsymbol{x}$-derivatives of order $\leq s$. We use $\|\cdot\|_s$ to denote the norm. In addition, we use $C(0,T;H^s)$, $AC(0,T;H^s)$ and $L^1(0,T;H^s)$ to denote the Banach spaces of H^s-valued continuous, absolutely continuous, and L^1-integrable functions on the time interval $[0,T]$, respectively.
For simplicity, we consider only the case where $r = 3$. Furthermore, we will often use the following well-known fact (see, e.g., [14]).

Lemma 3.2
Let s_1, s_2 be two non-negative integers and $s_3 = \min\{s_1, s_2, s_1 + s_2 - \sigma_d\} \geq 0$ where $\sigma_d = \lfloor d/2 \rfloor + 1 = 2$ for our two-dimensional case $d = 2$. Then the product of functions from H^{s_1} and H^{s_2} is in H^{s_3}, i.e.

$$H^{s_1} H^{s_2} \subset H^{s_3}.$$

where the inclusion symbol $\subset$ indicates the continuity of the embedding.

Lemma 3.3
Assume $s \geq \sigma_d$ and $\bar{\boldsymbol{u}} \in H^{s+5}$ with $\operatorname{div} \bar{\boldsymbol{u}} = 0$. Then there exists a finite $T_* > 0$, depending only on $\|\bar{\boldsymbol{u}}\|_{s+5}$, such that $f_\epsilon^3 \in C(0, T_*; H^s), \|\boldsymbol{u}_\epsilon^3\|_s = O(\epsilon)$ and $\hat{R}_3 \in L^1(0, T_*; H^s)$ with $\int_0^{T_*} \|\hat{R}_3(t)\|_s dt = O(1)$ as $\epsilon \to 0$.

Proof: Since $f_\epsilon^3 = \sum_{k=0}^4 \epsilon^k f_k, \boldsymbol{u}_\epsilon^3 = \langle 1, \boldsymbol{V} f_\epsilon^3 \rangle$ and

$$\hat{R}_3 = \frac{\partial f_3}{\partial t} + \boldsymbol{V} \cdot \nabla f_4 + \epsilon \left(\frac{\partial f_4}{\partial t} - \mathcal{A}^\dagger f_{quad}^{eq}(f_3, f_3) \right),$$

it suffices to show that there exists $T_* > 0$, depending only on $\|\bar{\boldsymbol{u}}\|_{s+5}$, such that

$$\begin{gathered} f_1, f_2 \in C(0, T_*; H^s), \qquad f_3 \in AC(0, T_*; H^s), \\ f_4 \in AC(0, T_*; H^s) \cap L^1(0, T_*; H^{s+1}). \end{gathered} \tag{27}$$

Note that $f_0 = f^*$ is independent of $(t, \boldsymbol{x})$ and thereby in $C(0, T_*; H^s)$. In addition, Lemma 3.2 can be used to show that the quadratic term $f_{quad}^{eq}(f_3, f_3)$ is in $C(0, T_*; H^s)$ if so is f_3. To show (27), we consider the equations for $\boldsymbol{u}_1, \rho_2$, and $\boldsymbol{u}_3, \rho_4$.
Denote by Π the orthogonal projection of L^2 onto its closed subspace consisting of all solenoidal vectors. Then the equations for $(\boldsymbol{u}_1, \rho_2)$ can be rewritten as

$$\begin{aligned} \partial_t \boldsymbol{u}_1 + \Pi(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1) &= \nu \Delta \boldsymbol{u}_1, & \Delta \rho_2 &= -3 \operatorname{div}(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1), \\ \boldsymbol{u}_1(0, \boldsymbol{x}) &= \bar{\boldsymbol{u}}(\boldsymbol{x}), & \int_\Omega \rho_2(t, \boldsymbol{x})\, d\boldsymbol{x} &= 0. \end{aligned} \tag{28}$$

Because $\bar{\boldsymbol{u}} \in H^{s+5}$ with $\operatorname{div} \bar{\boldsymbol{u}} = 0$, by Lemma 5.2 we take $T_* = T'$, which depends only on $\|\bar{\boldsymbol{u}}\|_{s+5}$, and have

$$\begin{aligned} \boldsymbol{u}_1 \in& AC(0, T_*; H^{s+4}) \cap C(0, T_*; H^{s+5}) \cap L^1(0, T_*; H^{s+6}), \\ \rho_2 \in& L^1(0, T_*; H^{s+6}). \end{aligned} \tag{29}$$

This implies that $\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1 \in AC(0, T_*; H^{s+3})$, since

$$\|\partial_t (\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1)\|_{s+3} \leq C \|\partial_t \boldsymbol{u}_1\|_{s+3} \|\boldsymbol{u}_1\|_{s+4} + C \|\boldsymbol{u}_1\|_{s+3} \|\partial_t \boldsymbol{u}_1\|_{s+4}$$

due to Lemma 3.2. Thus, from the equations in (28) and the familiar fact $\|\rho_2\|_2 \leq C \|\Delta \rho_2\|$ we see that

$$\rho_2 \in AC(0, T_*; H^{s+4}) \qquad \text{and} \qquad \partial_t \boldsymbol{u}_1 \in AC(0, T_*; H^{s+2}). \tag{30}$$

Similarly, we have $\partial_t(\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1)\in AC(0,T_*;H^{s+1})$ and differentiating the equations in (28) with respect to t gives

$$\partial_t\rho_2\in AC(0,T_*;H^{s+2})\quad\text{and}\quad\partial_t^2\boldsymbol{u}_1\in AC(0,T_*;H^s);\tag{31}$$

moreover, $\partial_t^2(\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1)\in AC(0,T_*;H^{s-1})$ and

$$\partial_t^2\rho_2\in AC(0,T_*;H^s).\tag{32}$$

Now (29) and (30) immediately give

$$\begin{aligned}&f_1=3\boldsymbol{u}_1\cdot\boldsymbol{V}f^*\in AC(0,T_*;H^{s+4})\cap L^1(0,T_*;H^{s+6}),\\&\partial_tf_1\in AC(0,T_*;H^{s+2}).\end{aligned}\tag{33}$$

Recall that $f_2=\rho_2f^*+f_{quad}^{eq}(f_1,f_1)-\mathcal{A}^\dagger(\boldsymbol{V}\cdot\nabla f_1)$. By using Lemma 3.2, it is easy to see from (29) and (30) that

$$\begin{aligned}f_{quad}^{eq}(f_1,f_1)\in&AC(0,T_*;H^{s+4})\cap C(0,T_*;H^{s+5}),\\\partial_tf_{quad}^{eq}(f_1,f_1)\in&AC(0,T_*;H^{s+2}).\end{aligned}$$

Thus it follows from (29), (30), (33) and (31) that

$$\begin{aligned}&f_2\in AC(0,T_*;H^{s+3})\cap L^1(0,T_*;H^{s+5}),\\&\partial_tf_2\in AC(0,T_*;H^{s+1}).\end{aligned}\tag{34}$$

Next we turn to the equations for $\boldsymbol{u}_3$ and ρ_4:

$$\begin{aligned}\operatorname{div}\boldsymbol{u}_3=-\partial_t\rho_2,\qquad&\partial_t\boldsymbol{u}_3+2\operatorname{div}\boldsymbol{u}_3\otimes\boldsymbol{u}_1+\nabla\rho_4/3=\nu\Delta\boldsymbol{u}_3+\boldsymbol{G}_3,\\\boldsymbol{u}_3(0,\boldsymbol{x})=0,\qquad&\int_\Omega\rho_4(t,\boldsymbol{x})\,d\boldsymbol{x}=0.\end{aligned}$$

Let ϕ be such that $\Delta\phi=-\partial_t\rho_2$ and $\int_\Omega\phi(t,\boldsymbol{x})\,d\boldsymbol{x}=0$. It follows from (30)-(32) that

$$\phi\in AC(0,T_*;H^{s+4})\cap L^1(0,T_*;H^{s+6}),\qquad\partial_t\phi\in AC(0,T_*;H^{s+2}).$$

Set $\boldsymbol{w}=\boldsymbol{u}_3-\nabla\phi$ and $p=\rho_4/3+\partial_t\phi+\nu\partial_t\rho_2$. Then we have

$$\begin{aligned}\operatorname{div}\boldsymbol{w}=\operatorname{div}\boldsymbol{u}_3-\Delta\phi=\operatorname{div}\boldsymbol{u}_3+\partial_t\rho_2=0,\\\partial_t\boldsymbol{w}+2\operatorname{div}\boldsymbol{w}\otimes\boldsymbol{u}_1+\nabla p=\nu\Delta\boldsymbol{w}+\boldsymbol{G}_3-2\operatorname{div}(\nabla\phi\otimes\boldsymbol{u}_1),\\\boldsymbol{w}(0,\boldsymbol{x})=-\nabla\phi(0,\boldsymbol{x})\in H^{s+3},\qquad\int_\Omega p(t,\boldsymbol{x})\,d\boldsymbol{x}=0.\end{aligned}\tag{35}$$

This is the Oseen problem with an external force $\boldsymbol{h}=\boldsymbol{G}_3-2\operatorname{div}(\nabla\phi\otimes\boldsymbol{u}_1)$. Recall that $\boldsymbol{G}_3=\operatorname{div}\langle 1,\boldsymbol{V}\otimes\boldsymbol{V}g_3\rangle$ with

$$g_3=-\mathcal{A}^\dagger\left(\frac{\partial f_2}{\partial t}-\boldsymbol{V}\cdot\nabla\mathcal{A}^\dagger\left(\frac{\partial f_1}{\partial t}-\boldsymbol{V}\cdot\nabla f_2\right)\right).$$

We see from (33)-(34) that $\boldsymbol{h} \in AC(0,T_*;H^s) \cap L^1(0,T_*;H^{s+2})$. Thus Lemma 5.1 gives

$$\boldsymbol{w} \in AC(0,T_*;H^{s+1}) \cap L^1(0,T_*;H^{s+3}), \qquad p \in L^1(0,T_*;H^{s+3}).$$

On the other hand, $\boldsymbol{w} \otimes \boldsymbol{u}_1 \in AC(0,T_*;H^{s+1})$ follows from Lemma 3.2 and (29). Taking divergence of (35) gives $\Delta p = \operatorname{div}(\boldsymbol{h} - 2\operatorname{div}\boldsymbol{w} \otimes \boldsymbol{u}_1)$. Thus we also have

$$p \in AC(0,T_*;H^{s+1}).$$

Recall that $\nabla\phi \in AC(0,T_*;H^{s+3}) \cap L^1(0,T_*;H^{s+5})$ and that $\partial_t\phi + \nu\partial_t\rho_2 \in AC(0,T_*;H^{s+2}) \cap L^1(0,T_*;H^{s+4})$. Then we have

$$\boldsymbol{u}_3 = \boldsymbol{w} + \nabla\phi, \quad \rho_4/3 = p - (\partial_t\phi + \nu\partial_t\rho_2) \in AC(0,T_*;H^{s+1}) \cap L^1(0,T_*;H^{s+3}).$$

Together with (33) and (34), this gives

$$\begin{aligned} f_3 &= 3\boldsymbol{u}_3 \cdot \boldsymbol{V} f^* - \mathcal{A}^\dagger\left(\frac{\partial f_1}{\partial t} + \boldsymbol{V} \cdot \nabla f_2\right) \in AC(0,T_*;H^{s+1}) \cap L^1(0,T_*;H^{s+3}), \\ f_4 &= \rho_4 f^* + 2f^{eq}_{quad}(f_1,f_3) \\ &\qquad\qquad -\mathcal{A}^\dagger\left(\frac{\partial f_2}{\partial t} + \boldsymbol{V} \cdot \nabla f_3\right) \in AC(0,T_*;H^s) \cap L^1(0,T_*;H^{s+2}). \end{aligned}$$

Hence (27) is verified. ■

We conclude this section with a more detailed description of f^3_ϵ.

Lemma 3.4
The truncated expansion f^3_ϵ coincides up to terms of order ϵ^3 with the Chapman-Enskog distribution $F_{CE}(p,\boldsymbol{u})$ corresponding to the solution $(\boldsymbol{u},p)$ of the Navier Stokes equation (3),

$$F_{CE}(p,\boldsymbol{u}) = F^{eq}_1(1,\epsilon\boldsymbol{u}) + \epsilon^2\left(3p - \frac{9}{2}\nu S[\boldsymbol{u}] : (\boldsymbol{V} \otimes \boldsymbol{V} - |\boldsymbol{V}|^2/2)\right) f^*,$$

where $S_{ij}[\boldsymbol{u}] = \partial_{x_j}u_i + \partial_{x_i}u_j$ is the viscous stress tensor and $:$ denotes the matrix scalar product $A : B = \sum_{ij} A_{ij}B_{ij}$.

Proof: According to our construction, $f^3_\epsilon = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \mathcal{O}(\epsilon^3)$ with

$$f_0 = f^*, \quad f_1 = 3\boldsymbol{u} \cdot \boldsymbol{V} f^*, \quad f_2 = 3pf^* + f^{eq}_{quad}(f_1,f_1) - \mathcal{A}^\dagger(\boldsymbol{V} \cdot \nabla f_1).$$

Since $f_0 + \epsilon f_1 = F^{eq}_{lin}(1,\epsilon\boldsymbol{u})$ and $\epsilon^2 f^{eq}_{quad}(f_1,f_1) = F^{eq}_{quad}(\epsilon\boldsymbol{u},\epsilon\boldsymbol{u})$, we thus have

$$f^3_\epsilon = F^{eq}_1(1,\epsilon\boldsymbol{u}) + \epsilon^2(3pf^* - \mathcal{A}^\dagger(\boldsymbol{V} \cdot \nabla f_1) + \mathcal{O}(\epsilon^3).$$

An explicit calculation of $\mathcal{A}^\dagger(\boldsymbol{V} \cdot \nabla f_1)$ yields

$$\mathcal{A}^\dagger(\boldsymbol{V} \cdot \nabla f_1) = \nabla\boldsymbol{u} : \mathcal{A}^\dagger(3\boldsymbol{V} \otimes \boldsymbol{V} f^*) = 9\nu\nabla\boldsymbol{u} : (\boldsymbol{V} \otimes \boldsymbol{V} - |\boldsymbol{V}|^2/2)f^*.$$

Since $\boldsymbol{v} \otimes \boldsymbol{v} - |\boldsymbol{v}|^2/2$ is a symmetric matrix, we can replace the Jacobian $\nabla\boldsymbol{u}$ also by its symmetric part $(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)/2 = S[\boldsymbol{u}]/2$ without changing the matrix scalar product. This completes the proof. ■

4 Justification of formal approximations

Having constructed formal asymptotic approximations f_ϵ^r for initial-value problems of (4), we prove in this section the validity of the approximations. The main result is

Theorem 4.1
Let $s \geq \sigma_d = 2$ be an integer, $\bar{\boldsymbol{u}} \in H^{s+5}$ with $\operatorname{div} \bar{\boldsymbol{u}} = 0$, and $\bar{p}$ the solution of $\Delta p = -\operatorname{div}(\bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}})$, $\int_\Omega p \, d\boldsymbol{x} = 0$. Then the lattice Boltzmann model (4), with $\mathcal{A}$ of the form (6), $\lambda_4 = \lambda_5 = 1/(3\nu)$, $\lambda_6 = \lambda_9$, and initial data

$$f^\epsilon|_{t=0} = F_1^{eq}(1, \epsilon\bar{\boldsymbol{u}}) + \epsilon^2 \left(3\bar{p} - \frac{9}{2}\nu S[\bar{\boldsymbol{u}}] : (\boldsymbol{V} \otimes \boldsymbol{V} - |\boldsymbol{V}|^2/2)\right) f^*$$

has a unique solution $f^\epsilon \in C(0, T_*; H^s)$. Moreover, there exists $K > 0$ such that for sufficiently small $\epsilon > 0$

$$\|f_\epsilon^3(t) - f^\epsilon(t)\|_s \leq K\epsilon^3, \qquad t \in [0, T_*].$$

In particular, the velocity field $\boldsymbol{u}_\epsilon/\epsilon = \langle 1, \boldsymbol{V} f^\epsilon \rangle / \epsilon$ coincides with the solution $\boldsymbol{u}$ of the Navier-Stokes equation (3) up to order ϵ^2 and $(\langle 1, f^\epsilon \rangle - 1)/(3\epsilon^2)$ with the pressure p up to $O(\epsilon)$:

$$\|\boldsymbol{u} - \boldsymbol{u}_\epsilon/\epsilon\|_s \leq K\epsilon^2, \qquad \|p - (\langle 1, f^\epsilon \rangle - 1)/(3\epsilon^2)\|_s \leq K\epsilon.$$

The proof of Theorem 4.1.
Since $f^\epsilon(0, \cdot) \in H^s$ with $s > d/2 = 1$, by the local existence theory for IVPs of symmetrizable hyperbolic systems (see [14]), there is a time interval $[0, T]$ so that (4) has a unique H^s-solution

$$f^\epsilon \in C([0, T], H^s).$$

Define

$$T_\epsilon = \sup\left\{T > 0 : f^\epsilon \in C([0, T], H^s)\right\}. \tag{36}$$

Namely, $[0, T_\epsilon)$ is the maximal time interval of H^s existence. Thanks to the *convergence-stability lemma* in [21, 22], it suffice to prove the error estimate for $t \in [0, \min\{T_*, T_\epsilon\})$. Indeed, once the estimate is proved, the lemma can be used to show $T_\epsilon > T_*$.
To this end, we compute from equations (4) and (26) that the error $E = f_\epsilon^3 - f^\epsilon$ satisfies

$$\frac{\partial E}{\partial t} + \frac{1}{\epsilon}\boldsymbol{V} \cdot \nabla E = \frac{J(f_\epsilon^3) - J(f^\epsilon)}{\epsilon^2} + \epsilon^3 \hat{R}_3.$$

We differentiate this equation with ∇^α (in $\boldsymbol{x}$) for a multi-index α satisfying $|\alpha| \leq s$ to get with $E_\alpha = \nabla^\alpha E$

$$\frac{\partial E_\alpha}{\partial t} + \frac{1}{\epsilon}\boldsymbol{V} \cdot \nabla E_\alpha = \frac{1}{\epsilon^2} J_0 E_\alpha + F_\alpha + H_\alpha, \tag{37}$$

where

$$F_\alpha = \frac{1}{\epsilon^2}\nabla^\alpha \left(J(f^3_\epsilon) - J(f^\epsilon) - J_0 E\right), \qquad H_\alpha = \epsilon^3 \nabla^\alpha \hat{R}_3$$

For the sake of clarity, we divide the following arguments into lemmas.

Lemma 4.2
Under the conditions of Theorem 4.1, we have

$$\frac{d}{dt}\int_\Omega \langle B_0 E_\alpha, E_\alpha\rangle \, d\boldsymbol{x} + C\frac{\|P_{II}E_\alpha\|^2}{\epsilon^2} \le C\epsilon^3 \|E_\alpha\| \|\nabla^\alpha \hat{R}_3\| + C\epsilon^2\|F_\alpha\|^2.$$

Here $P_{II} = \sum_{k=4}^9 P_k$, and C denotes a generic constant.

Proof: Applying B_0 to equation (37) as well as $\langle \cdot, E_\alpha\rangle$ we find (using the fact that B_0 and B_0V_j are multiplication operators and thus self-adjoint)

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\langle B_0 E_\alpha, E_\alpha\rangle + \frac{1}{2\epsilon}\sum_{j=1}^d \frac{\partial}{\partial x_j}\langle B_0 V_j E_\alpha, E_\alpha\rangle \\ =&\frac{1}{\epsilon^2}\langle B_0 J_0 E_\alpha, E_\alpha\rangle + \langle B_0 F_\alpha, E_\alpha\rangle + \langle B_0 H_\alpha, E_\alpha\rangle .\end{aligned} \tag{38}$$

Thanks to relations (16) and the fact that $\lambda_1, \lambda_2, \lambda_3 = 0$, it follows that

$$\langle B_0 J_0 E_\alpha, E_\alpha\rangle = -\sum_{k=4}^9 \lambda_k \langle P_k E_\alpha, P_k E_\alpha\rangle \le -\lambda_{min}\langle P_{II}E_\alpha, P_{II}E_\alpha\rangle$$

where $\lambda_{min} = \min\{\lambda_k : k = 4,\dots,9\}$. Setting $P_I = \sum_{k=1}^3 P_k$, (17) implies $P_I F_\alpha \equiv 0$. Thanks to (16), we have

$$\langle B_0 F_\alpha, E_\alpha\rangle = \sum_{k=4}^9 \langle P_k F_\alpha, P_k E_\alpha\rangle \le \frac{\lambda_{min}}{2}\frac{\langle P_{II}E_\alpha, P_{II}E_\alpha\rangle}{\epsilon^2} + C\epsilon^2\langle F_\alpha, F_\alpha\rangle . \tag{39}$$

Finally,

$$\langle B_0 H_\alpha, E_\alpha\rangle \le C|E_\alpha||H_\alpha|.$$

Thus, integrating (38) with respect to $\boldsymbol{x}$ over Ω, the result follows. ■

The next Lemma is used to estimate F_α.

Lemma 4.3
Set $\triangle(t) = \|E(t)\|_s/\epsilon = \|f^3_\epsilon(t) - f^\epsilon(t)\|_s/\epsilon$. Then we have under the conditions of Theorem 4.1 for $|\alpha| \le s$

$$\epsilon\|F_\alpha(t)\| \le C(1+\triangle(t))\|E(t)\|_s. \tag{40}$$

Proof: Observe

$$J(f_\epsilon^3) - J(f^\epsilon) - J_0 E = \int_0^1 \big(DJ(f(\theta)) - J_0\big) E \, d\theta$$

with $f(\theta) = f_\epsilon^3 + (1-\theta)(f^\epsilon - f_\epsilon^3)$. From (15), the definition of F_{quad}^{eq}, and Lemma 3.2, it follows that

$$\|(DJ(f(\theta)) - J_0)E\|_s \le C\|\boldsymbol{u}(\theta)\|_s \|E\|_s, \qquad \boldsymbol{u}(\theta) = \langle 1, \boldsymbol{V} f(\theta)\rangle$$

Since $\|\boldsymbol{u}(\theta)\|_s \le \|\boldsymbol{u}_\epsilon^3\|_s + \|\boldsymbol{u}^\epsilon - \boldsymbol{u}_\epsilon^3\|_s \le C\epsilon + C\epsilon\triangle$, we obtain

$$\|\nabla^\alpha \left(J(f_\epsilon^3) - J(f^\epsilon) - J_0 E\right)\| \le C\int_0^1 \|\boldsymbol{u}(\theta)\|_s \, d\theta \, \|E\|_s \le C\epsilon(1+\triangle)\|E\|_s.$$

Hence $\|F_\alpha\| \le \frac{C(1+\triangle)}{\epsilon}\|E\|_s$ and (40) follows. ∎

Substituting (40) into the inequality in Lemma 4.2 yields

$$\frac{d}{dt}\int_\Omega \langle B_0 E_\alpha, E_\alpha\rangle \, d\boldsymbol{x} \le C\epsilon^3 \|E_\alpha\| \|\nabla^\alpha \hat{R}_3\| + C(1+\triangle^2)\|E\|_s^2. \tag{41}$$

Note that $C^{-1}\langle E_\alpha, E_\alpha\rangle \le \langle B_0 E_\alpha, E_\alpha\rangle \le C\langle E_\alpha, E_\alpha\rangle$. We integrate (41) from 0 to T with $[0,T] \subset [0, \min\{T_\epsilon, T_*\})$ to obtain

$$\|E_\alpha(T)\|^2 \le C\epsilon^6 + C\epsilon^3 \int_0^T \|E_\alpha(t)\| \|\nabla^\alpha \hat{R}_3(t)\| dt + C\int_0^T \left(1+\triangle^2\right)\|E(t)\|_s^2 dt.$$

Here we have used $\|E(0)\|_s = O(\epsilon^3)$. Summing up this inequality for the multi-index α with $|\alpha| \le s$, we get

$$\|E(T)\|_s^2 \le C\epsilon^6 + C\epsilon^3 \int_0^T \|E(t)\|_s \|\hat{R}_3(t)\|_s dt + C\int_0^T \left(1+\triangle^2\right)\|E(t)\|_s^2 dt.$$

Denote by $F(T)$ the *square root* of the right-hand side of the last inequality. We have $\|E(t)\|_s \le F(t), F(0) = O(\epsilon^3)$ and

$$F(t)F'(t) = C\epsilon^3 \|E(t)\|_s \|\hat{R}_3(t)\|_s + C(1+\triangle^2)\|E(t)\|_s^2.$$

Moreover, we have

$$F'(t) \le C\epsilon^3 \|\hat{R}_3(t)\|_s + C(1+\triangle^2)F(t). \tag{42}$$

Recall that $\int_0^{T_*} \|\hat{R}_3(t)\|_s dt = O(1)$. We apply Gronwall's lemma to (42) to obtain

$$F(T) \le C\epsilon^3 \exp\left[C\int_0^T \left(1+\triangle^2\right) dt\right]. \tag{43}$$

Since $F \geq \|E\|_s = \epsilon\triangle$, it follows from (43) that

$$\triangle(T) \leq C\epsilon^2 \exp\left[C\int_0^T \left(1+\triangle^2\right) dt\right] \equiv \Phi(T). \tag{44}$$

Thus,

$$\Phi'(t) = C(1+\triangle^2)\Phi(t) \leq C\Phi(t) + C\Phi^3(t).$$

Applying the nonlinear Gronwall-type inequality in [20] to the last inequality yields

$$\Phi(t) \leq \exp(CT_*),$$

for $t \in [0, \min\{T_\epsilon, T_*\})$ if we choose ϵ so small that

$$\Phi(0) = C\epsilon^2 \leq e^{-CT_*}.$$

Because of (44), there exists a constant c, independent of ϵ, such that

$$\triangle(T) \leq c \tag{45}$$

for any $T \in [0, \min\{T_\epsilon, T_*\})$. Finally, the theorem is concluded from (43) with (45) and $\|E\|_s \leq F$. This completes the proof of Theorem 4.1.

5 Appendix

Here we slightly modify a proof in [13] to formulate an existence theorem for the Oseen problem

$$\begin{aligned} \operatorname{div} \boldsymbol{u} = 0, \qquad & \frac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1 + \nabla p = \nu\Delta\boldsymbol{u} + \boldsymbol{h}, \\ \boldsymbol{u}(0,\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}), \qquad & \int_\Omega p(t,\boldsymbol{x})\, d\boldsymbol{x} = 0. \end{aligned} \tag{46}$$

in a *periodic* domain. Here $\boldsymbol{u}_1, \boldsymbol{h}, \bar{\boldsymbol{u}}$ are given functions which are periodic in $\boldsymbol{x}$, and $\operatorname{div} \bar{\boldsymbol{u}} = 0$.

Lemma 5.1
Let $m \geq 0$ be an integer and $T > 0$ a real number. Assume $\boldsymbol{u}_1 \in C(0,T;H^{s+1})$ with $s \geq \max\{m, \sigma_d\}$, $\boldsymbol{h} \in L^1(0,T;H^m)$ and $\bar{\boldsymbol{u}} \in H^m$ with $\operatorname{div} \bar{\boldsymbol{u}} = 0$. Then the Oseen problem (46) has a unique solution $(\boldsymbol{u}, p)$ satisfying

$$\begin{aligned} \boldsymbol{u} \in & AC(0,T;H^{m-1}) \cap C(0,T;H^m) \cap L^1(0,T;H^{m+1}), \\ p \in & L^1(0,T;H^{m+1}). \end{aligned}$$

Proof: Denote by H^m_σ the closed subspace of H^m consisting of all solenoidal vectors. We decouple (46) as

$$\begin{aligned} \frac{d\boldsymbol{u}}{dt} + A\boldsymbol{u} = \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1) + \Pi\boldsymbol{h}(t), \qquad & \boldsymbol{u}(0,\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}), \\ \Delta p = \operatorname{div}(\boldsymbol{h} - \operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1), \qquad & \int_\Omega p(t,\boldsymbol{x})\, d\boldsymbol{x} = 0. \end{aligned} \tag{47}$$

Here $A = -\nu\Delta$ is a nonnegative self-adjoint operator in H_σ; Π is the orthogonal projection of H^m onto H^m_σ; and $\boldsymbol{F}$ is a bilinear operator

$$\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1) = -\Pi \operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1.$$

We firstly show the existence of $\boldsymbol{u}$. According to [13], it suffices to construct a solution to the integral equation

$$\boldsymbol{u}(t) \equiv G\boldsymbol{u}(t) = e^{-tA}\bar{\boldsymbol{u}} + \int_0^t e^{-(t-s)A}[\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1) + \Pi\boldsymbol{h}]ds. \tag{48}$$

To this end we shall use the method of contraction map.
For simplicity we write $X_m = C(0, T'; H^m_\sigma), Y_m = L^1(0, T'; H^m_\sigma)$ and set $Z = X_m \cap Y_{m+1}$. Here $T' > 0$ is to be determined later. For the norm in Z we choose

$$\|\boldsymbol{w}\|_Z = \max\{\|\boldsymbol{w}\|_{X_m}, L^{-1}\|\boldsymbol{w}\|_{Y_{m+1}}\},$$

where $L > 0$ is also to be determined later.
Since Π has norm one in any H^m, we use Lemma 3.2 to obtain

$$\begin{aligned}\|\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1)\|_m \leq& \|\operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1\|_m\\ \leq& C(\|\boldsymbol{u}\|_m\|\nabla\boldsymbol{u}_1\|_s + \|\boldsymbol{u}_1\|_s\|\nabla\boldsymbol{u}\|_m)\\ \leq& C\|\boldsymbol{u}\|_{m+1}\|\boldsymbol{u}_1\|_{s+1} = C\|\boldsymbol{u}\|_{m+1}.\end{aligned}$$

Thus, we have

$$\|\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1)\|_{Y_m} \leq C\|\boldsymbol{u}\|_{Y_{m+1}}. \tag{49}$$

Next we compute $G\boldsymbol{u} - G\boldsymbol{w}$ for $\boldsymbol{u}, \boldsymbol{w} \in Z$. We have

$$G\boldsymbol{u}(t) - G\boldsymbol{w}(t) = \int_0^t e^{-(t-s)A}F(\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{u}_1)ds.$$

Since e^{-tA} has norm one as an operator in H^m, we conclude

$$\|G\boldsymbol{u} - G\boldsymbol{w}\|_{X_m} \leq \|\boldsymbol{F}(\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{u}_1)\|_{Y_m} \leq C\|\boldsymbol{u} - \boldsymbol{w}\|_{Y_{m+1}}. \tag{50}$$

Since e^{-tA} has norm $(\pi t)^{-1/2}$ as an operator from H^m to H^{m+1}, it follows that $\|G\boldsymbol{u} - G\boldsymbol{w}\|_{m+1}$ is majorized by the convolution of $\|\boldsymbol{F}(\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{u}_1)\|_m$ and $(\pi t)^{-1/2}$. Hence

$$\|G\boldsymbol{u} - G\boldsymbol{w}\|_{Y_{m+1}} \leq 2(T'/\pi)^{1/2}C\|\boldsymbol{u} - \boldsymbol{w}\|_{Y_{m+1}}. \tag{51}$$

We now take $L = 2(T'/\pi)^{1/2}$. Recalling the definition of $\|\cdot\|_Z$ and comparing (50) and (51), we thus obtain

$$\|G\boldsymbol{u} - G\boldsymbol{w}\|_Z \leq CL\|\boldsymbol{u} - \boldsymbol{w}\|_Z. \tag{52}$$

A similar (and simpler) computation gives

$$\|G\boldsymbol{0}\|_Z \leq B \equiv \|\bar{\boldsymbol{u}}\|_m + \int_0^{T'} \|\boldsymbol{h}(t)\|_m dt.$$

These results show that G maps Z into itself. Moreover, if T' is sufficiently small, we have $CL < 1$.
With such choices of T' and L, G maps Z into Z. At the same time, we see from (52) that G is a strict contraction map on Z. Therefore G has a unique fixed point $\boldsymbol{u}$ in Z, which is a local solution of the integral equation (48).
Since T' depends only on $\|\boldsymbol{u}_1\|_{s+1}$, the solution can be directly extended to $[0, T]$. Moreover, from the equation in (47) and the estimate in (49) we see that $\boldsymbol{u} \in AC(0, T; H^{m-1})$.
Finally, we turn to the Poisson equation in (47) for p. Since $\|\nabla p\|$ is a norm equivalent to $\|p\|_1$ in the closed subspace $S \subset H^1$:

$$S := \left\{ p \in H^1 : \int_\Omega p(\boldsymbol{x})\, d\boldsymbol{x} = 0 \right\},$$

the Poisson equation has a unique solution $p \in S$ satisfying

$$\|p\|_{m+1} \le C\| \operatorname{div} (\boldsymbol{h} - \operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1)\|_{m-1} \le C(\|\boldsymbol{h}\|_m + \|\boldsymbol{u}_1\|_{s+1}\|\boldsymbol{u}\|_{m+1}).$$

Therefore p is in $L^1(0, T; H^{m+1})$. ■

Combining the above argument for the Poisson equation and the proof in [13], we are in a position to state the following existence theorem for the Navier-Stokes equation

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \frac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u} + \nabla p = \nu \Delta \boldsymbol{u} + \boldsymbol{h}, \qquad \boldsymbol{u}(0, \boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x})$$

in a *periodic* domain.

Lemma 5.2
Let $m \ge \sigma_d$ be an integer and $T > 0$ a real number. Assume $\boldsymbol{h} \in L^1(0, T; H^m)$ and $\bar{\boldsymbol{u}} \in H^m$ with $\operatorname{div} \bar{\boldsymbol{u}} = 0$. Then there is a $T' \in (0, T]$ such that the Navier-Stokes equation has a unique solution $(\boldsymbol{u}, p)$ satisfying

$$\begin{aligned} \boldsymbol{u} \in & AC(0, T'; H_\sigma^{m-1}) \cap C(0, T'; H_\sigma^m) \cap L^1(0, T'; H_\sigma^{m+1}), \\ p \in & L^1(0, T'; H^{m+1}). \end{aligned}$$

We mention that, although there are global-in-time existence theories for 2-dimensional incompressible Navier-Stokes equation, we do not know a reference for H^m solutions. The latter matches the local H^m existence theory for symmetrizable hyperbolic systems. On the other hand, the above result is appropriate for our purpose and a stronger one is not in the interest of this paper.

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