Multiscale Deformation Analysis by Cauchy-Navier Wavelets

M.K.Abeyratne, W. Freeden[1], C. Mayer

University of Kaiserslautern
Geomathematics Group
67653 Kaiserslautern
P.O. Box 3049
Germany

phone:	++49 631 205-2852
fax:	++49 631 205-4736
email:	abeyratn@mathematik.uni-kl.de
	freeden@mathematik.uni-kl.de
	cmayer@mathematik.uni-kl.de
www:	http://www.mathematik.uni-kl.de/~wwwgeo

Abstract

A geoscientifically relevant wavelet approach is established for the classical (inner) displacement problem corresponding to a regular surface (such as sphere, ellipsoid, actual earth's surface). Basic tools are the limit and jump relations of (linear) elastostatics. Scaling functions and wavelets are formulated within the framework of the vectorial Cauchy–Navier equation. Based on appropriate numerical integration rules a pyramid scheme is developed providing fast wavelet transform (FWT). Finally multiscale deformation analysis is investigated numerically for the case of a spherical boundary.

Key words. Cauchy-Navier equation, (inner) displacement problem, limit and jump relations, Cauchy-Navier scaling function and wavelet.

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[1]Author to whom correspondence should be addressed.

1 Introduction

First we recapitulate some results known from the theory of elasticity: we will always regard the inner space Σ_{int} of a closed surface Σ as a fixed reference configuration of a body. By a *deformation* of $\overline{\Sigma_{int}}$ we mean a one-to-one c^1-function $z : \overline{\Sigma_{int}} \to \mathbb{R}^3$ such that $det(\nabla \otimes z) > 0$. The function $u : \overline{\Sigma_{int}} \to \mathbb{R}^3$, defined by $u(x) = z(x) - x, \;\; x \in \overline{\Sigma_{int}}$, is called the *displacement* of $\overline{\Sigma_{int}}$ relative to the deformation z. The tensor field $(\nabla \otimes u)(x)$ is called the *displacement gradient*. The *(infinitesimal) strain tensor* is defined by $\mathbf{e} = \mathbf{\frac{1}{2}}((\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^{\mathbf{T}})$ as the symmetric part of the *displacement gradient*, while the antisymmetric part is used to define the *(infinitesimal) rotation tensor* $\mathbf{d}$ as $\mathbf{d} = \frac{1}{2}((\nabla \otimes u) - (\nabla \otimes u)^T)$. While $\mathbf{d}$ describes a 'rigid' displacement field, $\mathbf{e}$ is responsible for the 'non-rigid' displacements. According to Kirchhoff's Theorem (see e.g., [21]) if two displacement fields u and u' correspond to the same strain field, then $u = u' + w$ where w is a rigid displacement field. One calls $trace(\mathbf{e}) = \nabla \cdot \mathbf{u}$ the (elastic) *dilatation*. Thus the dilatation are determined by the diagonal elements of $\mathbf{e}$, the remaining elements of $\mathbf{e}$ prescribe torsions. Every displacement field can be decomposed into a pure torsion (i.e. $\nabla \cdot u = 0$) and a pure dilatation (i.e. $\nabla \wedge u = 0$).

An elastic body in a strained configuration performs by definition a tendency of recovering its original form: this tendency is materialized by a field of forces on each part of the body by the other parts. This field of internal forces called (elastic) *stress*, is due to the interaction of the molecules of the body which have been removed from their relative position of equilibrium and to recover it, following the principle of action and interaction. If x is a point of a (regular) surface element in Σ_{int} with unit normal ν, then the *stress vector* $s_\nu(x) = T_\nu(u)(x)$ is the force per unit area at x exerted by the portion of Σ_{int} on the side of the surface element in Σ_{int} towards $\nu(x)$ on the portion of Σ_{int} on the other side. For time-independent behavior and in the absence of body stress fields there exists a symmetric tensor field $\mathbf{s}$, called *stress tensor field*, such that $s_\nu = \mathbf{s}\nu$ for each vector ν and $\nabla(\mathbf{s}a) = 0$ for each fixed $a \in \mathbb{R}^3$ (for more details see e.g., [12],[22]).

Hooke's law relates the stress to strain, i.e. linear elasticity of the body implies that for each $x \in \Sigma_{int}$ there exists a linear transformation $\mathbf{C}$ from the space of all tensors into the space of all symmetric tensors such that $\mathbf{s} = \mathbf{Ce}$. The linear theory of elasticity is based on the *strain-displacement* relation

$$\mathbf{e} = \frac{1}{2}((\nabla \otimes u) + (\nabla \otimes u)^T), \tag{1.1}$$

the *stress-strain* relation

$$\mathbf{s} = \mathbf{Ce} \tag{1.2}$$

and the *equation of equilibrium*

$$div\; \mathbf{s} + b = 0, \tag{1.3}$$

where b is the body force field in Σ_{int}. The above equations imply the *displacement equation of equilibrium* in Σ_{int}

$$div\ \mathbf{C}(\nabla \otimes u) + b = 0. \tag{1.4}$$

For given $\mathbf{C}$ and b, this is a coupled linear system of partial differential equations for the fields u, $\mathbf{e}$ and $\mathbf{s}$. If the material is *isotropic*, $\mathbf{C}$ is given by

$$\mathbf{C}\mathbf{e} = 2\mu\mathbf{e} + \lambda(trace\ \mathbf{e})\mathbf{i}, \tag{1.5}$$

where the scalars λ and μ are called the Lamé moduli. Moreover, if the material is *homogeneous*, λ and μ are constants (typical requirements imposed on λ and μ are $\mu > 0,\ \ 3\lambda + 2\mu > 0$(see, for example, [16])). Therefore, in the homogeneous isotropic case, observing the identities

$$\nabla \cdot (\mu(\nabla \otimes u)) = \mu\Delta u, \quad \nabla \cdot (\mu(\nabla \otimes u)^T) = 0, \quad \nabla \cdot (\lambda(\nabla \cdot u)\mathbf{i}) = \lambda\nabla(\nabla \cdot u), \tag{1.6}$$

we are led to the displacement equation of equilibrium in the form

$$\mu\Delta u + (\lambda + \mu)\nabla\nabla u + b = 0. \tag{1.7}$$

Finally, assuming that the body force field b vanishes, this equation can be reduced to the so-called *Cauchy-Navier equation* in Σ_{int}

$$\mu\Delta u + (\lambda + \mu)\nabla\nabla u = 0. \tag{1.8}$$

This equation plays the same part in the theory of elasticity as the Laplace equation in the theory of harmonic functions and it formally reduces to it for $\mu = 1,\ \lambda = -1$. The Cauchy-Navier equation allows an equivalent formulation in Σ_{int}

$$\Delta u + \sigma\nabla\nabla \cdot u = 0, \tag{1.9}$$

where $\sigma = (1 - 2\rho)^{-1},\ \ \rho = \lambda/2(\lambda + \mu),\ \ \mu \neq 0$. ρ is the Poisson ratio. For simplicity we let

$$\Diamond u = \mu\Delta u + (\lambda + \mu)\nabla\nabla u = 0 \tag{1.10}$$

in Σ_{int}. It is easy to show that the displacement field u is biharmonic and its divergence and curl are harmonic. This yields a deep relation between linear elasticity and potential theory (see e.g., [17]).

The layout of this report on multiscale deformation analysis by Cauchy-Navier wavelets is as follows: after a brief sketch of the theory of linear elasticity given in the Introduction (Chapter 1), we deal with some preliminary concepts of elastic potentials in Chapter 2. In analogy to the classical potential theoretic case we discuss (in Chapter 3) the limit and jump relations within the framework of the Hilbert space of square-integrable vector fields on a regular surface Σ. The uniqueness, existence and regularity of the solution of the displacement boundary-value problem of elastostatics are discussed in Chapter 4. Next (in Chapter 5) a wavelet approach is introduced based on the layer potentials and their operator formulation in the nomenclature of the Hilbert space of square integrable vector fields on the regular boundary Σ. We introduce the so-called (Cauchy-Navier) Σ-scaling functions and wavelets. The wavelet transform and the reconstruction formulae both in continuous and discrete formulations are explicitly written down. The geomathematically relevant (inner) three-dimensional displacement boundary-value problem of elastostatics is treated within the multiscale structure of Cauchy-Navier wavelets. Finally, Chapter 6 is devoted to numerical applications of wavelet approximation on the sphere.

2 Preliminaries

We begin by introducing some preliminaries that will be used throughout this report.

2.1 Notation

$\mathbb{R}^3$ denotes the three-dimensional (real) Euclidean space. We consistently write $x, y, \ldots$ for the elements of $\mathbb{R}^3$. In components we have the representation $x = x_1\epsilon^1 + x_2\epsilon^2 + x_3\epsilon^3$, where the vectors $\epsilon^1, \epsilon^2, \epsilon^3$ form the canonical orthonormal basis in $\mathbb{R}^3$. The inner product, vector product and the tensor product between x and y, respectively, are defined as usual by

$$x \cdot y = x^T y = \sum_{i=1}^{3} x_i y_i, \tag{2.1}$$

$$x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_3 - y_3y_1)^T, \tag{2.2}$$

$$x \otimes y = xy^T = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix}. \tag{2.3}$$

Furthermore, the Euclidean norm of x is denoted by $|x|$, i.e., $|x| = (x \cdot x)^{1/2}$. The unit sphere in $\mathbb{R}^3$ is denoted by Ω. More explicitly, $\Omega = \{\xi \in \mathbb{R}^3 | |\xi| = 1\}$.

2.2 Regular Surfaces

A surface $\Sigma \subset \mathbb{R}^3$ is called *regular* if it satisfies the following properties:

(i) Σ divides $\mathbb{R}^3$ uniquely into the bounded region Σ_{int} (*inner space*) and the unbounded region Σ_{ext} (*outer space*) given by $\Sigma_{ext} = \mathbb{R}^3 \setminus \overline{\Sigma_{int}}$, $\overline{\Sigma_{int}} = \Sigma_{int} \cup \Sigma$,

(ii) Σ is a closed and compact surface free of double points,

(iii) Σ_{int} contains the origin,

(iv) Σ is locally of class $C^{(2)}$.

Given a regular surface then there exist positive constants α and β such that

$$\alpha < \sigma^{\inf} = \inf_{x\in\Sigma} |x| \leq \sup_{x\in\Sigma} |x| = \sigma^{\sup} < \beta. \tag{2.4}$$

Ω_α and Ω_β denote the spheres of radii α and β, respectively. As usual, $\Omega_\beta^{int}, \Omega_\beta^{ext}$ (resp. $\Omega_\alpha^{int}, \Omega_\alpha^{ext}$) denote the inner and outer spaces of Ω_β (resp. Ω_α).

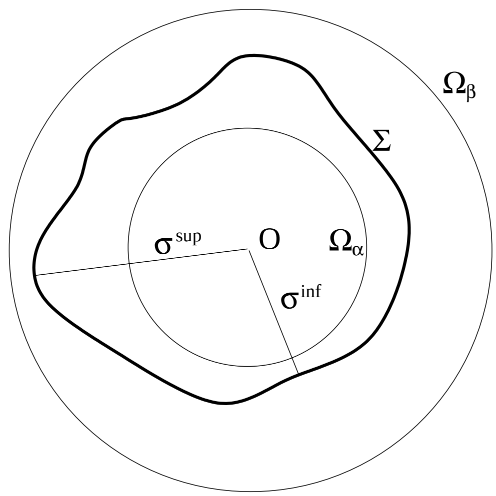


Figure 1: Configuration of a regular surface

A vector field f possessing k continuous derivatives is said to be of class $c^{(k)}$, $0 \leq k \leq \infty$. $c^{(0)}(\Sigma)$ $(= c(\Sigma))$ is the class of continuous vector fields f defined on Σ. The space $c(\Sigma)$ is a complete normed space endowed with the norm $\|f\|_{c(\Sigma)} = \sup_{x\in\Sigma} |f(x)|$. In $c(\Sigma)$ we have the inner product $(\cdot,\cdot)_{\ell^2(\Sigma)}$ corresponding to the norm

$$\|f\|_{\ell^2(\Sigma)} = \left(\int_\Sigma |f(x)|^2 d\omega(x)\right)^{\frac{1}{2}}, \tag{2.5}$$

where $d\omega$ represents the surface element on Σ. Furthermore, for each $f \in c(\Sigma)$, we have the norm-estimate

$$\|f\|_{\ell^2(\Sigma)} \leq \|\Sigma\| \|f\|_{c(\Sigma)}, \quad \|\Sigma\| = \left(\int_\Sigma d\omega\right)^{\frac{1}{2}}. \tag{2.6}$$

By $\ell^2(\Sigma)$ we denote the space of (Lebesgue) square-integrable vector fields on Σ. $\ell^2(\Sigma)$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{\ell^2(\Sigma)}$ and a

Banach space with respect to $\|\cdot\|_{\ell^2(\Sigma)}$. $\ell^2(\Sigma)$ is the completion of $c(\Sigma)$ with respect to the norm $\|\cdot\|_{\ell^2(\Sigma)}$, i.e.

$$\ell^2(\Sigma) = \overline{c(\Sigma)}^{\|\cdot\|_{\ell^2(\Sigma)}}. \tag{2.7}$$

$pot(\Sigma_{int})$ denotes the space of potentials $u \in c^{(2)}(\Sigma)$ satisfying the Cauchy-Navier equation $\Diamond u = \mu\Delta u + (\lambda+\mu)\nabla\nabla u = 0$ in Σ_{int} (with λ, μ being fixed). With $pot(\overline{\Sigma_{int}})$ we denote the space of all vector fields $u : \overline{\Sigma_{int}} \longrightarrow \mathbb{R}^3$ satisfying the properties

(1) $u \in c^{(2)}(\Sigma_{int}) \cap c(\overline{\Sigma_{int}})$,

(2) $u|_{\Sigma_{int}} \in pot(\Sigma_{int})$.

Moreover, $pot(\Sigma_{ext})$ denotes the space of all vector fields $u \in c^{(2)}(\Sigma_{ext})$ satisfying the Cauchy-Navier equation in Σ_{ext} and being regular at infinity, i.e. $|u(x)| = o(1)$.

3 Potential Operators

Elastostatics may be formulated by a vector potential theory which closely parallels classical scalar potential theory. As a matter of fact, the displacement vector corresponds to the scalar harmonic function, whereas the traction vector corresponds to the normal derivative. Well-known integral formulae parallel the Gauss flux theorem, Betti's and Somigliana's formulae parallel Green's formulae. Moreover, vector potentials may be constructed in close orientation to the scalar single- and double-layer potentials. The resulting boundary integral equations show analogous properties to the scalar boundary integral equations. As a consequence the fundamental existence-uniqueness theorems of classical elastostatics can be formulated in analogy to the corresponding theorems of harmonic function theory. For more details the reader is referred to [16], which gives the theoretical treatment of the vector theory. Further theoretical aspects can be found in many books, for example, [12], [15], [20].

At each point x of a regular surface Σ we can construct a normal $\nu(x)$ pointing into the outer space Σ_{ext}. The set

$$\Sigma(\tau) = \{x_\tau \in \mathbb{R}^3 \mid x_\tau = x + \tau\nu(x), \;\; x \in \Sigma\}, \tag{3.1}$$

generates a *parallel surface* which is exterior for $\tau > 0$ and interior for $\tau < 0$. It is known that, if $|\tau|$ is sufficiently small the parallel surface is regular and the normal to one parallel surface is normal to the other.
The matrix $\Gamma(x)$, $x \in \mathbb{R}^3$ with $|x| \neq 0$, given by

$$\Gamma(x) = \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\left(\left(\epsilon^i\cdot\epsilon^k + \frac{(\lambda+\mu)}{\lambda+3\mu}\frac{(x\cdot\epsilon^i)(x\cdot\epsilon^k)}{|x|^2}\right)\frac{1}{|x|}\right)_{i,k=1,2,3} \tag{3.2}$$

is constituted by the so-called fundamental solutions $\Gamma_k(x) = \Gamma(x)\epsilon^k$, $k = 1,2,3$, associated to the operator $\Diamond$ (cf. [17]).

The operator

$$N = \frac{1}{\lambda + 3\mu}\left(2\mu(\lambda + 2\mu)\frac{\partial}{\partial\nu} + (\lambda + \mu)(\lambda + 2\mu)\nu \text{ div} + \mu(\lambda + \mu)\nu \times \text{rot}\right) \tag{3.3}$$

is called the *(pseudo-)stress operator*. Furthermore, $N_x\Gamma_k(x)$, $x \in \mathbb{R}^3$ with $|x| \neq 0$, is given by

$$N_x\Gamma_k(x) = \left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x|}\right)\Lambda_k(x), \tag{3.4}$$

where

$$\Lambda_k(x) = \frac{2\mu}{\lambda + 3\mu}\left(\epsilon^k + \frac{3(\lambda + \mu)}{2\mu}\frac{(\epsilon^k \cdot x)x}{|x|^2}\right), \quad k = 1, 2, 3. \tag{3.5}$$

We let

$$\Lambda(x) = \left(\Lambda_i(x) \cdot \epsilon^k\right)_{i,k} \quad i, k = 1, 2, 3. \tag{3.6}$$

Assuming $|\tau|$ to be sufficiently small we define the so-called potential operators $P(\tau)$, $P_N(\tau)$ and $N_P(\tau)$, respectively, by the following integrals

$$P(\tau)g(x) = \int_\Sigma \Gamma(x_\tau - y)g(y)\ d\omega(y) \tag{3.7}$$

$P(\tau)$: operator of the *single layer potential* on Σ for values on $\Sigma(\tau)$,

$$P_N(\tau)g(x) = \int_\Sigma \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x_\tau - y|}\right)\Lambda(x_\tau - y)g(y)\ d\omega(y) \tag{3.8}$$

$P_N(\tau)$: operator of the *double layer potential* on Σ for values on $\Sigma(\tau)$,

$$\begin{aligned} N_P(\tau)g(x) &= \int_\Sigma \left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x_\tau - y|}\right)\Lambda(x_\tau - y)g(y)\ d\omega(y) \\ &= N_x \int_\Sigma \Gamma(x_\tau - y)g(y)d\omega(y) \end{aligned} \tag{3.9}$$

$N_P(\tau)$: *N-derivative of the single layer potential* on Σ for values on $\Sigma(\tau)$.

The operators $P(\tau)$, $P_N(\tau)$, $N_P(\tau)$ form mappings from $\ell^2(\Sigma)$ into $c(\Sigma)$ provided that $|\tau|$ is sufficiently small. Furthermore, the integrals formally defined by

$$P(0)g(x) = \int_\Sigma \Gamma(x - y)g(y)\ d\omega(y) \tag{3.10}$$

$$P_N(0)g(x) = \int_\Sigma \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x - y|}\right)\Lambda(x - y)g(y)\ d\omega(y) \tag{3.11}$$

$$N_P(0)g(x) = \int_\Sigma \left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x - y|}\right)\Lambda(x - y)g(y)\ d\omega(y) \tag{3.12}$$

exist and define linear bounded operators $P(0), P_N(0), N_P(0)$ mapping $\ell^2(\Sigma)$ into $c(\Sigma)$.

As mentioned before, the potential operators in elastostatics behave near the boundary much like the ordinary harmonic potential operators. In particular, *limit formulae* and *jump relations* can be formulated in close orientation to the potential theoretic case. To be more explicit, let I be the identity operator in $\ell^2(\Sigma)$. For all $\tau > 0$ sufficiently small the operators $L_i^{\pm}(\tau)$ $i = 1,2,3$, and $J_i(\tau)$, $i = 1,2,3,4,5$, are defined by

$$L_1^{\pm}(\tau) = P(\pm\tau) - P(0), \tag{3.13}$$
$$L_2^{\pm}(\tau) = P_N(\pm\tau) - P_N(0) \mp 2\pi I, \tag{3.14}$$
$$L_3^{\pm}(\tau) = N_P(\pm\tau) - N_P(0) \pm 2\pi I, \tag{3.15}$$

$$J_1(\tau) = P(\tau) - P(-\tau), \tag{3.16}$$
$$J_2(\tau) = P_N(\tau) - P_N(-\tau) - 4\pi I, \tag{3.17}$$
$$J_3(\tau) = N_P(\tau) - N_P(-\tau) + 4\pi I, \tag{3.18}$$
$$J_4(\tau) = P_N(\tau) + P_N(-\tau) - 2P_N(0), \tag{3.19}$$
$$J_5(\tau) = N_P(\tau) + N_P(-\tau) - 2N_P(0), \tag{3.20}$$

respectively. Then for all $g \in c(\Sigma)$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \|L_i^{\pm}(\tau)g\|_{c(\Sigma)} = 0 \quad i = 1,2,3, \tag{3.21}$$

and

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \|J_i(\tau)g\|_{c(\Sigma)} = 0, \quad i = 1,2,3,4,5. \tag{3.22}$$

In addition, the adjoint operators with respect to the inner product $(\cdot,\cdot)_{\ell^2(\Sigma)}$ are bounded, linear operators with respect to the norm $\|\cdot\|_{c(\Sigma)}$ (see e.g. [2],[17]).

Theorem 3.1 *For all $g \in \ell^2(\Sigma)$*

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \|L_i^{\pm}(\tau)g\|_{\ell^2(\Sigma)} = 0, \quad i = 1,2,3, \tag{3.23}$$

and

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \|J_i(\tau)g\|_{\ell^2(\Sigma)} = 0 \quad i = 1,2,3,4,5. \tag{3.24}$$

Proof. We use a modification of a technique due to [18]. Denote by $T(\tau)$ one of the operators $L_i^{\pm}(\tau)$ $i = 1,2,3$, $J_i(\tau)$, $i = 1,2,3,4,5$. Let $T^*(\tau)$ be the adjoint operator with respect to the inner product $(\cdot,\cdot)_{\ell^2}(\Sigma)$. According to the Cauchy-Schwarz inequality we find

$$\|T(\tau)g\|^2_{\ell^2(\Sigma)} \leq \|g\|_{\ell^2(\Sigma)}\|T^*(\tau)T(\tau)\|_{\ell^2(\Sigma)}. \tag{3.25}$$

Therefore it follows that

$$\begin{aligned}\|T(\tau)g\|^{2^2}_{\ell^2(\Sigma)} &\leq \|g\|^2_{\ell^2(\Sigma)}\|T^*(\tau)T(\tau)\|^2_{\ell^2(\Sigma)} \\ &\leq \|g\|^2_{\ell^2(\Sigma)}\|g\|_{\ell^2(\Sigma)}\|(T^*(\tau)T(\tau))^2 g\|^2_{\ell^2(\Sigma)}.\end{aligned} \tag{3.26}$$

Induction states that for all $n \geq 2$,

$$\|T(\tau)g\|^{2^n}_{\ell^2(\Sigma)} \leq \|g\|^{2^n-1}_{\ell^2(\Sigma)}\|(T^*(\tau)T(\tau))^{2^{n-1}} g\|_{\ell^2(\Sigma)}. \tag{3.27}$$

Because of the boundedness of the operators $T^*(\tau)$ and $T(\tau)$ with respect to $\|\cdot\|_{c(\Sigma)}$ there exists a positive constant D such that

$$\|T(\tau)g\|^{2^n}_{\ell^2(\Sigma)} \leq \|\Sigma\| D^{2^n}\|g\|^{2^n-1}_{\ell^2(\Sigma)}\|g\|_{c(\Sigma)}, \;\; \|\Sigma\| = (\int_\Sigma d\omega)^{1/2}. \tag{3.28}$$

Thus, for all $n \geq 2$ and $g \in c(\Sigma)$ with $g \neq 0$, we obtain

$$\frac{\|T(\tau)g\|_{\ell^2(\Sigma)}}{\|g\|_{\ell^2(\Sigma)}} \leq D\left(\frac{\|\Sigma\|\|g\|_{c(\Sigma)}}{\|g\|_{\ell^2(\Sigma)}}\right)^{2^{-n}}. \tag{3.29}$$

Letting n tend to infinity we obtain for all $g \neq 0$

$$\lim_{n\to\infty}\left(\frac{\|g\|_{c(\Sigma)}}{\|g\|_{\ell^2(\Sigma)}}\right)^{2^{-n}} = 1. \tag{3.30}$$

This shows that $\|T(\tau)\|_{\ell^2(\Sigma)} \leq D$ for all $g \in c(\Sigma)$. Since $c(\Sigma)$ is a dense linear subspace of $\ell^2(\Sigma)$, we are able to extend the operator $T(\tau)$ from $c(\Sigma)$ to $\ell^2(\Sigma)$ without enlarging its norm (cf. [1],[3],[19]). Therefore, $T(\tau)$ is bounded with respect to $\|\cdot\|_{\ell^2(\Sigma)}$ and we have

$$\|T(\tau)\|_{\ell^2(\Sigma)} \leq \sqrt{\|T(\tau)\|_{c(\Sigma)}\|T^*(\tau)\|_{c(\Sigma)}}. \tag{3.31}$$

Hence it follows that $\|T(\tau)\|_{\ell^2(\Sigma)} \to 0$ as $\tau \to 0,\ \tau > 0$. □

4 Uniqueness, Existence, and Regularity

In the notations given above the homogeneous isotropic elastic displacement boundary-value problem can be formulated as follows: given $f \in c(\Sigma)$ find a vector fields $u \in pot(\overline{\Sigma_{int}})$ satisfying the boundary condition $u|_\Sigma = f$. As it is well-known, the boundary-value problem has a unique solution (see e.g. [16]). In order to prove the existence we use the double layer potential

$$\begin{aligned}u(x) =& P_N(0)g(x) \\ =& \int_\Sigma \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}\right)\Lambda(x-y)g(y)d\omega(y), \;\; g \in c(\Sigma).\end{aligned} \tag{4.1}$$

Observing the discontinuity of the double layer potential we obtain from (3.18)

$$f(x) = -2\pi g(x) + \int_\Sigma \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}\right)\Lambda(x-y)g(y)\ d\omega(y). \tag{4.2}$$

for all $x \in \Sigma$. The resulting integral equation $-f = (2\pi I - P_N(0))g,\ g \in c(\Sigma)$ fulfills all standard Fredholm theorems.

The homogeneous integral equation $(2\pi I - P_N(0))g = 0$ has no solution different from $g = 0$. Thus, the solution of the boundary-value problem exists and is representable by a double layer potential as indicated in (4.1). For details the reader is referred to [17]. The operator $T = 2\pi I - P_N(0)$ and its adjoint operator T^* (with respect to the scalar product $(\cdot,\cdot)_{\ell^2(\Sigma)}$) form mappings from $c(\Sigma)$ into $c(\Sigma)$ which are linear and bounded with respect to the norm $\|\cdot\|_{c(\Sigma)}$. The operators T, T^* in $c(\Sigma)$ are injective and, by the Fredholm alternative, bijective in the Banach space $c(\Sigma)$. Consequently, by the open mapping theorem (see [23]) the operators T^{-1}, T^{*-1} are linear and bounded with respect to $\|\cdot\|_{c(\Sigma)}$. Moreover, $(T^*)^{-1} = (T^{-1})^*$. But this implies that both T^{-1} and $(T^*)^{-1}$ are bounded with respect to the norm $\|\cdot\|_{\ell^2(\Sigma)}$ in $c(\Sigma)$. As we have shown, for a given $f \in c(\Sigma)$, there exists a vector field $g \in c(\Sigma)$ determined by (4.2) such that u is representable in the form (4.1). Suppose that $\mathcal{K}$ is a subset of Σ_{int} with $\mathrm{dist}(\mathcal{K},\Sigma) > 0$. Then Cauchy-Schwarz inequality applied to (4.1) gives for each $x \in \mathcal{K}$

$$|u(x)| \leq \left(\int_\Sigma \sum_{k=1}^{3} \left| \left(\frac{\partial}{\partial\nu(y)} \frac{1}{|x-y|}\right) \Lambda_k(x-y) \right|^2 d\omega(y)\right)^{\frac{1}{2}} \left(\int_\Sigma |g(y)|^2 d\omega(y)\right)^{\frac{1}{2}}. \tag{4.3}$$

But this means that

$$\sup_{x\in\mathcal{K}} |u(x)| \leq E\ \|g\|_{\ell^2(\Sigma)}, \tag{4.4}$$

where

$$E = \sup_{x\in\mathcal{K}} \left(\int_\Sigma \sum_{k=1}^{3} \left| \left(\frac{\partial}{\partial\nu(y)} \frac{1}{|x-y|}\right) \Lambda_k(x-y) \right|^2 d\omega(y)\right)^{\frac{1}{2}}. \tag{4.5}$$

In connection with (4.2) this implies the existence of a positive constant B (depending on Σ and $\mathcal{K}$) such that

$$\sup_{x\in\mathcal{K}} |u(x)| \leq E\|T^{-1}f\|_{\ell^2(\Sigma)} \leq B\|f\|_{\ell^2(\Sigma)}. \tag{4.6}$$

Summarizing our result we obtain the following regularity condition.

THEOREM 4.1 *Let u be a vector field of class pot$(\overline{\Sigma_{int}})$ and $\mathcal{K}$ a subset of Σ_{int} with $(\mathcal{K},\Sigma) > 0$. Then*

$$\sup_{x\in\mathcal{K}} |u(x)| \leq B\left(\int_\Sigma |u(x)|^2\ d\omega(x)\right)^{1/2}. \tag{4.7}$$

5 Cauchy-Navier Wavelets

Next we introduce the so-called scaling and wavelet functions. Essential tools are the layer potentials introduced in Chapter 3.

THEOREM 5.1 *For* $f \in \ell^2(\Sigma)$,

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_\Sigma \mathbf{\Phi}_\tau^{(i)}(x,y) f(y)\, d\omega(y) = \begin{cases} \int_\Sigma \Gamma(x-y) f(y)\, d\omega(y) & ,\ i=1 \\ 0 & ,\ i=4 \\ f(x) & ,\ i=2,3,5,6 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}\right)\Lambda(x-y) f(y)\, d\omega(y) & ,\ i=7 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x-y|}\right)\Lambda(x-y) f(y)\, d\omega(y) & ,\ i=8, \end{cases} \tag{5.1}$$

where

$$\begin{aligned}
\mathbf{\Phi}_\tau^{(1)}(x,y) =&\ \Gamma(x_\tau - y),\\
\mathbf{\Phi}_\tau^{(2)}(x,y) =&\ \frac{1}{2\pi}\left(\left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x_\tau-y|}\right)\Lambda(x_\tau-y) - \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}\right)\Lambda(x-y)\right),\\
\mathbf{\Phi}_\tau^{(3)}(x,y) =&-\frac{1}{2\pi}\left(\left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x_\tau-y|}\right)\Lambda(x_\tau-y) - \left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x-y|}\right)\Lambda(x-y)\right),\\
\mathbf{\Phi}_\tau^{(4)}(x,y) =&\ \Gamma(x_\tau - y) - \Gamma(x_{-\tau}-y),\\
\mathbf{\Phi}_\tau^{(5)}(x,y) =&\ \frac{1}{4\pi}\left(\left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x_\tau-y|}\right)\Lambda(x_\tau-y) - \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x_{-\tau}-y|}\right)\Lambda(x_{-\tau}-y)\right),\\
\mathbf{\Phi}_\tau^{(6)}(x,y) =&-\frac{1}{4\pi}\left(\left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x_\tau-y|}\right)\Lambda(x_\tau-y) - \left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x_{-\tau}-y|}\right)\Lambda(x_{-\tau}-y)\right),\\
\mathbf{\Phi}_\tau^{(7)}(x,y) =&\frac{1}{2}\left(\left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x_\tau-y|}\right)\Lambda(x_\tau-y) + \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x_{-\tau}-y|}\right)\Lambda(x_{-\tau}-y)\right),\\
\mathbf{\Phi}_\tau^{(8)}(x,y) =&\frac{1}{2}\left(\left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x_\tau-y|}\right)\Lambda(x_\tau-y) + \left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x_{-\tau}-y|}\right)\Lambda(x_{-\tau}-y)\right),
\end{aligned}$$

$\tau > 0,\ \ (x,y) \in \Sigma\times\Sigma.$

5.1 Scaling and Wavelet Functions

For $\tau > 0$ and $i = 1,\dots,8$, the family $\{\mathbf{\Phi}_\tau^{(i)}\}_{\tau>0}$ of kernels $\mathbf{\Phi}_\tau^{(i)} : \Sigma\times\Sigma \longrightarrow \mathbb{R}^{3\times 3}$ is called a *(Cauchy-Navier)* $\Sigma-$*scaling function of type* i. Moreover, $\mathbf{\Phi}_1^{(i)} : \Sigma\times\Sigma \longrightarrow \mathbb{R}^{3\times 3}$ (i.e. with $\tau = 1$) is called the mother kernel of the (Cauchy-Navier) $\Sigma-$ scaling function of type i. Correspondingly, for $\tau > 0$ and $i = 1,\dots,8$, the family $\{\mathbf{\Psi}_\tau^{(i)}\}_{\tau>0}$ of kernels $\mathbf{\Psi}_\tau^{(i)} : \Sigma\times\Sigma \longrightarrow \mathbb{R}^{3\times 3}$ given by

$$\mathbf{\Psi}_\tau^{(i)}(x,y) = -(\alpha(\tau))^{-1}\frac{d}{d\tau}\mathbf{\Phi}_\tau^{(i)}(x,y), \quad x,y\in\Sigma, \tag{5.2}$$

is called a *(Cauchy-Navier)* $\Sigma-$*wavelet function of type* i. Moreover, $\mathbf{\Psi}_1^{(i)}$: $\Sigma\times\Sigma \longrightarrow \mathbb{R}^{3\times 3}$ defines the so-called mother kernel of the (Cauchy-Navier) $\Sigma-$wavelet of type i. It should be noted that (5.2) is called the (scale continuous) $\Sigma-$scaling equation. The factor $\alpha(\tau)^{-1}$ can be chosen in an appropriate way. For simplicity, throughout the remainder of this article, we will use $\alpha(\tau) = \tau^{-1}$.

Definition 5.1 *Let $\{\mathbf{\Phi}_\tau^{(i)}\}_{\tau>0}$ be a $\Sigma-$scaling function of type i. Then the associated $\Sigma-$wavelet transform of type i $(WT)^{(i)}:\ell^2(\Sigma)\longrightarrow \ell^2((0,\infty),\Sigma)$ of a function $f\in\ell^2(\Sigma)$ is defined by*

$$(WT)^{(i)}(f)(\tau,x)=\int_\Sigma \mathbf{\Psi}_\tau^{(i)}(x,y)f(y)\ d\omega(y). \tag{5.3}$$

According to our definitions we obtain explicit formulae for the $\Sigma-$scaling and $\Sigma-$wavelet functions by means of the ordinary differential equation

$$\mathbf{\Psi}_\tau^{(i)}(x,y)=-\tau\frac{d}{d\tau}\mathbf{\Phi}_\tau^{(i)}(x,y),\quad i=1,\ldots,8. \tag{5.4}$$

In particular, $\mathbf{\Phi}_\tau^{(5)}(\cdot,\cdot)$ reads as follows.

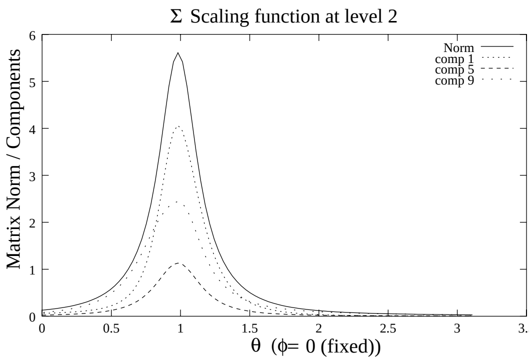
Lemma 5.1 *For $x,y\in\Sigma$,*

$$\begin{aligned}
&\mathbf{\Phi}_\tau^{(5)}(x,y)=\\
&\frac{1}{4\pi}\Bigg\{\left(\frac{2\mu}{\lambda+3\mu}\right)\big((x-y)\cdot\nu(y)\big)\left(\frac{1}{|x_\tau-y|^3}-\frac{1}{|x_{-\tau}-y|^3}\right)\mathbf{i}_3\\
&+\left(\frac{2\mu}{\lambda+3\mu}\right)\big(\nu(x)\cdot\nu(y)\big)\left(\frac{\tau}{|x_\tau-y|^3}+\frac{\tau}{|x_{-\tau}-y|^3}\right)\mathbf{i}_3\\
&+\left(\frac{3(\lambda+\mu)}{\lambda+3\mu}\right)\big((x-y)\cdot\nu(y)(x-y)\otimes(x-y)\big)\left(\frac{1}{|x_\tau-y|^5}-\frac{1}{|x_{-\tau}-y|^5}\right)\\
&+\left(\frac{3(\lambda+\mu)}{\lambda+3\mu}\right)\big((x-y)\cdot\nu(y)(\nu(x)\otimes\nu(y)\big)\left(\frac{\tau^2}{|x_\tau-y|^5}-\frac{\tau^2}{|x_{-\tau}-y|^5}\right)\\
&+\left(\frac{3(\lambda+\mu)}{\lambda+3\mu}\right)\big(\nu(x)\cdot\nu(y)((x-y)\otimes\nu(x)+\\
&\qquad\qquad \nu(x)\otimes(x-y)\big)\left(\frac{\tau^2}{|x_\tau-y|^5}-\frac{\tau^2}{|x_{-\tau}-y|^5}\right)\\
&+\left(\frac{3(\lambda+\mu)}{\lambda+3\mu}\right)\big((x-y)\cdot\nu(y)((x-y)\otimes\nu(x)+\\
&\qquad\qquad \nu(x)\otimes(x-y)\big)\left(\frac{\tau}{|x_\tau-y|^5}+\frac{\tau}{|x_{-\tau}-y|^5}\right)\\
&+\left(\frac{3(\lambda+\mu)}{\lambda+3\mu}\right)\big(\nu(x)\cdot\nu(y)(x-y)\otimes(x-y)\big)\left(\frac{\tau}{|x_\tau-y|^5}+\frac{\tau}{|x_{-\tau}-y|^5}\right)\\
&+\left(\frac{3(\lambda+\mu)}{\lambda+3\mu}\right)\big(\nu(x)\cdot\nu(y)\nu(x)\otimes\nu(y)\big)\left(\frac{\tau^3}{|x_\tau-y|^5}+\frac{\tau^3}{|x_{-\tau}-y|^5}\right)\Bigg\},
\end{aligned} \tag{5.5}$$

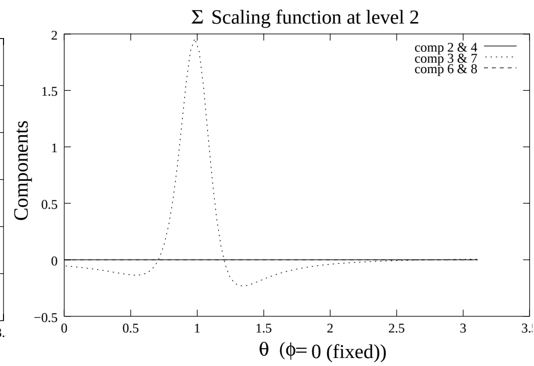
where $\mathbf{i}_3$ denotes the 3×3 identity matrix.

5.2 Scale Continuous Reconstruction Formula

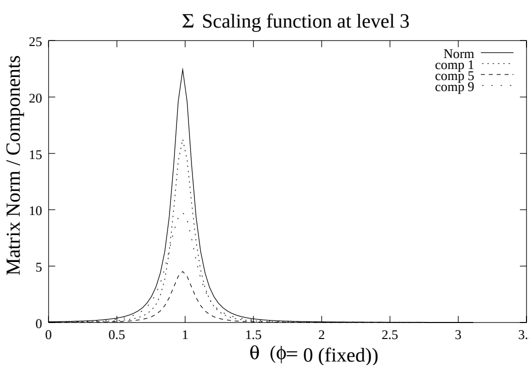
It is not difficult to show that the $\Sigma-$wavelet functions $\mathbf{\Psi}_\tau^{(i)}$, $i=1,\ldots,8$, behave (componentwise) like $O(\tau^{-1})$, hence the convergence of the integrals occurring in the next theorem is guaranteed.



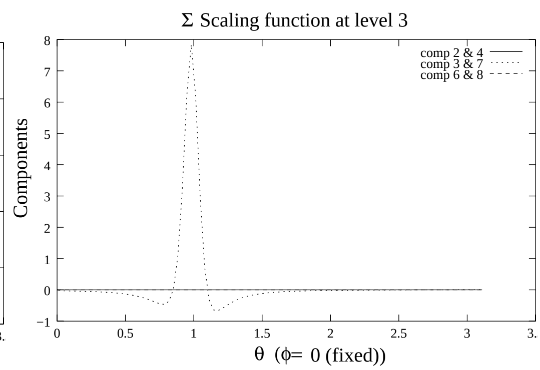
(a) Diagonal/Norm



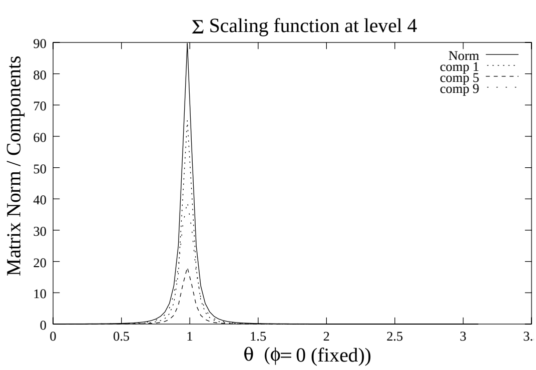
(b) Non-diagonal



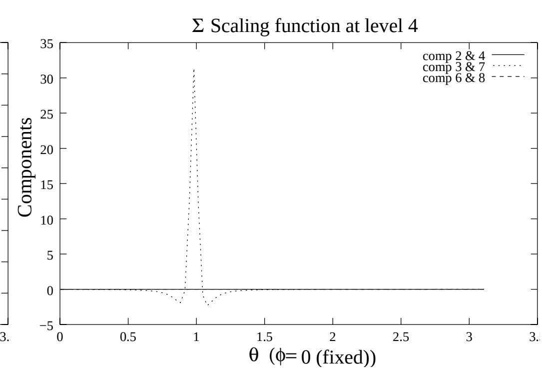
(c) Diagonal/Norm



(d) Non-diagonal



(e) Diagonal/Norm



(f) Non-diagonal

Figure 2: Matrix norm, diagonal and non-diagonal components of the $\Sigma-$scaling function $\mathbf{\Phi}_\tau^{(5)}(\cdot,\cdot)$ for $\tau = 2^{-j}, \quad j = 2, 3, 4$

Theorem 5.2 *Let* $\{\mathbf{\Phi}_\tau^{(i)}\}_{\tau>0}$ *be a* $\Sigma-$*scaling function of type* i. *Suppose that* f *is of class* $\ell^2(\Sigma)$. *Then the reconstruction formula*

$$\int_0^\infty (WT)^{(i)}(f)(\tau,x)\frac{d\tau}{\tau} = \begin{cases} \int_\Sigma \Gamma(x-y)f(y)\ d\omega(y) & , i=1 \\ 0 & , i=4 \\ f(x) & , i=2,3,5,6 \\ \int_\Sigma \big(\frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}\big)\Lambda(x-y)f(y)\ d\omega(y) & , i=7 \\ \int_\Sigma \big(\frac{\partial}{\partial\nu(x)}\frac{1}{|x-y|}\big)\Lambda(x-y)f(y)\ d\omega(y) & , i=8 \end{cases} \tag{5.6}$$

holds in the sense of $\|\cdot\|_{\ell^2(\Sigma)}$.

Proof. Let $R>0$ be arbitrary. Taking the identity

$$\mathbf{\Phi}_R^{(i)}(x,y) = \int_R^\infty \mathbf{\Psi}_\tau^{(i)}(x,y)\ \frac{d\tau}{\tau}, \qquad x,y\in\Sigma \tag{5.7}$$

we obtain

$$\begin{aligned}\int_R^\infty (WT)^{(i)}(f)(\tau,x)\ \frac{d\tau}{\tau} &= \int_R^\infty \int_\Sigma \mathbf{\Psi}_\tau^{(i)}(x,y)f(y)\ d\omega(y)\frac{d\tau}{\tau} \\ &= \int_\Sigma \left(\int_R^\infty \mathbf{\Psi}_\tau^{(i)}(x,y)\frac{d\tau}{\tau}\right) f(y)\ d\omega(y) \\ &= \int_\Sigma \mathbf{\Phi}_R^{(i)}(x,y)f(y)\ d\omega(y).\end{aligned}$$

Letting R tend to 0 we get the desired result. □

Next we are interested in formulating the wavelet transform and the reconstruction formula by using the so-called 'shift' and 'dilation' operators. We define the $x-$shift and $\tau-$dilation of a mother kernel, respectively, by

$$T_x : \mathbf{\Psi}_1^{(i)} \longmapsto T_x\mathbf{\Psi}_1^{(i)} = \mathbf{\Psi}_{1,x}^{(i)}(\cdot) = \mathbf{\Psi}_1^{(i)}(x,\cdot), \quad x\in\Sigma, \tag{5.8}$$

$$D_\tau : \mathbf{\Psi}_1^{(i)} \longmapsto D_\tau\mathbf{\Psi}_1^{(i)} = \mathbf{\Psi}_\tau^{(i)}, \quad \tau>0. \tag{5.9}$$

Consequently, we obtain by composition

$$T_x D_\tau \mathbf{\Psi}_1^{(i)} = T_x\mathbf{\Psi}_\tau^{(i)} = \mathbf{\Psi}_{\tau,x}^{(i)}(\cdot) = \mathbf{\Psi}_\tau^{(i)}(x,\cdot), \quad i=1,\dots,8. \tag{5.10}$$

We can thus state the following theorem.

Theorem 5.3 *For* $x\in\Sigma$ *and* $f\in\ell^2(\Sigma)$

$$\lim_{\substack{R\to0\\R>0}} \int_\Sigma \mathbf{\Phi}_R^{(i)}(x,y)f(y)\ d\omega(y) = \begin{cases} \int_\Sigma \Gamma(x-y)f(y)\ d\omega(y) & , i=1 \\ 0 & , i=4 \\ f(x) & , i=2,3,5,6 \\ \int_\Sigma \big(\frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}\big)\Lambda(x-y)f(y)\ d\omega(y) & , i=7 \\ \int_\Sigma \big(\frac{\partial}{\partial\nu(x)}\frac{1}{|x-y|}\big)\Lambda(x-y)f(y)\ d\omega(y) & , i=8 \end{cases} \tag{5.11}$$

and

$$\int_0^\infty \int_\Sigma \mathbf{\Psi}_\tau^{(i)}(x,y) f(y)\ d\omega(y) \frac{d\tau}{\tau} = \begin{cases} \int_\Sigma \Gamma(x-y) f(y)\ d\omega(y) & , i=1 \\ 0 & , i=4 \\ f(x) & , i=2,3,5,6 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(y)} \frac{1}{|x-y|}\right) \Lambda(x-y) f(y)\ d\omega(y) & , i=7 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(x)} \frac{1}{|x-y|}\right) \Lambda(x-y) f(y)\ d\omega(y) & , i=8 \end{cases} \tag{5.12}$$

hold in the sense of $\|\cdot\|_{\ell^2(\Sigma)}$.

5.3 Scale Discrete Reconstruction Formula

Let $(\tau_j)_{j\in\mathbb{Z}}$ denote a sequence of numbers satisfying the properties

$$\lim_{\tau\to\infty} \tau_j = 0, \qquad \lim_{\tau\to-\infty} \tau_j = \infty, \tag{5.13}$$

(for example, $\tau_j = 2^{-j},\ \ j\in\mathbb{Z}$). Given a $\Sigma-$scaling function $\{\mathbf{\Phi}_\tau^{(i)}\}_{\tau>0}$ of type i, we define the (scale) discretized $\Sigma-$scaling function of type i by $\{\mathbf{\Phi}_{\tau_j}^{(i)}\}_{j\in\mathbb{Z}}$. Then we are led to the following result.

THEOREM 5.4 *For* $f\in\ell^2(\Sigma)$*, the limit*

$$\lim_{j\to\infty} \int_\Sigma \mathbf{\Phi}_{\tau_j}^{(i)}(x,y) f(y)\ d\omega(y) = \begin{cases} \int_\Sigma \Gamma(x-y) f(y)\ d\omega(y) & , i=1 \\ 0 & , i=4 \\ f(x) & , i=2,3,5,6 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(y)} \frac{1}{|x-y|}\right) \Lambda(x-y) f(y)\ d\omega(y) & , i=7 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(x)} \frac{1}{|x-y|}\right) \Lambda(x-y) f(y)\ d\omega(y) & , i=8 \end{cases} \tag{5.14}$$

holds in the $\|\cdot\|_{\ell^2(\Sigma)}-$*sense.*

DEFINITION 5.2 *Let* $\{\mathbf{\Phi}_{\tau_j}^{(i)}\}_{j\in\mathbb{Z}}$ *be a discretized* $\Sigma-$*scaling function of type* i*. Then the (scale) discretized* $\Sigma-$*wavelet function of type* i *is defined by*

$$\mathbf{\Psi}_{\tau_j}^{(i)}(x,y) = \int_{\tau_{j+1}}^{\tau_j} \mathbf{\Psi}_\tau^{(i)}(x,y) \frac{d\tau}{\tau}, \qquad j\in\mathbb{Z},\ \ x,y\in\Sigma,\ \ i=1,\ldots,8. \tag{5.15}$$

With the definition of $\mathbf{\Psi}_\tau^{(i)}$ we immediately obtain that

$$\mathbf{\Psi}_{\tau_j}^{(i)}(x,y) = -\int_{\tau_{j+1}}^{\tau_j} \tau \frac{d}{d\tau} \mathbf{\Phi}_\tau^{(i)}(x,y) \frac{d\tau}{\tau} = \mathbf{\Phi}_{\tau_{j+1}}^{(i)} - \mathbf{\Phi}_{\tau_j}^{(i)}, \quad x,y\in\Sigma. \tag{5.16}$$

The equation (5.16) is called the (scale) discretized $\Sigma-$scaling equation of type i. It should be remarked that, with a suitably chosen τ_j, formula (5.16) can easily be used to formulate the $\Sigma-$wavelet function in a discrete form once the $\Sigma-$scaling function has been given. To be more specific, assume that f is a vector field of class $\ell^2(\Sigma)$ and consider the discretized $\Sigma-$scaling equation of type i. Then, for $J \in \mathbb{Z}$ and $N \in \mathbb{N}$, we have

$$\int_\Sigma \mathbf{\Phi}^{(i)}_{\tau_{J+N}}(x,y)f(y)\ d\omega(y) = \int_\Sigma \mathbf{\Phi}^{(i)}_{\tau_J}(x,y)f(y)\ d\omega(y) + \sum_{j=J}^{J+N-1} \int_\Sigma \mathbf{\Psi}^{(i)}_{\tau_j}(x,y)f(y)\ d\omega(y), \quad x \in \Sigma. \tag{5.17}$$

By taking into account the property (5.15) together with Theorem 5.3 we find the following theorem.

THEOREM 5.5 *Let $\{\mathbf{\Phi}^{(i)}_{\tau_j}\}_{j\in\mathbb{Z}}$ be a (scale) discretized $\Sigma-$ scaling function of type i. Then the multiscale representation of a function $f \in \ell^2(\Sigma)$*

$$\int_\Sigma \mathbf{\Phi}^{(i)}_{\tau_J}(x,y)f(y)d\omega(y) + \sum_{j=J}^{\infty} \int_\Sigma \mathbf{\Psi}^{(i)}_{\tau_j}(x,y)f(y)\ d\omega(y)$$

$$= \begin{cases} \int_\Sigma \Gamma(x-y)f(y)\ d\omega(y) & , i=1 \\ 0 & , i=4 \\ f(x) & , i=2,3,5,6 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}\right)\Lambda(x-y)f(y)\ d\omega(y) & , i=7 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x-y|}\right)\Lambda(x-y)f(y)\ d\omega(y) & , i=8 \end{cases} \tag{5.18}$$

holds for all $J \in \mathbb{Z}$ (in the sense of $\|\cdot\|_{\ell^2(\Sigma)}$).

Now defining the *so-called (scale) discretized $\Sigma-$wavelet transform of type i* by

$$(WT)^i(f)(\tau_j;x) = \int_\Sigma \mathbf{\Psi}^{(i)}_{\tau_j;x}(y)f(y)\ d\omega(y), \quad x \in \Sigma, \tag{5.19}$$

we are able to derive the following corollary.

COROLLARY 5.1 *Let $\{\mathbf{\Phi}^{(i)}_{\tau_j}\}_{j\in\mathbb{Z}}$ be a (scale) discretized $\Sigma-$scaling function of type i. Then, for all $f \in \ell^2(\Sigma)$,*

$$\sum_{j=-\infty}^{\infty} (WT)^i(f)(\tau_j;x) = \begin{cases} \int_\Sigma \Gamma(x-y)f(y)\ d\omega(y) & , i=1 \\ 0 & , i=4 \\ f(x) & , i=2,3,5,6 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|}\right)\Lambda(x-y)f(y)\ d\omega(y) & , i=7 \\ \int_\Sigma \left(\frac{\partial}{\partial\nu(x)}\frac{1}{|x-y|}\right)\Lambda(x-y)f(y)\ d\omega(y) & , i=8 \end{cases} \tag{5.20}$$

holds in the $\|\cdot\|_{\ell^2(\Sigma)}-$sense.

5.4 Scale and Detail Spaces

As in the spherical theory of wavelets (see [6],[7],[9] for more details), the operators $P^{(i)}_{\tau_j}$, $R^{(i)}_{\tau_j}$ defined by

$$P^{(i)}_{\tau_j} = \int_\Sigma \mathbf{\Phi}^{(i)}_{\tau_j}(\cdot,y)f(y)\ d\omega(y), \quad f \in \ell^2(\Sigma), \tag{5.21}$$

$$R^{(i)}_{\tau_j} = \int_\Sigma \mathbf{\Psi}^{(i)}_{\tau_j}(x\cdot,y)f(y)\ d\omega(y), \quad f \in \ell^2(\Sigma) \tag{5.22}$$

may be referred as *low pass filter* and *band pass filter*, respectively. The scale spaces $V_{\tau_j}^{(i)}$ and details spaces $W_{\tau_j}^{(i)}$ of type i are defined by

$$V_{\tau_j}^{(i)} = \{P_{\tau_j}^{(i)}(f) \mid f \in \ell^2(\Sigma)\}, \tag{5.23}$$
$$W_{\tau_j}^{(i)} = \{R_{\tau_j}^{(i)}(f) \mid f \in \ell^2(\Sigma)\}, \tag{5.24}$$

respectively. It is clear that

$$P_{\tau_{J+1}}^{(i)}(f) = P_{\tau_J}^{(i)}(f) + R_{\tau_J}^{(i)}, \quad J \in \mathbb{Z}. \tag{5.25}$$

Consequently,

$$V_{\tau_{J+1}}^{(i)} = V_{\tau_J}^{(i)} + W_{\tau_J}^{(i)}, \tag{5.26}$$
$$V_{\tau_{J+1}}^{(i)} = V_{\tau_{J_0}}^{(i)} + \sum_{j=0}^{J} W_{\tau_j}^{(i)}. \tag{5.27}$$

However, it should be remarked that the sum (5.26), in general, is neither direct nor orthogonal. Furthermore,

$$\overline{\bigcup_{j=-\infty}^{\infty}}^{\|\cdot\|_{\ell^2(\Sigma)}} V_{\tau_j}^{(i)} = \ell^2(\Sigma), \quad i = 2, 3, 5, 6, \tag{5.28}$$
$$\overline{\bigcup_{j=-\infty}^{\infty}}^{\|\cdot\|_{\ell^2(\Sigma)}} V_{\tau_j}^{(1)} = P(0)(\ell^2(\Sigma)), \tag{5.29}$$
$$\overline{\bigcup_{j=-\infty}^{\infty}}^{\|\cdot\|_{\ell^2(\Sigma)}} V_{\tau_j}^{(7)} = P_N(0)(\ell^2(\Sigma)), \tag{5.30}$$
$$\overline{\bigcup_{j=-\infty}^{\infty}}^{\|\cdot\|_{\ell^2(\Sigma)}} V_{\tau_j}^{(8)} = N_P(0)(\ell^2(\Sigma)). \tag{5.31}$$

The notion of a multiresolution analysis is prescribed by the following definition.

DEFINITION 5.3 *A family of subspaces* $\{V_\tau^{(i)}\}_{\tau \in (0,\infty)} \subset \ell^2(\Sigma), \quad i \in \{1, \ldots, 8\}$, *is called multiresolution analysis if it satisfies the following properties:*

(i) $\{0\} \subset V_\tau^{(i)} \subset V_{\tau'}^{(i)} \subset \ell^2(\Sigma)$ *for* $0 \leq \tau' \leq \tau \leq \infty$,

(ii) $\{\lim_{\tau\to\infty}(\int_\Sigma \mathbf{\Phi}_\tau^{(i)}(x,y) f(y)\, d\omega(y)) | f \in \ell^2(\Sigma)\} = \{0\}$,

(iii) $\overline{\{f \in \ell^2(\Sigma) | f \in V_\tau^{(i)}(\Sigma) \text{ for some } \tau \in (0,\infty)\}}^{\|\cdot\|_{\ell^2(\Sigma)}} = \ell^2(\Sigma)$.

The framework of a multiresolution analysis can be provided by using the so-called $\ell^2 -$ *closure* properties of the Cauchy-Navier vector fields (see e.g., [5]) and taking the Fourier series representation of the scaling function. To be more specific, the multiresolution can be characterized as follows (cf. [8],[10]).

THEOREM 5.6 *Let Σ be a regular surface. If the $\Sigma-$scaling function $\mathbf{\Phi}_\tau^{(i)}$ satisfies the property*

$$\|\mathbf{\Phi}_\tau^{(i)}\|_{\ell^2(\Sigma\times\Sigma)} \leq \|\mathbf{\Phi}_{\tau'}^{(i)}\|_{\ell^2(\Sigma\times\Sigma)}, \quad \tau' < \tau, \quad i = 2,3,5,6, \tag{5.32}$$

then the scale spaces $V_\tau^{(i)}(\Sigma)$ satisfy the inclusion

$$V_\tau^{(i)}(\Sigma) \subset V_{\tau'}^{(i)}(\Sigma). \tag{5.33}$$

5.5 A Tree Algorithm

In what follows, we present a particular scheme which utilizes the computational process of the reconstruction and decomposition of the wavelet approximation. This is known as *a pyramid scheme* i.e. a tree algorithm that provides a recursive process to compute the integrals $P_{\tau_j}^{(i)}(f)$ and $R_{\tau_j}^{(i)}(f)$ on different levels, starting from an initial approximation of a given $f \in \ell^2(\Sigma)$ without falling upon the original vector field f in each step.

Let $\{(y_k^{N_j}, w_k^{N_j})\}$ be an appropriate integration rule on Σ with given nodes $y_k^{N_j} \in \Sigma$ and weights $w_k^{N_j} \in \mathbb{R}$. Assume that for sufficiently large $J \in \mathbb{N}$ there exist coefficient vectors $a_k^{N_J} \in \mathbb{R}^3,\ k = 1,\ldots,N_J$, such that

$$P_{\tau_J}^{(i)}(f)(x) = \sum_{k=1}^{N_J} \mathbf{\Phi}_{\tau_J}^{(i)}(x, y_k) a_k^{N_J}, \quad i = 1,\ldots,8,\ x \in \Sigma. \tag{5.34}$$

Now we want to introduce an algorithm to obtain the coefficients $a^{N_j} = (a_1^{N_j},\ldots,a_{N_j}^{N_j}) \in \mathbb{R}^3 \times \mathbb{R}^{N_j},\ \ j = J_0,\ldots,J$, such that

(a) the vector a^{N_j} is obtainable from $a^{N_{j+1}},\ \ j = J_0,\ldots,J-1$

(b) the expressions $P_{\tau_j}^{(i)}(f)(x),\ \ R_{\tau_{j-1}}^{(i)}(f)(x)$ can be written as

$$P_{\tau_j}^{(i)}(f)(x) = \sum_{k=1}^{N_j} \mathbf{\Phi}_{\tau_j}^{(i)}(x, y_k^{N_j}) a_k^{N_j}, \quad j = J_0,\ldots,J, \tag{5.35}$$

$$R_{\tau_{j-1}}^{(i)}(f)(x) = \sum_{k=1}^{N_{j-1}} \mathbf{\Psi}_{\tau_{j-1}}^{(i)}(x, y_k^{N_j}) a_k^{N_j-1}, \quad j = J_0+1,\ldots,J. \tag{5.36}$$

For this scheme, we use appropriately chosen approximate integration rules such that $P_{\tau_j}^{(i)}(f)$ and $R_{\tau_j}^{(i)}(f)$ can be represented by

$$P_{\tau_j}^{(i)}(f)(x) \approx \sum_{k=1}^{N_j} w_k^{N_j} \mathbf{\Phi}_{\tau_j}^{(i)}(x, y_k^{N_j}) f(y_k^{N_j}), \tag{5.37}$$

$$R_{\tau_{j-1}}^{(i)}(f)(x) \approx \sum_{k=1}^{N_{j-1}} w_k^{N_{j-1}} \mathbf{\Psi}_{\tau_{j-1}}^{(i)}(x, y_k^{N_j}) f(y_k^{N_{j-1}}), \tag{5.38}$$

where $\{(y_k^{N_j}, w_k^{N_j}) \in \Sigma \times \mathbb{R}\}$ are the prescribed integration points and nodes and $' \approx'$ means that the remainder terms can be neglected.

The tree algorithm (pyramid scheme) can be divided into two parts namely the initial step and the pyramid step. For the *initial step* we consider (5.34) that $J \in \mathbb{N}$ sufficiently large and thus we see that

$$a_k^{N_J} = w_k^{N_J} f(y_k^{N_J}), \quad k = 1, \ldots, N_J \; . \tag{5.39}$$

The aim of the pyramid step is to construct a^{N_j} from $a^{N_{j+1}}$ by recursion. At this point, it is essential to assume that there exist (tensor) kernel functions $\Xi_j^{(i)} : \Sigma \times \Sigma \longrightarrow \mathbb{R}^{3\times 3}$ such that

$$\mathbf{\Phi}_{\tau_j}^{(i)}(x,y) \approx \int_\Sigma \mathbf{\Phi}_{\tau_j}^{(i)}(x,z) \Xi_j^{(i)}(z,y) \; d\omega(z), \tag{5.40}$$

$$\Xi_j^{(i)}(x,y) \approx \int_\Sigma \Xi_j^{(i)}(x,z) \Xi_{j+1}^{(i)}(z,y) \; d\omega(z) \tag{5.41}$$

for $j = J_0, \ldots, J$. A reasonable choice for $\Xi_j^{(i)}$ is

$$\Xi_j^{(i)} = \mathbf{\Phi}_{\tau_{J+L}}^{(i)}, \quad j = J_0, \ldots, J; \; i \in \{2,3,5,6\}$$

with $L \in \mathbb{N}$ suitably large. By the aforementioned approximate integration rules we obtain

$$\begin{aligned} \int_\Sigma \mathbf{\Phi}_{\tau_j}^{(i)}(x,y) f(y) \; d\omega(y) &\approx \int_\Sigma \left[\int_\Sigma \mathbf{\Phi}_{\tau_j}^{(i)}(x,z) \Xi_j^{(i)}(z,y) \; d\omega(z) \right] f(y) \; d\omega(y) \\ &= \int_\Sigma \mathbf{\Phi}_{\tau_j}^{(i)}(x,z) \left[\int_\Sigma \Xi_j^{(i)}(z,y) f(y) d\omega(y) \right] \; d\omega(z) \\ &\approx \sum_{k=1}^{N^j} \mathbf{\Phi}_{\tau_j}^{(i)}(x, y_k^{N_j}) a_k^{N_j}, \end{aligned} \tag{5.42}$$

where

$$a_k^{N_j} = w_k^{N_j} \int_\Sigma \Xi_j^{(i)}(y_k^{N_j}, y) f(y) \; d\omega(y), \quad j = J_0, \ldots, J. \tag{5.43}$$

Hence, in connection with (5.34), we find

$$\begin{aligned} a_k^{N_j} &= w_k^{N_j} \int_\Sigma \Xi_j^{(i)}(y_k^{N_j}, y) f(y) \; d\omega(y) \\ &\approx w_k^{N_j} \int_\Sigma \left[\int_\Sigma \Xi_j^{(i)}(y_k^{N_j}, x) \Xi_{j+1}^{(i)}(x,y) \; d\omega(y) \right] f(y) \; d\omega(y) \\ &\approx w_k^{N_j} \int_\Sigma \sum_{l=1}^{N_{j+1}} \Xi_j^{(i)}(y_k^{N_j}, y_l^{N_{j+1}}) \Xi_{j+1}^{(i)}(y_l^{N_{j+1}}, y) f(y) \; d\omega(y) \\ &= w_k^{N_j} \sum_{l=1}^{N_{j+1}} \Xi_j^{(i)}(y_k^{N_j}, y_l^{N_{j+1}}) a_l^{N_{j+1}} \end{aligned} \tag{5.44}$$

for $j = J-1, \ldots, J_0$ and $k = 1, \ldots, N_j$.
We see that once the coefficients a^{N_j} are calculated, the coefficients $a^{N_{j-1}}$ are obtained by (5.44). Starting from an initial value a^{N_J} all the coefficient vectors can be calculated recursively in. Note that the coefficients a^{N_J} in the initial step do not depend on the choice of $\Xi^{(i)}_J = \mathbf{\Phi}^{(i)}_{\tau_{J+L}}$. Furthermore, the functions $\Xi^{(i)}_j$, $j = J_0, \ldots, J$, can be chosen independently of the $\Sigma-$scaling function used in the integrals $P^{(i)}_{\tau_j}(f)$ and $R^{(i)}_{\tau_j}(f)$.

Finally, with the proposed pyramid scheme prescribed by (5.39) and (5.44), the decomposition and reconstruction process of the wavelet approximation can be illustrated briefly as follows:

$$\begin{array}{ccccccccc} f & \rightarrow & a^{N_J} & \rightarrow & a^{N_{J-1}} & \rightarrow \cdots \rightarrow & a^{N_{J_0+1}} & \rightarrow & a^{N_{J_0}} \\ & & \downarrow & & \downarrow & & \downarrow & & \swarrow \quad \searrow \\ & & R^{(i)}_{\tau_J}(f) & & R^{(i)}_{\tau_{J-1}}(f) & & R^{(i)}_{\tau_{J_0+1}}(f) & & R^{(i)}_{\tau_{J_0}}(f) \quad P^{(i)}_{\tau_{J_0}}(f) \end{array}$$

(*decomposition scheme*)

$$\begin{array}{ccccccc} a^{N_{J_0}} & & a^{N_{J_0+1}} & & a^{N_{J-1}} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ R^{(i)}_{\tau_{J_0}}(f) & & R^{(i)}_{\tau_{J_0+1}}(f) & & R^{(i)}_{\tau_{J-1}}(f) & & \\ & \searrow & & \searrow & & \searrow & \\ P^{(i)}_{\tau_{J_0}}(f) & \rightarrow + \rightarrow & P^{(i)}_{\tau_{J_0+1}}(f) & \rightarrow + \cdots + \rightarrow & P^{(i)}_{\tau_{J-1}}(f) & \rightarrow + \rightarrow & P^{(i)}_{\tau_J}(f) \end{array}$$

(*reconstruction scheme*).

5.6 Multiscale Solution of the (Inner) Displacement Boundary-Value Problem

In what follows, we discuss the solution of the (inner) displacement boundary-value problem by means of the wavelet approximation techniques derived in preceding chapters. The existence, uniqueness and the regularity of the solutions of such problems are known from Chapter 4.

For given $f \in c(\Sigma)$ the solution $u \in pot(\overline{\Sigma_{int}})$ with $u|_\Sigma = f$ (more accurately, $u^- = f$) of the (inner) displacement problem can be expressed uniquely by a double layer potential

$$u(x) = \int_\Sigma \Big(\frac{\partial}{\partial\nu(y)} \frac{1}{|x-y|}\Big)\Lambda(x-y)g(y)\ d\omega(y), \qquad g \in \ell^2(\Sigma). \tag{5.45}$$

The corresponding integral equation reads as follows

$$2\pi I - P_N(0))g = -f, \qquad g \in \ell^2(\Sigma). \tag{5.46}$$

More explicitly,

$$2\pi g(x) - \int_\Sigma (\frac{\partial}{\partial\nu(y)}\frac{1}{|x-y|})\Lambda(x-y)g(y)\ d\omega(y) = -f(x),\ \ g\in\ell^2(\Sigma),\ \ x\in\Sigma. \tag{5.47}$$

To approximate u given by the double layer potential in (5.45), we use the concept stated in Theorem 5.1. In accordance with this approach we are able to rewrite (5.47) approximately in the form

$$2\pi g(x) - \int_\Sigma \mathbf{\Phi}^{(7)}_{\tau_L}(x,y)g(y)\ d\omega(y) = -f(x), \quad x\in\Sigma, \tag{5.48}$$

provided that $L\in\mathbb{N}$ is sufficiently large. Once the boundary integral equation (5.46) is solved, the density function g is inserted into (5.45) and, thereby the (approximate) solution u is obtained in Σ_{int}. Like in many cases of boundary integral equations, there is, in general, no straightforward way of constructing the unknown function g. It is, therefore, necessary to apply a suitable approximation method. In this respect we again go back to Theorem 5.1 that enables us to formulate

$$g(y) = \lim_{j\to\infty}\int_\Sigma \mathbf{\Phi}^{(5)}_{\tau_j}(y,z)g(z)\ d\omega(y). \tag{5.49}$$

Using an appropriate numerical integration technique, an approximation of g of level J, denoted by g_J, can be expressed by

$$\begin{aligned} g_J(y) &= \sum_{l=1}^{N_J} w_l^{N_J}\mathbf{\Phi}^{(5)}_{\tau_J}(y,y_l^{N_J})g(y_l^{N_J}) \\ &= \sum_{l=1}^{N_J} \mathbf{\Phi}^{(5)}_{\tau_J}(y,y_l^{N_J})a_l^{N_J}, \end{aligned} \tag{5.50}$$

where $w_l^{N_J},\ \ l=1,\ldots,N_J$ are the integration weights corresponding to the nodal points $y_l^{N_J}\in\Sigma,\ l=1,\ldots,N_J$ and $a_l^{N_J}\in\mathbb{R}^3, l=1,\ldots,N_J$.

The unknowns $a_l^{N_J}\in\mathbb{R}^3, l=1,\ldots,N_J$, are deducible from (5.46) by solving a system of linear equations obtained by a suitable approximation method such as collocation, Galerkin procedure, least square approximation etc. (see for example, [4]). In consequence we are led to the following system of equations for the unknowns $a_l^{N_J}, l=1,\ldots,N_J$,

$$\sum_{l=1}^{N_J}\left(2\pi\mathbf{\Phi}^{(5)}_{\tau_J}(y_m^{N_J},y_l^{N_J}) - \sum_{k=1}^{N_J} w_k^{N_J}\mathbf{\Phi}^{(5)}_{\tau_J}(y_m^{N_J},y_k^{N_J})\mathbf{\Phi}^{(7)}_{\tau_L}(y_k^{N_J},y_l^{N_J})\right)a_l^{N_J} = -f(y_m^{N_J}), \tag{5.51}$$

$m=1,\ldots,N_J$.

However, such a consideration leads to a system of linear equations with a 'full' matrix which seems to require much computational work for the definition of the matrix as well as for the solution. In this context, taking into consideration the localization behavior of the kernel functions, suitable accelerating techniques such as panel clustering, domain decomposition etc. can efficiently be used (see e.g. [11],[13],[14]). However, further modifications of such techniques relevant to this particular consideration will be needed. This is a challenge for future work.

In this respect, a variant of our tree algorithm comes into play: once the starting values (see (5.50)) $a^{N_J} = (a_1^{N_J}, \ldots, a_{N_J}^{N_J})^T \in \mathbb{R}^{3\times N_J}$ are given, the coefficients $a^{N_j} = (a_1^{N_j}, \ldots, a_{N_j}^{N_j})^T \in \mathbb{R}^{3\times N_j}$, $j = J_0, \ldots, J-1$, are obtained by the recursion formula

$$a_k^{N_j} = w_k^{N_j} \sum_{l=1}^{N_{j+1}} \Xi_j^{(i)}(y_k^{N_j}, y_l^{N_{j+1}}) a_l^{N_{j+1}}, \quad k = 1, \ldots, N_j. \tag{5.52}$$

The corresponding approximate integrals are obtained by

$$P_{\tau_j}^{(i)}(g)(x) \approx \sum_{k=1}^{N_j} \mathbf{\Phi}_{\tau_j}^{(5)}(x, y_k^{N_j}) a_k^{N_j}, x \in \Sigma, \; j = J_0, \ldots, J, \tag{5.53}$$

and

$$R_{\tau_{j-1}}^{(i)}(g)(x) \approx \sum_{k=1}^{N_j} \mathbf{\Psi}_{\tau_{j-1}}^{(5)}(x, y_k^{N_j}) a_k^{N_j}, x \in \Sigma, \; j = J_0+1, \ldots, J, \tag{5.54}$$

where

$$R_{\tau_{j-1}}^{(i)}(g)(x) = P_{\tau_j}^{(i)}(g)(x) - P_{\tau_{j-1}}^{(i)}(g)(x). \tag{5.55}$$

Hence, we finally arrive at the following theorem for the (inner) displacement boundary-value problems of the Cauchy-Navier theory.

THEOREM 5.7 *Let Σ be a regular surface. For given $f \in c(\Sigma)$, let u be the potential of class pot$(\overline{\Sigma_{int}})$ with $u^- = f$. Then the function $f_J \in c(\Sigma)$ given by*

$$\begin{aligned} f_J(x) =& 2\pi \sum_{l=1}^{N_{J_0}} \mathbf{\Phi}_{\tau_{J_0}}^{(i)}(x, y_l^{N_{J_0}}) a_l^{N_{J_0}} + \sum_{j=J_0}^{J-1} 2\pi \sum_{l=1}^{N_j} \mathbf{\Psi}_{\tau_j}^{(i)}(x, y_l^{N_j}) a_l^{N_j} \\ &- \sum_{l=1}^{N_{J_0}} \left(\int_\Sigma \mathbf{\Phi}_{\tau_L}^{(7)}(x,y) \mathbf{\Phi}_{\tau_{J_0}}^{(i)}(y, y_l^{N_{J_0}}) \; d\omega(y) \right) a_l^{N_{J_0}} \\ &- \sum_{j=J_0}^{J-1} \sum_{l=1}^{N_j} \left(\int_\Sigma \mathbf{\Phi}_{\tau_L}^{(7)}(x,y) \mathbf{\Psi}_{\tau_j}^{(i)}(y, y_l^{N_j}) \; d\omega(y) \right) a_l^{N_j}, \end{aligned} \tag{5.56}$$

$x \in \Sigma$, represents a $J-$scale approximation of $f \in c(\Sigma)$, in the sense of $\|\cdot\|_{\ell^2(\Sigma)}$, where $i = 2, 3, 5, 6$ and $L \in \mathbb{N}$ is sufficiently large. Furthermore,

$$\begin{aligned} u_J = \sum_{l=1}^{N_{J_0}} \int_\Sigma \mathbf{\Phi}_{\tau_L}^{(7)}(\cdot, y) \mathbf{\Phi}_{\tau_{J_0}}^{(i)}(y, y_l^{N_{J_0}}) \; d\omega(y) a_l^{N_{J_0}} \\ + \sum_{j=J_0}^{J-1} \sum_{l=1}^{N_J} \int_\Sigma \mathbf{\Phi}_{\tau_L}^{(7)}(\cdot, y) \mathbf{\Psi}_{\tau_J}^{(i)}(y, y_l^{N_J}) \; d\omega(y) a_l^{N_j} \end{aligned} \tag{5.57}$$

represents a $J-$scale approximation of u in the sense of $\|\cdot\|_{c(\Sigma)}$ for every $\mathcal{K} \subset \Sigma_{int}$ with $(\mathcal{K}, \Sigma) > 0$.

In other words, locally uniform approximation on proper subsets of Σ_{int} is established by means of Cauchy-Navier wavelets.

6 Numerical Examples

Next, we present some sample examples for the geoscientifically important case of a sphere (i.e. $\Sigma = \Omega$).

6.1 Spherical Approximation of Vector Fields by Layer Potentials

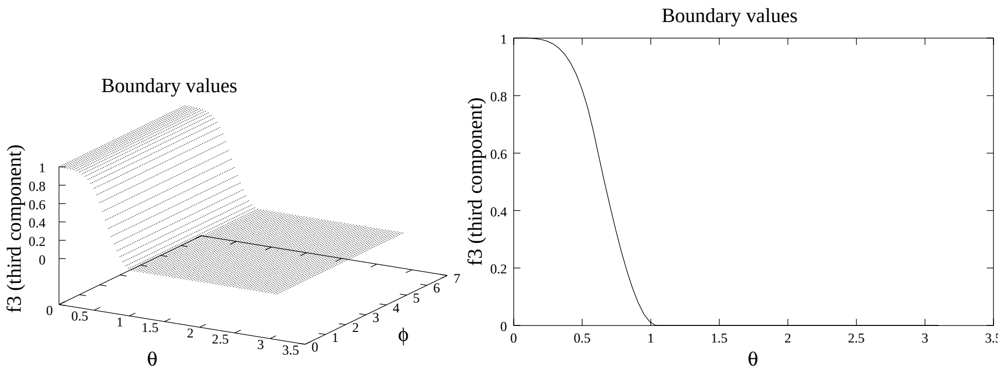
For this purpose, we first consider the vector field $f : \Omega \longrightarrow \mathbb{R}^3$ given by

$$f(x) = \begin{cases} 0\epsilon^3 & , -1 \leq \xi \cdot \epsilon^3 \leq h \\ \frac{3}{2}\left(\frac{h-\xi\cdot\epsilon^3}{h-1}\right)^2 \epsilon^3 & , h \leq \xi \cdot \epsilon^3 \leq \frac{2+h}{3} \\ \left(1 - 3\left(\frac{\xi\cdot\epsilon^3 - 1}{h-1}\right)^2\right)\epsilon^3 & , \frac{2+h}{3} \leq \xi \cdot \epsilon^3 \leq 1 \end{cases} \tag{6.1}$$

for $x \in \Omega$, $\xi = x/|x|$, $h = 1/2$.

The third component of the boundary function f is illustrated in Figure 3.

We are particularly interested in approximating (the third component) of the vector function f by our wavelet approach based on layer potentials (as prescribed above). Figure 4 shows the sectional illustration of approximations of the boundary function corresponding to the $\Sigma-$scaling function $\mathbf{\Phi}^{(5)}_{\tau_j}$ for different levels, i.e. $\tau_j = 2^{-j}$, $j = 1, 2, 3, 4$. Note that, in each evaluation step, a sufficiently large number of equiangular longitude-latitude grid points on the unit sphere have been used in order to avoid oscillations in the approximation process.



(a) (b)

Figure 3: Functional values of f (third component): (a) on a longitude-latitude grid of points on Ω (b) one-dimensional sectional illustration

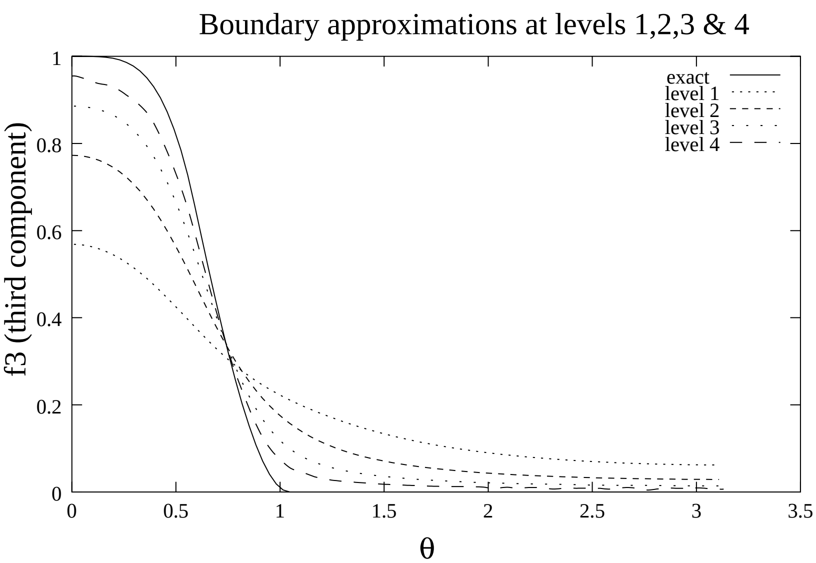


Figure 4: Sectional illustration of the scale approximations of f (third component) associated to the $\Sigma-$scaling function $\mathbf{\Phi}^{(5)}_{\tau_j}$ for $j = 1, 2, 3, 4$

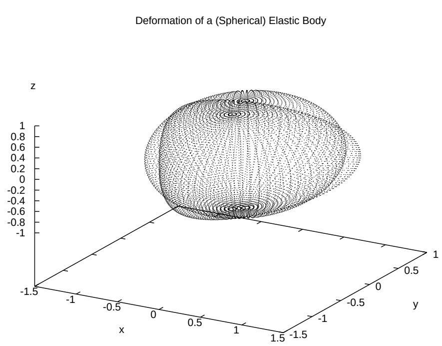
In accordance with Theorem 4.1 it may be expected that our multiscale procedure also provide a good approximation of $u \in pot(\overline{\Omega_{int}})$ with $u|_\Omega = f$ inside Ω. However, we did not make effort to make a more detailed quantitative description.

6.2 Solution of the Displacement Problem Corresponding to a Sphere

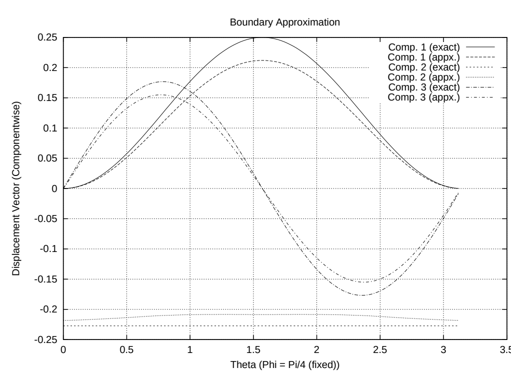
Finally our multiscale method will be considered for a simple inner displacement boundary-value problem of which the boundary is assumed to be the unit sphere Ω and an analytical solution is explicitly known in $\overline{\Omega_{int}}$. In contrast to the example discussed in Chapter 6.1 this one enables us to check the accuracy of the scale approximation in Ω_{int}.
To be more explicit, we consider the solution $u \in pot(\overline{\Omega_{int}})$ corresponding to the boundary field $u|_\Omega = f$ given by

$$f(x) = \frac{x_1 x_2}{2}\epsilon^1 - \frac{\sigma}{2(\sigma+3)}(x_1^2 + x_2^2 + x_3^2)\epsilon^2 + \frac{x_2 x_3}{2}\epsilon^3, \quad \sigma \neq 0. \tag{6.2}$$

We choose, in particular, $\sigma = 2.5$ (i.e. $\lambda = 2$, $\mu = 3$). Figure 5(b) shows the evaluated scale approximations of the boundary displacements using the Ω-scaling function of type 5.



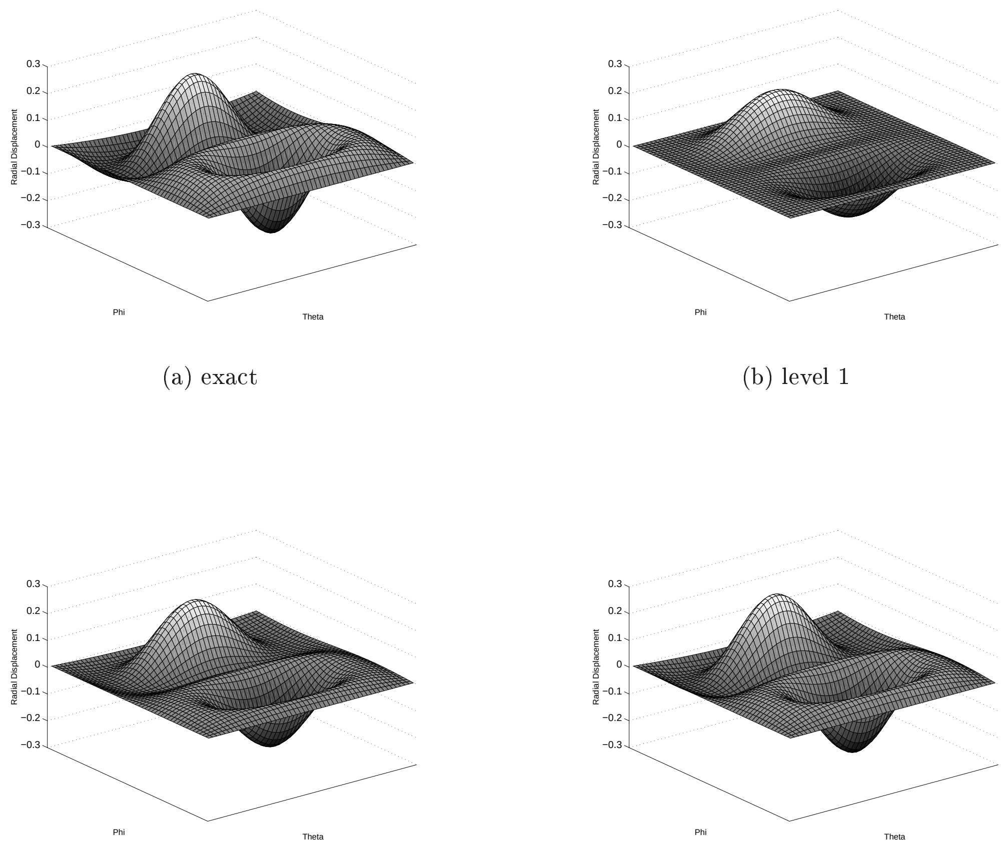
(a)



(b)

Figure 5: (a) Reference (spherical) and deformed configuration (b) Sectional illustration of the approximation of the field $u|_\Omega = f$.

Figure 6 shows that the scale approximations of the radial displacements on Ω become closer to the exact solution for increasing orders of scales (and simultaneously chosen increasing numbers of nodal points on Ω).



(a) exact (b) level 1

(c) level 2 (d) level 3

Figure 6: Exact and the approximations of radial displacements for the levels $1, 2$ and 3.

7 Concluding Remarks

In this paper we introduced a multiscale method for solving the inner displacement problem using Cauchy-Navier wavelets. The method is particularly suitable for the deformation analysis corresponding to geoscientifically relevant boundaries (such as sphere, ellipsoid, actual earth's surface etc) that involve efficient rules of numerical integration. The principal idea of the Cauchy-Navier wavelets is based on the classical limit and jump relations of elastostatics. In conclusion, this paper can be viewed as a first attempt to 'short-wavelength modelling', i.e. high resolution of the fine structure of displacement fields. The method is restricted to the homogeneous and isotropic case of linear elasticity, hence, it needs to be formulated under more complex (geo-)physical assumptions. Nevertheless, we believe that the 'zoom-in' procedure as presented here will become a flexible and useful technique of microstructural analysis of elastic fields (such as the earth's displacement field) on (geoscientifically relevant) regular boundaries.

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